# 多元复合函数的链式求导法则

/* Chain Rules for Several Variables*/

一元复合函数 $y = f(u),\ u = \varphi(x)$

求导法则 $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$$

微分法则 $\mathrm{d}y = f'(u)\,\mathrm{d}u = f'(u)\varphi'(x)\,\mathrm{d}x$

## 本节内容:

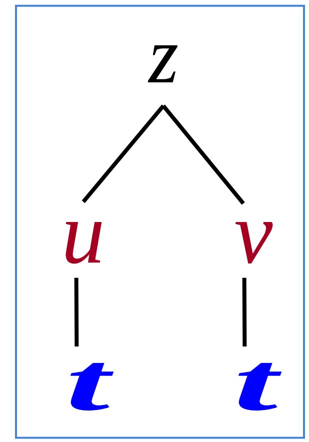
一、多元复合函数求导的链式法则

二、全微分形式不变性

# 一、多元复合函数求导的链式法则

**定理.** 若函数 $u=u(t),\ v=v(t)$ 在点 $t$ 可导, $z=f(u,v)$ 在点 $(u,v)$ 处偏导连续，则复合函数 $z=f(u(t),v(t))$ 在点 $t$ 可导, 且有链式法则

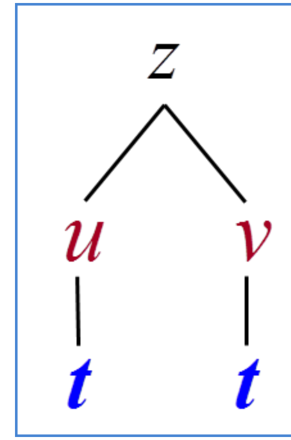
$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}$$




**证:** 设 $t$ 取增量 $\Delta t$, 则相应中间变量有增量 $\Delta u, \Delta v$,

$$\Delta z=\frac{\partial z}{\partial u}\Delta u+\frac{\partial z}{\partial v}\Delta v+o(\rho)\quad \left(\rho=\sqrt{(\Delta u)^2+(\Delta v)^2}\ \right)$$

$$\frac{\Delta z}{\Delta t}=\frac{\partial z}{\partial u}\frac{\Delta u}{\Delta t}+\frac{\partial z}{\partial v}\frac{\Delta v}{\Delta t}+\frac{o(\rho)}{\Delta t}\quad \left(\rho=\sqrt{(\Delta u)^2+(\Delta v)^2}\ \right)$$




令 $\Delta t\to 0$, 则有 $\Delta u\to 0, \Delta v\to 0,$

$$\frac{\Delta u}{\Delta t}\to\frac{\mathrm{d}u}{\mathrm{d}t},\qquad \frac{\Delta v}{\Delta t}\to\frac{\mathrm{d}v}{\mathrm{d}t}$$

$$\frac{o(\rho)}{\Delta t}=\frac{o(\rho)}{\rho}\sqrt{\left(\frac{\Delta u}{\Delta t}\right)^2+\left(\frac{\Delta v}{\Delta t}\right)^2}\to 0$$

**($\Delta t<0$ 时, 根式前加“—”号)**

$$\boxed{\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}}$$

**( 全导数公式 )**

**说明**：若定理中 $f(u,v)$ 在点 $(u,v)$ **偏导数连续** 减弱为 **偏导数存在**，则定理结论**不一定成立**.

**例如**， $z = f(u,v) = \begin{cases} \dfrac{u^2 v}{u^2+v^2}, & u^2+v^2 \neq 0 \\ 0, & u^2+v^2 = 0 \end{cases}$

$u = t,\ v = t$

**易知** $\left.\dfrac{\partial z}{\partial u}\right|_{(0,0)} = f_u(0,0) = 0$， $\left.\dfrac{\partial z}{\partial v}\right|_{(0,0)} = f_v(0,0) = 0$

