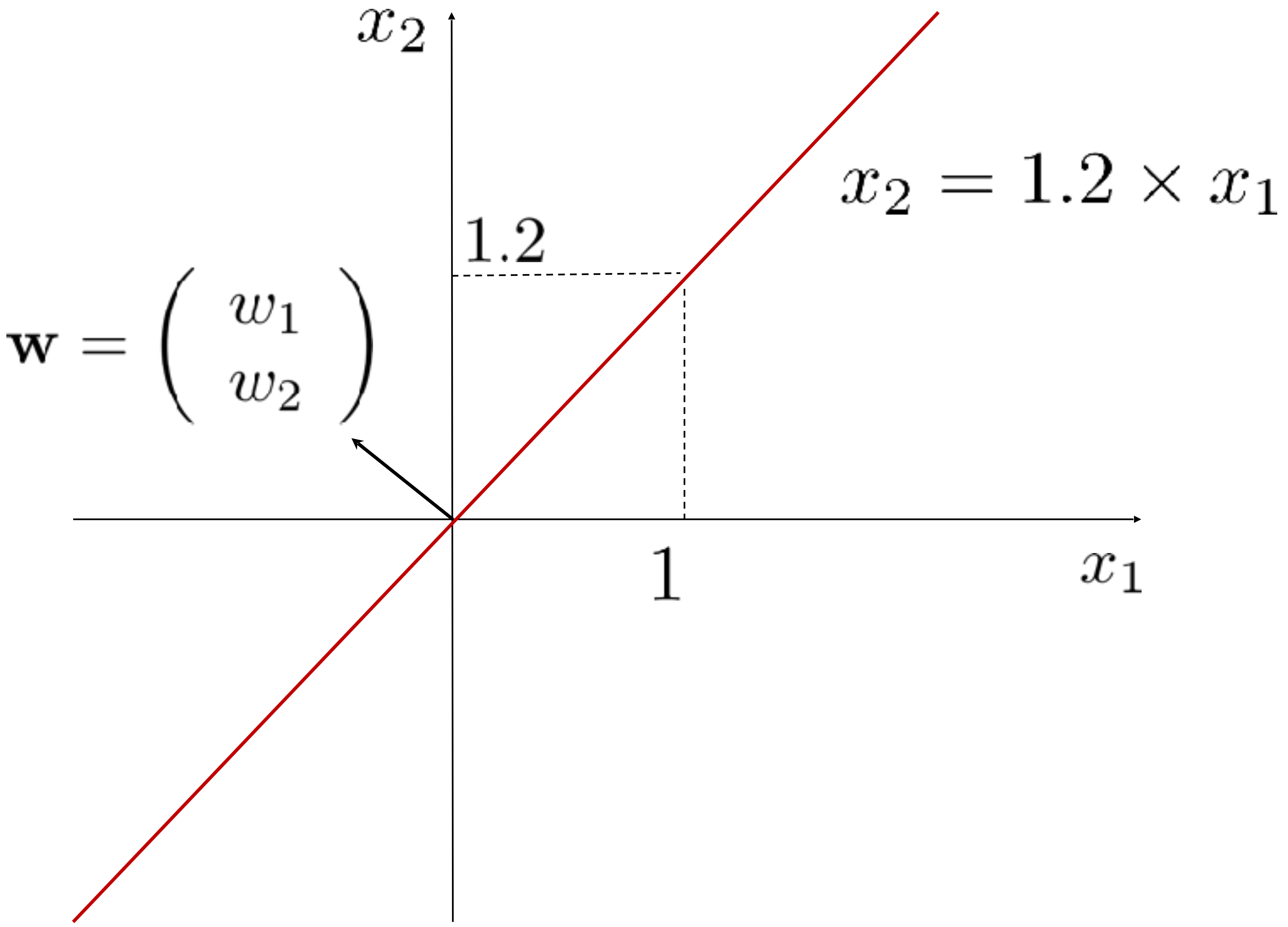
Lecture 9 pre-video

Decision boundaries

Orienting yourself

Line and vector

an example:

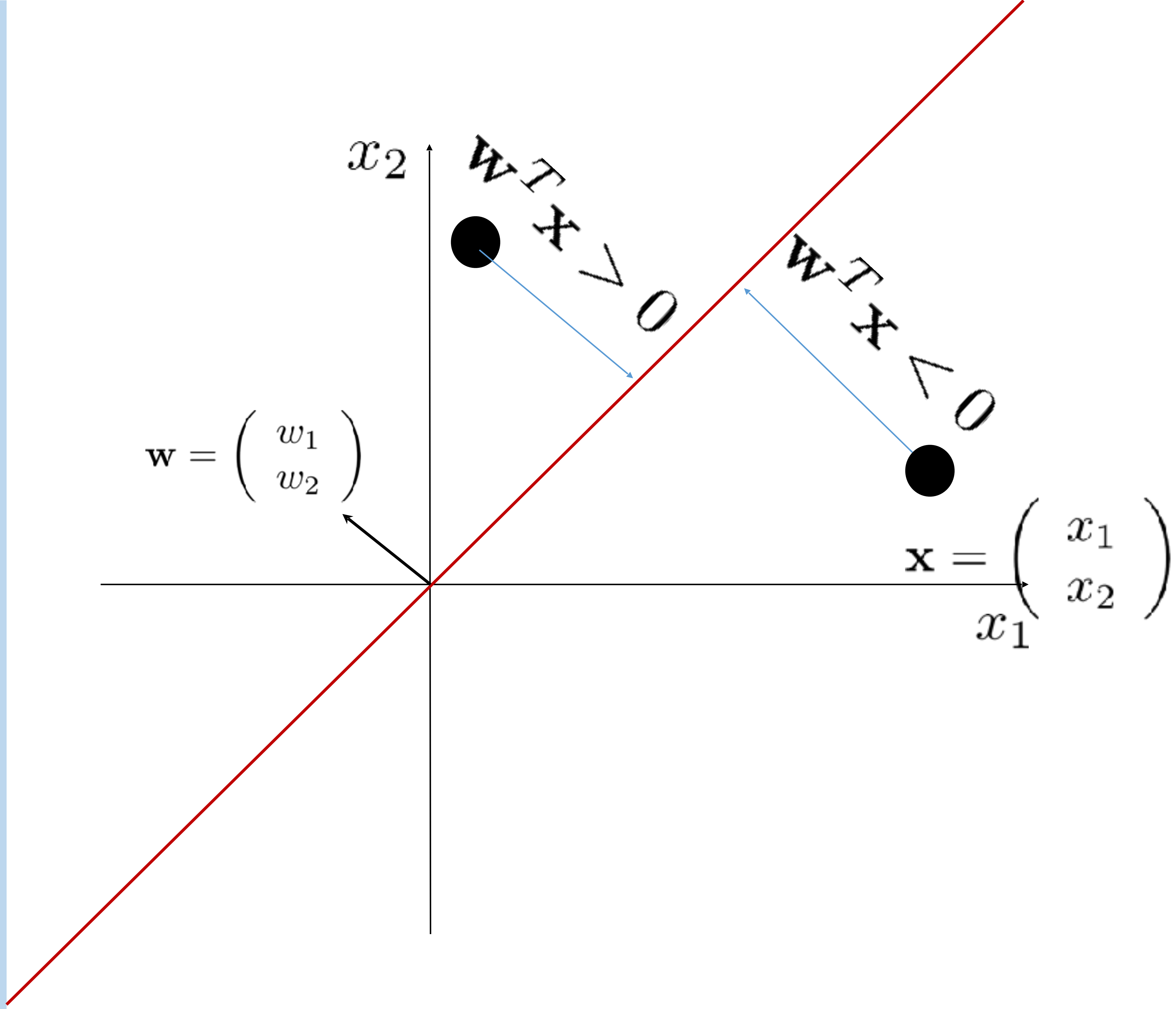


$\mathbf{w}$ is the normal direction of the line

Often: $||\mathbf{w}||_2 = 1$: a unit vector

$$\mathbf{w} = \begin{bmatrix} \frac{-1}{\sqrt{2.44}} \\ \frac{1.2}{\sqrt{2.44}} \end{bmatrix}$$

Distance to the decision boundary



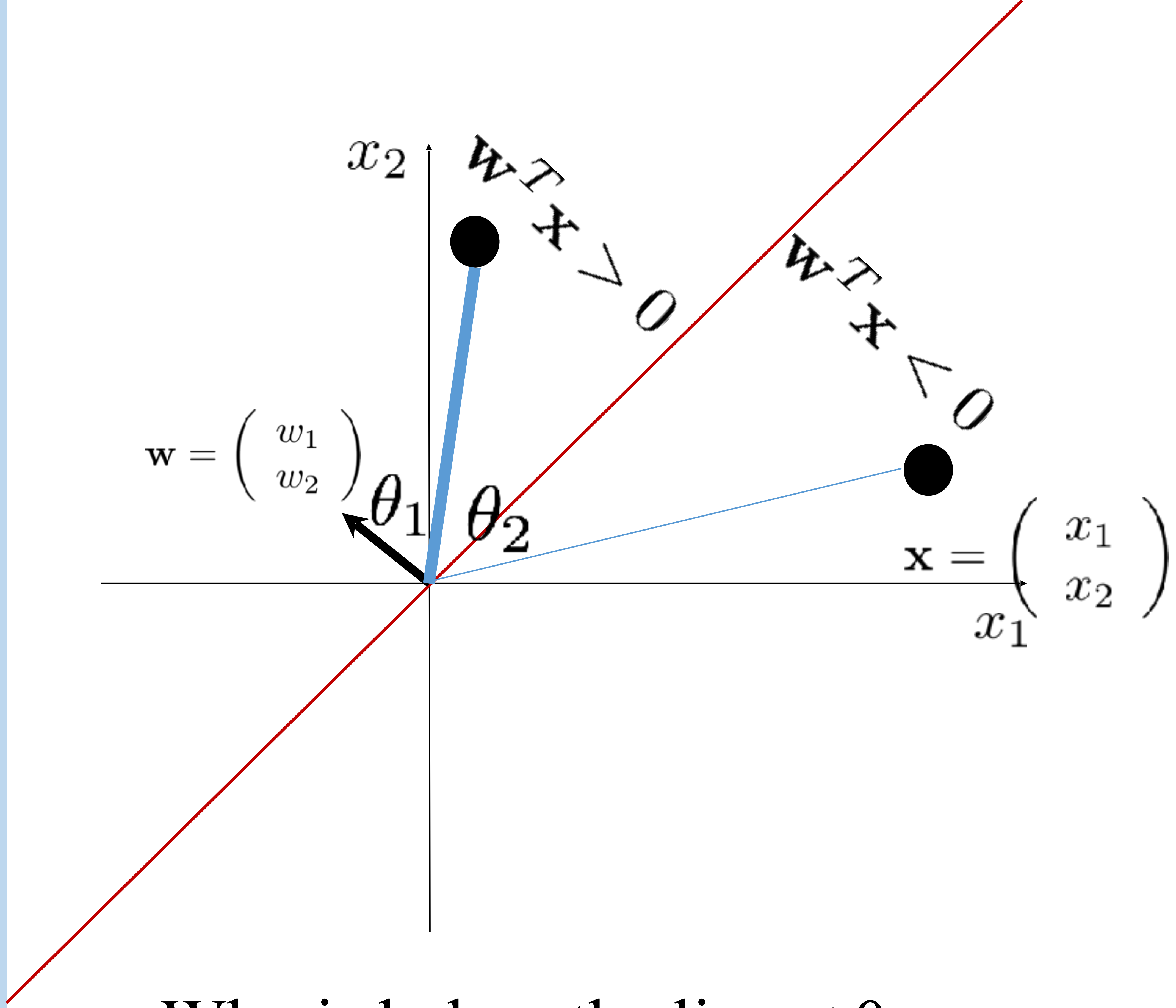
The distance (signed) of any point $\mathbf{x}$ to the line is:

$$\mathbf{w}^T\mathbf{x} \equiv < \mathbf{w}, \mathbf{x} >$$

$\mathbf{w}^T\mathbf{x} > 0$: above the line

$\mathbf{w}^T\mathbf{x} < 0$: below the line

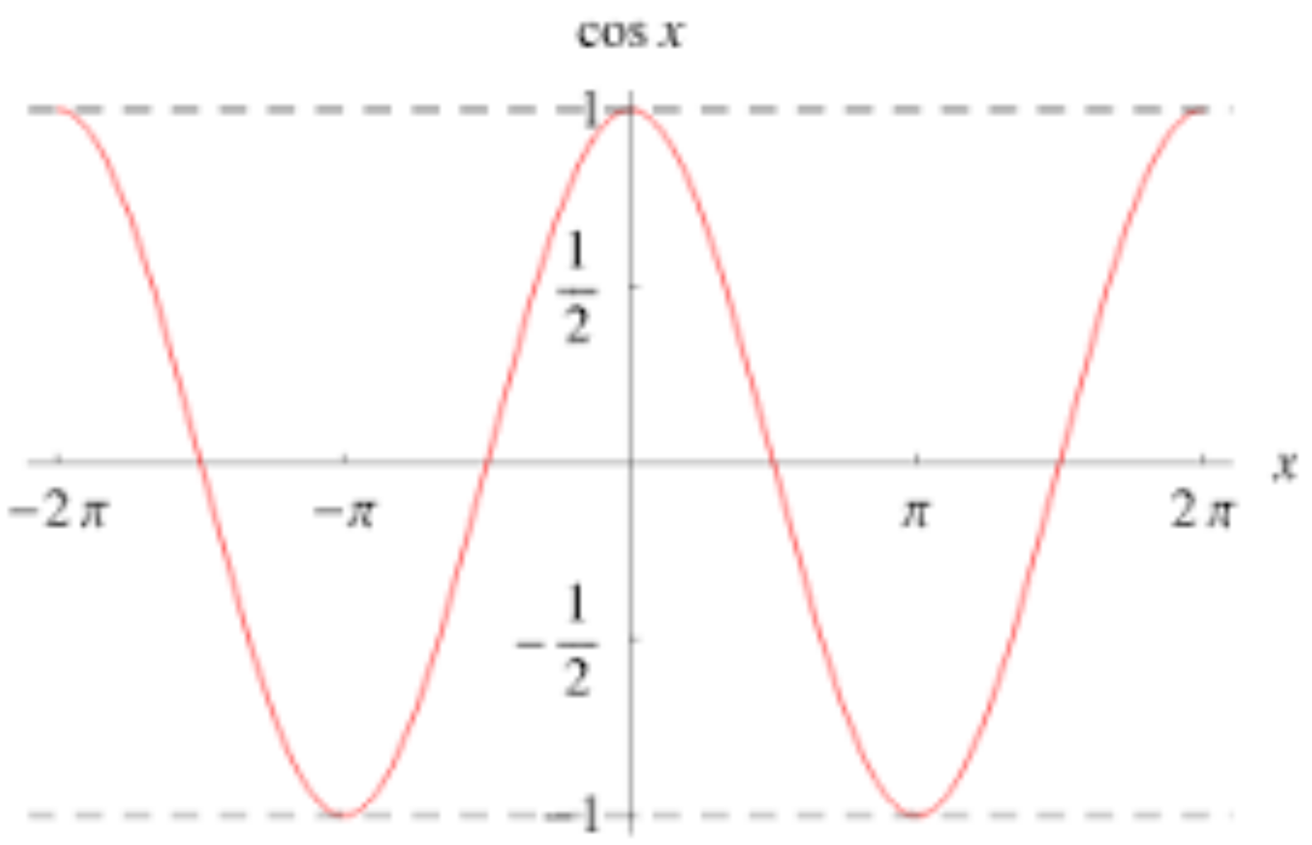
Distance to the decision boundary



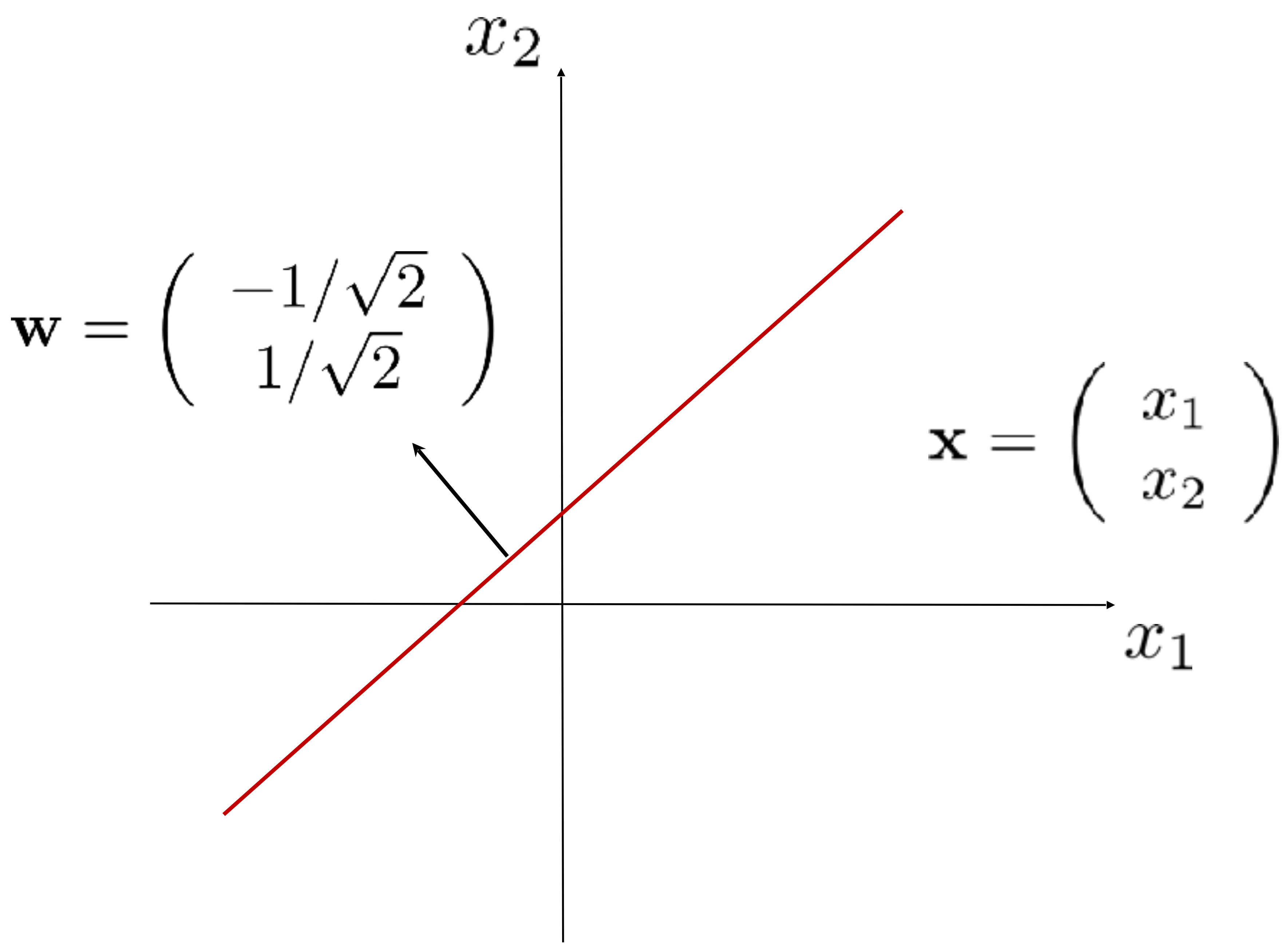
Why is below the line < 0
while above the line is > 0?

$$\mathbf{w}^T\mathbf{x} \equiv < \mathbf{w}, \mathbf{x} >$$
$$\mathbf{w}^T\mathbf{x} = \|\mathbf{w}\| \cdot \|\mathbf{x}\| \cdot \cos\theta$$

It depends on the angle formed by w and x



Can the line in red be the decision boundary of the classifier w shown to the right?



