§6_2_1 偏 导 数

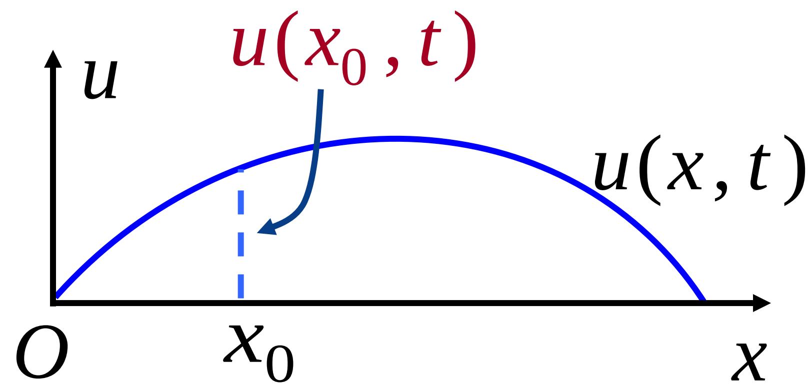
/* Partial Derivatives*/

一、偏导数概念及其计算

二、高阶偏导数

一、偏导数定义及其计算法

引例. 研究弦在点 x_0 处的振动速度与加速度，就是将振幅 $u(x,t)$ 中的 x 固定于 x_0 处，求 $u(x_0,t)$ 关于 t 的一阶导数与二阶导数，即 $u(x,t)$ 对 t 的一阶、二阶偏导数在 $x=x_0$ 处的偏导数值.



定义1. 设函数 $z = f(x, y)$ 在点 (x_0, y_0) 的某邻域内有定义, 且极限 $\lim\limits_{\Delta x \to 0} \dfrac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$ 存在, 则称此极限为函数 $z = f(x, y)$ 在点 (x_0, y_0) 对 x 的偏导数, 记为 $\left.\dfrac{\partial z}{\partial x}\right|_{(x_0, y_0)}$; $\left.\dfrac{\partial f}{\partial x}\right|_{(x_0, y_0)}$; $z_x\big|_{(x_0, y_0)}$; $f_x(x_0, y_0)$; $f_1'(x_0, y_0)$.

注1: $f_x(x_0,y_0)=\lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$

$$=\frac{\mathrm{d}}{\mathrm{d}x}f(x,y_0)\Big|_{x=x_0}$$

同样可得对 y 偏导数:

$$f_y(x_0,y_0)=\lim_{\Delta y\to 0}\frac{f(x_0,y_0+\Delta y)-f(x_0,y_0)}{\Delta y}$$

$$=\frac{\mathrm{d}}{\mathrm{d}y}f(x_0,y)\Big|_{y=y_0}$$

记为 $\dfrac{\partial z}{\partial y}\Big|_{(x_0,y_0)}$; $\dfrac{\partial f}{\partial y}\Big|_{(x_0,y_0)}$; $z_y\big|_{(x_0,y_0)}$; $f_y(x_0,y_0)$;

$f_2'(x_0,y_0)$.

注2:

若函数 $z = f(x, y)$ 在区域 D 内每一点 (x, y) 处对 x 或 y 偏导数存在，则该偏导数称为偏导函数，也简称为偏导数，记为

$$\frac{\partial z}{\partial x},\ \frac{\partial f}{\partial x},\ z_x,\ f_x(x,y),\ f_1'(x,y)$$

$$\frac{\partial z}{\partial y},\ \frac{\partial f}{\partial y},\ z_y,\ f_y(x,y),\ f_2'(x,y)$$

偏导数的概念可以推广到二元以上的函数.

例如, 三元函数 $u = f(x, y, z)$ 在点 (x, y, z) 处对 x 的偏导数定义为

$$f_x(x, y, z) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$$

$$f_y(x, y, z) = ?$$

(请自己写出)

$$f_z(x, y, z) = ?$$

二元函数偏导数的几何意义

$$z = f(x, y)$$

$$\left.\frac{\partial z}{\partial x}\right|_M = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