**但复合函数** $z = f(t,t) = \dfrac{t}{2}$

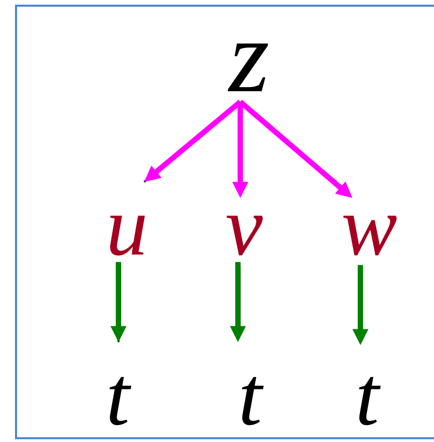
$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{1}{2} \neq \frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t} = 0\cdot 1 + 0\cdot 1 = 0$$

**推广**:设下面所涉及的函数都可微.

**1) 中间变量多于两个的情形.**

**例如,** $z = f(u, v, w)$,

$$\boldsymbol{u = u(t),\ v = v(t),\ w = w(t)}$$




$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\partial z}{\partial w}\cdot\frac{\mathrm{d}w}{\mathrm{d}t}$$

$$= f_1'\varphi' + f_2'\psi' + f_3'\omega'$$

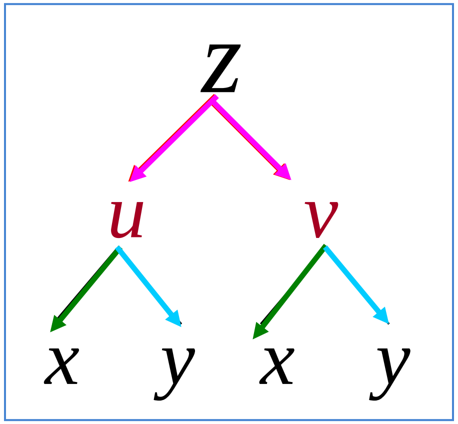
**推广**:设下面所涉及的函数都可微.

**2) 中间变量是多元函数的情形.**

**例如,** $z = f(u,v),\ u = u(x,y), v = v(x,y)$

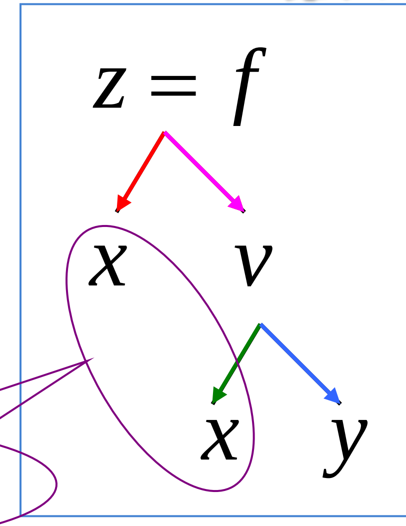
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = f_1'\varphi_1' + f_2'\psi_1'$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = f_1'\varphi_2' + f_2'\psi_2'$$



## 3) 中间变量也是自变量的情形.

**例如,** $z = f(x,v)$, $v = \psi(x,y)$




$$\boxed{\frac{\partial z}{\partial x}} = \boxed{\frac{\partial f}{\partial x}} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} = f_1' + f_2'\psi_1'$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} = f_2'\psi_2'$$

父子变量

**注意:** 这里 $\frac{\partial z}{\partial x}$ 与 $\frac{\partial f}{\partial x}$ 不同,

**记忆：链路乘, 分路加**
**父子变量换记法**

$\frac{\partial z}{\partial x}$ 表示 $f(x,\psi(x,y))$ 固定 $y$ 对 $x$ (自变量)求导

$\frac{\partial f}{\partial x}$ 表示 $f(x,v)$ 固定 $v$ 对 $x$ (中间变量)求导

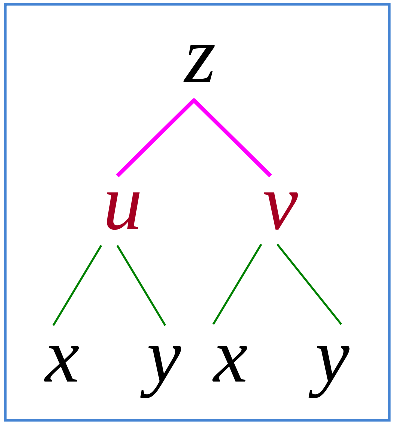
**例1.** 设 $z = \mathrm{e}^u \sin v$，$u = xy$，$v = x + y$，求 $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$.