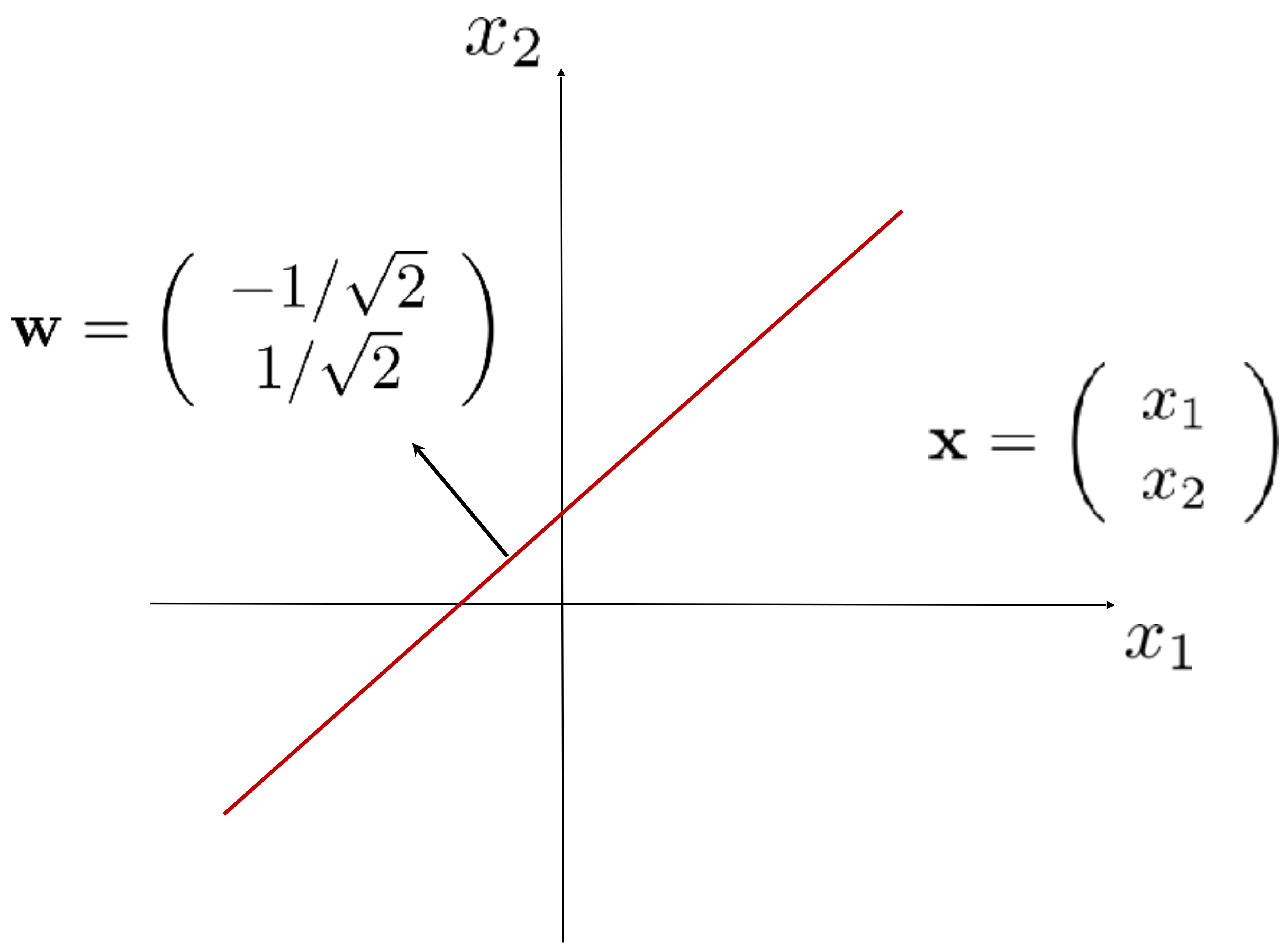
$$\mathbf{w} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \qquad y = \begin{cases} +1 & if\ \mathbf{w}^T\mathbf{x} > 0 \\ -1 & if\ \mathbf{w}^T\mathbf{x} < 0 \end{cases}$$

A. Yes

B. No

C. It depends

Can the line in red be the decision boundary of the classifier w shown to the right?



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A. Yes

B. No

C. It depends

Decision boundary

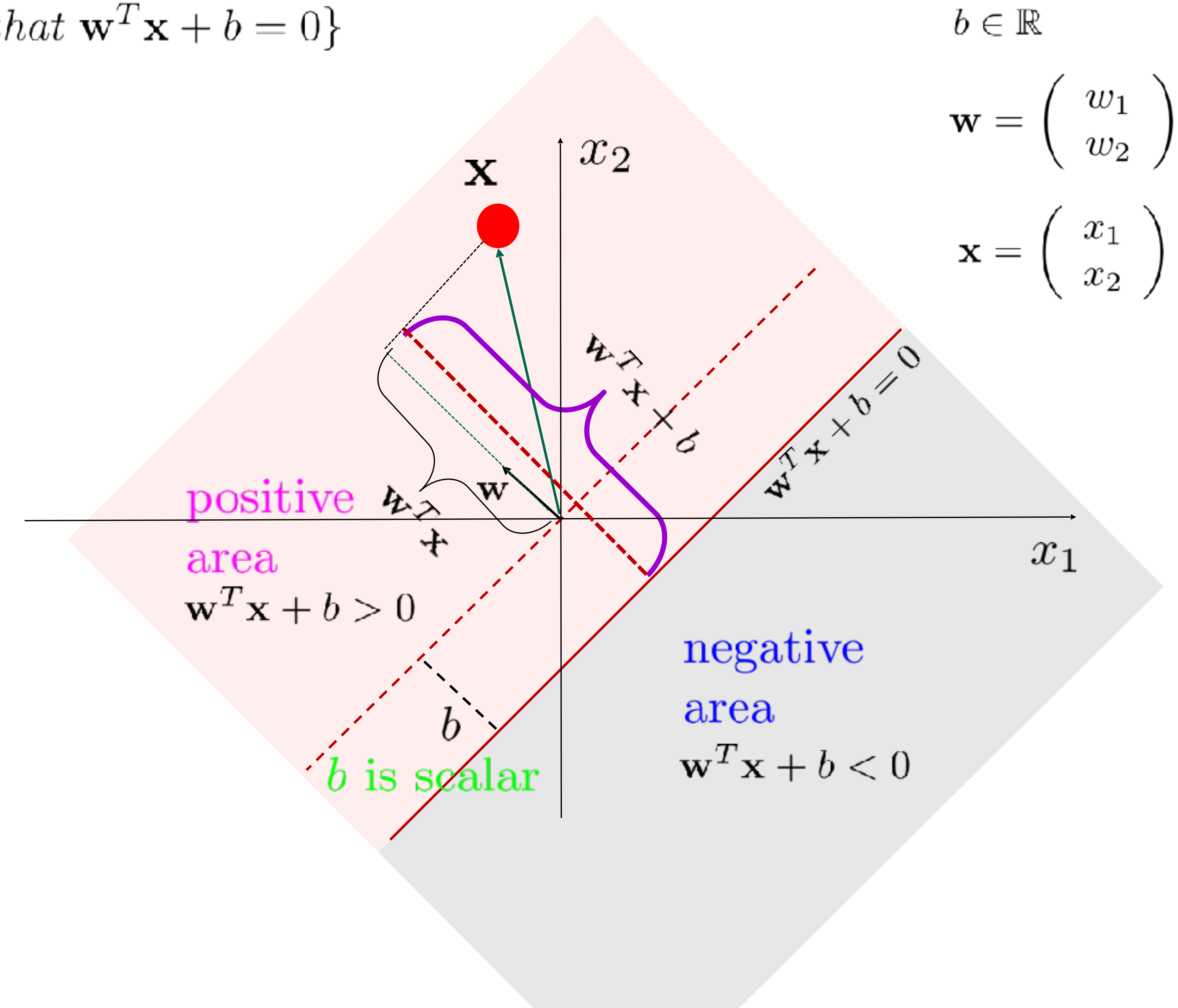
Decision boudary:

$\{\mathbf{x}; \forall \mathbf{x} \text{ } such \text{ } that \text{ } \mathbf{w}^T\mathbf{x} + b = 0\}$

$b \in \mathbb{R}$

$$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$



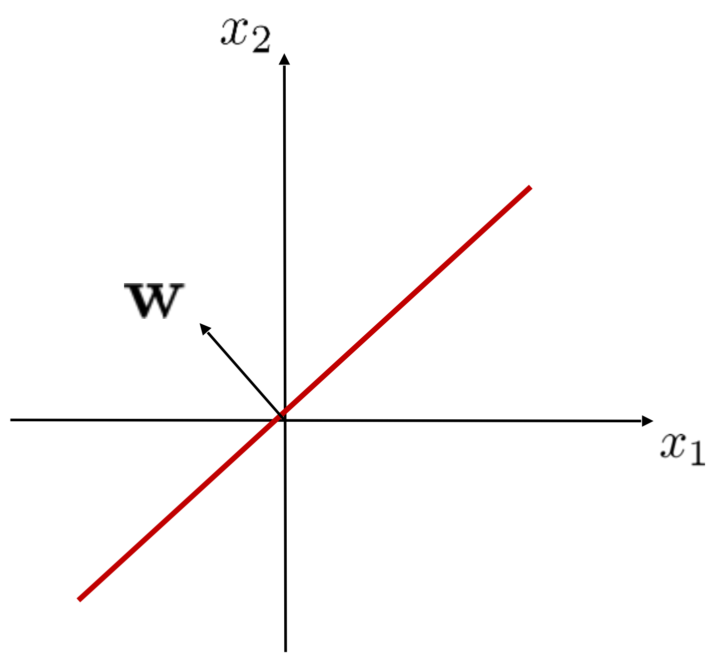
The distance (signed) of any point $\mathbf{x}$ to the decision boundary is: $\mathbf{w}^T\mathbf{x} + b$

Decision boundary

When b=0:

$\{\mathbf{x}; \forall \mathbf{x} \textit{ such that } \mathbf{w}^T\mathbf{x} = 0\}$

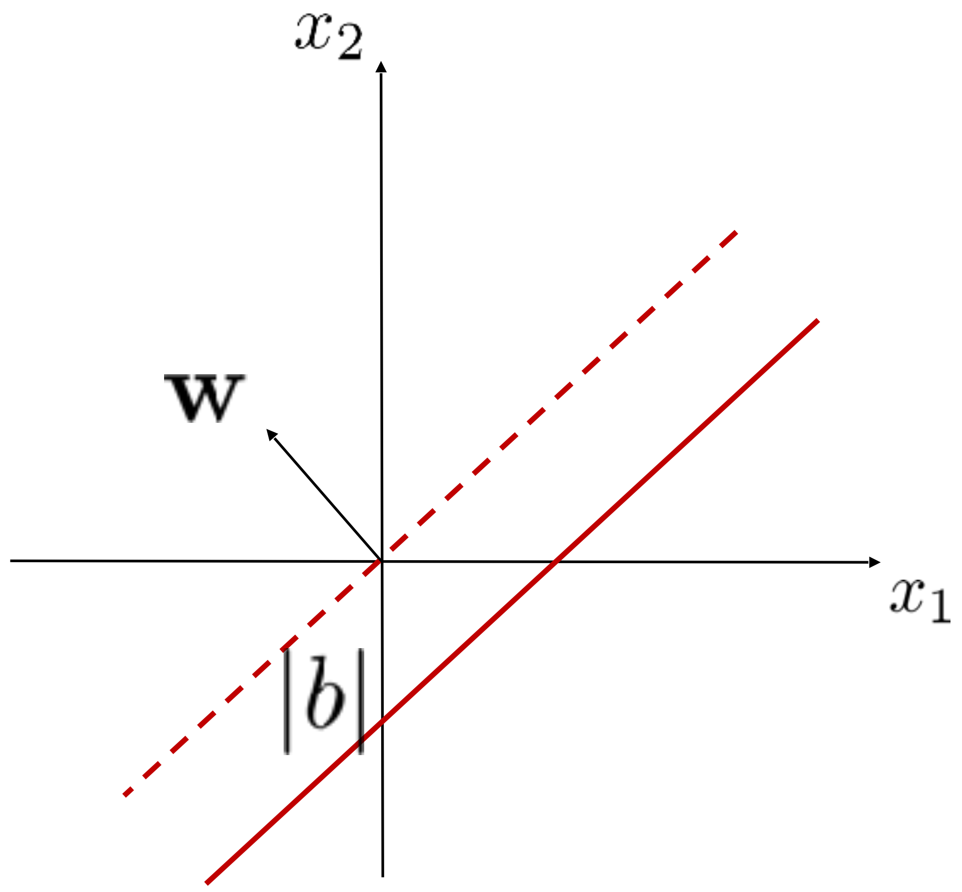
The decision boundary always goes through the origin.



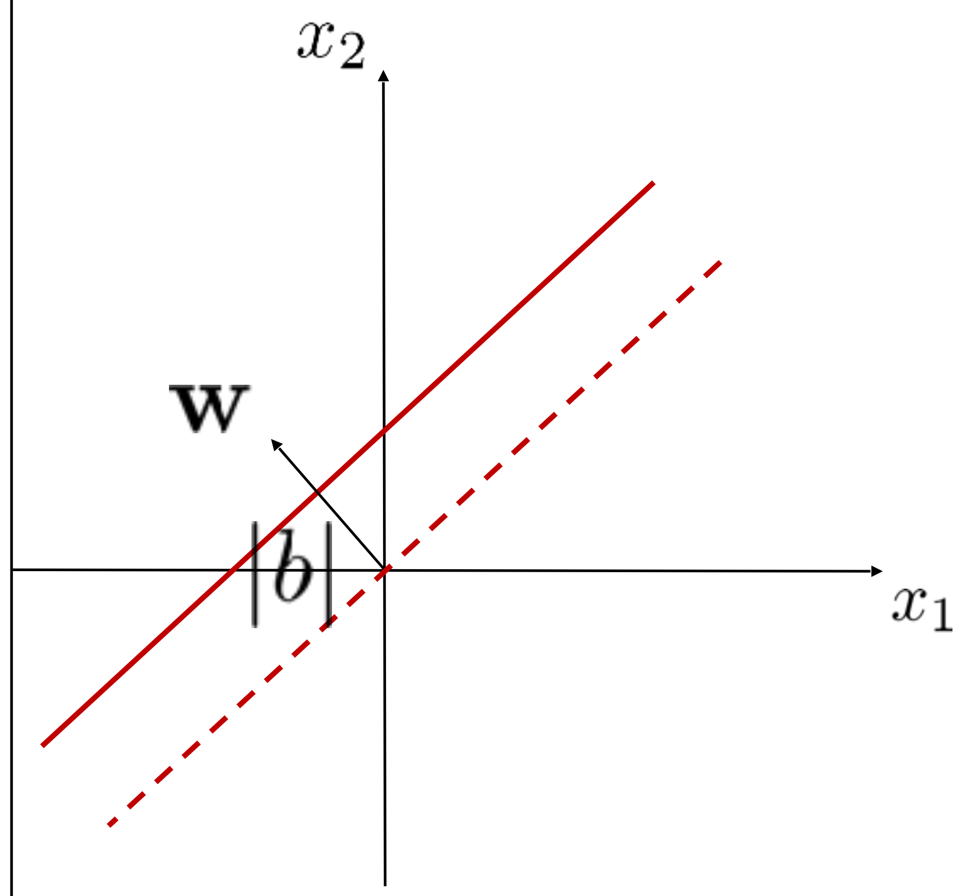
Decision boudary:

$\{\mathbf{x}; \forall \mathbf{x} \textit{ such that } \mathbf{w}^T\mathbf{x} + b = 0\}$

When b>0 the decision boundary is moved along the opposite direction of w.



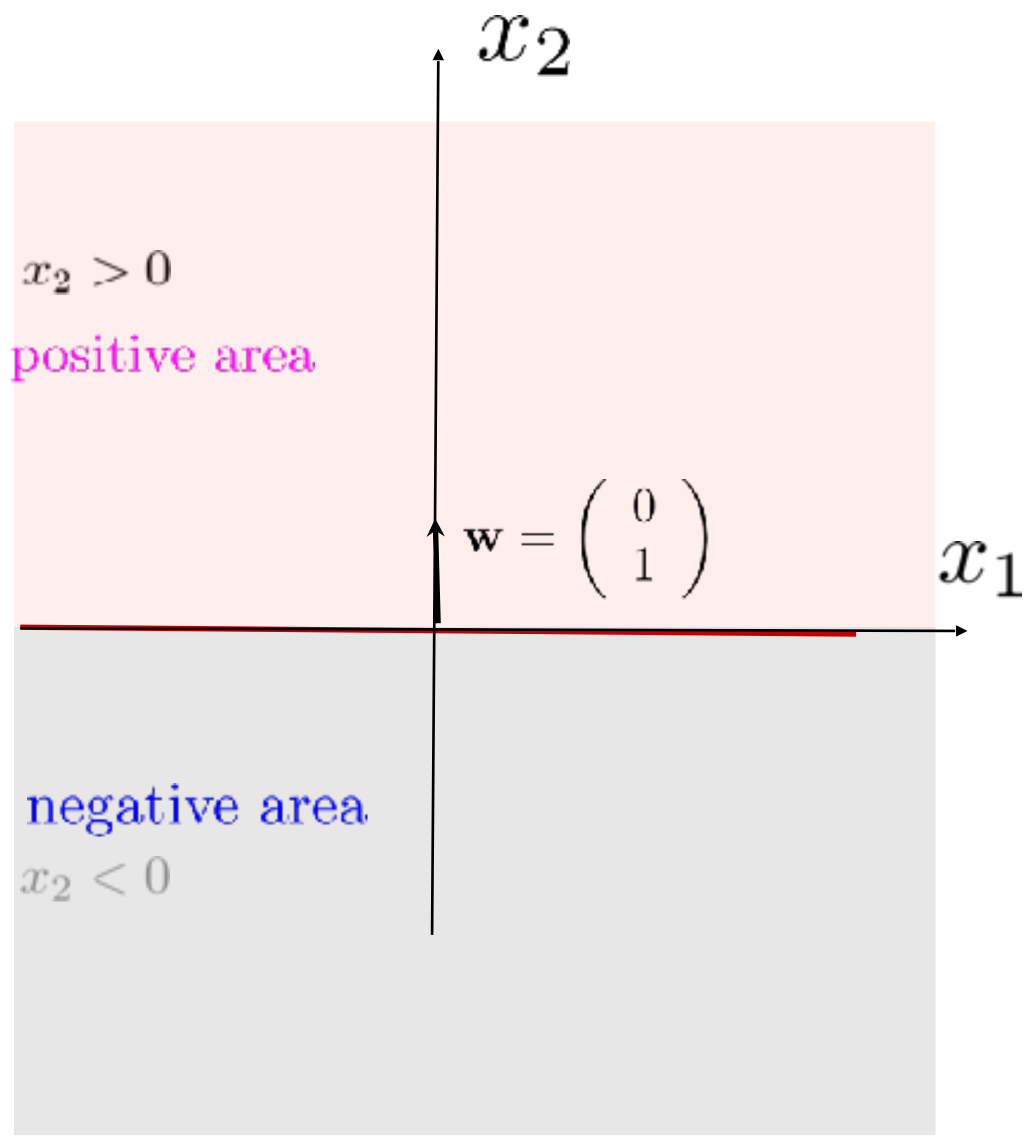
When b<0 the decision boundary is moved along the same direction of w.