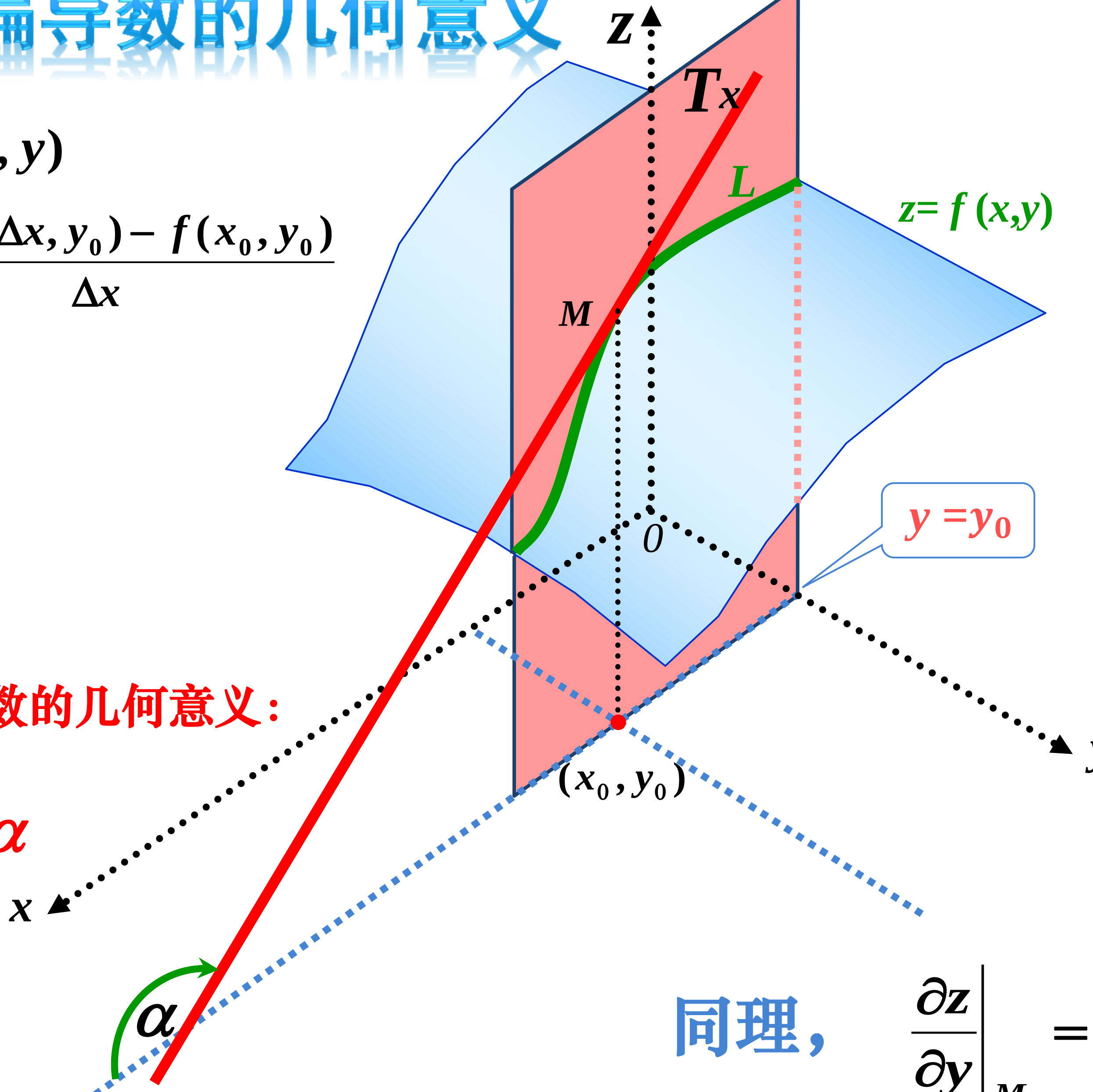
固定$y = y_0$

得曲线

$$L: \begin{cases} z = f(x, y) \\ y = y_0 \end{cases}$$

由一元函数导数的几何意义：

$$\left.\frac{\partial z}{\partial x}\right|_M = \tan\alpha$$



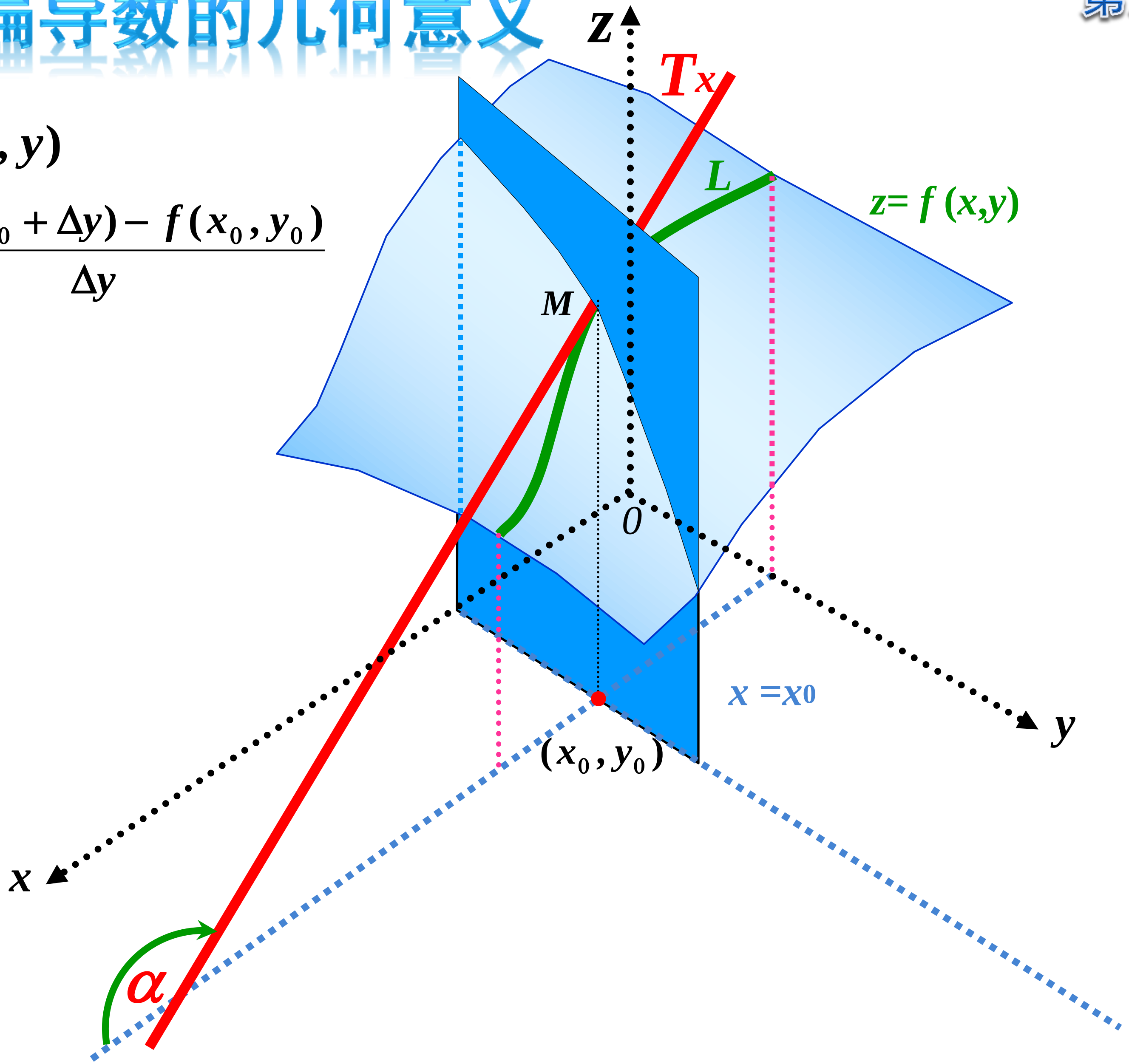
同理，$\left.\dfrac{\partial z}{\partial y}\right|_M = ?$

二元函数偏导数的几何意义

$$z = f(x, y)$$

$$\left.\frac{\partial z}{\partial y}\right|_{M} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

固定 $x = x_0$



二元函数偏导数的几何意义

$$z = f(x, y)$$

$$\left.\frac{\partial z}{\partial y}\right|_M = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

