第一节

向量及其运算

四、两向量的数量积/*Scalar Product*/

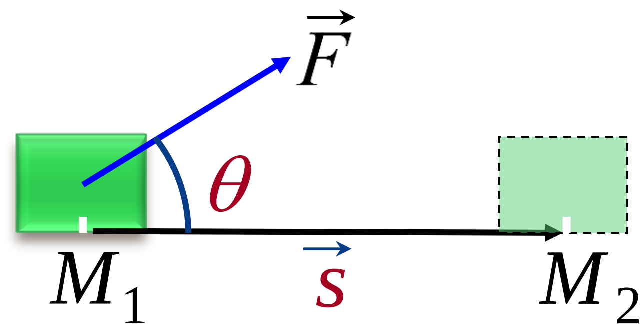
五、两向量的向量积/*Vector Product*/

六、向量的混合积/*Mixed Product*/

四、两向量的数量积

引例. 设一物体在常力 $\vec{F}$ 作用下,沿与力夹角为θ的直线移动, 位移为$\vec{s}$, 则力$\vec{F}$ 所作的功为

$$W = |\vec{F}|\cos\theta \cdot |\vec{s}| = |\vec{F}||\vec{s}|\cos\theta$$



$$\boxed{W = \vec{F}\cdot\vec{s}}$$

1. 定义

设向量 $\vec{a}, \vec{b}$ 的夹角为θ, 称

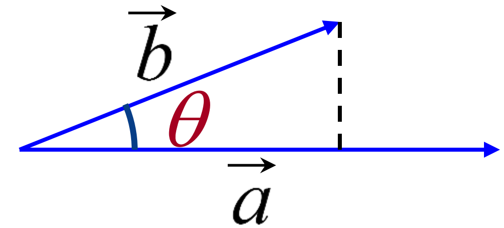
$$|\vec{a}||\vec{b}|\cos\theta \overset{\text{记作}}{=\!=\!=} \vec{a}\cdot\vec{b}$$

为$\vec{a}$与$\vec{b}$的**数量积**

(点积/*Doc Product*/;内积/*Inner Product*/).

当$\vec{a}\neq\vec{0}$时，$\vec{b}$在$\vec{a}$上的投影为

$$|\vec{b}|\cos\theta \xlongequal{\text{记作}} \mathrm{Prj}_{\vec{a}}\vec{b}$$



故 $\vec{a}\cdot\vec{b}=|\vec{a}|\,\mathrm{Prj}_{\vec{a}}\vec{b}$

同理，当$\vec{b}\neq\vec{0}$时，

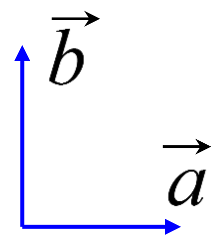
$$\vec{a}\cdot\vec{b}=|\vec{b}|\,\mathrm{Prj}_{\vec{b}}\vec{a}$$

2. 性质

(1) $\vec{a}\cdot\vec{a}=|\vec{a}|^2$

(2) $\vec{a},\vec{b}$为两个非零向量，则有

$$\vec{a}\cdot\vec{b}=0 \iff \vec{a}\perp\vec{b}$$



$\vec{a}\neq\vec{0},\ \vec{b}\neq\vec{0}$

则 $\vec{a}\cdot\vec{b}=0$

$\Updownarrow$

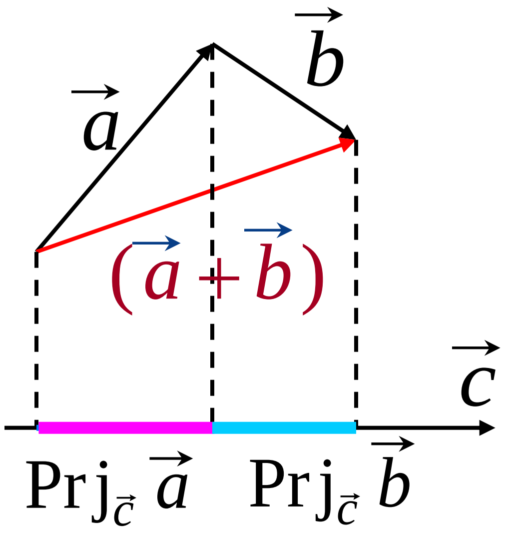
$(\widehat{\vec{a},\vec{b}})=\dfrac{\pi}{2}$

$$\boxed{\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta}$$

3. 运算律

(1) 交换律 $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$

(2) 分配律 $(\vec{a}+\vec{b})\cdot\vec{c}=\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}$



证 事实上, 当 $\vec{c}=\vec{0}$ 时, 显然成立;

当 $\vec{c}\neq\vec{0}$ 时, $(\vec{a}+\vec{b})\cdot\vec{c}$

$$=|\vec{c}|\,\mathrm{Prj}_{\vec{c}}(\vec{a}+\vec{b})$$

$$=|\vec{c}|\,(\mathrm{Prj}_{\vec{c}}\,\vec{a}+\mathrm{Prj}_{\vec{c}}\,\vec{b})$$

$$=|\vec{c}|\,\mathrm{Prj}_{\vec{c}}\,\vec{a}+|\vec{c}|\,\mathrm{Prj}_{\vec{c}}\,\vec{b}=\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}$$

3. 运算律

(1) 交换律 $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$

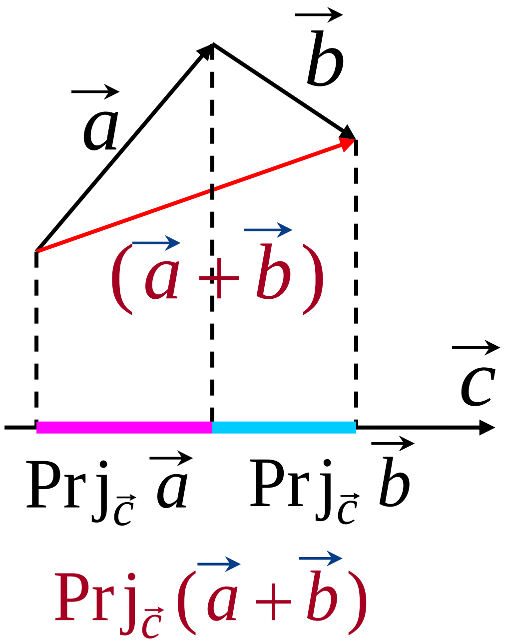
(2) 分配律 $(\vec{a}+\vec{b})\cdot\vec{c}=\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}$