Some typical examples

Assuming $\mathbf{w}$ being normalized: $||\mathbf{w}||_2 = \sqrt{w_1^2 + w_2^2} = 1$.

$$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \qquad b \in \mathbb{R} \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

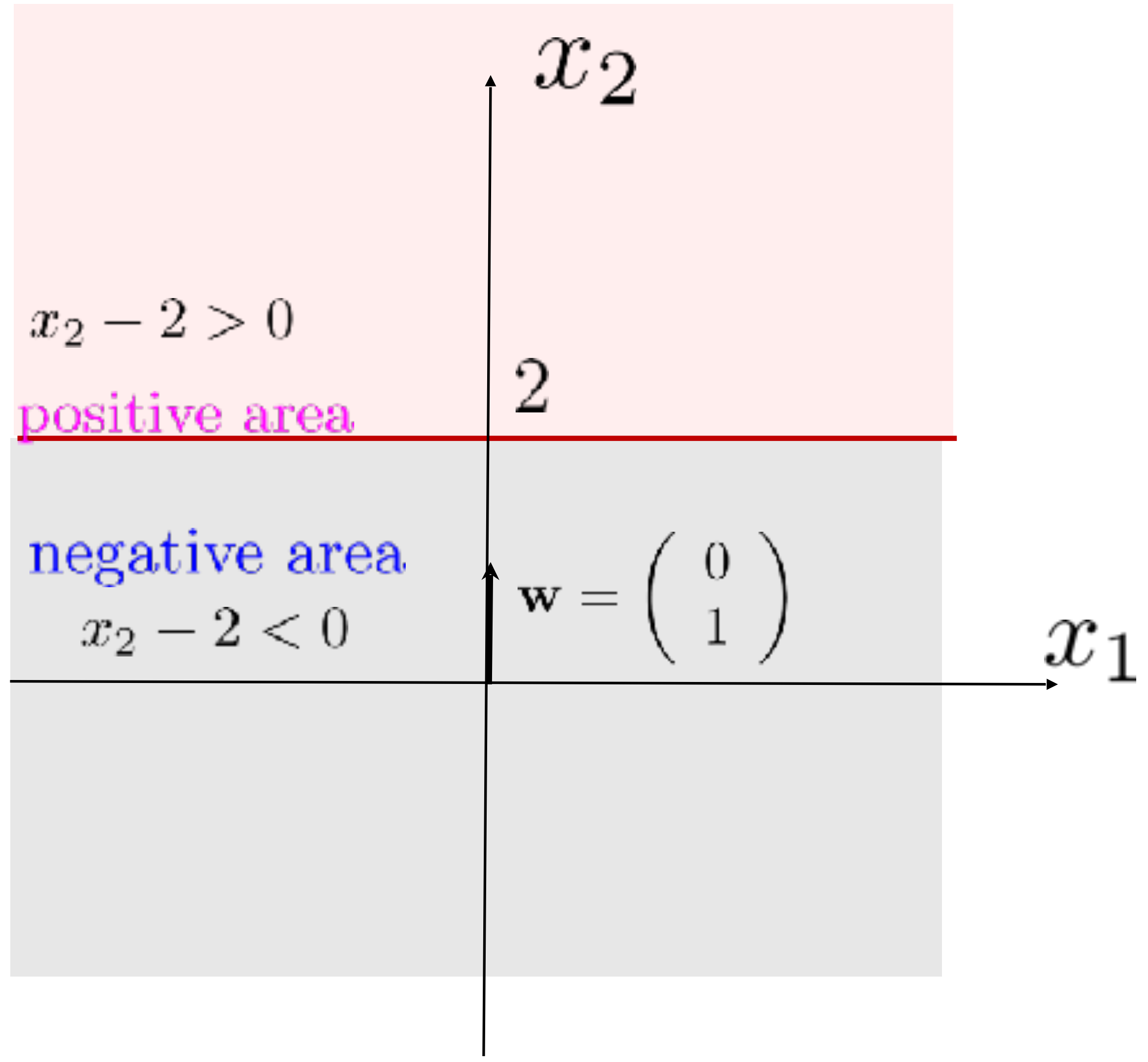


$$\mathbf{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$b = 0$$

Decision boundary:

$$0 \times x_1 + 1 \times x_2 = 0$$
$$\Downarrow$$
$$x_2 = 0$$



$$\mathbf{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$b = -2$$

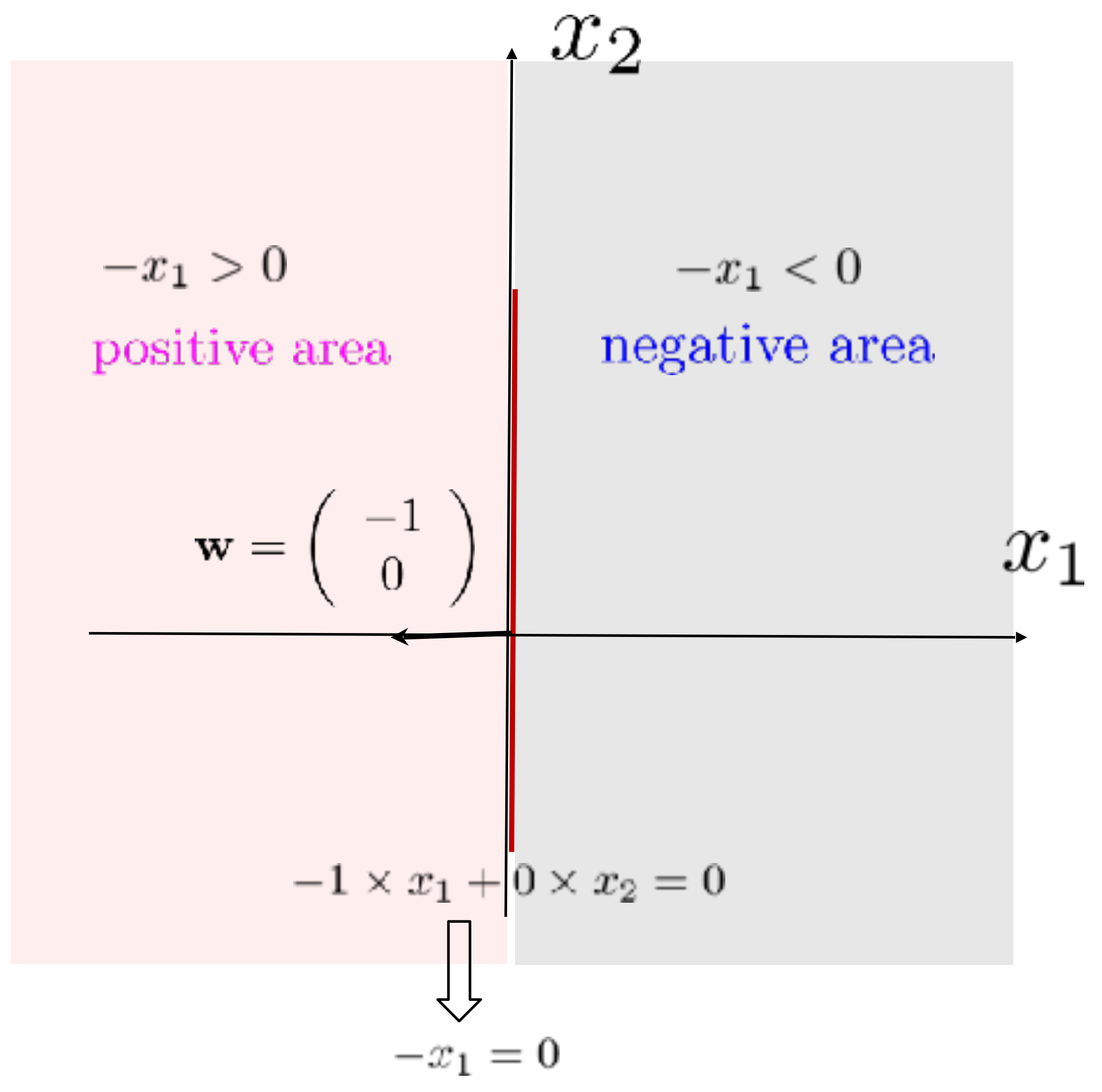
Decision boundary:

$$0 \times x_1 + 1 \times x_2 - 2 = 0$$
$$\Downarrow$$
$$x_2 - 2 = 0$$

Some typical examples

Assuming $\mathbf{w}$ being normalized: $||\mathbf{w}||_2 = \sqrt{w_1^2 + w_2^2} = 1.$

$$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \qquad b \in \mathbb{R} \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

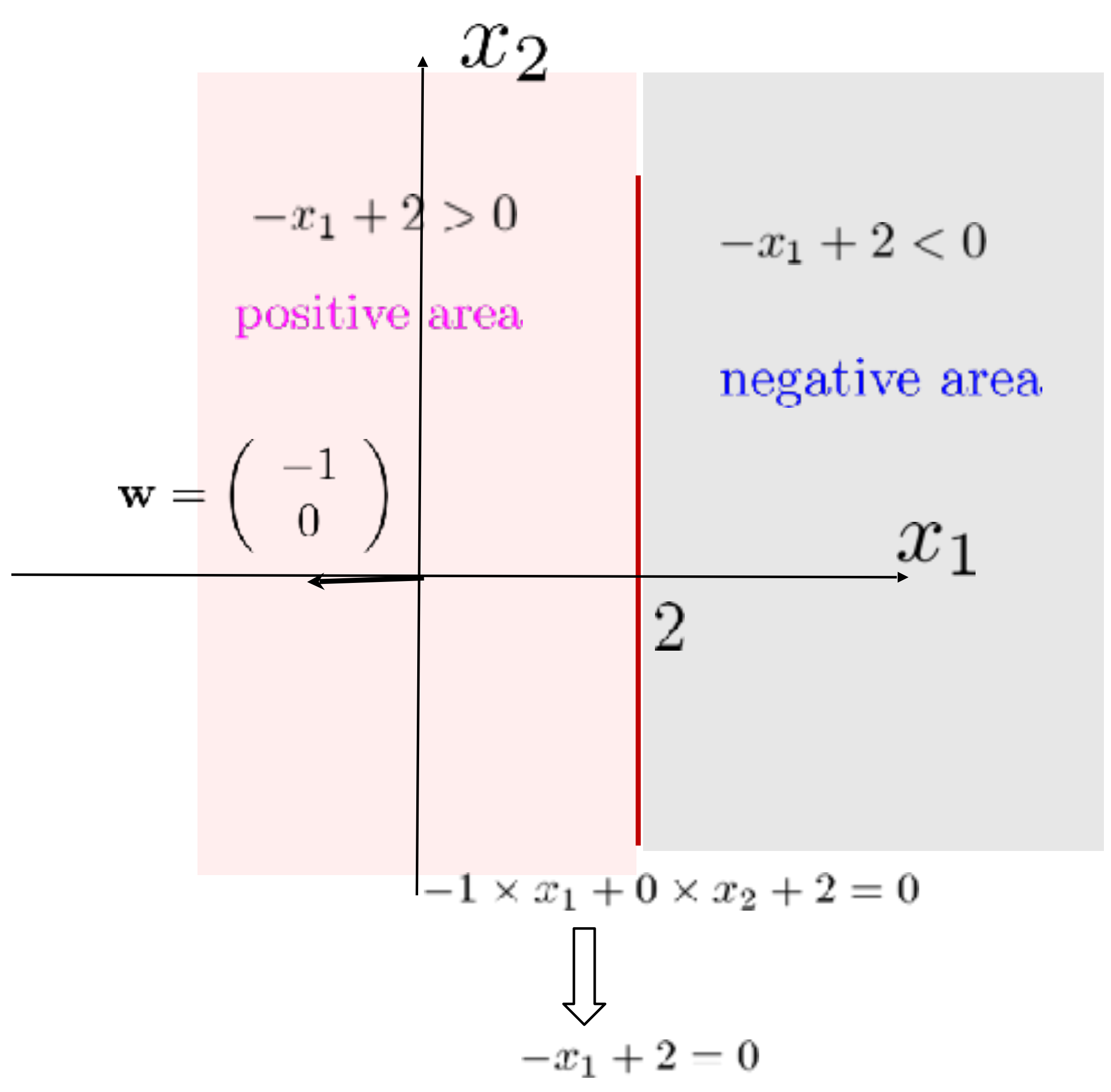


$\mathbf{w} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$b = 0$

Decision boundary:

$$-1 \times x_1 + 0 \times x_2 = 0$$
$$\Downarrow$$
$$-x_1 = 0$$



$\mathbf{w} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

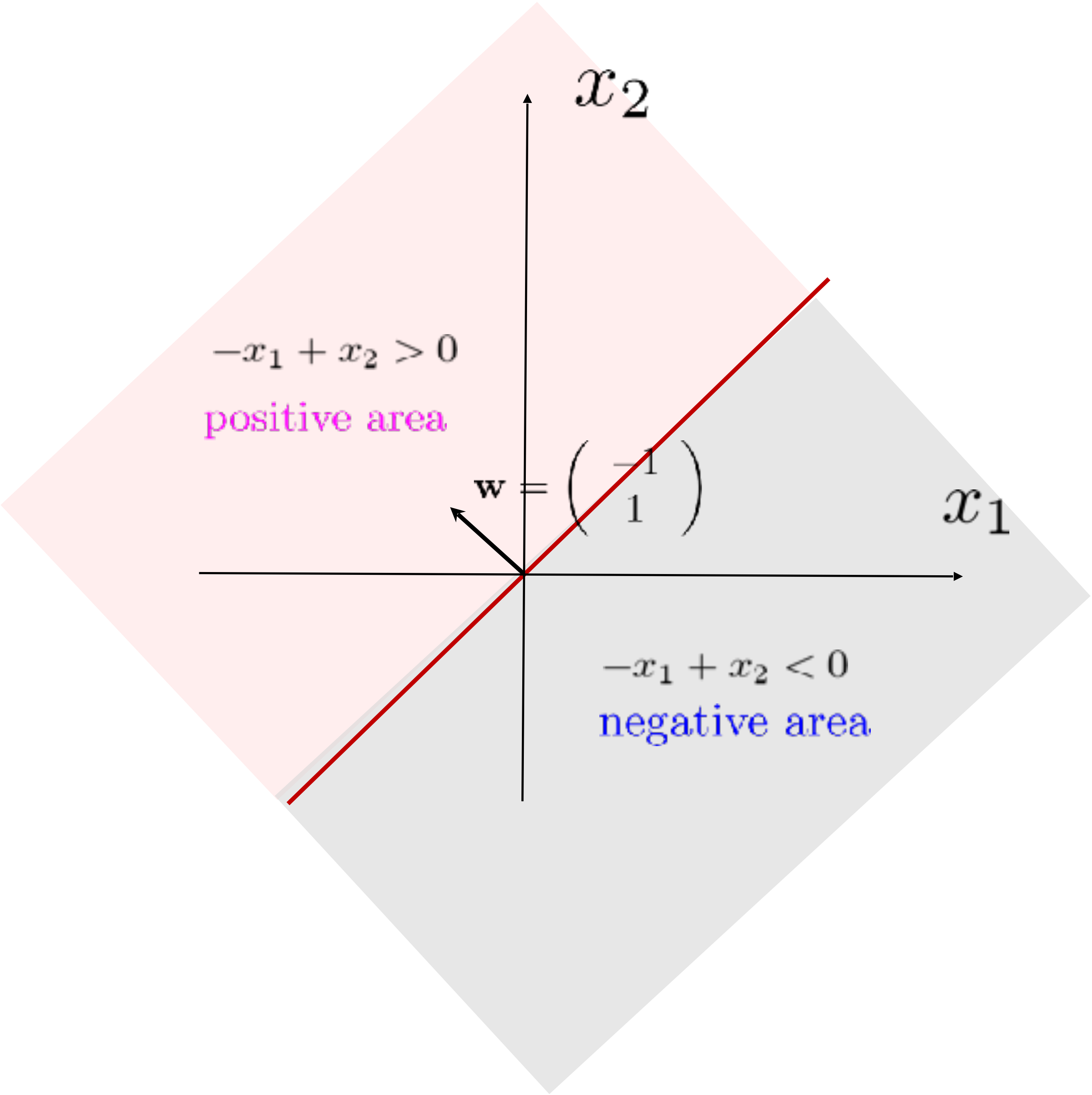
$b = 2$

Decision boundary:

$$-1 \times x_1 + 0 \times x_2 + 2 = 0$$
$$\Downarrow$$
$$-x_1 + 2 - 0$$

Some typical examples

$$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \qquad b \in \mathbb{R} \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