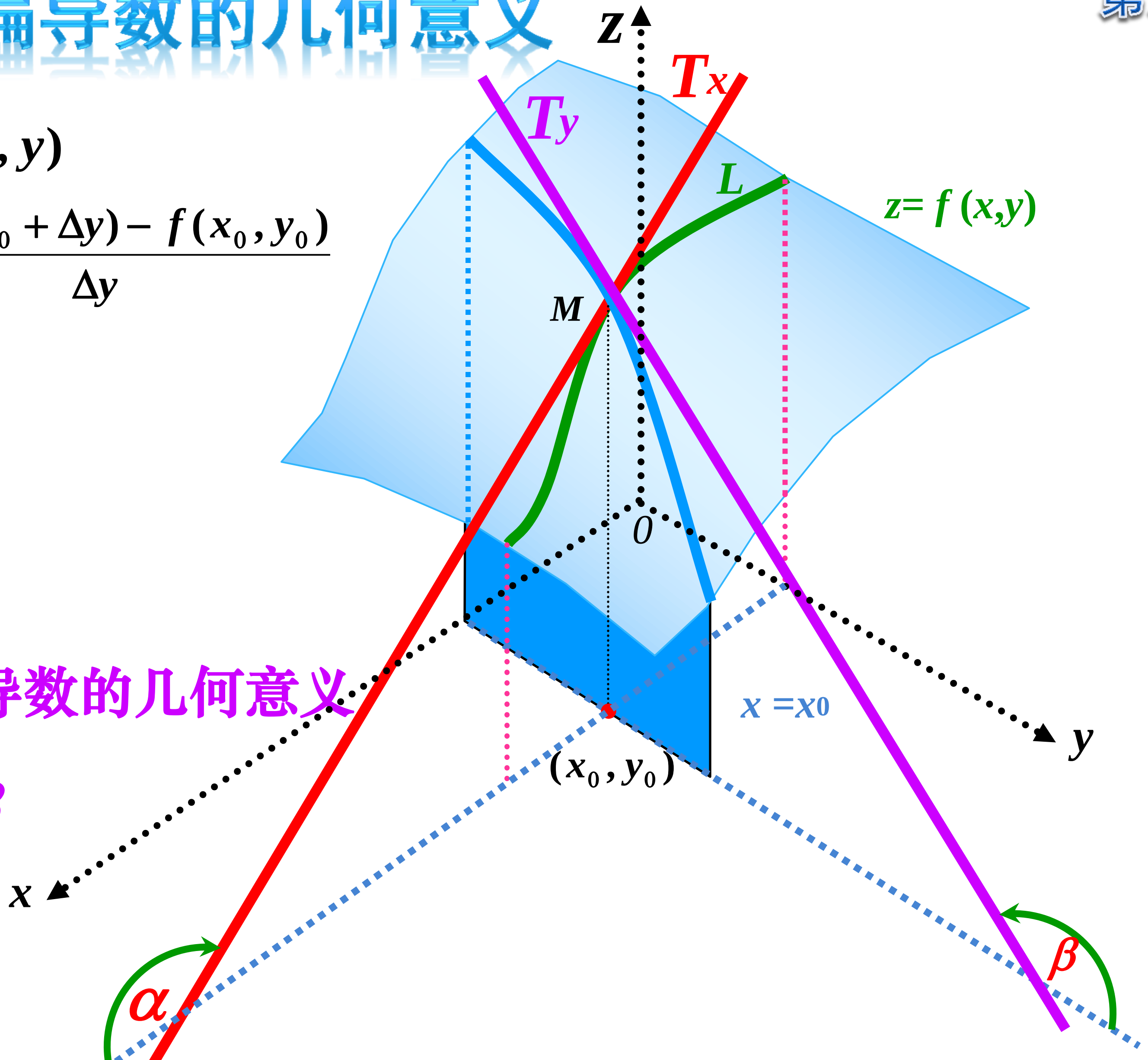
固定 $x = x_0$

得曲线

$$\begin{cases} z = f(x, y) \\ x = x_0 \end{cases}$$

由一元函数导数的几何意义

$$\left.\frac{\partial z}{\partial y}\right|_M = \tan\beta$$



注3: 偏导数存在 $\not\Rightarrow$ 此点连续

例如, $$z = f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}$$

显然 $$f_x(0,0) = \frac{\mathrm{d}}{\mathrm{d}x} f(x,0) \Big|_{x=0} = 0$$

$$f_y(0,0) = \frac{\mathrm{d}}{\mathrm{d}y} f(0,y) \Big|_{y=0} = 0$$

在上节已证 $f(x,y)$ 在点(0, 0)并不连续!