(3) 结合律 $(\lambda,\mu$为实数$)$

$$(\lambda\vec{a})\cdot\vec{b}=\vec{a}\cdot(\lambda\vec{b})=\lambda(\vec{a}\cdot\vec{b})$$

$$(\lambda\vec{a})\cdot(\mu\vec{b})=\lambda(\vec{a}\cdot(\mu\vec{b}))$$

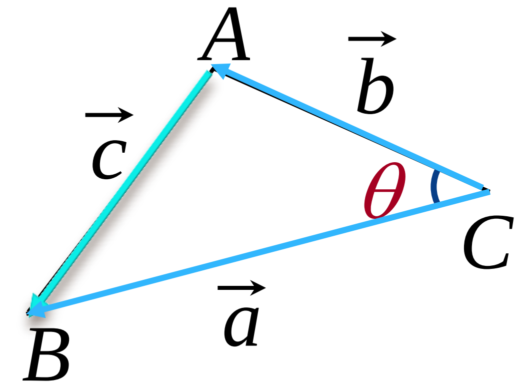
$$=\lambda\mu(\vec{a}\cdot\vec{b})$$



例1. 证明三角形余弦定理

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

证: 如图. 设



$$\overrightarrow{CB} = \vec{a},\quad \overrightarrow{CA} = \vec{b},\quad \overrightarrow{AB} = \vec{c}$$

则 $\vec{c} = \vec{a} - \vec{b}$

$$|\vec{c}|^2 = (\vec{a} - \vec{b})\cdot(\vec{a} - \vec{b}) = \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} - 2\vec{a}\cdot\vec{b}$$

$$= |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$$

$$a = |\vec{a}|,\ b = |\vec{b}|,\ c = |\vec{c}|$$

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

4. 数量积的坐标表示[计算公式]

设 $\vec{a}=a_x\vec{i}+a_y\vec{j}+a_z\vec{k}$, $\vec{b}=b_x\vec{i}+b_y\vec{j}+b_z\vec{k}$, 则

$$\vec{a}\cdot\vec{b}=(a_x\vec{i}+a_y\vec{j}+a_z\vec{k})\cdot(b_x\vec{i}+b_y\vec{j}+b_z\vec{k})$$

$$\vec{i}\cdot\vec{i}=\vec{j}\cdot\vec{j}=\vec{k}\cdot\vec{k}=1,\quad \vec{i}\cdot\vec{j}=\vec{j}\cdot\vec{k}=\vec{k}\cdot\vec{i}=0$$

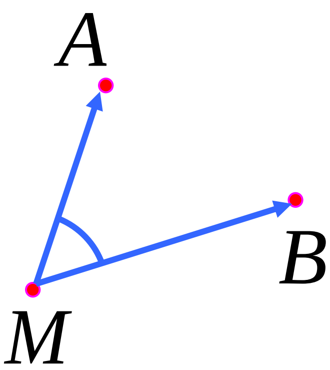
$$\vec{a}\cdot\vec{b}=a_xb_x+a_yb_y+a_zb_z$$

两向量的夹角公式

当 $\vec{a}$, $\vec{b}$ 为非零向量时, 由于 $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$, 得

$$\cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}=\frac{a_xb_x+a_yb_y+a_zb_z}{\sqrt{a_x^2+a_y^2+a_z^2}\ \sqrt{b_x^2+b_y^2+b_z^2}}$$

例2. 已知三点 $M(1,1,1)$, $A(2,2,1)$, $B(2,1,2)$, 求 $\angle AMB$.



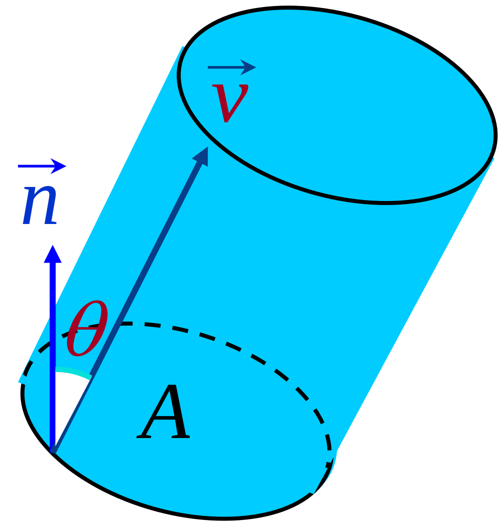
解: $\overrightarrow{MA}=(1,\ 1,\ 0)$, $\overrightarrow{MB}=(1,\ 0,\ 1)$

则 $$\cos\angle AMB=\frac{\overrightarrow{MA}\cdot\overrightarrow{MB}}{|\overrightarrow{MA}||\overrightarrow{MB}|}$$

$$=\frac{1+0+0}{\sqrt{2}\,\sqrt{2}}=\frac{1}{2}$$

故 $$\angle AMB=\frac{\pi}{3}$$

例3. 设均匀流速为 $\vec{v}$ 的流体流过一个面积为 A 的平面域, 且 $\vec{v}$ 与该平面域的单位垂直向量 $\vec{n}$ 的夹角为 θ, 求单位时间内流过该平面域的流体的质量 P (流体密度为 ρ).