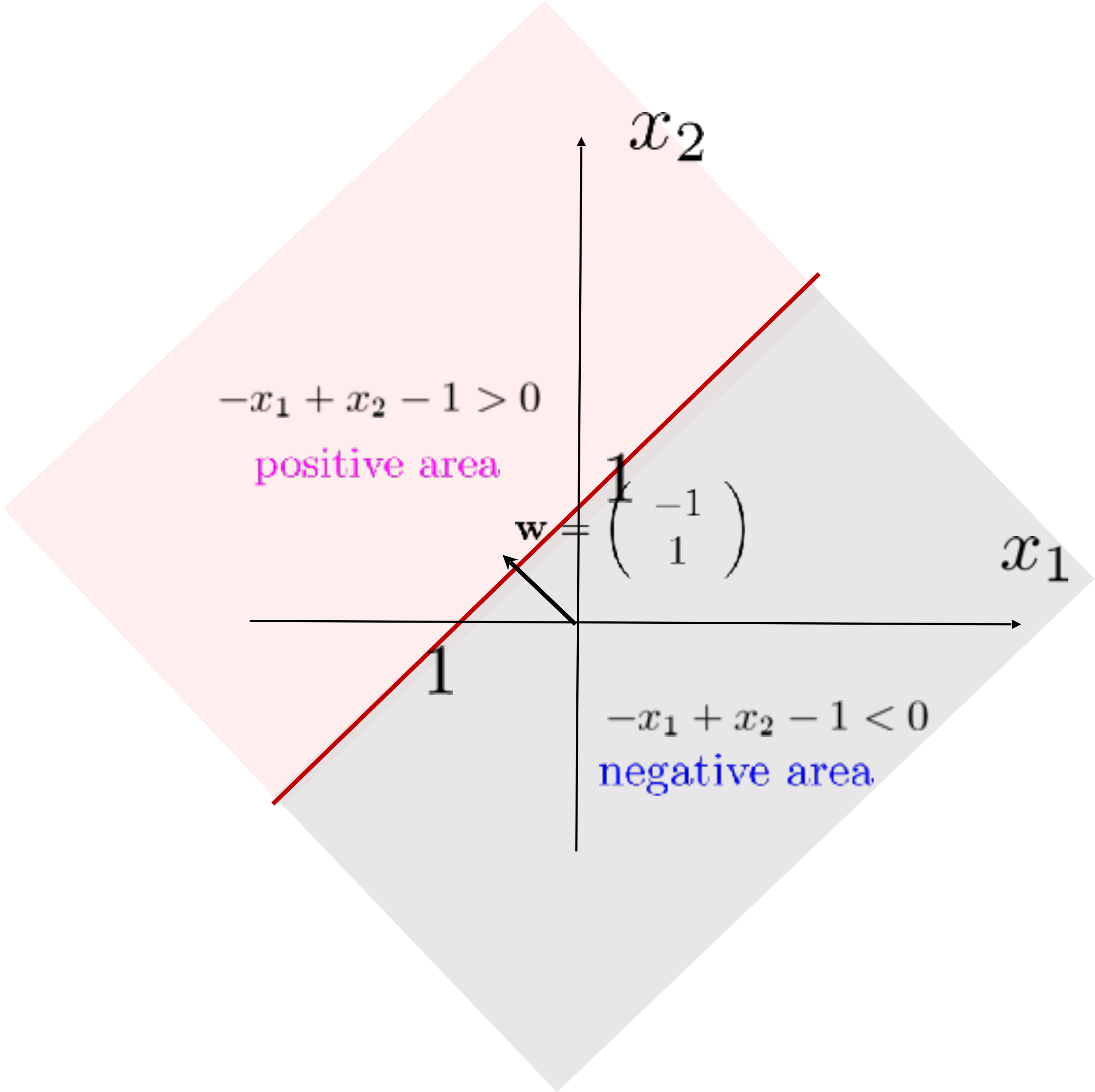


$$\mathbf{w} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$b = 0$

Decision boundary:

$$-1 \times x_1 + 1 \times x_2 = 0$$
$$\Downarrow$$
$$-x_1 + x_2 = 0$$



$$\mathbf{w} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$b = -1$

Decision boundary:

$$-1 \times x_1 + 1 \times x_2 - 1 = 0$$
$$\Downarrow$$
$$-x_1 + x_2 - 1 = 0$$

Take home message

Any data sample (point) lying on the decision boundary receives a classification decision that is equally positive and negative.

The decision boundary of a linear classifier is a hyper-plane.

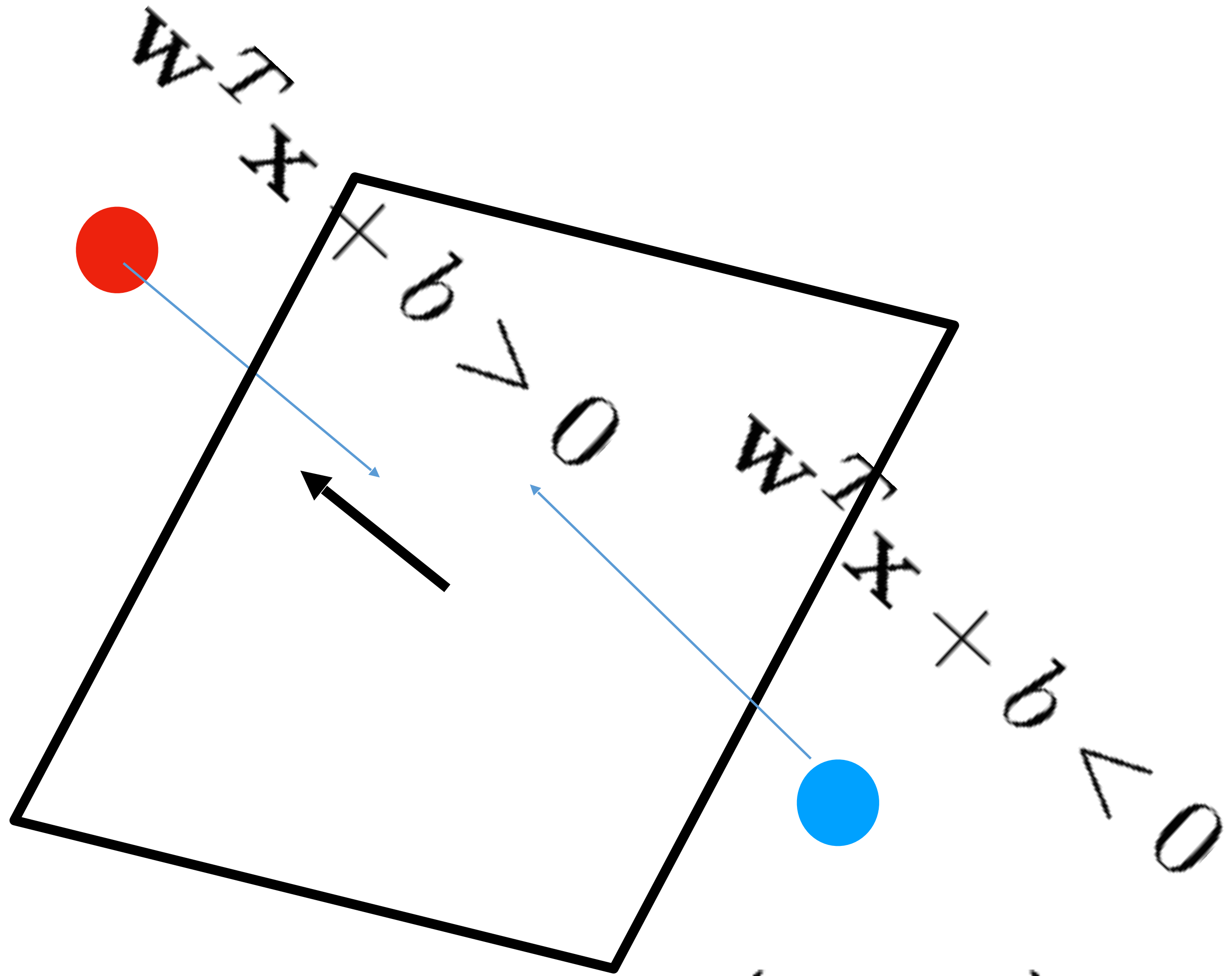
The model parameter **w** is along the normal direction of the decision boundary, pointing to the positive samples.

The bias terms, b (scalar), refers to as the translation (shift) of the decision boundary.

Distance to the decision boundary

in an arbitrary vector space

$$\mathbf{w} = \begin{pmatrix} w_1 \\ . \\ . \\ w_n \end{pmatrix}$$



$\mathbf{w} \in \mathbb{R}^n$

$b \in \mathbb{R}$

$\mathbf{x} \in \mathbb{R}^n$

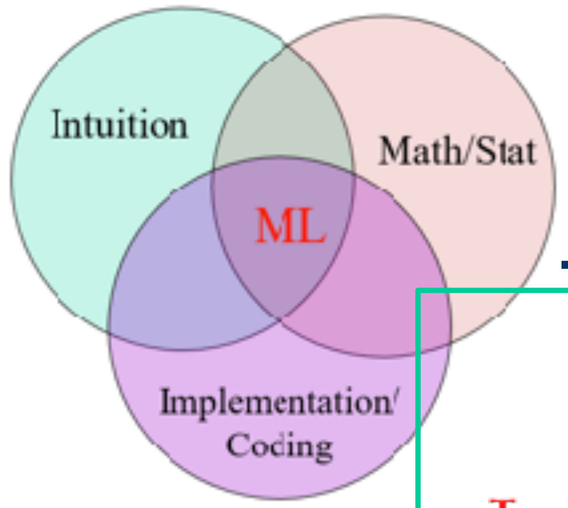
$$\mathbf{x} = \begin{pmatrix} x_1 \\ . \\ . \\ x_n \end{pmatrix}$$

The distance (signed) of any point $\mathbf{x}$ to the hyper-plane is:

$$\mathbf{w}^T\mathbf{x} + b \equiv <\mathbf{w}, \mathbf{x}> +b \equiv \mathbf{w} \cdot \mathbf{x} + b$$

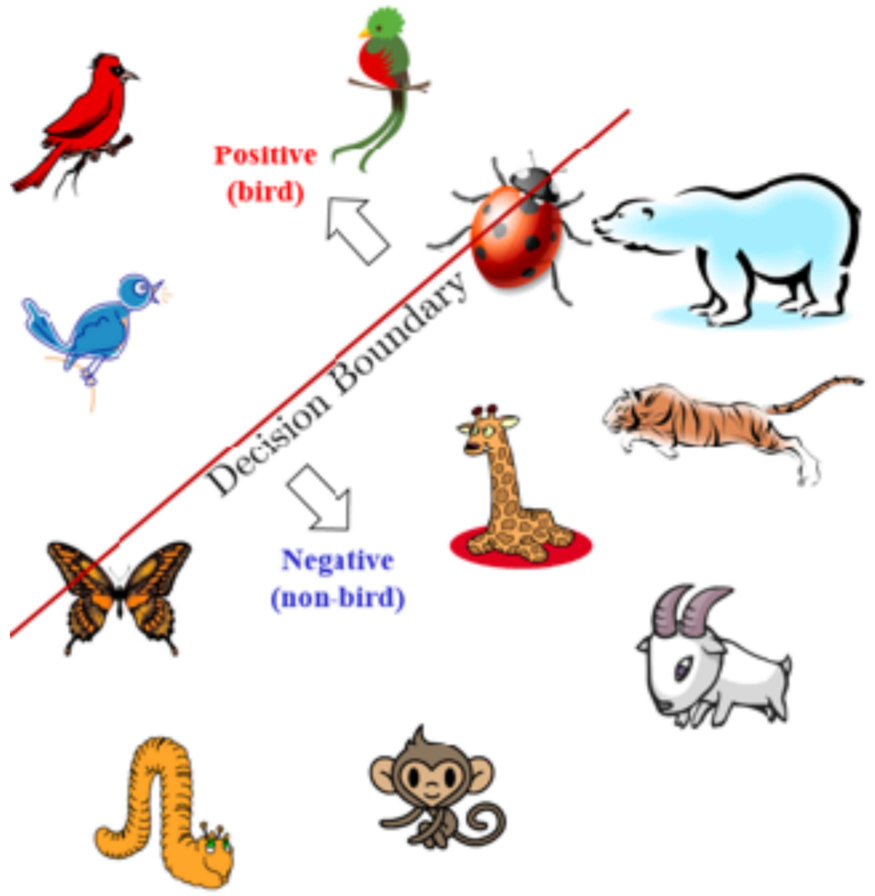
$\mathbf{w}^T\mathbf{x} + b < 0$: below the hyper-plane

$\mathbf{w}^T\mathbf{x} + b > 0$: above the hyper-plane



Recap: Decision Boundary

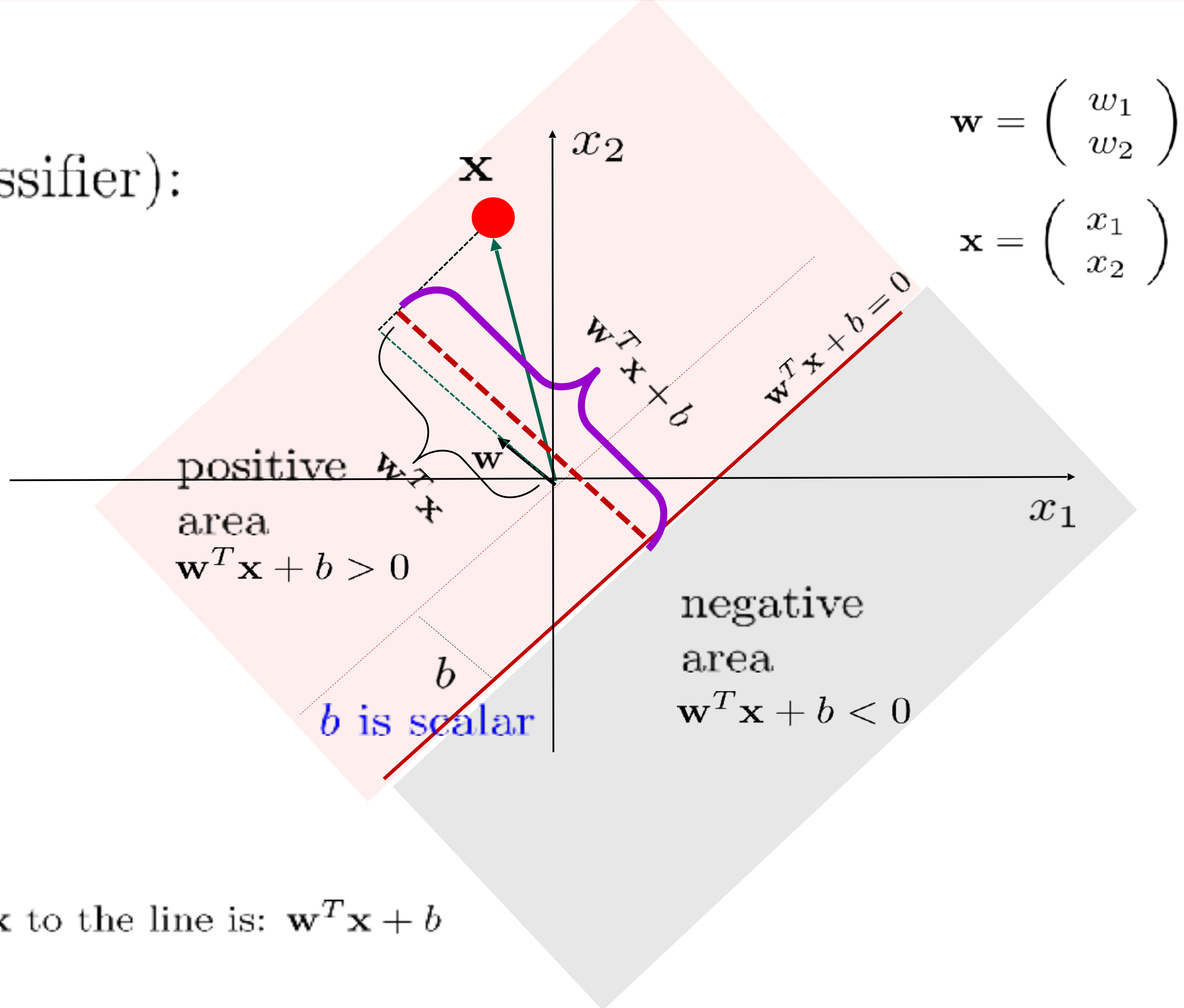
Intuition: Decision boundary of a discriminative classifier is a set that consists of possible samples that are on the border (typically 50% – 50%) of the separation between the positive and negative areas (for two-class classification).



Math:

Decision boudary (for a linear classifier):

$\{\mathbf{x}; \forall \mathbf{x}\ such\ that\ \mathbf{w}^T\mathbf{x} + b = 0\}$



The distance (signed) of any point $\mathbf{x}$ to the line is: $\mathbf{w}^T\mathbf{x} + b$

Lecture 9 pre-video

Logit + its derivatives

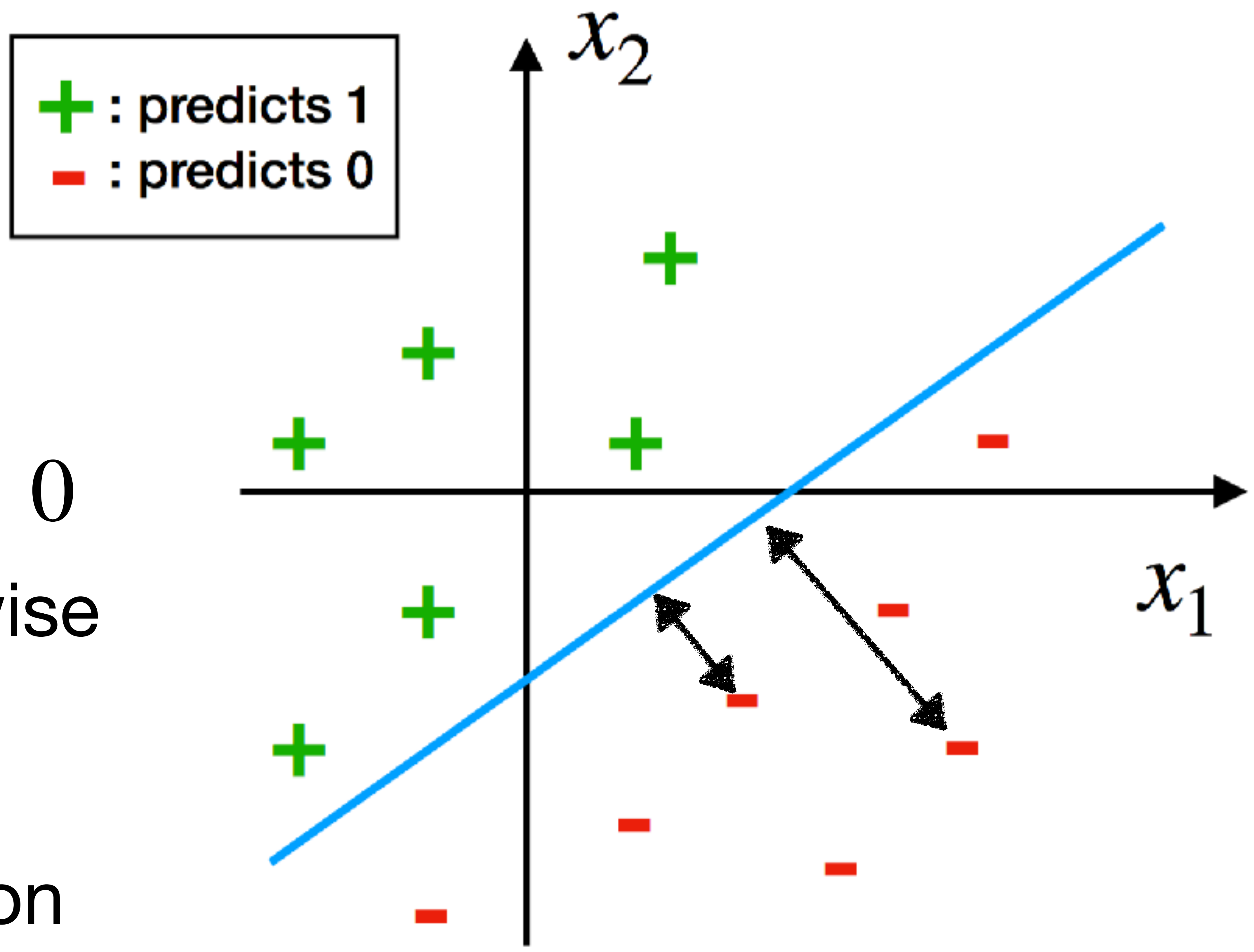
Distance as Probability

- Classification $f(x; w) = \begin{cases} 1 \text{ if } w^T x \geq 0 \\ -1 \text{ otherwise} \end{cases}$

- Our predictions are only +1 or -1

- What if we want to make our prediction a probability?