注3: 偏导数存在 $\underset{\nLeftarrow}{\nRightarrow}$ 此点连续

例如, $z = f(x,y) = \sqrt{x^2 + y^2}$ 在点(0, 0)连续;

但是 $f_x(0,0) = \lim\limits_{\Delta x \to 0} \dfrac{f(0+\Delta x,0) - f(0,0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{\sqrt{(\Delta x)^2}}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} \dfrac{|\Delta x|}{\Delta x}$ 不存在；

同理 $f_y(0,0)$不存在.

例1. 求 $z = x^2 + 3xy + y^2$ 在点(1, 2) 处的偏导数.

解法1 $\frac{\partial z}{\partial x} = 2x + 3y, \quad \frac{\partial z}{\partial y} = 3x + 2y$

先求后代

$$\therefore \quad \left.\frac{\partial z}{\partial x}\right|_{(1,2)} = 2\cdot 1 + 3\cdot 2 = 8, \quad \left.\frac{\partial z}{\partial y}\right|_{(1,2)} = 3\cdot 1 + 2\cdot 2 = 7$$

解法2 $z\big|_{y=2} = x^2 + 6x + 4, \quad \left.\frac{\partial z}{\partial x}\right|_{(1,\ 2)} = (2x+6)\big|_{x=1} = 8$

$$z\big|_{x=1} = 1 + 3y + y^2, \quad \left.\frac{\partial z}{\partial y}\right|_{(1,\ 2)} = (3+2y)\big|_{y=2} = 7$$

先代后求

例2. **设** $z = x^y \ (x > 0,$ 且 $x \neq 1)$，**求证**

$$\frac{x}{y}\frac{\partial z}{\partial x} + \frac{1}{\ln x}\frac{\partial z}{\partial y} = 2z$$

证: $\because \ \dfrac{\partial z}{\partial x} = yx^{y-1}, \quad \dfrac{\partial z}{\partial y} = x^y \ln x$

$$\therefore \ \frac{x}{y}\frac{\partial z}{\partial x} + \frac{1}{\ln x}\frac{\partial z}{\partial y} = x^y + x^y = 2z$$

例3. 求 $u=\sin(x+y^2-e^z)$ 的偏导数.

解:

$$\frac{\partial u}{\partial x}=\cos(x+y^2-e^z)$$

$$\frac{\partial u}{\partial y}=\cos(x+y^2-e^z)\cdot 2y$$

$$\frac{\partial u}{\partial z}=\cos(x+y^2-e^z)\cdot(-e^z)$$

例4. 已知理想气体的状态方程 $pV = RT$ (R 为常数),

求证: $\dfrac{\partial p}{\partial V}\cdot\dfrac{\partial V}{\partial T}\cdot\dfrac{\partial T}{\partial p} = -1$

证: $p = \dfrac{RT}{V}$, $\dfrac{\partial p}{\partial V} = -\dfrac{RT}{V^2}$

等温条件下,
压强关于容积的变化率

$V = \dfrac{RT}{p}$, $\dfrac{\partial V}{\partial T} = \dfrac{R}{p}$

等压条件下,
容积关于温度的变化率

$T = \dfrac{pV}{R}$, $\dfrac{\partial T}{\partial p} = \dfrac{V}{R}$

等容条件下,
温度关于压强的变化率

$$\therefore\ \frac{\partial p}{\partial V}\cdot\frac{\partial V}{\partial T}\cdot\frac{\partial T}{\partial p} = -\frac{RT}{pV} = -1$$

例4. 已知理想气体的状态方程 $pV = RT$ (**R** 为常数),

求证: $$\frac{\partial p}{\partial V} \cdot \frac{\partial V}{\partial T} \cdot \frac{\partial T}{\partial p} = -1$$

证: $$p = \frac{RT}{V}, \quad \frac{\partial p}{\partial V} = -\frac{RT}{V^2}$$

$$V = \frac{RT}{p}, \quad \frac{\partial V}{\partial T} = \frac{R}{p} \qquad \therefore \frac{\partial p}{\partial V} \cdot \frac{\partial V}{\partial T} \cdot \frac{\partial T}{\partial p} = -\frac{RT}{pV} = -1$$

$$T = \frac{pV}{R}, \quad \frac{\partial T}{\partial p} = \frac{V}{R}$$

说明: 此例表明,偏导数记号是一个整体记号, 不能看作分子与分母的商.

