EECS 475 HW2. Yuanwen Liu. 04/19/2020

4.1 4.2 4.5 5.2 5.9

4.1 (a). $\vec{C}$. $N\times N$. symmetric.

Show that if $\vec{C}$ has all eigen-values then the quantity $\vec{z}^T\vec{C}\vec{z}\geq 0$ for all $\vec{z}$.

$$\vec{C}=\vec{V}\vec{D}\vec{V}^T$$

$$\Rightarrow \vec{V}^T\vec{C}\vec{V}=\vec{D}$$

Let $\vec{Y}=\vec{V}^{-1}\vec{z}$.

$$\vec{z}=\vec{V}\vec{Y}$$

$$\vec{z}^T\vec{C}\vec{z}=(\vec{V}\vec{Y})^T\vec{C}(\vec{V}\vec{Y})$$
$$=\vec{Y}^T\vec{V}^T\vec{C}\vec{V}\vec{Y}$$
$$=\vec{Y}^T\vec{D}\vec{Y}$$
$$=\sum_{i=1}^{n} d_i y_i^2$$

Since $d_i\geq 0$. $(i=1,2,\dots n)$

$$\sum_{i=1}^{n} d_i y_i^2\geq 0.$$

$$\vec{z}^T\vec{C}\vec{z}\geq 0$$

(b) Since $\vec{z}^T\vec{C}\vec{z}\geq 0$. for all $\vec{z}$ and $\vec{C}\vec{x}=d\vec{x}$. where $\vec{x}$ is the eigen-vector for eigen-value d.

$$\vec{x}^T\vec{C}\vec{x}=\vec{x}^T d\vec{x}\geq 0$$

All d must be larger than 0.

(c) $g(\vec{w})=a+\vec{b}^T\vec{w}+\vec{w}^T\vec{C}\vec{w}$

$$\nabla^2 g(\vec{w})=2\vec{C}$$
$$=\begin{bmatrix}2&2\\2&2\end{bmatrix}$$

$$\det(dE-2\vec{C})=0$$

$$\det\begin{bmatrix}d-2&-2\\-2&d-2\end{bmatrix}=0$$

$$(d-2)^2-4=0$$

$$d^2-4d=0$$

$$d_1=0\quad d_2=4$$

$$d_1\geq 0.\quad d_2\geq 0.$$

Thus, this quadratic function holds convexity.

(d) If all eigenvalues of $\vec{C}+d\vec{I}_{N\times N}$ can be set to positive, it should holds that

$$\vec{z}^T(\vec{C}+d\vec{I}_{N\times N})\vec{z}>0 \text{ for all } \vec{z}.$$

Let $\vec{Y}=\vec{V}^{-1}\vec{z}$, $\vec{z}=\vec{V}\vec{Y}$

$$(\vec{V}\vec{Y})^T(\vec{V}\vec{D}\vec{V}^T+d\vec{I}_{N\times N})\vec{V}\vec{Y}>0$$

$$\vec{Y}^T\vec{V}^T\vec{V}\vec{D}\vec{V}^T\vec{V}\vec{Y}+\vec{Y}^T\vec{V}^T d\vec{I}_{N\times N}\vec{V}\vec{Y}>0$$

Since $\vec{V}^T=\vec{V}^{-1}$ $\quad\vec{V}^T\vec{V}=\vec{I}$

$$\vec{Y}^T\vec{D}\vec{Y}+\vec{Y}^T d\vec{I}_{N\times N}\vec{Y}>0.$$

$$\vec{Y}^T(\vec{D}+d\vec{I}_{N\times N})\vec{Y}>0$$

$\vec{D}$ is the diagonal matrix and the diagonal elements are eigen-value of $\vec{C}$.

If $\vec{Y}^T(\vec{D}+d\vec{I}_{N\times N})\vec{Y}>0$, the diagonal elements of $(\vec{D}+d\vec{I}_{N\times N})$ should larger than 0.