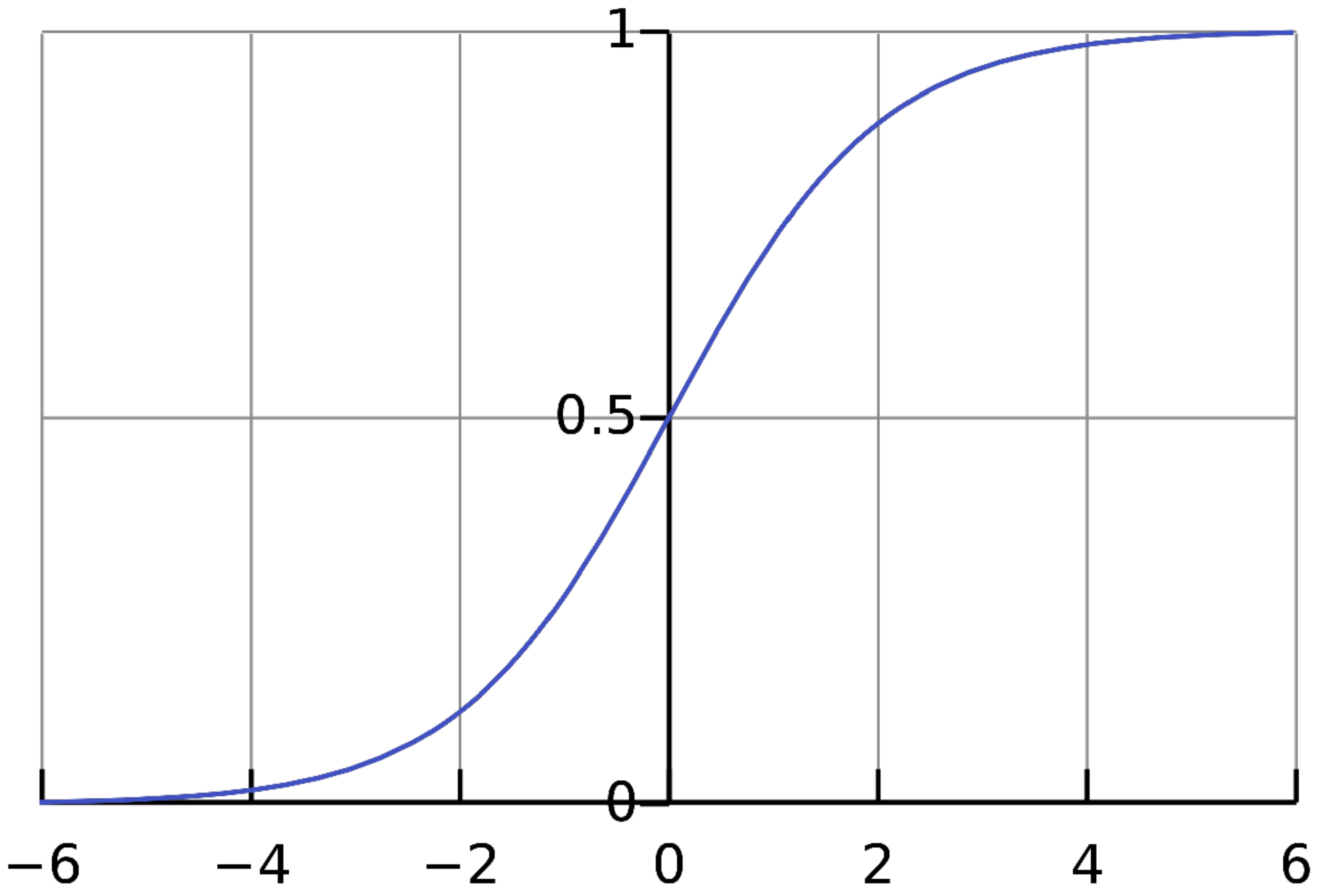
- $f(x; w) = p(y = +1 \,|\, x; w)$
  or equivalently
  $f(x; w) = -p(y = -1 \,|\, x; w)$



Logistic function a.k.a. logit

$$f(x) = \frac{1}{1+e^{-x}} = \frac{e^x}{e^x+1}$$

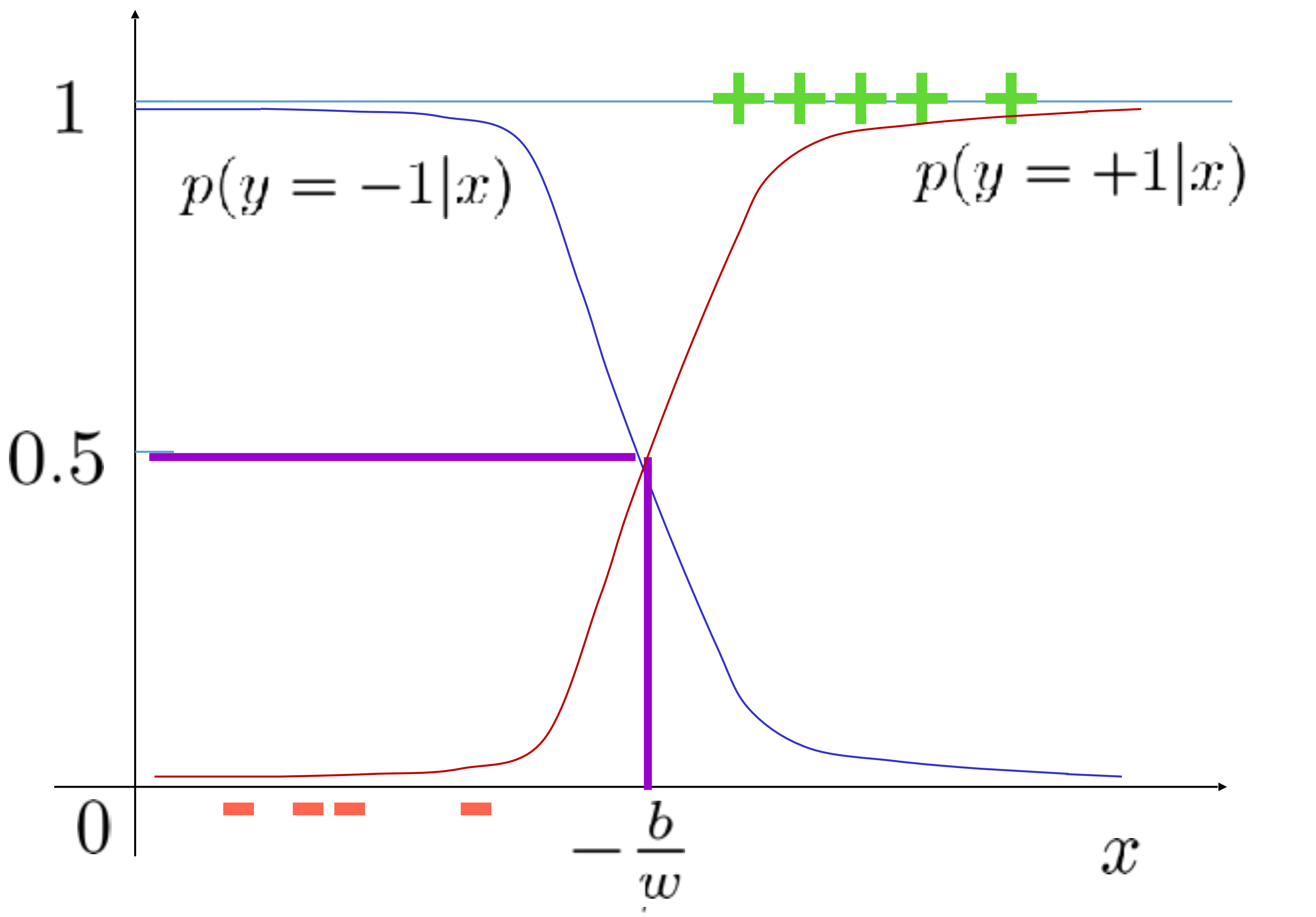
Always in range 0 - 1 => probability



Logistic regression classifier

$x, w, b \in \mathbb{R}$

Let's look at the simplest case where x is a scalar:



We have: $f(x;w,b) = \begin{cases} +1 & if\ w \times x + b \geq 0 \\ -1 & otherwise \end{cases}$.

Probability of a sample being a positive case

$$p(y=+1|x) = \frac{e^{w \times x + b}}{1+e^{w \times x + b}}$$

$$p(y=-1|x) = 1 - p(y=+1|x)$$

$$= 1 - \frac{e^{wx+b}}{1+e^{wx+b}}$$

$$= \frac{1+e^{wx+b}}{1+e^{wx+b}} - \frac{e^{wx+b}}{1+e^{wx+b}}$$

Probability of a sample being a negative case

$$p(y=-1|x) = \frac{1}{1+e^{w \times x + b}}$$

Logistic regression function

The main mathematical convenience of the logistic regression function!

$$p(y=+1|x) = \frac{e^{wx+b}}{1+e^{wx+b}}$$

$$= \frac{e^{wx+b}}{1+e^{wx+b}} \cdot \frac{e^{-(wx+b)}}{e^{-(wx+b)}}$$

$$= \frac{1}{e^{-(wx+b)}+1}$$

$$p(y=+1|\mathbf{x}) = \frac{1}{1+e^{-(\mathbf{w}^T\mathbf{x}+b)}}$$

$$p(y=-1|\mathbf{x}) = \frac{1}{1+e^{(\mathbf{w}^T\mathbf{x}+b)}} \qquad y \in \{-1,+1\}$$

$$\boxed{p(y|\mathbf{x}) = \frac{1}{1+e^{-y(\mathbf{w}^T\mathbf{x}+b)}}}$$

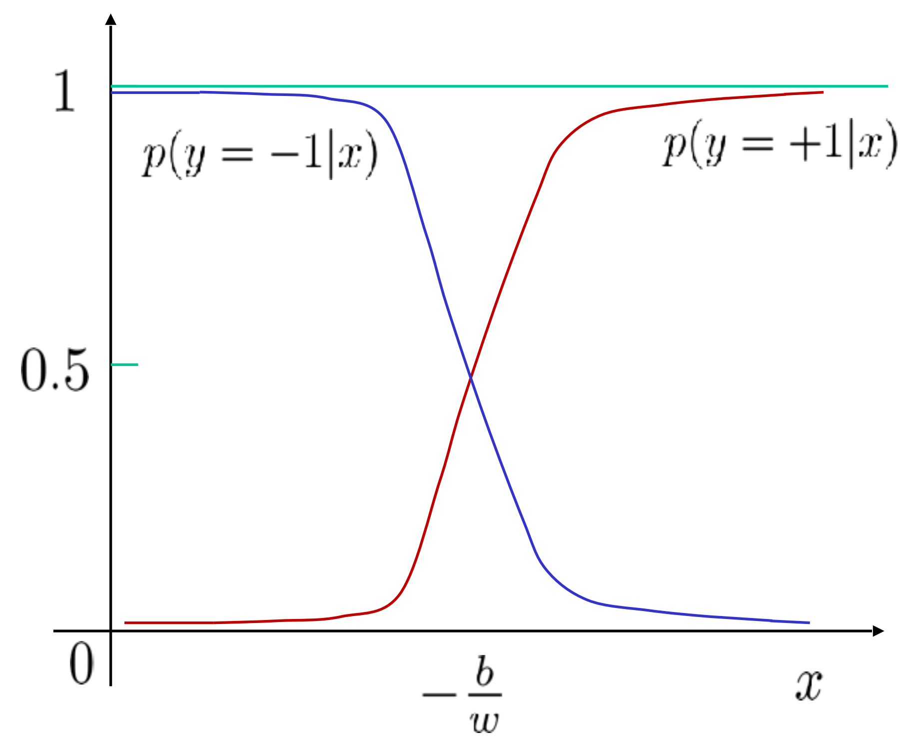
A general form, independent of the value of y!

Training a logistic regression classifier

$S_{training} = \{(-1.1, -1), (3.2, +1), (2.5, -1), (5.0, +1), (4.3, +1)\}$

$$p(y = +1|\mathbf{x}) = \frac{1}{1+e^{-(\mathbf{w}^T\mathbf{x}+b)}}$$

$$p(y = -1|\mathbf{x}) = \frac{1}{1+e^{(\mathbf{w}^T\mathbf{x}+b)}}$$



$$p(y_i|\mathbf{x}_i) = \frac{1}{1+e^{-y_i(\mathbf{w}^T\mathbf{x}_i+b)}}$$

Train a logistic regression classifier $f(\mathbf{x}) = \begin{cases} +1 & if\ \frac{1}{1+e^{-(\mathbf{w}^T\mathbf{x}+b)}} \geq 0.5 \\ -1 & otherwise \end{cases}$;

Intuition: find the best parameters $(\mathbf{w}, b)^*$ to maximize the probabilities of fitting the ground-truth label y_i for each $\mathbf{x}_i$.

Math: $(\mathbf{w}, b)^* = \arg\max_{(\mathbf{w},b)} \prod_{i=1}^{n} \frac{1}{1+e^{-y_i(\mathbf{w}^T\mathbf{x}_i+b)}}$

Training a logistic regression classifier

Intuition: find the best parameters $(\mathbf{w}, b)^*$ to maximize the probabilities of fitting the ground-truth label y_i for each $\mathbf{x}_i$.

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$$(\mathbf{w}, b)^* = \arg\max_{(\mathbf{w},b)} \prod_{i=1}^{n} \frac{1}{1+e^{-y_i(\mathbf{w}^T\mathbf{x}_i+b)}}$$

$$= \arg\max_{(\mathbf{w},b)} \ln\left(\prod_{i=1}^{n} \frac{1}{1+e^{-y_i(\mathbf{w}^T x_i+b)}}\right)$$

$$= \arg\min_{(\mathbf{w},b)} \sum_{i=1}^{n} - \ln\left(\frac{1}{1+e^{-y_i \times (\mathbf{w}^T\mathbf{x}_i+b)}}\right)$$

Logistic regression

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Slides in this presentation are from material kindly provided by Zhuowen Tu and others credited at those slides

Distance as Probability

- Classification $f(x; w) = \begin{cases} 1 \text{ if } w^T x \geq 0 \\ -1 \text{ otherwise} \end{cases}$
- Our predictions are only +1 or -1
- What if we want to make our prediction a probability?
- $f(x; w) = p(y = +1 \mid x; w)$
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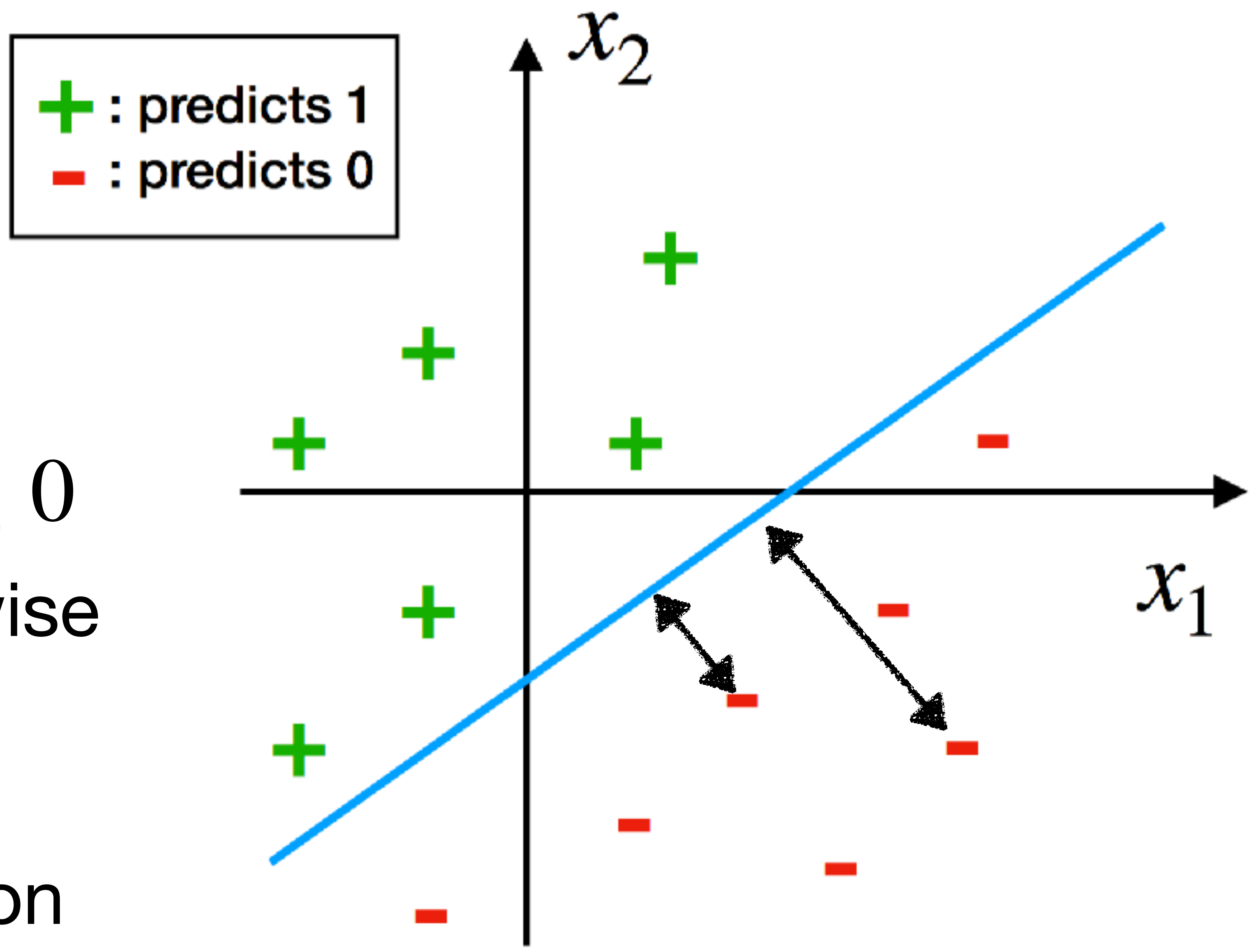