注4: $$f_x(x,y)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$$

$$\text{偏增量}\left\{\begin{array}{l}\Delta_x z=f(x+\Delta x,y)-f(x,y)\approx f_x(x,y)\Delta x\\ \Delta_y z=f(x,y+\Delta y)-f(x,y)\approx f_y(x,y)\Delta y\end{array}\right\}\text{偏微分}$$

二、高阶偏导数

设 $z=f(x,y)$在区域 D 内存在偏导数

$$\frac{\partial z}{\partial x}=f_x(x,y)\,,\qquad \frac{\partial z}{\partial y}=f_y(x,y)$$

若这两个偏导数仍存在偏导数,则称它们是 $z=f(x,y)$ 的二阶偏导数. 按求导顺序不同, 有下列四个二阶偏导数:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}=f_{xx}(x,y);\quad \boxed{f''_{11}}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x\partial y}=f_{xy}(x,y)\quad \boxed{f''_{12}}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y\partial x}=f_{yx}(x,y);\quad \boxed{f''_{21}}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y^2}=f_{yy}(x,y)\quad \boxed{f''_{22}}$$

类似可以定义更高阶的偏导数.

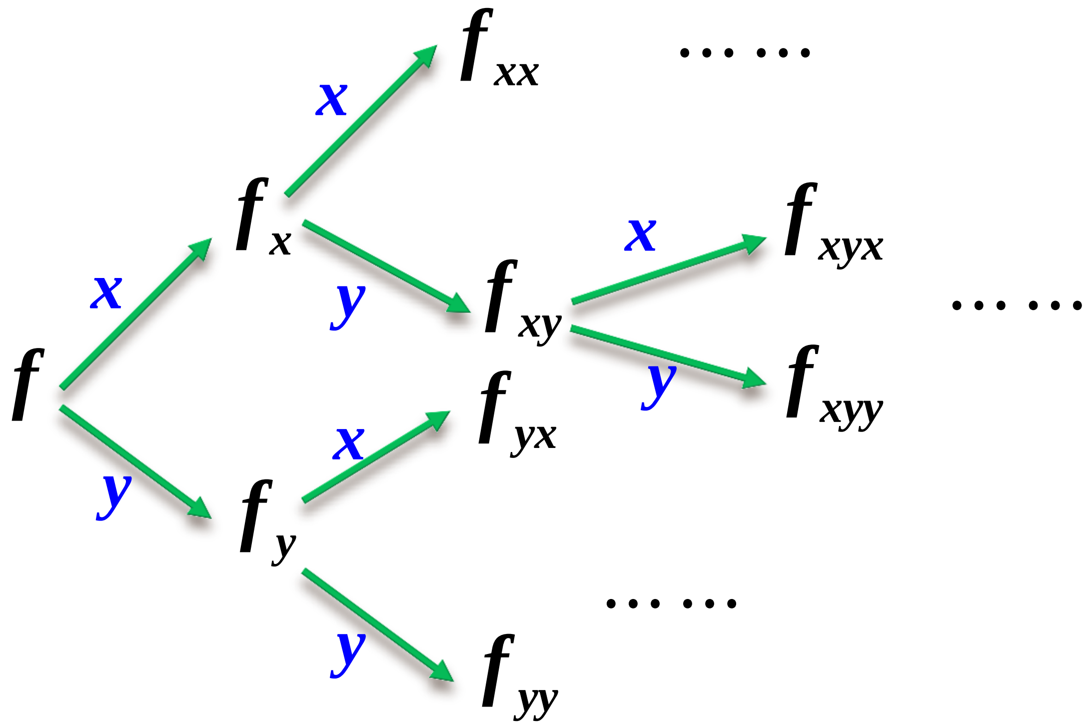
例如，$z=f(x,y)$ 关于 x 的三阶偏导数为

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right)=\frac{\partial^3 z}{\partial x^3}$$

$z=f(x,y)$ 关于 x 的 $n-1$ 阶偏导数, 再关于 y 的一阶偏导数为

$$\frac{\partial}{\partial y}\left(\frac{\partial^{n-1} z}{\partial x^{n-1}}\right)=\frac{\partial^n z}{\partial x^{n-1}\partial y}$$

高阶偏导树型图



例5. 求函数 $z = x^y (x > 0 且 x \neq 1)$ 的二阶偏导数及 $\dfrac{\partial^3 z}{\partial y \partial x^2}$.