**解:**

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= \mathrm{e}^u \sin v \cdot y + \mathrm{e}^u \cos v \cdot 1$$

$$= \mathrm{e}^{xy}[y \cdot \sin(x+y) + \cos(x+y)]$$




$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$= \mathrm{e}^u \sin v \cdot x + \mathrm{e}^u \cos v \cdot 1$$

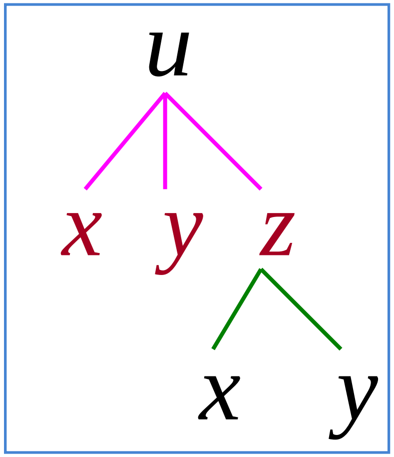
$$= \mathrm{e}^{xy}[x \cdot \sin(x+y) + \cos(x+y)]$$

**例2.** 设 $u=f(x,y,z)=\mathrm{e}^{x^2+y^2+z^2}$, $z=x^2\sin y$, 求 $\dfrac{\partial u}{\partial x},\dfrac{\partial u}{\partial y}$.

**解:** $$\frac{\partial u}{\partial x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x}$$

$$=2x\mathrm{e}^{x^2+y^2+z^2}+2z\mathrm{e}^{x^2+y^2+z^2}\cdot 2x\sin y$$

$$=2x(1+2x^2\sin^2 y)\mathrm{e}^{x^2+y^2+x^4\sin^2 y}$$




$$\frac{\partial u}{\partial y}=\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y}$$

$$=2y\mathrm{e}^{x^2+y^2+z^2}+2z\mathrm{e}^{x^2+y^2+z^2}\cdot x^2\cos y$$

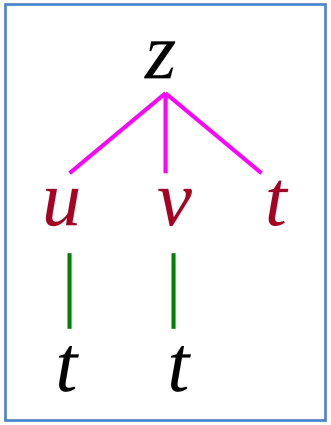
$$=2(y+x^4\sin y\cos y)\mathrm{e}^{x^2+y^2+x^4\sin^2 y}$$

**例3.** 设 $z = uv + \sin t$, $u = e^t$, $v = \cos t$, 求全导数 $\dfrac{dz}{dt}$.

**解:** $$\frac{dz}{dt} = \frac{\partial z}{\partial u}\cdot\frac{du}{dt} + \frac{\partial z}{\partial v}\cdot\frac{dv}{dt} + \frac{\partial z}{\partial t}$$

$$= v e^t - u\sin t + \cos t$$

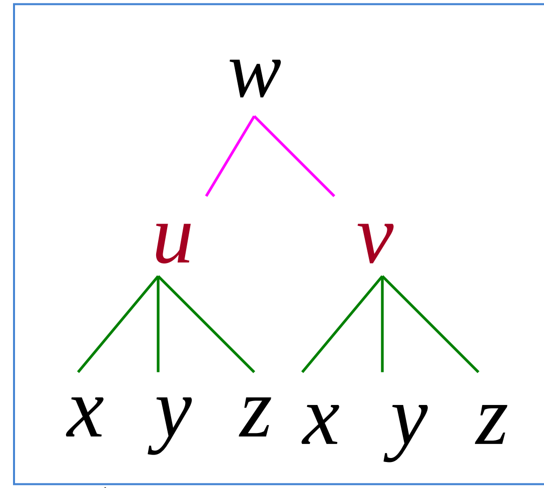
$$= e^t(\cos t - \sin t) + \cos t$$




**注意：多元抽象复合函数**求导在偏微分方程变形与验证解的问题中经常遇到, 下列两个例题有助于掌握这方面问题的**求导技巧**与**常用导数符号**.

**例4.** 设 $w = f(x+y+z, xyz)$, $f$ 具有二阶连续偏导数,

求 $\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x \partial z}$.