解: $P = \rho A |\vec{v}| \cos\theta$

$\vec{n}$ 为单位向量

$= \rho A |\vec{v}||\vec{n}| \cos\theta$

$= \rho A \vec{v} \cdot \vec{n}$

单位时间内流过的体积为:

底面积为 A 斜高为 $|v|$ 的斜柱体的体积 $A|\vec{v}|\cos\theta$

五、两向量的向量积

引例. 利用扳手拧螺母时的力矩分析

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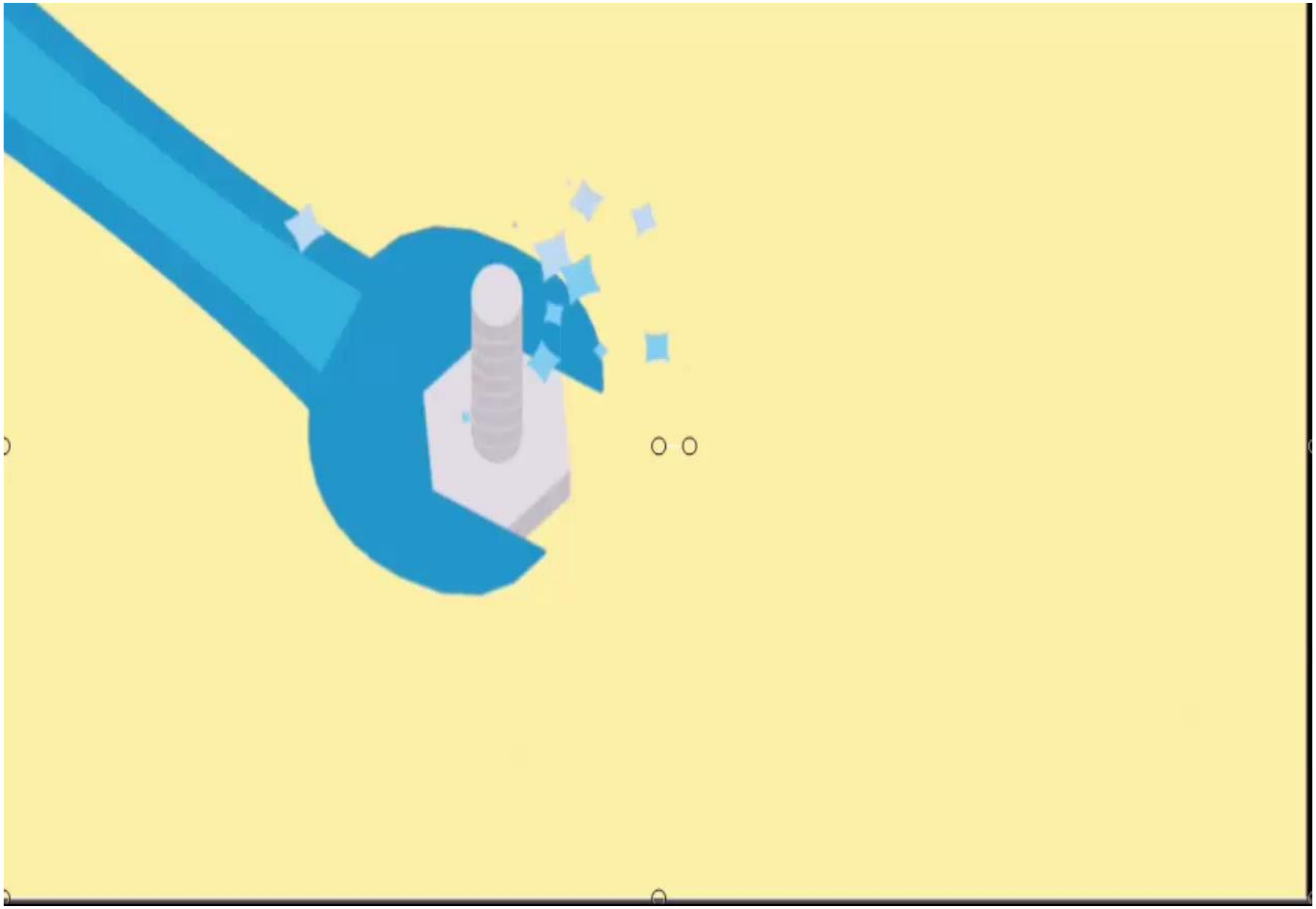
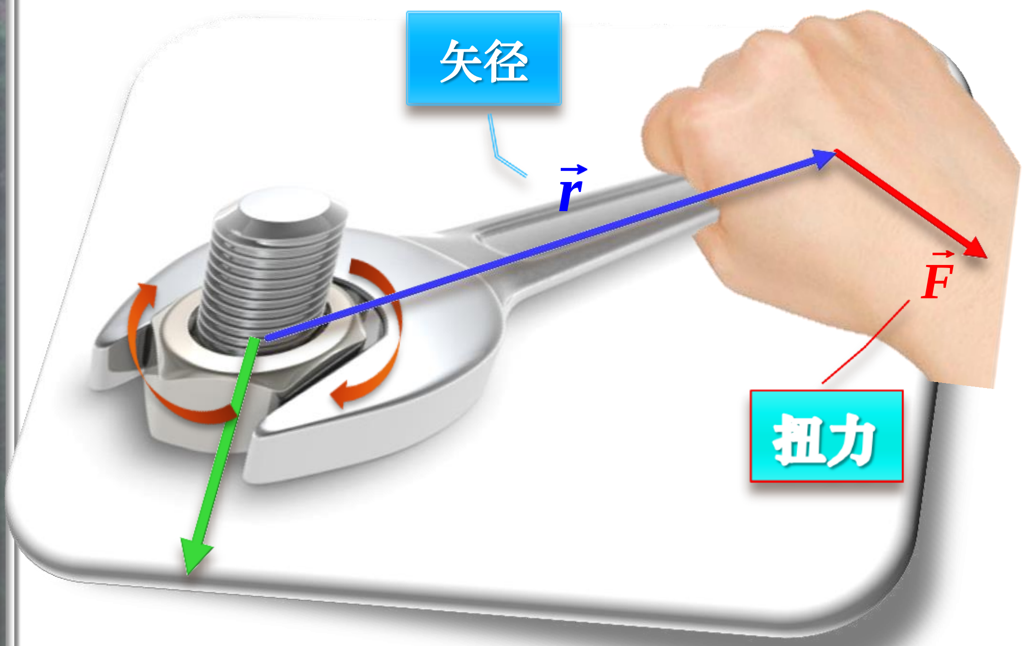