解：

$$\frac{\partial z}{\partial x} = y x^{y-1} \qquad \frac{\partial z}{\partial y} = x^y \ln x$$

$$\frac{\partial^2 z}{\partial x^2} = y(y-1)x^{y-2} \qquad \frac{\partial^2 z}{\partial x \partial y} = \underline{x^{y-1}(1 + y \ln x)}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \underline{x^{y-1}(1 + y \ln x)} \qquad \frac{\partial^2 z}{\partial y^2} = x^y (\ln x)^2$$

$$\frac{\partial^3 z}{\partial y \partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y \partial x}\right) = x^{y-2} y(1 - \ln x + y \ln x)$$

注意: 此处 $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}$，但这一结论并不总成立.

例如, $f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$

$$f_x(x,y)=\begin{cases} \boxed{y\dfrac{x^4+4x^2y^2-y^4}{(x^2+y^2)^2}}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$$

$$f_y(x,y)=\begin{cases} \boxed{x\dfrac{x^4-4x^2y^2-y^4}{(x^2+y^2)^2}}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$$

$$\boldsymbol{f_x(0,\Delta y)=-\Delta y\,,\ \ f_y(\Delta x,0)=\Delta x}$$

例如, $f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$

$$f_x(0,\Delta y)=-\Delta y\,,\quad f_y(\Delta x,0)=\Delta x$$

$$\left.\begin{aligned} f_{xy}(0,0)&=\lim_{\Delta y\to 0}\frac{\boxed{f_x(0,\Delta y)}-f_x(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{-\Delta y}{\Delta y}=-1 \\ f_{yx}(0,0)&=\lim_{\Delta x\to 0}\frac{\boxed{f_y(\Delta x,0)}-f_y(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x}=1 \end{aligned}\right\}\text{二者不等}$$

定理. 若 $f_{xy}(x,y)$和 $f_{yx}(x,y)$都在点(x_0,y_0)**连续**,则

$$f_{xy}(x_0,y_0)=f_{yx}(x_0,y_0)$$

(证明略)

本定理对 n 元函数的高阶混合导数也成立.

例如, 对三元函数 $u=f(x,y,z)$, 当三阶混合偏导数在点 (x,y,z) **连续**时, 有

与次序无关

$$f_{xyz}(x,y,z)=f_{yzx}(x,y,z)=f_{zxy}(x,y,z)$$

$$=f_{xzy}(x,y,z)=f_{yxz}(x,y,z)=f_{zyx}(x,y,z)$$

定理. 若 $f_{xy}(x,y)$和 $f_{yx}(x,y)$都在点(x_0,y_0)**连续**,则

$$f_{xy}(x_0,y_0)=f_{yx}(x_0,y_0)$$

(证明略)

本定理对 n 元函数的高阶混合导数也成立.

说明: 因为初等函数的偏导数仍为初等函数, 而初等函数在其定义区域内是连续的, 故初等函数在定义区域内高阶偏导数与求导顺序无关, 求偏导时可按方便次序.

例6. 证明函数 $u(x,y)=\ln\sqrt{x^2+y^2}$ 满足**拉普拉斯方程**

$$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

证: $$\frac{\partial u}{\partial x}=\frac{1}{\sqrt{x^2+y^2}}\frac{2x}{2\sqrt{x^2+y^2}}=\frac{x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{x^2+y^2-x\cdot 2x}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2}$$

利用对称性, 有 $$\frac{\partial^2 u}{\partial y^2}=\frac{x^2-y^2}{(x^2+y^2)^2}$$

所以 $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$ *Laplace*算子: $\Delta=\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}$

内容小结

1. 偏导数的概念及有关结论

- 定义; 记号; 几何意义
- 函数在一点偏导数存在 $\overset{\times}{\rightleftarrows}$ 函数在此点连续
- 混合偏导数连续 ⟶ 与求导顺序无关

2. 偏导数的计算方法

- 求一点处偏导数的方法 $\left\{\begin{array}{l}\text{先代后求}\\ \text{先求后代}\\ \text{利用定义}\end{array}\right.$
- 求高阶偏导数的方法 —— 逐次求导法

(与求导顺序无关时, 应选择方便的求导顺序)