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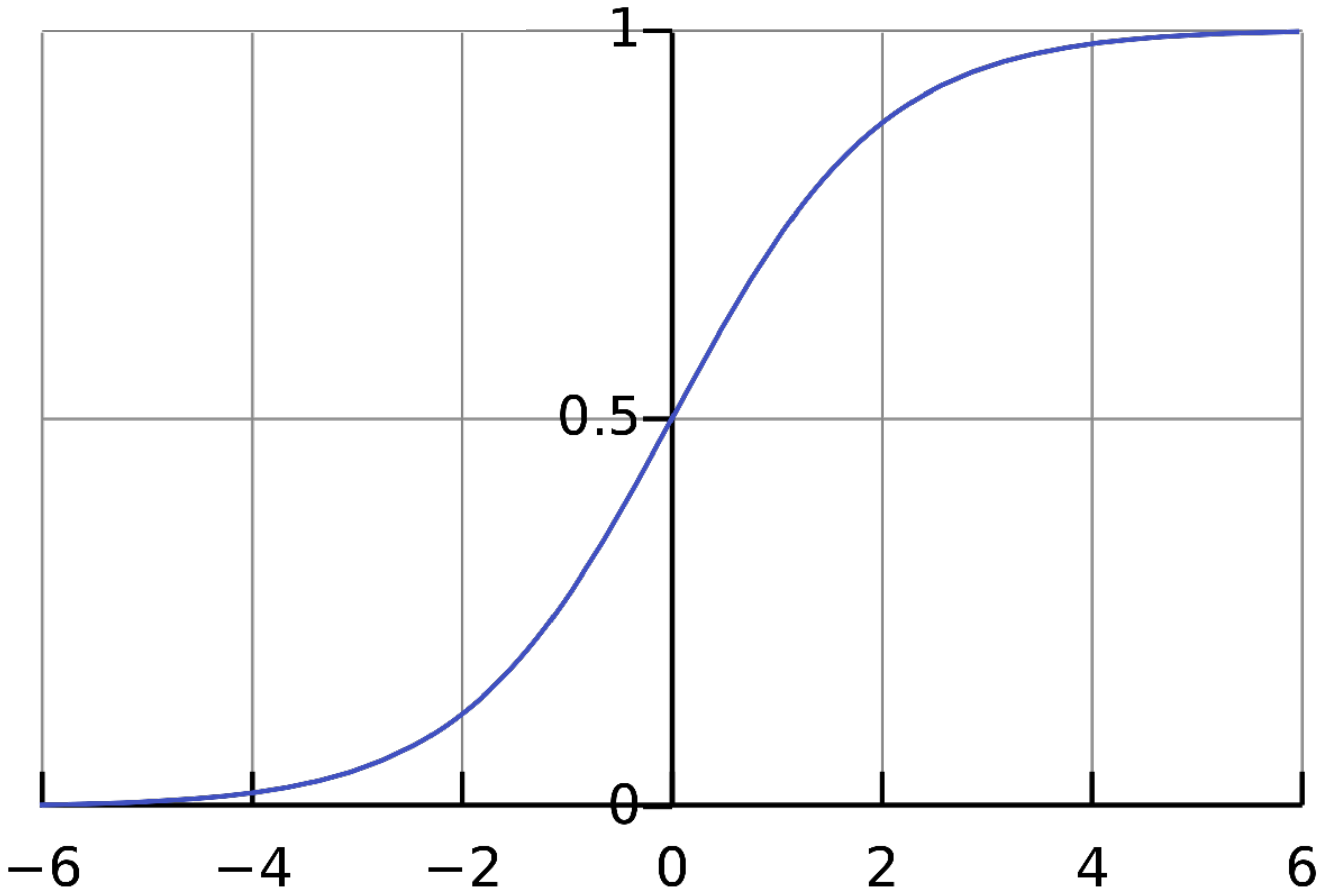
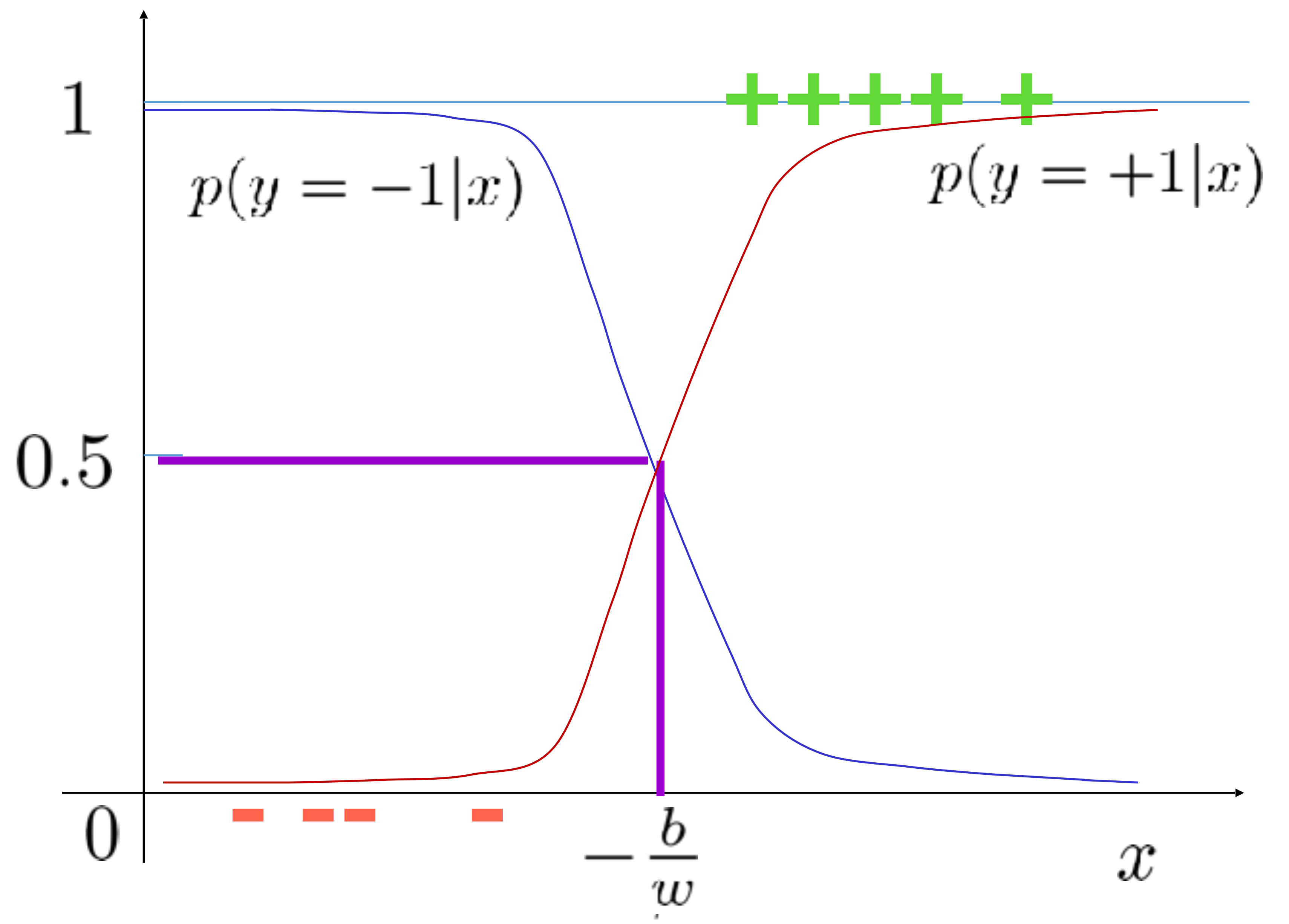


Logistic regression classifier

$x, w, b \in \mathbb{R}$

Let's look at the simplest case where x is a scalar:



We have: $f(x; w, b) = \begin{cases} +1 & if\ w \times x + b \geq 0 \\ -1 & otherwise \end{cases}.$

Probability of a sample being a positive case

$$p(y=+1|x) = \frac{1}{1+e^{-(w \times x + b)}}$$

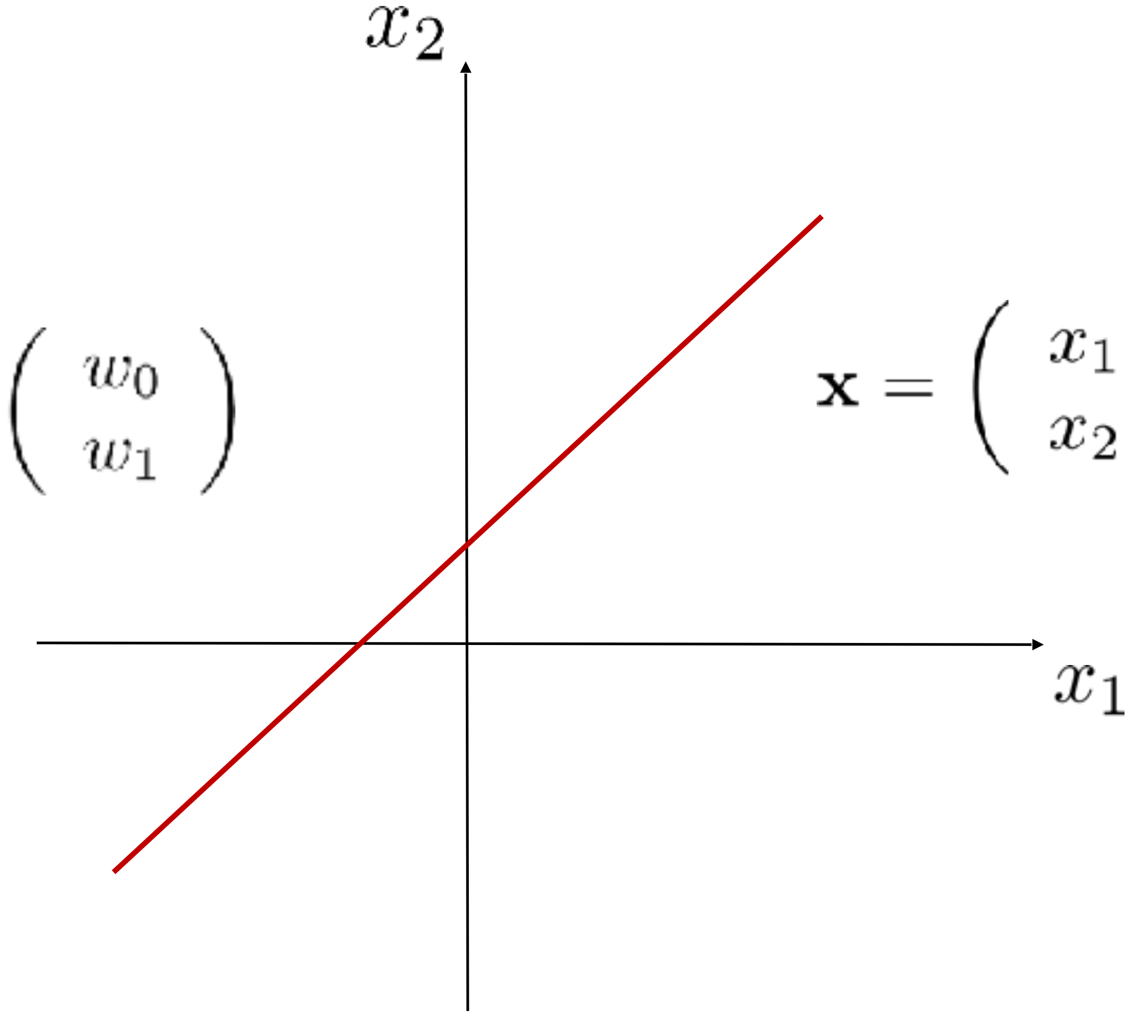
Probability of a sample being a negative case

$$p(y=-1|x) = \frac{1}{1+e^{w \times x + b}}$$

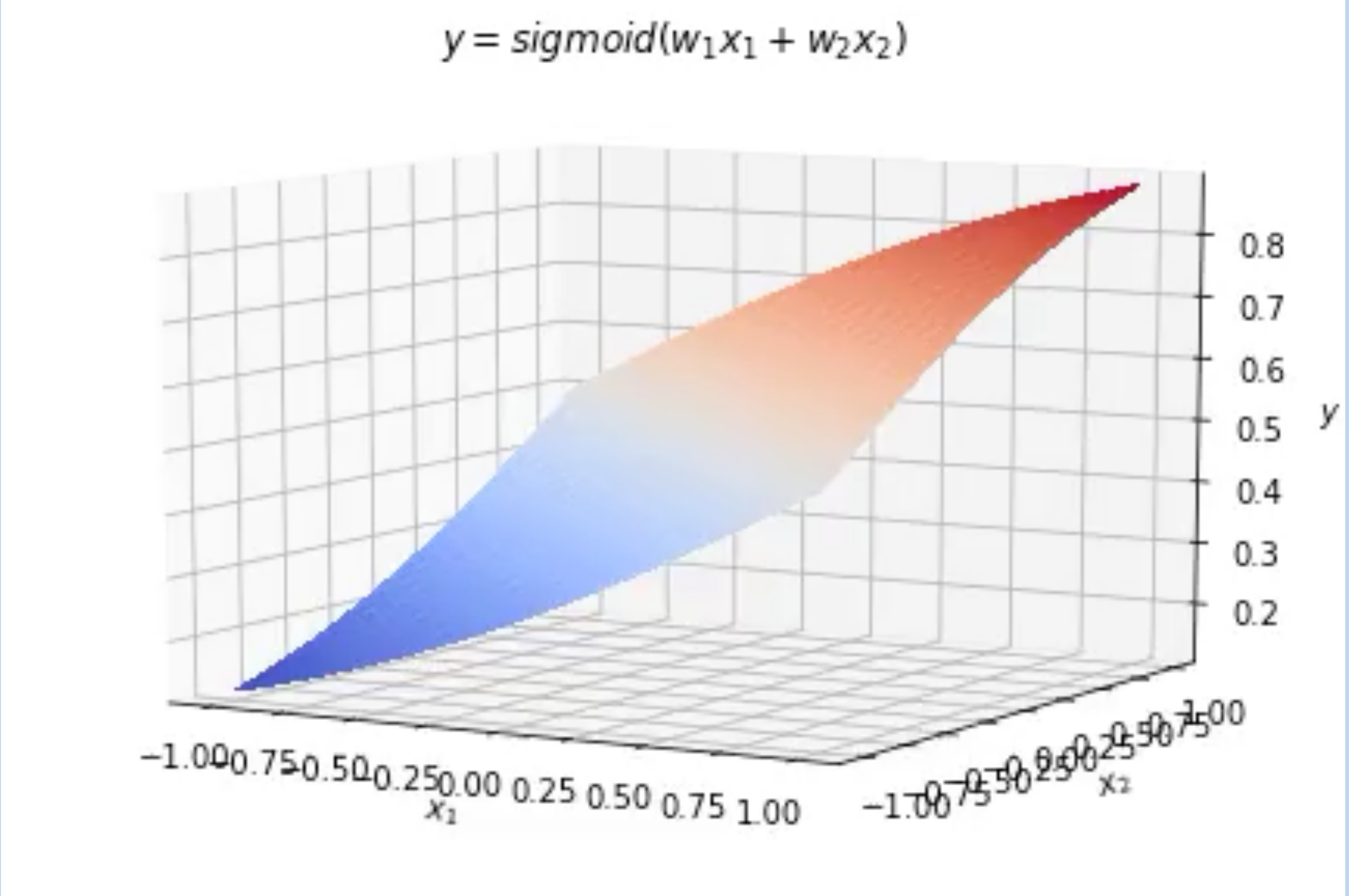
Logistic regression classifier (2D case)

$$\mathbf{w} = \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}$$

b



$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$



We have: $f(\mathbf{x};\mathbf{w},b) = \begin{cases} +1 & if \;\; \mathbf{w}\cdot\mathbf{x}+b \geq 0 \\ -1 & otherwise \end{cases}.$

sigmoid function: $\sigma(v) = \frac{1}{1+e^{(-v)}}.$

$$p(y=+1|\mathbf{x}) = \sigma\big(\mathbf{w}\cdot\mathbf{x}+b\big)$$

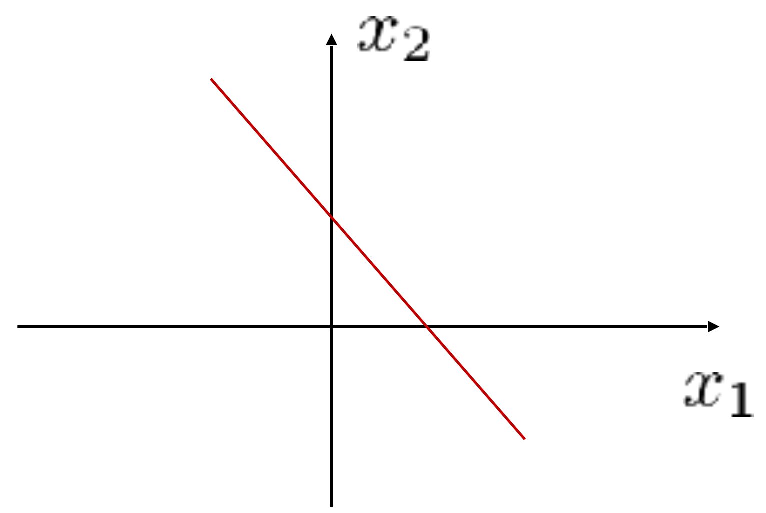
$$p(y=-1|\mathbf{x}) = \sigma\big(-(\mathbf{w}\cdot\mathbf{x}+b)\big)$$

Decision boundary for a logistic regression classifier?

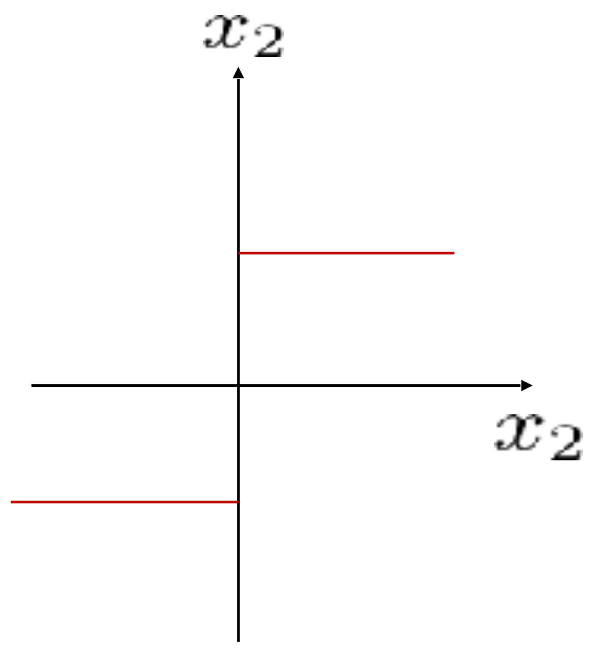
$$\mathbf{w} = \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$f(\mathbf{x}; \mathbf{w}; b) = \begin{cases} +1 & if \ \frac{1}{1+e^{-(\mathbf{x}\cdot\mathbf{w}+b)}} \geq 0.5 \\ -1 & otherwise \end{cases}.$$

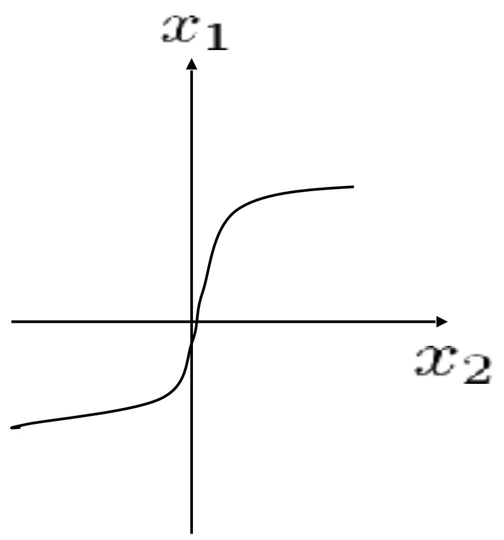
A.