Assuming that the smallest eigen-value of $\vec{C}$ is a, the smallest d should be

$$d>-a$$

If d is an integer, $d=-a+1$

4.2.

(a) XX^T is a symmetric matrix.

~~So it can be represented as~~

$\vec{z}^T \vec{x} \vec{x}^T \vec{z}$

$= (\vec{x}^T \vec{z})^T \vec{x}^T \vec{z}$

$= \|\vec{x}^T \vec{z}\|^2 \geq 0$

According to 4.1. if $\vec{z}^T \vec{x} \vec{x}^T \vec{z} \geq 0$, all the eigen-value of $\vec{x}\vec{x}^T$ will larger or equal to 0.

(b) Similarly, $\sum_{p=1}^{P} \delta_p \vec{x}_p \vec{x}_p^T$ is a symmetric matrix.

$\vec{z}^T \sum_{p=1}^{P} \delta_p \vec{x}_p \vec{x}_p^T \vec{z}$

$= \sum_{p=1}^{P} \delta_p (\vec{z}^T \vec{x}_p \vec{x}_p^T \vec{z})$

According to (a), $\vec{z}^T \vec{x}_p \vec{x}_p^T \vec{z} \geq 0$

If $\delta_p \geq 0$ for each p.

$\sum_{p=1}^{P} \delta_p (\vec{z}^T \vec{x}_p \vec{x}_p^T \vec{z}) \geq 0$

Thus, all eigen-values of $\sum_{p=1}^{P} \delta_p \vec{x}_p \vec{x}_p^T$ are non-negative.

(c). $\vec{z}^T \left(\sum_{p=1}^{P} \delta_p \vec{x}_p \vec{x}_p^T + \lambda \vec{I}_{N\times N}\right) \vec{z}$

$= \sum_{p=1}^{P} \delta_p (\vec{z}^T \vec{x}_p \vec{x}_p^T \vec{z}) + \vec{z}^T \lambda I_{N\times N} \vec{z}$

$= \sum_{p=1}^{P} \delta_p (\vec{z}^T \vec{x}_p \vec{x}_p^T \vec{z}) + \sum_{i=1}^{n} \lambda z_i^2$

If $\delta_p \geq 0$ and $\lambda > 0$.

$\sum_{p=1}^{P} \delta_p (\vec{z}^T \vec{x}_p \vec{x}_p^T \vec{z}) + \sum_{i=1}^{n} \lambda z_i > 0$

Thus, the eigen-values of $\sum_{p=1}^{P} \delta_p \vec{x}_p \vec{x}_p^T + \lambda I_{N\times N}$ must be positive.

4.5 (a) $g(\vec{w}) = \log(1+e^{\vec{w}^T\vec{w}})$

$\nabla g(\vec{w}) = \dfrac{e^{\vec{w}^T\vec{w}} \cdot 2\vec{w}}{1+e^{\vec{w}^T\vec{w}}} = 0$

Since $e^{\vec{w}^T\vec{w}} \neq 0$ for all $\vec{w}$

$2\vec{w} = 0 \Rightarrow \vec{w} = 0$

$\vec{w} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is the stationary point.

(b) $\nabla^2 g(\vec{w}) = \dfrac{4e^{\vec{w}^T\vec{w}} \cdot \vec{w}\vec{w}^T + 2e^{\vec{w}^T\vec{w}}(1+e^{\vec{w}^T\vec{w}})}{(1+e^{\vec{w}^T\vec{w}})^2}$

Since $4(e^{\vec{w}^T\vec{w}} + 2e^{\vec{w}^T\vec{w}}) > 0$

$2e^{\vec{w}^T\vec{w}} > 0 \Rightarrow 2e^{\vec{w}^T\vec{w}}(1+e^{\vec{w}^T\vec{w}}) > 0$

According to Exercise 4.2

$\nabla^2 g(\vec{w})$'s eigen-values will be non-negative. So $g(\vec{w})$ is convex.