五、两向量的向量积

描述螺母在扭力作用下的运动轨迹特征

引例. 利用扳手拧螺母时的力矩分析



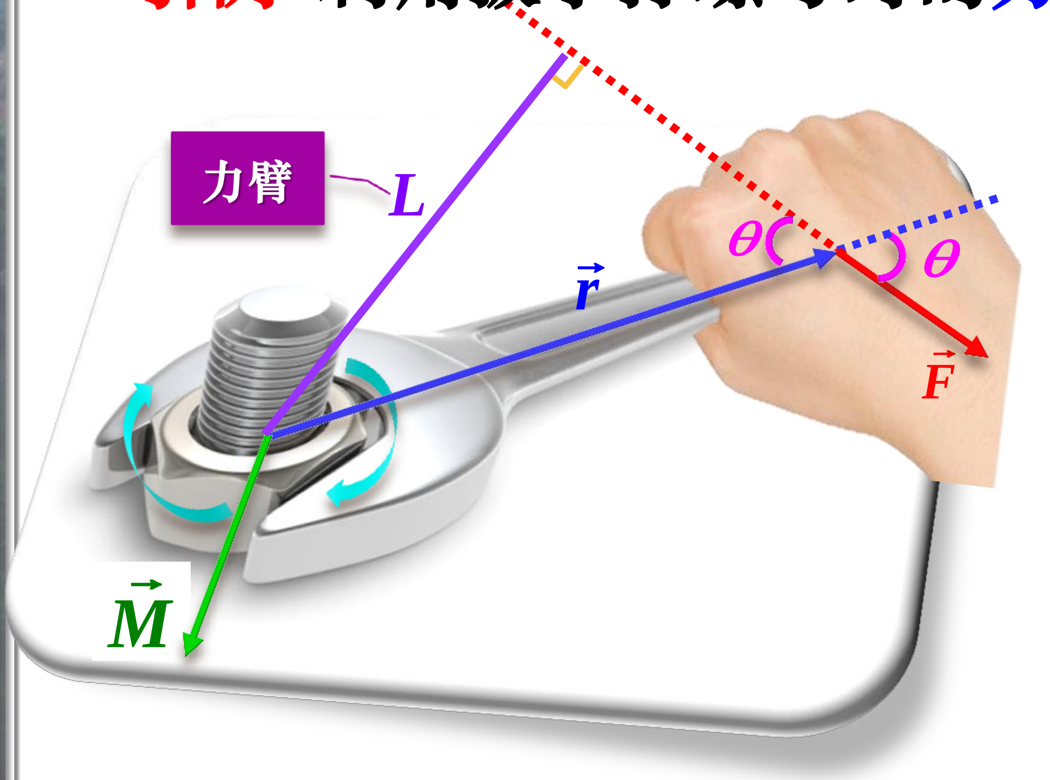
- 当扭力顺时针向右后方拉动时，使螺母在顺时针旋转的同时即可沿螺丝钉向下移动.

满足右手规则：
由$\vec{r}$到$\vec{F}$握拳时大拇指方向.