**解:** 令 $u = x+y+z$ , $v = xyz$, 则

$$w = f(u, v)$$

$$\frac{\partial w}{\partial x} = f_1' \cdot 1 + f_2' \cdot yz = f_1'(u, v) + f_2'(u, v) yz$$

$$= \underline{f_1'(x+y+z, xyz)} + \underline{yz\, f_2'(x+y+z, xyz)}$$

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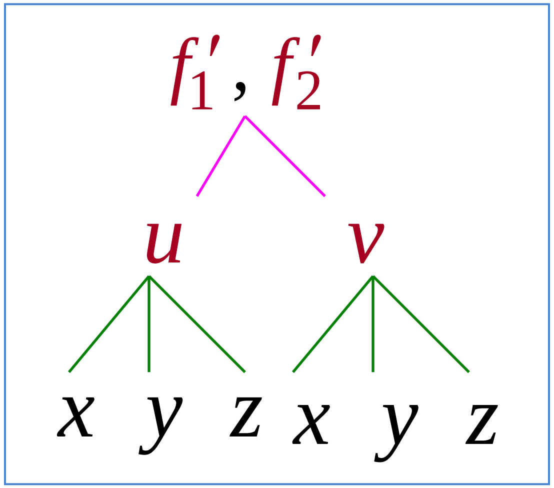
为简便起见, 应用记号 $f_1' = \dfrac{\partial f}{\partial u}$

**例4.** 设 $w = f(x+y+z, xyz)$，$f$ 具有二阶连续偏导数，求 $\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x \partial z}$.




**解:** $w = f(u, v)$ $\quad \dfrac{\partial w}{\partial x} = f_1' \cdot 1 + f_2' \cdot yz$

$u = x + y + z, v = xyz,$

$$\frac{\partial^2 w}{\partial x \partial z} = f_{11}'' \cdot 1 + f_{12}'' \cdot xy + y f_2' + yz[f_{21}'' \cdot 1 + f_{22}'' \cdot xy]$$

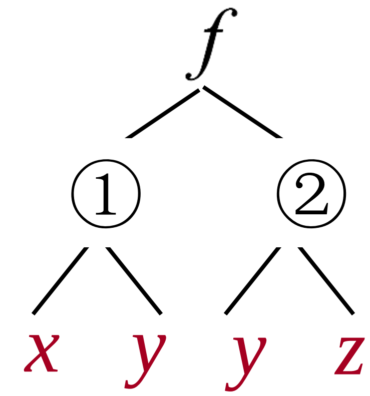
$$= f_{11}'' + y(x+z) f_{12}'' + xy^2 z f_{22}'' + y f_2'$$

为简便起见，应用记号 $f_1' = \dfrac{\partial f}{\partial u}$，$f_{11}'' = \dfrac{\partial^2 f}{\partial u^2}$,

$f_{12}'' = \dfrac{\partial^2 f}{\partial u \partial v}$, ……

# P82 题8(2)

$$u = f\left(\frac{x}{y}, \frac{y}{z}\right) = f(v, w)$$



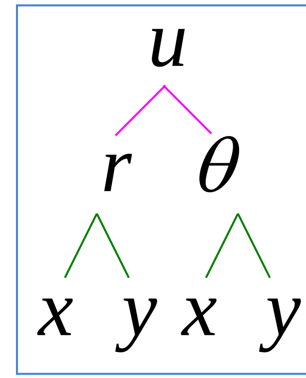

$$\frac{\partial u}{\partial x} = f_1' \cdot \frac{1}{y} = \frac{1}{y} f_1'$$

$$\frac{\partial u}{\partial y} = f_1' \cdot \left(-\frac{x}{y^2}\right) + f_2' \cdot \frac{1}{z} = -\frac{x}{y^2} f_1' + \frac{1}{z} f_2'$$

$$\frac{\partial u}{\partial z} = f_2' \cdot \left(-\frac{y}{z^2}\right) = -\frac{y}{z^2} f_2'$$