B.



C.

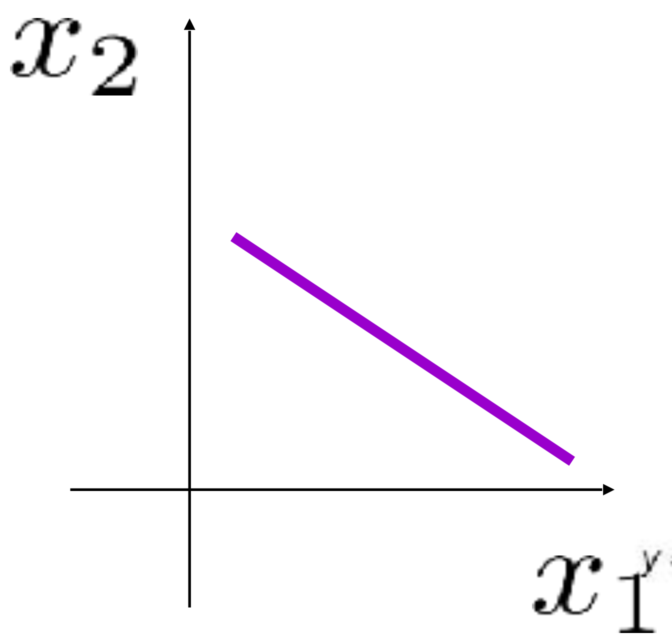


D. None of above.

Logistic regression function

$$p(y=+1|\mathbf{x}) = \frac{1}{1+e^{-(\mathbf{w}\cdot\mathbf{x}+b)}}$$

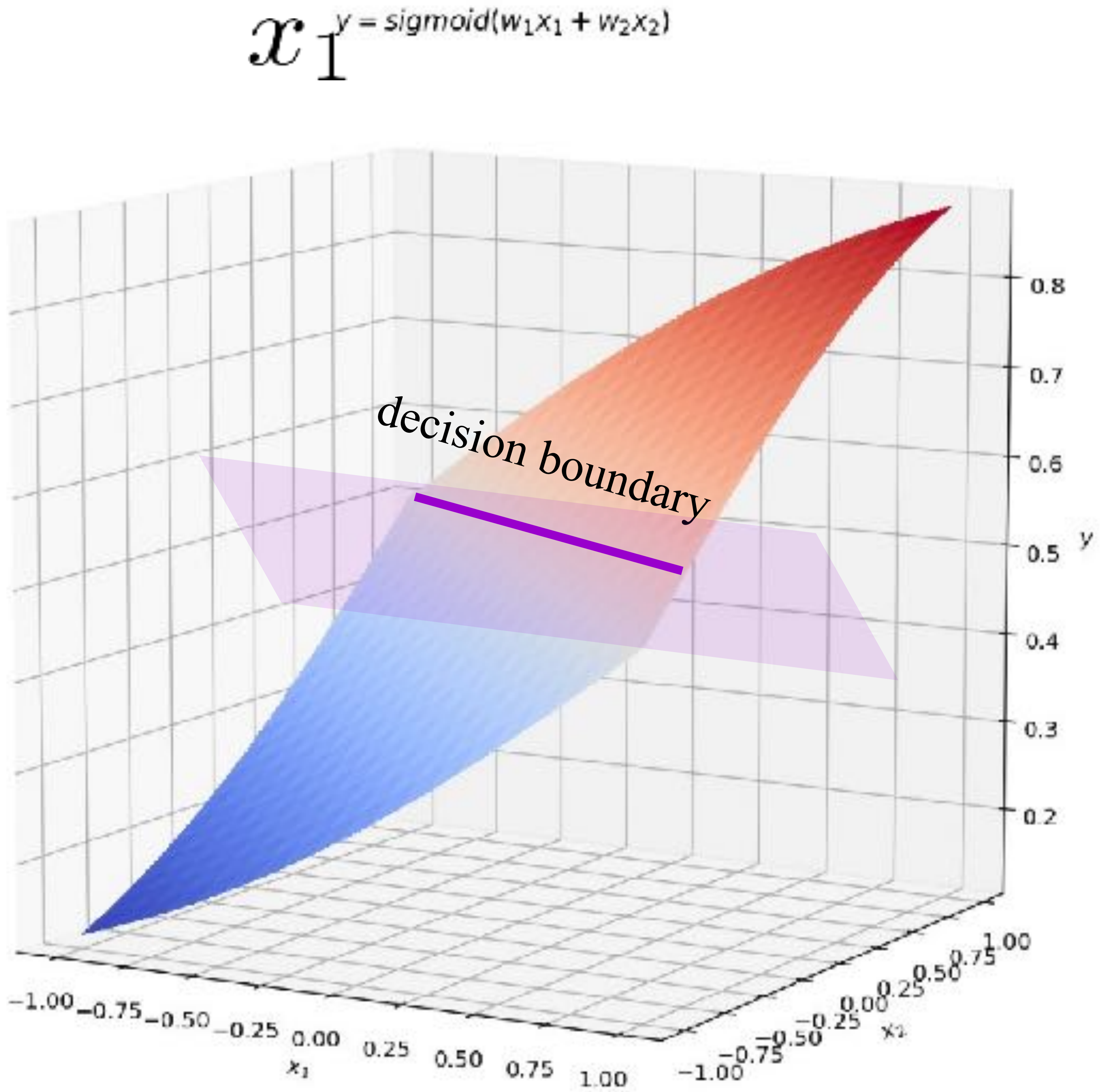
$$p(y=-1|\mathbf{x}) = \frac{1}{1+e^{(\mathbf{w}\cdot\mathbf{x}+b)}}$$



$$\mathbf{x},\mathbf{w} \in \mathbb{R}^m$$

$$b \in \mathbb{R}$$

$$y \in \{-1,+1\}$$



Training a logistic regression classifier

$$S_{training} = \{(\mathbf{x}_i, y_i), i = 1..n\}$$

$$\mathbf{x}_i \in \mathbb{R}^m, i = 1..n \qquad y_i \in \{-1, +1\}, i = 1..n$$

$$p(y_i|\mathbf{x}_i) = \frac{1}{1+e^{-y_i(\mathbf{w}^T\mathbf{x}_i+b)}}$$

Model parameters:

$\mathbf{w} \in \mathbb{R}^m$

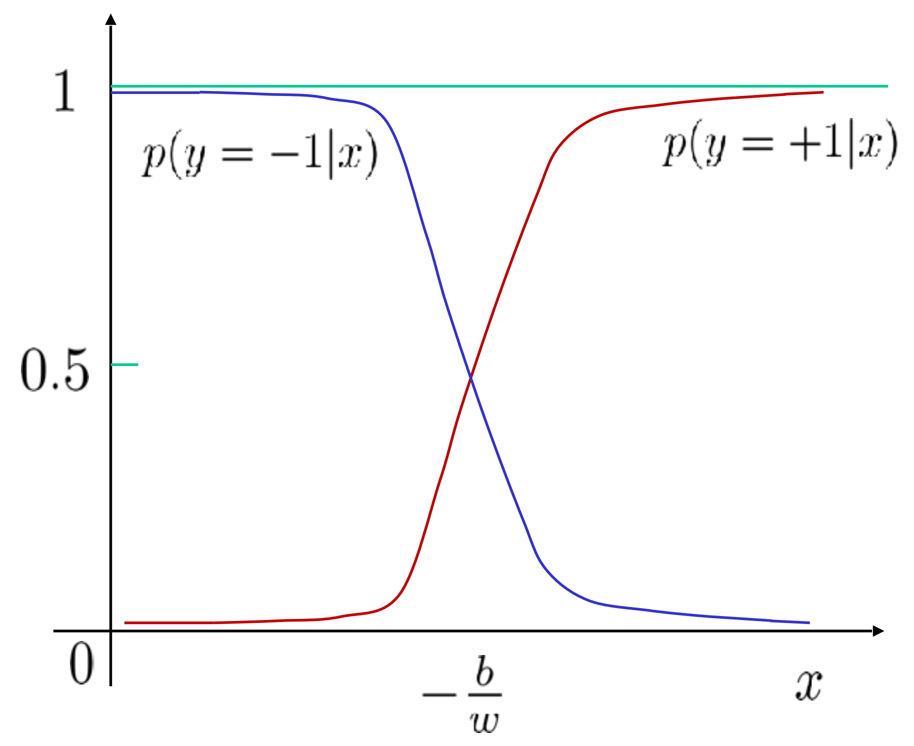
$b \in \mathbb{R}$

Training a logistic regression classifier

$$S_{training} = \{(-1.1, -1), (3.2, +1), (2.5, -1), (5.0, +1), (4.3, +1)\}$$

$$p(y = +1|\mathbf{x}) = \frac{1}{1+e^{-(\mathbf{w}^T\mathbf{x}+b)}}$$

$$p(y = -1|\mathbf{x}) = \frac{1}{1+e^{(\mathbf{w}^T\mathbf{x}+b)}}$$



$$p(y_i|\mathbf{x}_i) = \frac{1}{1+e^{-y_i(\mathbf{w}^T\mathbf{x}_i+b)}}$$

Train a logistic regression classifier $f(\mathbf{x}) = \begin{cases} +1 & if \ \frac{1}{1+e^{-(\mathbf{w}^T\mathbf{x}+b)}} \geq 0.5 \\ -1 & otherwise \end{cases}$;

Intuition: find the best parameters $(\mathbf{w}, b)^*$ to maximize the probabilities of fitting the ground-truth label y_i for each $\mathbf{x}_i$.

Math: $(\mathbf{w}, b)^* = \arg\max_{(\mathbf{w},b)} \prod_{i=1}^{n} \frac{1}{1+e^{-y_i(\mathbf{w}^T\mathbf{x}_i+b)}}$

Training a logistic regression classifier

Intuition: find the best parameters $(\mathbf{w}, b)^*$ to maximize the probabilities of fitting the ground-truth label y_i for each $\mathbf{x}_i$.

Math: $(\mathbf{w}, b)^* = \arg\max_{(\mathbf{w},b)} \prod_{i=1}^{n} \frac{1}{1+e^{-y_i(\mathbf{w}^T\mathbf{x}_i+b)}}$

$$(\mathbf{w}, b)^* = \arg\max_{(\mathbf{w},b)} \prod_{i=1}^{n} \frac{1}{1+e^{-y_i(\mathbf{w}^T\mathbf{x}_i+b)}}$$

$$= \arg\max_{(\mathbf{w},b)} \ln\left(\prod_{i=1}^{n} \frac{1}{1+e^{-y_i(\mathbf{w}^T x_i+b)}}\right)$$

$$= \arg\min_{(\mathbf{w},b)} \sum_{i=1}^{n} -\ln\left(\frac{1}{1+e^{-y_i\times(\mathbf{w}^T\mathbf{x}_i+b)}}\right)$$

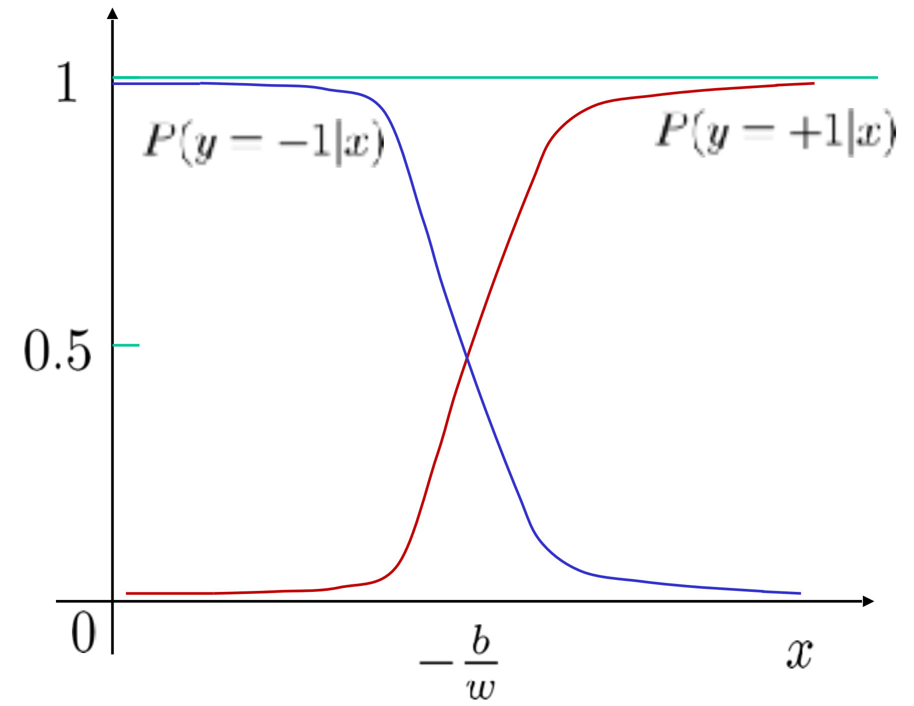
Minus sign moves inside the sum and ln!

$$= \arg\min_{(w,b)} \sum_{i=1}^{n} \ln(1+e^{-y_i\times(w\times x_i+b)})$$

Training a logistic regression classifier

$$S_{training} = \{(-1.1,-1),(3.2,+1),(2.5,-1),(5.0,+1),(4.3,+1)\} \quad y_i \in \{-1,+1\}, i = 1..n$$

$$p(y_i|\mathbf{x}_i) = \frac{1}{1+e^{-y_i(\mathbf{w}^T\mathbf{x}_i+b)}}$$



$$(\mathbf{w}, b)^* = \arg\max_{(\mathbf{w},b)} \prod_{i=1}^{n} [p(y_i|\mathbf{x}_i)]$$

$$(w,b)^* = \arg\min_{(w,b)} -\sum_{i=1}^{n} \ln\left(\frac{1}{1+e^{-y_i\times(w\times x_i+b)}}\right) = \arg\min_{(w,b)} \sum_{i=1}^{n} \ln(1+e^{-y_i\times(w\times x_i+b)})$$

$$(w,b)^* = \arg\min_{(w,b)} [\ln(1+e^{(-1.1w+b)}) + \ln(1+e^{-(3.2w+b)}) +$$

$$\ln(1+e^{(2.5w+b)}) + \ln(1+e^{-(5.0w+b)}) + \ln(1+e^{-(4.3w+b)})]$$

Training a MULTIVARIATE logistic regression classifier

$$\mathbf{x}_i \in \mathbb{R}^m, i = 1..n \qquad y_i \in \{-1, +1\}, i = 1..n$$

Model parameters: $\mathbf{w} \in \mathbb{R}^m$ and $b \in \mathbb{R}$

$$p(y_i|\mathbf{x}_i) = \frac{1}{1+e^{-y_i(\mathbf{w}^T\mathbf{x}_i+b)}}$$

Intuition: find the best parameters $(\mathbf{w}, b)^*$ to maximize the probabilities of fitting the ground-truth label y_i for each x_i.

Math: $(\mathbf{w}, b)^* = \arg\max_{(\mathbf{w},b)} \prod_{i=1}^{n} \frac{1}{1+e^{-y_i(\mathbf{w}^T\mathbf{x}_i+b)}}$

$$(\mathbf{w}, b)^* = \arg\min_{(\mathbf{w},b)} \sum_{i=1}^{n} -\ln\left(\frac{1}{1+e^{-y_i(\mathbf{w}^T\mathbf{x}_i+b)}}\right)$$

$$= \arg\min_{\mathbf{w},b} \mathcal{L}(\mathbf{w}, b)$$

Derivative for the logistic regression classifier

$$\mathcal{L}(\mathbf{w}, b) = \sum_{i=1}^{n} \ln(1 + e^{-y_i(\mathbf{w}^T \mathbf{x}_i + b)})$$

$$p(y_i|\mathbf{x}_i) = \frac{1}{1+e^{-y_i(\mathbf{w}^T \mathbf{x}_i + b)}}$$

$$\frac{\partial \mathcal{L}(\mathbf{w}, b)}{\partial \mathbf{w}} = \sum_{i=1}^{n} \frac{\partial \ln(1+e^{-y_i(\mathbf{w}^T \mathbf{x}_i + b)})}{\partial \mathbf{w}}$$

$$= \sum_{i=1}^{n} \frac{\frac{\partial(1+e^{-y_i(\mathbf{w}^T \mathbf{x}_i + b)})}{\partial \mathbf{w}}}{1+e^{-y_i(\mathbf{w}^T \mathbf{x}_i + b)}}$$

Ln’ = 1/x; chain rule

$$= \sum_{i=1}^{n} \frac{e^{-y_i(\mathbf{w}^T \mathbf{x}_i + b)}(-y_i \mathbf{x}_i)}{1+e^{-y_i(\mathbf{w}^T \mathbf{x}_i + b)}}$$

Exp’ = exp; chain rule

$$= \sum_{i=1}^{n} \frac{(1+e^{-y_i(\mathbf{w}^T \mathbf{x}_i + b)} - 1)(-y_i \mathbf{x}_i)}{1+e^{-y_i(\mathbf{w}^T \mathbf{x}_i + b)}}$$