既有大小 又有方向

五、两向量的向量积

引例. 利用扳手拧螺母时的力矩分析



力矩大小

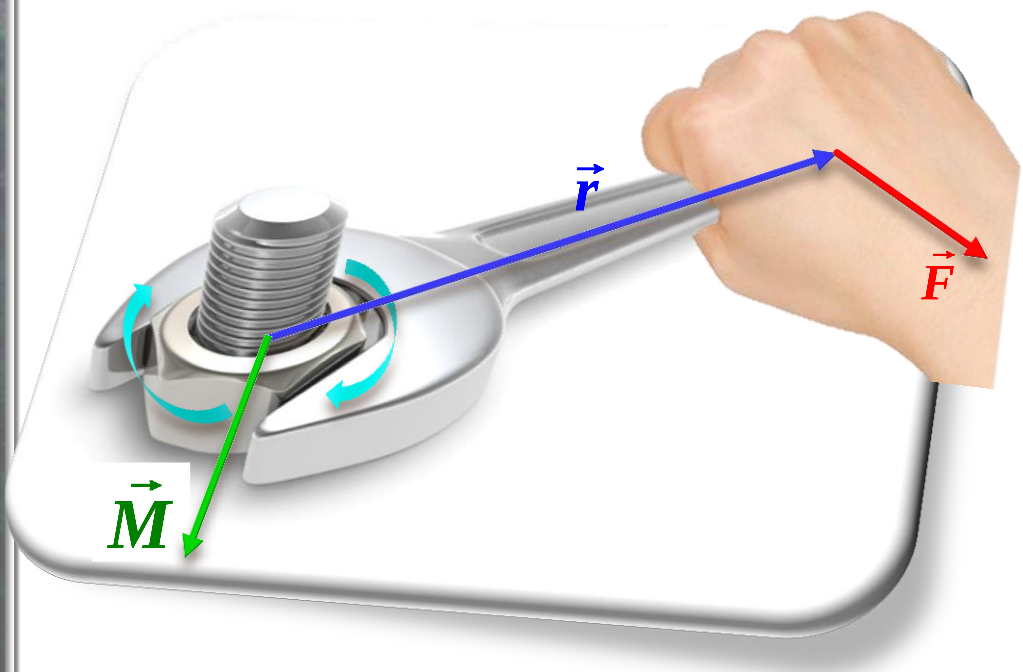
$= |$扭力$| \cdot$力臂

$$= |\vec{F}| \cdot |\vec{r}| \sin(\theta)$$

$$= |\vec{r}| \cdot |\vec{F}| \cdot \sin(\vec{r}, \vec{F})$$

五、两向量的向量积

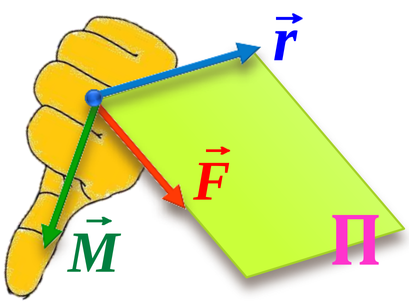
引例. 利用扳手拧螺母时的力矩分析



记作 $\vec{M} = \vec{r} \times \vec{F}$

大小 $|\vec{r}|\cdot|\vec{F}|\cdot\sin(\vec{r},\vec{F})$

方向 右手规则



1. 定义

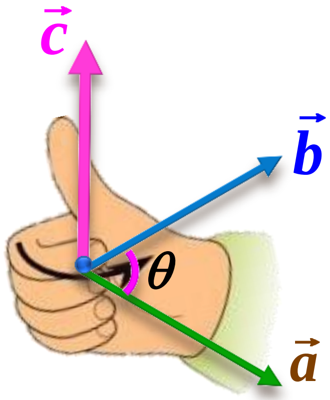
设$\vec{a},\vec{b}$的夹角为θ, 定义

$$\text{向量 } \vec{c} \begin{cases} \text{方向: } \vec{c}\perp\vec{a},\ \vec{c}\perp\vec{b} \text{ 且符合右手规则} \\ \text{模: } |\vec{c}| = |\vec{a}||\vec{b}|\sin\theta \end{cases}$$

称 $\vec{c}$ 为向量$\vec{a}$与$\vec{b}$的 向量积, 记作

$$\vec{c} = \vec{a}\times\vec{b}$$

(叉积/*Cross Product*/外积/*Outer Product*/)

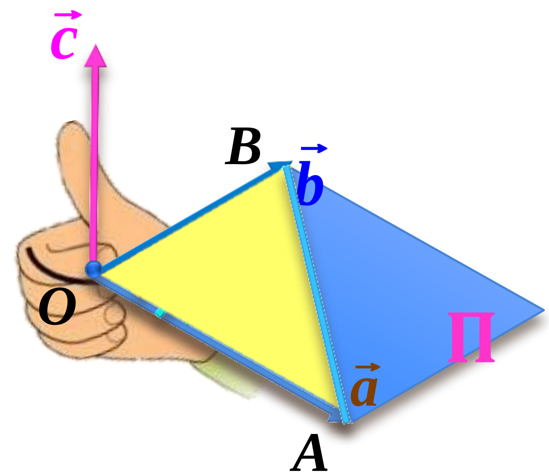


1. 定义

设 $\vec{a},\vec{b}$ 的夹角为 θ，定义

$$\text{向量 } \vec{c} \begin{cases} \text{方向: } \vec{c}\perp\vec{a},\ \vec{c}\perp\vec{b} \text{ 且符合右手规则} \\ \text{模: } |\vec{c}| = |\vec{a}|\,|\vec{b}|\sin\theta \end{cases}$$

称 $\vec{c}$ 为向量 $\vec{a}$ 与 $\vec{b}$ 的**向量积** $\vec{c}=\vec{a}\times\vec{b}$



- **向量叉积的几何意义：**

$$|\vec{c}| = |\vec{a}|\cdot|\vec{b}|\cdot\sin\theta = |\vec{a}|\,h = S(\square),$$

则 $S(\triangle OAB) = \dfrac{1}{2}|\vec{a}|\cdot|\vec{b}|\cdot\sin\theta$.

2. 性质

$$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$$

(1) $\vec{a}\times\vec{a} = \vec{0}$

(2) $\vec{a}, \vec{b}$**为非零向量, 则** $\vec{a}\times\vec{b} = \vec{0} \iff \vec{a} \,/\!/\, \vec{b}$

证明: 当$\vec{a}\neq\vec{0}, \vec{b}\neq\vec{0}$时,

$$\vec{a}\times\vec{b} = \vec{0} \iff |\vec{a}||\vec{b}|\sin\theta = 0$$

$$\iff \sin\theta = 0, \text{即 } \theta = 0 \text{ 或 } \pi \iff \vec{a} \,/\!/\, \vec{b}$$

3. 运算律

(1) $\vec{\boldsymbol{a}}\times\vec{\boldsymbol{b}} = -\vec{b}\times\vec{a}$

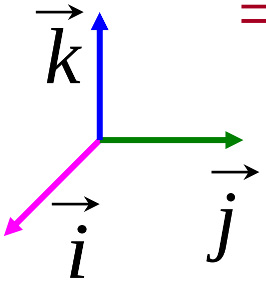
(2) **分配律** $(\vec{a}+\vec{b})\times\vec{c} = \vec{a}\times\vec{c}+\vec{b}\times\vec{c}$

(3) **结合律** $(\lambda\vec{a})\times\vec{b} = \vec{a}\times(\lambda\vec{b}) = \lambda(\vec{a}\times\vec{b})$

4.(1) 向量积的坐标表示式[计算公式]

设 $\vec{a}=a_x\vec{i}+a_y\vec{j}+a_z\vec{k}$, $\vec{b}=b_x\vec{i}+b_y\vec{j}+b_z\vec{k}$, 则

$$\vec{a}\times\vec{b}=(\underline{a_x}\vec{i}+\underline{a_y}\vec{j}+\underline{a_z}\vec{k})\times(b_x\vec{i}+b_y\vec{j}+b_z\vec{k})$$



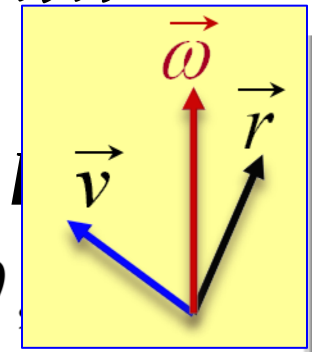
$$=\cancel{a_xb_x(\vec{i}\times\vec{i})}+\underline{a_xb_y(\vec{i}\times\vec{j})}+\underline{a_xb_z(\vec{i}\times\vec{k})}$$

$$+\underline{a_yb_x(\vec{j}\times\vec{i})}+\cancel{a_yb_y(\vec{j}\times\vec{j})}+\underline{a_yb_z(\vec{j}\times\vec{k})}$$

$$+\underline{a_zb_x(\vec{k}\times\vec{i})}+\underline{a_zb_y(\vec{k}\times\vec{j})}+\cancel{a_zb_z(\vec{k}\times\vec{k})}$$

$$=(a_yb_z-a_zb_y)\vec{i}+(a_zb_x-a_xb_z)\vec{j}+(a_xb_y-a_yb_x)\vec{k}$$

例5. 设刚体以等角速度 ω 绕 l 轴旋转,导出刚体上一点 M 的线速度 $\vec{v}$ 与角速度 $\vec{\omega}$ 的关系式.