**例5.** 设 $u=f(x,y)$ 二阶偏导数连续,求下列表达式在极坐标系下的形式 (1) $\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2$ (2) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$

**解:(1)已知** $x=r\cos\theta$ , $y=r\sin\theta$, **则**




$$r=\sqrt{x^2+y^2},\ \ \theta=\arctan\frac{y}{x}$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}=\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}$$

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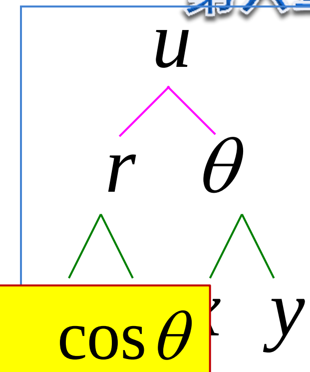
$$\frac{\partial r}{\partial x}=\frac{x}{r}=\cos\theta,\ \frac{\partial \theta}{\partial x}=\frac{\frac{-y}{x^2}}{1+(\frac{y}{x})^2}=\frac{-y}{x^2+y^2}=\frac{-r\sin\theta}{r^2}=\frac{-\sin\theta}{r}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$$




$$\frac{\partial r}{\partial y}=\frac{y}{r}=\sin\theta,\qquad \frac{\partial \theta}{\partial y}=\frac{\frac{1}{x}}{1+(\frac{y}{x})^2}=\frac{x}{x^2+y^2}=\frac{\cos\theta}{r}$$

$$=\frac{\partial u}{\partial r}\sin\theta+\frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}$$

$$\therefore\quad (\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2=(\frac{\partial u}{\partial r})^2+\frac{1}{r^2}(\frac{\partial u}{\partial \theta})^2$$

$$\boxed{\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}}$$

$$\boxed{r=\sqrt{x^2+y^2},\quad \theta=\arctan\frac{y}{x}}$$

已知 $\dfrac{\partial u}{\partial x}=\boxed{\dfrac{\partial u}{\partial r}\cos\theta-\dfrac{\partial u}{\partial \theta}\dfrac{\sin\theta}{r}}$

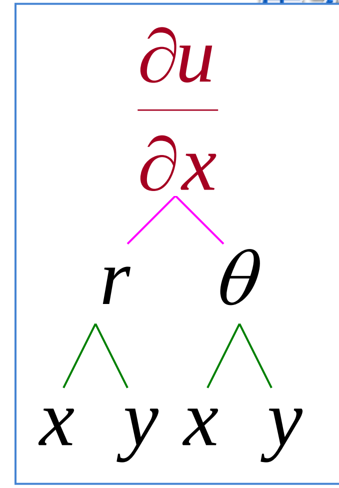
(2)
$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right)\cdot\cos\theta-\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial x}\right)\frac{\sin\theta}{r}$$

$$=\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial\theta}\frac{\sin\theta}{r}\right)\cdot\cos\theta$$

$$-\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial\theta}\frac{\sin\theta}{r}\right)\cdot\frac{\sin\theta}{r}$$

$$=\frac{\partial^2 u}{\partial r^2}\cos^2\theta-2\frac{\partial^2 u}{\partial r\partial\theta}\frac{\sin\theta\cos\theta}{r}+\frac{\partial^2 u}{\partial\theta^2}\frac{\sin^2\theta}{r^2}$$

$$+\frac{\partial u}{\partial\theta}\frac{2\sin\theta\cos\theta}{r^2}+\frac{\partial u}{\partial r}\frac{\sin^2\theta}{r}$$