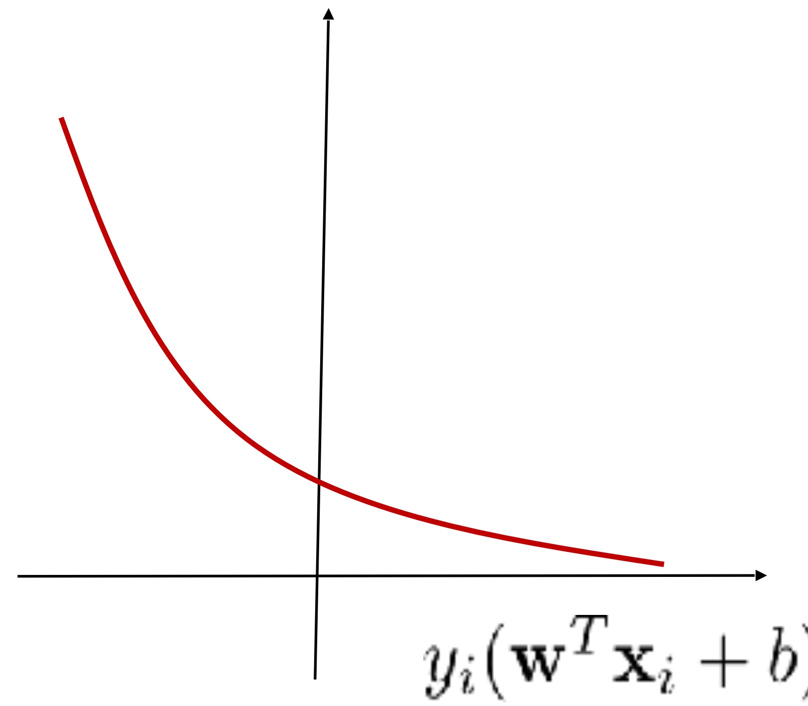
+1 - 1 to allow factoring below

$$= \sum_{i=1}^{n} \left(1 - \frac{1}{1+e^{-y_i(\mathbf{w}^T \mathbf{x}_i + b)}}\right)(-y_i \mathbf{x}_i)$$

$$= \sum_i -y_i \mathbf{x}_i (1 - p(y_i|\mathbf{x}_i))$$

Multivariate input

$$\mathcal{L}(\mathbf{w}, b) = \sum_{i=1}^{n} \ln(1 + e^{-y_i(\mathbf{w}^T\mathbf{x}_i + b)})$$



$$\boxed{\begin{aligned} \nabla_{\mathbf{w}}\mathcal{L}(\mathbf{w}, b) &= \sum_i \frac{-y_i\mathbf{x}_i e^{-y_i(\mathbf{w}^T\mathbf{x}_i+b)}}{1+e^{-y_i(\mathbf{w}^T\mathbf{x}_i+b)}} = \sum_i -y_i\mathbf{x}_i(1 - p(y_i|\mathbf{x}_i)) \\ \nabla_b\mathcal{L}(\mathbf{w}, b) &= \sum_i \frac{-y_i e^{-y_i(\mathbf{w}^T\mathbf{x}_i+b)}}{1+e^{-y_i(\mathbf{w}^T\mathbf{x}_i+b)}} = \sum_i -y_i(1 - p(y_i|\mathbf{x}_i)) \end{aligned}}$$

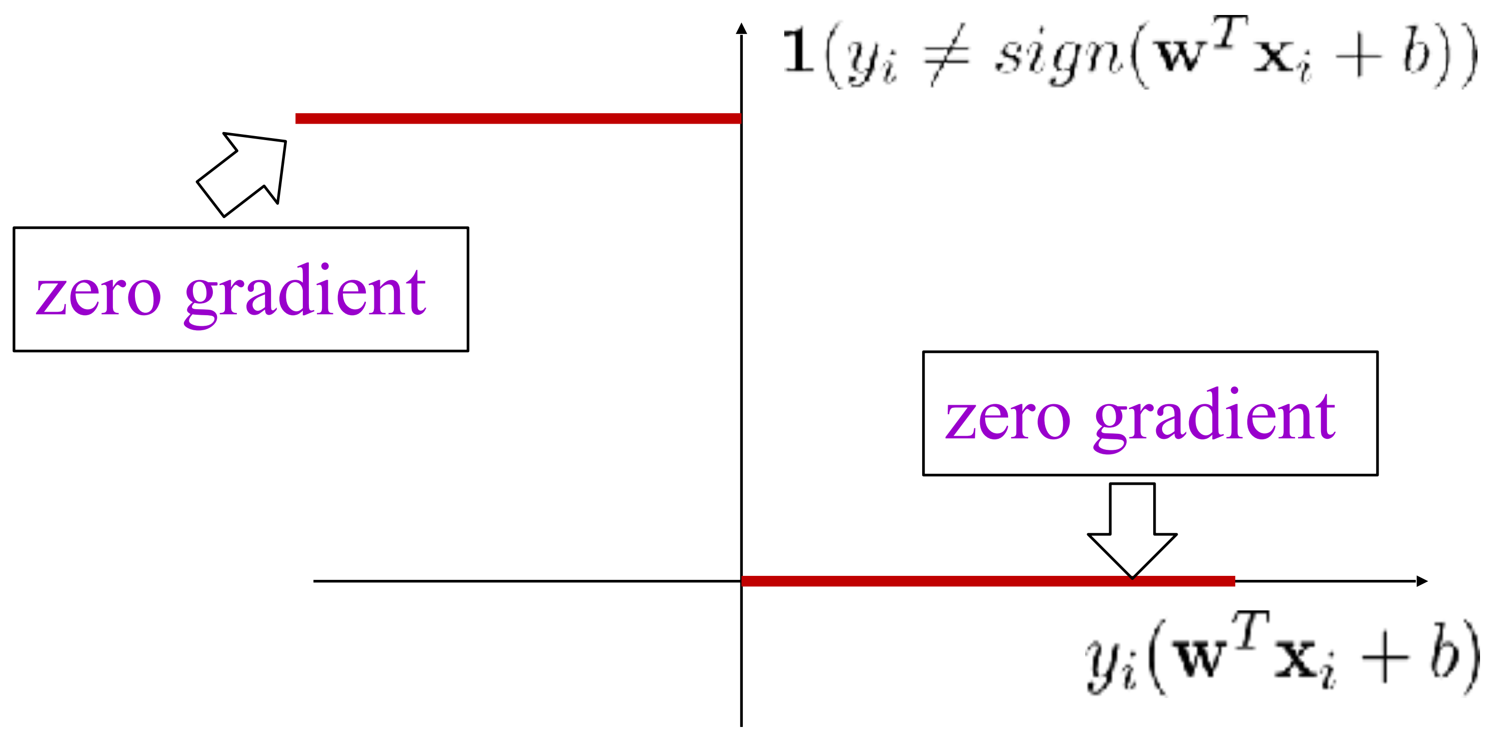
$$\mathbf{w}_{t+1} = \mathbf{w}_t - \lambda_t \times \nabla_{\mathbf{w}}\mathcal{L}(\mathbf{w}_t, b_t)$$

$$b_{t+1} = b_t - \lambda_t \times \nabla_b\mathcal{L}(\mathbf{w}_t, b_t)$$

Hard loss (error) function

Standard 0/1 loss (gradient 0 nearly everywhere, no gradient feedback):

Training: Minimize $\mathcal{L}(\mathbf{w}, b) = \sum_i \mathbf{1}(y_i \neq sign(\mathbf{w}^T\mathbf{x}_i + b))$



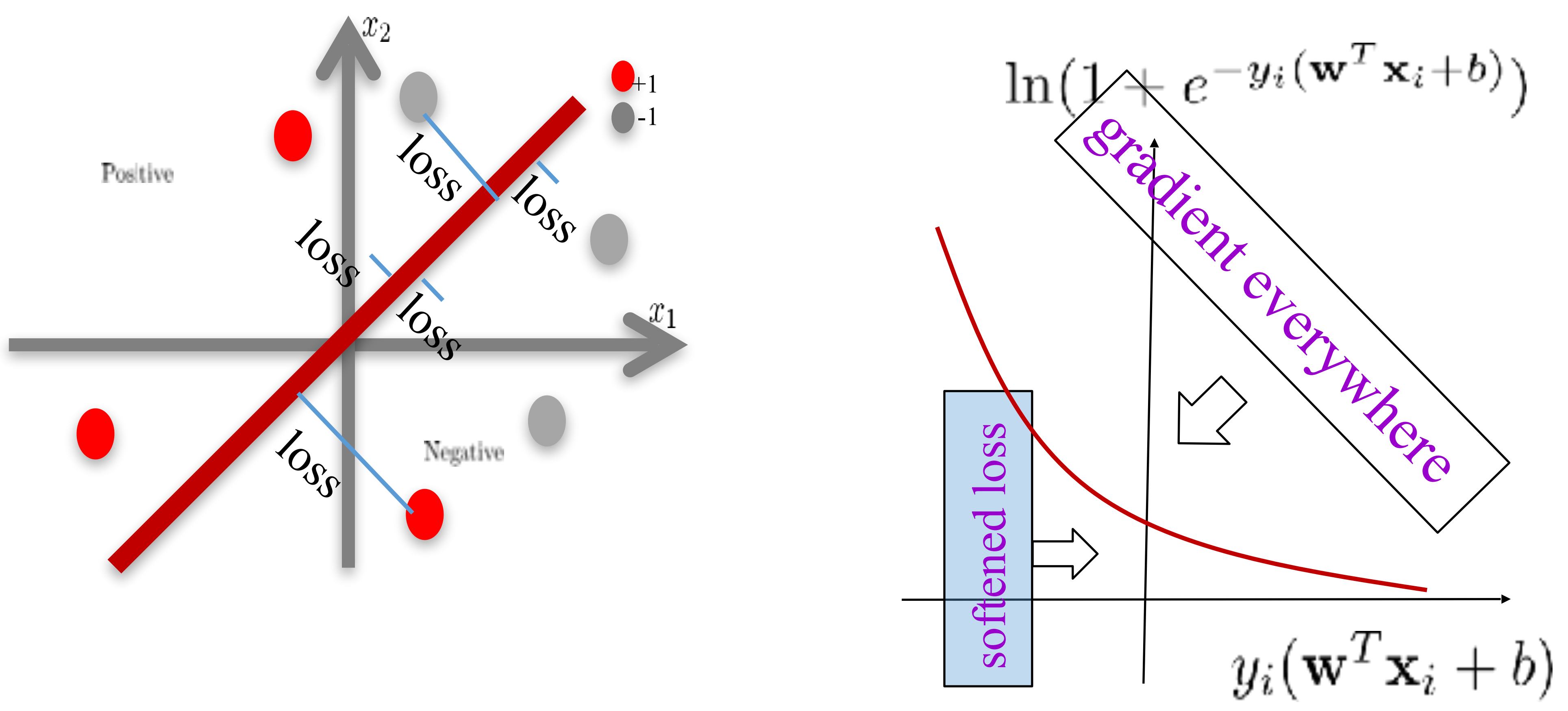
It is the most direct loss, but is also the hardest to minimize.

Zero gradient everywhere!

Soft loss (error) function

Loss used in logistic regression.

Training: minimize $\mathcal{L}(\mathbf{w}, b) = \sum_{i=1}^{n} \ln(1 + e^{-y_i(\mathbf{w}^T\mathbf{x}_i + b)})$



Every data point receives a loss (gradient everywhere).

A loss based on the distance to the decision boundary for wrong classification (has a gradient).

Used in logistic regression classifier.

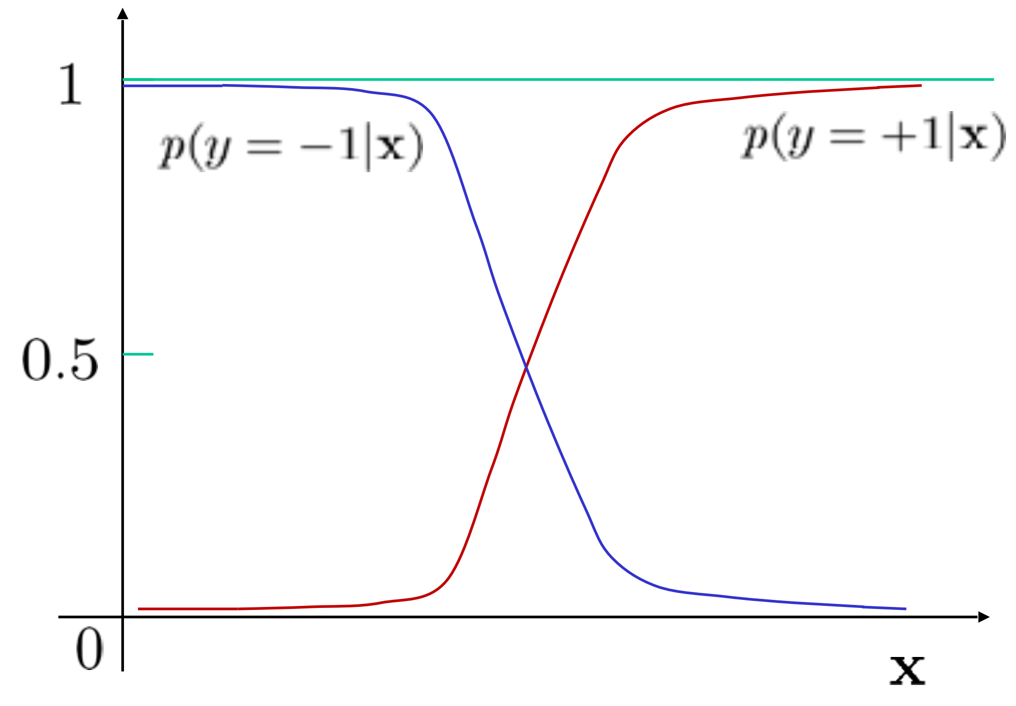
Logistic regression classifier

$$p(y_i|\mathbf{x}_i) = \frac{1}{1+e^{-y_i(\mathbf{w}\cdot\mathbf{x}_i+b)}}$$

$$\mathbf{x} \in \mathbb{R}^m$$

$$y \in \{-1,+1\}$$

$$f(\mathbf{x}) = \begin{cases} +1 & if\ \frac{1}{1+e^{-(\mathbf{w}\cdot\mathbf{x}+b)}} \geq 0.5 \\ -1 & otherwise \end{cases}$$



Pros:

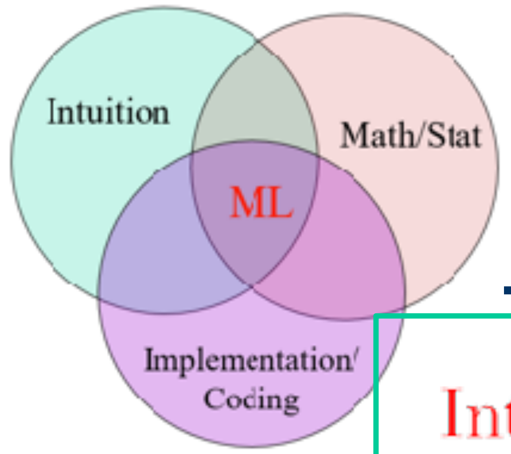
1. It is well-normalized.
2. Easy to turn into probability.
3. Easy to implement.

Cons:

1. Indirect loss function.
2. Dependent on good feature set.
3. Weak on feature selection.

Take home message

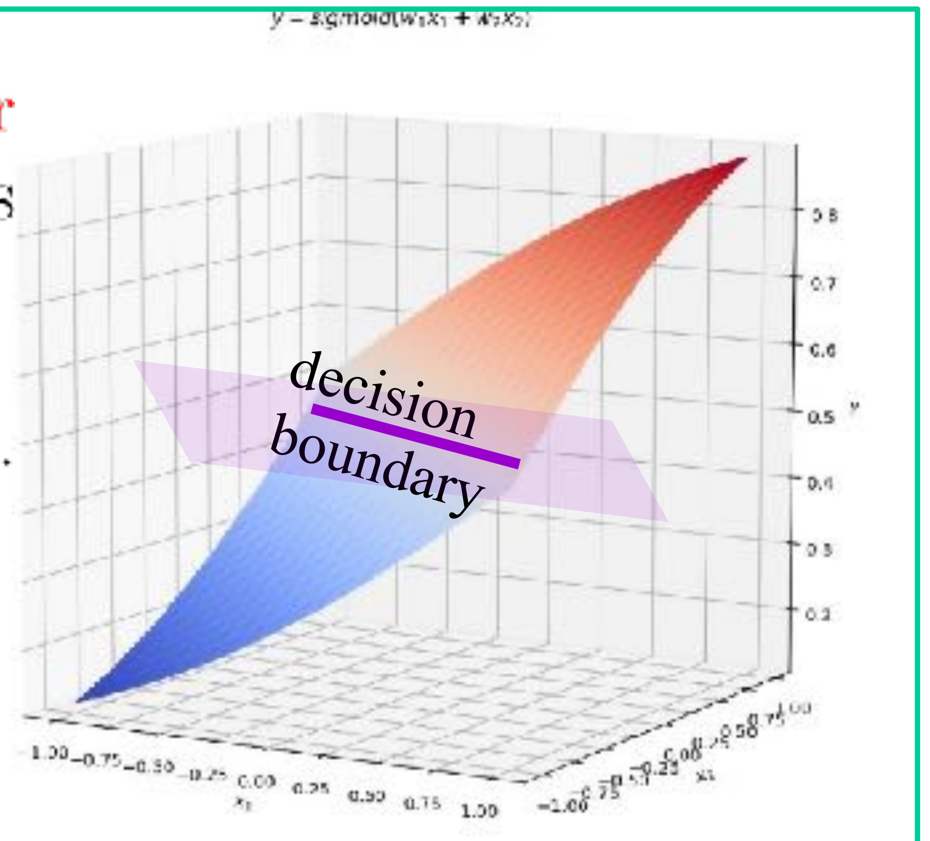
- Logistic regression classifier is still a linear classifier but with a probability output.
- It can be trained using a gradient descent algorithm.
- The "regression" refers to fitting the discriminative probabilities: $p(y|\mathbf{x})$
- It has been widely adopted in practice, especially in the modern deep learning era.



Recap: Logistic Regression Classifier

Intuition: Logistic regression classifier nicely turns a hard classification error (0 *or* 1) into a soft measure using the sigmoid function $\sigma(v) = \frac{1}{1+e^{-v}}$ which has three particuarly appealing properties:

- A soft measure that maps any value $v \in (-\infty, \infty)$ to a normalized $\rightarrow (0, 1)$.
- Nice gradient form.
- Convex function for the objective function in training.



Math:

$$p(y|\mathbf{x}) = \frac{1}{1+e^{-y(\mathbf{w}^T\mathbf{x}+b)}}$$

Training :

$$(\mathbf{w}, b)^* = \arg\min_{(\mathbf{w},b)} \mathcal{L}(\mathbf{w}, b) = \arg\min_{(\mathbf{w},b)} \sum_{i=1}^{n} \ln(1 + e^{-y_i(\mathbf{w}^T\mathbf{x}_i+b)})$$

$$\nabla_{\mathbf{w}}\mathcal{L}(\mathbf{w}, b) = \sum_i -y_i \times \mathbf{x}_i(1 - p(y_i|\mathbf{x}_i))$$

$$\nabla_b\mathcal{L}(\mathbf{w}, b) = \sum_i -y_i \times (1 - p(y_i|\mathbf{x}_i))$$

Recap: Logistic Regression Classifier

Implementation:

Gradient Descent Direction

(a) Pick a direction $\nabla\mathcal{L}(\mathbf{w}_t, b_t)$

(b) Pick a step size λ_t

(c) $\mathbf{w}_{t+1} = \mathbf{w}_t - \lambda_t \times \nabla\mathcal{L}_{\mathbf{w}_t}(\mathbf{w}_t, b_t)$ such that function decreases;
$b_{t+1} = b_t - \lambda_t \times \nabla\mathcal{L}_{b_t}(\mathbf{w}_t, b_t)$

(d) Repeat