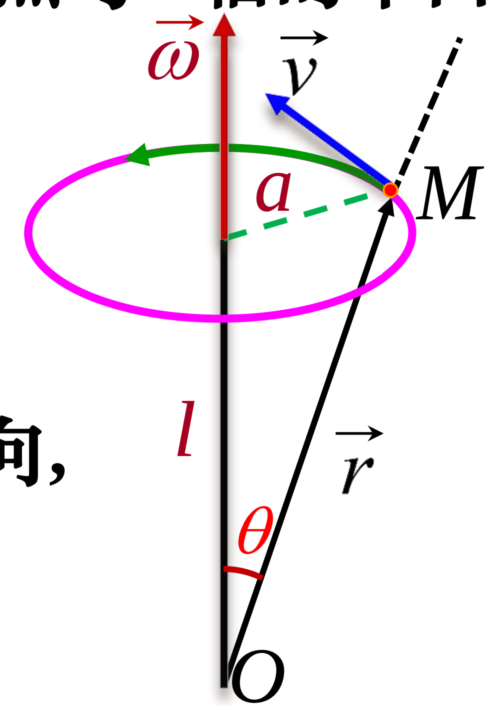


解: 在轴 l 上任取一点 O, 作**向径** $\overrightarrow{OM}=\vec{r}$, 它与 l

夹角为 θ, 则点 M 离开转轴的距离 $a=|\vec{r}|\sin\theta$,

点 M 的线速度 $\vec{v}$, 方向垂直于通过 M 点与 l 轴的平面,

$$|\vec{v}| = \omega a = \omega|\vec{r}|\sin\theta$$



引进一个**角速度**向量 $\vec{\omega}$, 使 $|\vec{\omega}|=\omega$,

方向用以描述刚体的转动特征，即

$\vec{r}\Rightarrow\vec{v}\Rightarrow\vec{\omega}$ 符合右手法则时旋转轴方向,

$\vec{\omega}=\vec{r}\times\vec{v}$, 同时 $\vec{v}=\vec{\omega}\times\vec{r}$

$$\vec{r}=\vec{v}\times\vec{\omega}$$

补充 观察二阶行列式 对角线法则

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

补充 观察三阶行列式 对角线法则

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$- a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

注意 红线上三元素的乘积冠以正号，蓝线上三元素的乘积冠以负号.

补充 观察三阶行列式

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

补充　观察三阶行列式

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

补充 观察三阶行列式

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

4.(2)向量积的行列式表示式[记忆法 P19]

$$\vec{a}\times\vec{b}=(a_yb_z-a_zb_y)\vec{i}-(a_xb_z-a_zb_x)\vec{j}+(a_xb_y-a_yb_x)\vec{k}$$

$$\vec{a}=(a_x,a_y,a_z)$$

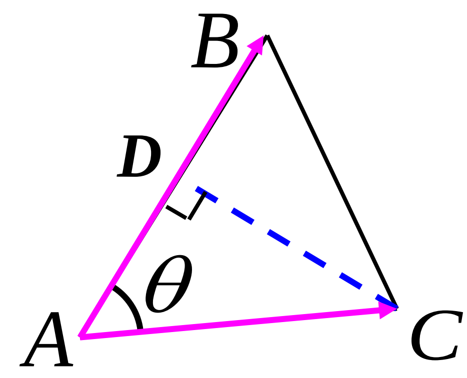
$$\vec{b}=(b_x,b_y,b_z)$$

$$=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}=\vec{i}\cdot M_{11}-\vec{j}\cdot M_{12}+\vec{k}\cdot M_{13}$$

$$\|$$

$$=\left(\begin{vmatrix}a_y & a_z\\ b_y & b_z\end{vmatrix},-\begin{vmatrix}a_x & a_z\\ b_x & b_z\end{vmatrix},\begin{vmatrix}a_x & a_y\\ b_x & b_y\end{vmatrix}\right)$$

例4. 已知三点 $A(1,2,3), B(3,4,5), C(2,4,7)$, 求三角形 ABC 的面积及高CD.



解: 如图所示:

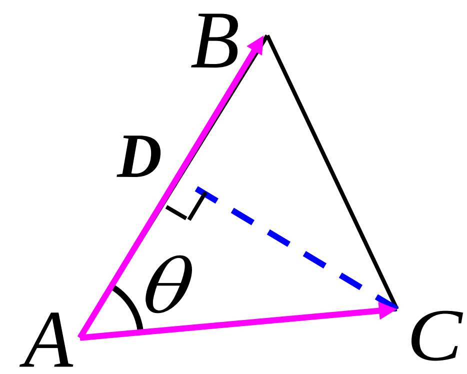
$$S_{\Delta ABC}=\frac{1}{2}|\overrightarrow{AB}||\overrightarrow{AC}|\sin\theta=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$$

$$=\frac{1}{2}\left|\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & 2 & 2\\ 1 & 2 & 4\end{vmatrix}\right|=\frac{1}{2}|(4,-6,\ 2)|$$

$$=\frac{1}{2}\sqrt{4^2+(-6)^2+2^2}=\sqrt{14}$$

例4. 已知三点 $A(1,2,3), B(3,4,5), C(2,4,7)$, 求三角形 ABC 的面积及高 CD.