注意利用已有公式

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial r^2}\cos^2\theta-\cancel{2\frac{\partial^2 u}{\partial r\partial\theta}\frac{\sin\theta\cos\theta}{r}}+\frac{\partial^2 u}{\partial\theta^2}\frac{\sin^2\theta}{r^2}$$

$$+\cancel{\frac{\partial u}{\partial\theta}\frac{2\sin\theta\cos\theta}{r^2}}+\frac{\partial u}{\partial r}\frac{\sin^2\theta}{r}$$

**同理可得**

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}\sin^2\theta+\cancel{2\frac{\partial^2 u}{\partial r\partial\theta}\frac{\sin\theta\cos\theta}{r}}+\frac{\partial^2 u}{\partial\theta^2}\frac{\cos^2\theta}{r^2}$$

$$-\cancel{\frac{\partial u}{\partial\theta}\frac{2\sin\theta\cos\theta}{r^2}}+\frac{\partial u}{\partial r}\frac{\cos^2\theta}{r}$$

$$\therefore\ \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$

$$=\frac{1}{r^2}\left[r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{\partial^2 u}{\partial\theta^2}\right]$$

# 二、全微分形式不变性

设函数 $z = f(u,v)$, $u = u(x,y)$, $v = v(x,y)$都可微,

则复合函数 $z = f(u(x,y), v(x,y))$的全微分为

$$\mathrm{d}z = \frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y = \frac{\partial z}{\partial u}\mathrm{d}u + \frac{\partial z}{\partial v}\mathrm{d}v$$

$$= \left(\underline{\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}} + \underline{\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}}\right)\mathrm{d}x + \left(\underline{\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}} + \underline{\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}}\right)\mathrm{d}y$$

$$= \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y\right) + \frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}\mathrm{d}x + \frac{\partial v}{\partial y}\mathrm{d}y\right)$$

可见无论 $u, v$ 是自变量还是中间变量, 其全微分表达形式都一样, 这性质叫做**全微分形式不变性**.

**例1.** 设 $z=e^{u}\sin v,\ u=xy,\ v=x+y$, 求 $\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$.

**解:**

$$dz=d(e^{u}\sin v)$$

$$=e^{u}\sin v\,du+e^{u}\cos v\,dv$$

**利用全微分形式不变性, 再解例1.**

$$=e^{xy}[\sin(x+y)\,d(xy)+\cos(x+y)\,d(x+y)]$$

$$=e^{xy}[\sin(x+y)(y\,dx+x\,dy)+\cos(x+y)(dx+dy)]$$

$$=e^{xy}[y\sin(x+y)+\cos(x+y)]dx$$

$$+e^{xy}[x\sin(x+y)+\cos(x+y)]dy$$

所以

$$\frac{\partial z}{\partial x}=e^{xy}[y\cdot\sin(x+y)+\cos(x+y)]$$

$$\frac{\partial z}{\partial y}=e^{xy}[x\cdot\sin(x+y)+\cos(x+y)]$$

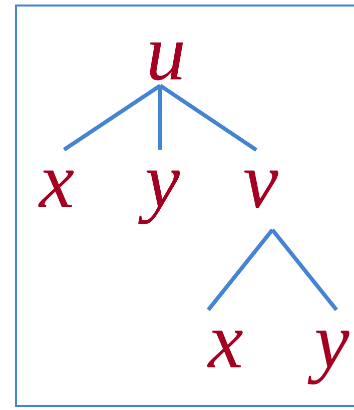
# 内容小结

## 1. 复合函数求导的链式法则

**“链路乘，分路加，父子变量换记法”**

**例如，** $u = f(x, y, v),\ v = \varphi(x, y),$

$$\frac{\partial u}{\partial x} = f_1' + f_3' \cdot \varphi_1' ; \quad \frac{\partial u}{\partial y} = f_2' + f_3' \cdot \varphi_2'$$




## 2. 全微分形式不变性

对 $z = f(u, v)$，不论 $u, v$ 是自变量还是中间变量，

$$\boxed{\mathrm{d}z = f_u(u, v)\,\mathrm{d}u + f_v(u, v)\,\mathrm{d}v}$$