章



解: 如图所示:

$$S_{\Delta ABC}=\sqrt{14}=\frac{1}{2}|\overrightarrow{AB}||\overrightarrow{CD}|$$

$$=\frac{1}{2}\sqrt{(3-1)^2+(4-2)^2+(5-3)^2}\,|\overrightarrow{CD}|$$

$$=\sqrt{3}\,|\overrightarrow{CD}|$$

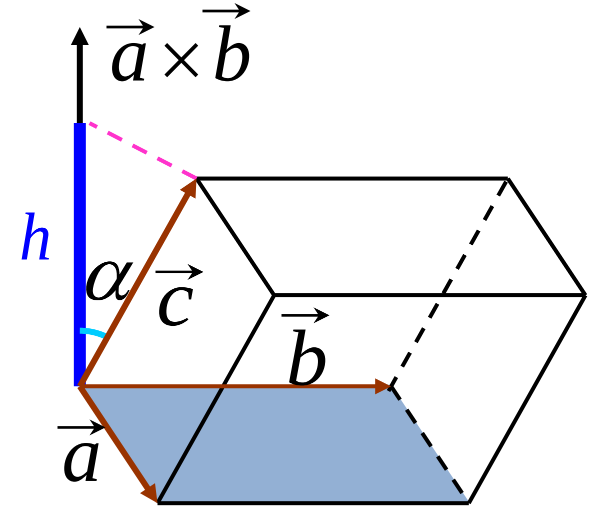
所以 $|\overrightarrow{CD}|=\sqrt{\frac{14}{3}}$

六、向量的混合积

1. 定义 已知三向量 $\vec{a},\vec{b},\vec{c}$, 称数量

$$(\vec{a}\times\vec{b})\cdot\vec{c} \xlongequal{\text{记作}} [\vec{a}\ \vec{b}\ \vec{c}]$$

为 $\vec{a},\vec{b},\vec{c}$ 的**混合积**.



2. 几何意义

混合积的绝对值表示: 以 $\vec{a},\vec{b},\vec{c}$ 为棱的平行六面体的体积. 其中

底面积 $A=|\vec{a}\times\vec{b}|$, 高 $h=|\vec{c}|\,|\cos\alpha|$

$$V=Ah=|\vec{a}\times\vec{b}|\,|\vec{c}|\,|\cos\alpha|=|(\vec{a}\times\vec{b})\cdot\vec{c}|=|[\vec{a}\ \vec{b}\ \vec{c}]|$$

3. 混合积的坐标表示

设 $\vec{a}=(a_x, a_y, a_z)$, $\vec{b}=(b_x, b_y, b_z)$, $\vec{c}=(c_x, c_y, c_z)$

$$\vec{a}\times\vec{b}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}=\left(\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix}, -\begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix}, \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}\right)$$

$$=\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix}\vec{i}-\begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix}\vec{j}+\begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}\vec{k}$$

$$[\vec{a}\,\vec{b}\,\vec{c}]=(\vec{a}\times\vec{b})\cdot\vec{c}=\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix}c_x-\begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix}c_y+\begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}c_z$$

$$=\begin{vmatrix} c_x & c_y & c_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}=\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

3. 混合积的坐标表示

设 $\vec{a}=(a_x, a_y, a_z)$, $\vec{b}=(b_x, b_y, b_z)$, $\vec{c}=(c_x, c_y, c_z)$

$$[\vec{a}\,\vec{b}\,\vec{c}]=(\vec{a}\times\vec{b})\cdot\vec{c}=\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} c_x - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} c_y + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} c_z$$

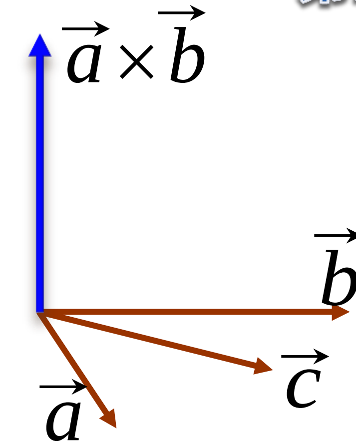
$$=\begin{vmatrix} c_x & c_y & c_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}=\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}=\begin{vmatrix} b_x & b_y & b_z \\ c_x & c_y & c_z \\ a_x & a_y & a_z \end{vmatrix}$$

(可用三阶行列式对角线法则展开式推出)

4. 性质

(1) 三个非零向量 $\vec{a}, \vec{b}, \vec{c}$

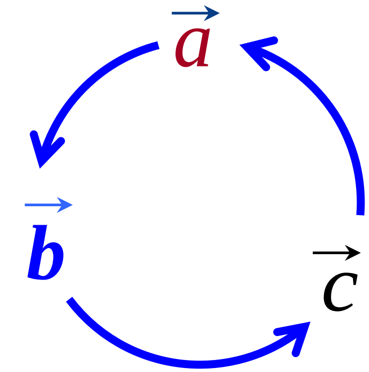
共面的**充要条件**是 $[\vec{a}\ \vec{b}\ \vec{c}] = 0$



(2) **轮换对称性**

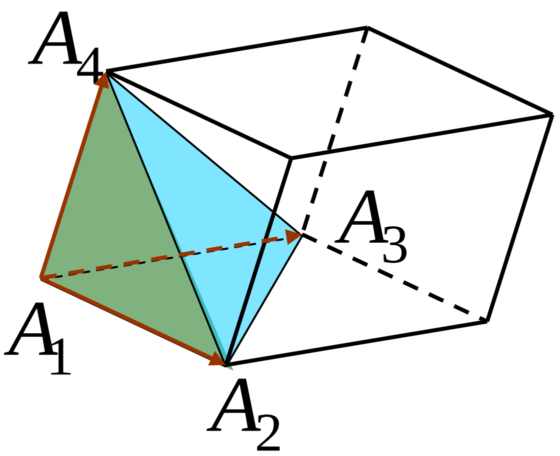
$$[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$$

$$\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = \begin{vmatrix} b_x & b_y & b_z \\ c_x & c_y & c_z \\ a_x & a_y & a_z \end{vmatrix} = \begin{vmatrix} c_x & c_y & c_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$



例6. 已知一四面体的顶点 $A_k(x_k, y_k, z_k)$ $(k=1,2,3,4)$, 求该四面体体积.

解: 已知四面体的体积等于以向量 $\overrightarrow{A_1A_2}$, $\overrightarrow{A_1A_3}$, $\overrightarrow{A_1A_4}$ 为棱的平行六面体体积的 $\frac{1}{6}$, 故



$$V = \frac{1}{6}\left| [\overrightarrow{A_1A_2}\ \overrightarrow{A_1A_3}\ \overrightarrow{A_1A_4}] \right|$$

$$= \frac{1}{6}\left| \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} \right|$$

例7. 已知$\vec{a}=\vec{i},\vec{b}=\vec{j}-2\vec{k},\vec{c}=2\vec{i}-2\vec{j}+\vec{k}$,求一单位向量$\vec{r}$,$\vec{r}\perp\vec{c}$,且$\vec{r}$与$\vec{a},\vec{b}$同时共面.

解: 假设$\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}=(x,y,z)$

(1) $|\vec{r}|=1 \Leftrightarrow x^2+y^2+z^2=1$

(2) $\vec{r}\perp\vec{c} \Leftrightarrow \vec{r}\cdot\vec{c}=0$

$\Leftrightarrow 2x-2y+z=0$

(3) $\vec{r}$与$\vec{a},\vec{b}$共面 $\Leftrightarrow [\vec{r}\,\vec{a}\,\vec{b}]=0 \Leftrightarrow \begin{vmatrix} x & y & z \\ 1 & 0 & 0 \\ 0 & 1 & -2 \end{vmatrix}=0$

即$\begin{cases} x^2+y^2+z^2=1 \\ 2x-2y+z=0 \\ 2y+z=0 \end{cases}$ 得$x=\frac{2}{3},y=\frac{1}{3},z=-\frac{2}{3}$,

或$x=-\frac{2}{3},y=-\frac{1}{3},z=\frac{2}{3}$.

内容小结

设 $\vec{a}=(a_x,a_y,a_z)$, $\vec{b}=(b_x,b_y,b_z)$, $\vec{c}=(c_x,c_y,c_z)$

1. 向量运算

加减: $\vec{a}\pm\vec{b}=(a_x\pm b_x, a_y\pm b_y, a_z\pm b_z)$

数乘: $\lambda\vec{a}=(\lambda a_x, \lambda a_y, \lambda a_z)$

点积: $\vec{a}\cdot\vec{b}=a_x b_x+a_y b_y+a_z b_z$

叉积: $\vec{a}\times\vec{b}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$

混合积: $[\vec{a}\ \vec{b}\ \vec{c}] = (\vec{a}\times\vec{b})\cdot\vec{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$

2. 向量关系

$$\vec{a}\,//\,\vec{b} \rightleftarrows \vec{a}\times\vec{b}=\vec{0} \rightleftarrows \frac{b_x}{a_x}=\frac{b_y}{a_y}=\frac{b_z}{a_z} \quad (a_x a_y a_z \neq 0)$$

$$\vec{a}\perp\vec{b} \rightleftarrows \vec{a}\cdot\vec{b}=0 \rightleftarrows a_x b_x + a_y b_y + a_z b_z = 0$$

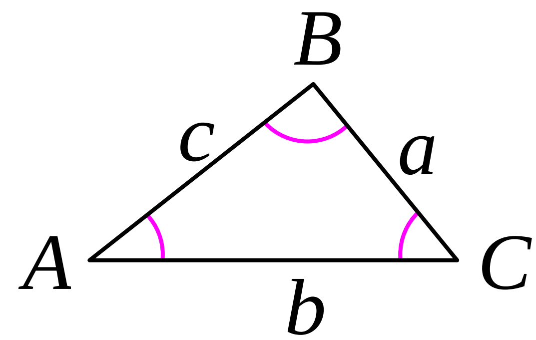
$$\vec{a},\vec{b},\vec{c}\text{共面} \rightleftarrows (\vec{a}\times\vec{b})\cdot\vec{c}=0$$

$$\rightleftarrows \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = 0$$

思考与练习

用向量方法证明正弦定理:

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$



证：由三角形面积公式

$$S_{\Delta ABC}=\frac{1}{2}\left|\overrightarrow{AC}\times\overrightarrow{AB}\right|$$

$$=\frac{1}{2}\left|\overrightarrow{BA}\times\overrightarrow{BC}\right|=\frac{1}{2}\left|\overrightarrow{CB}\times\overrightarrow{CA}\right|$$

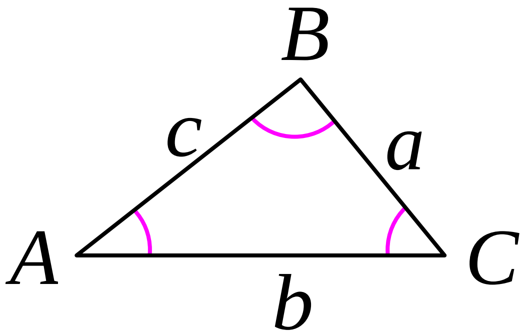
$$S_{\Delta ABC}=\frac{1}{2}\left|\overrightarrow{AC}\times\overrightarrow{AB}\right|=\frac{1}{2}\left|\overrightarrow{BA}\times\overrightarrow{BC}\right|=\frac{1}{2}\left|\overrightarrow{CB}\times\overrightarrow{CA}\right|$$

因

$$\left|\overrightarrow{AC}\times\overrightarrow{AB}\right|=b\cdot c\cdot\sin A$$

$$\left|\overrightarrow{BA}\times\overrightarrow{BC}\right|=c\cdot a\cdot\sin B$$

$$\left|\overrightarrow{CB}\times\overrightarrow{CA}\right|=a\cdot b\cdot\sin C$$



所以

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$