Scalp topographic mapping using spherical spline interpolation in EMSE v4.2

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We seek to solve the following problem:

Given a set of m voltage measurements, represented as the column vector $\mathbf{v} = \left[v_1,\ldots,v_m\right]^T$ at known angular locations $q\left(m_i\right)$, assumed to lie on the surface of a sphere, find the interpolated value $\tilde{v}_r$ at location $q\left(r\right)$, using spherical splines.

Following the method given in Perrin et al. (EEG Clin Neurophys 72:184-187 (1989), we want to solve for the (unknown) coefficient vector $\underset{(m+1)\times 1}{\mathbf{c}}$

$$\tilde{v}_J = \mathbf{c}^T \mathbf{g} \tag{1.1}$$

where the elements of g are obtained from the n$^{\text{th}}$ degree Legendre polynomial P_n

$$g_i\left(x_i\right) = \begin{cases} \dfrac{1}{4p} \displaystyle\sum_{n=1}^{N} \dfrac{2n+1}{n^4\left(n+1\right)^4} P_n\left(x_i\right) & 1 \le i \le m \\ 1 & i = \left(m+1\right) \end{cases} \tag{1.2}$$

and $x_i = \cos\left(q(r), q\left(m_i\right)\right)$.

We can find $\mathbf{c}$ as the solution to

$$\mathbf{v}' = \mathbf{G}\mathbf{c} \tag{1.3}$$

where $\mathbf{v}' = \left[v_1,\ldots,v_n,0\right]^T$ and

$$\mathbf{G} = \begin{bmatrix} & & & 1 \\ & g_{ij} = g\left(\cos\left(q\left(m_i\right), q\left(m_j\right)\right)\right) & & . \\ & & & . \\ 1 & . & . & 1 \end{bmatrix} \tag{1.4}$$

Then

$$\mathbf{c} = \mathbf{G}^{-1}\mathbf{v}' \tag{1.5}$$

In practice, the inversion of G is sensitive to errors (e.g. the electrode locations do not really lie on the surface of a sphere), so the solution should be regularized. IN EMSE v4.2, we regularize by singular value truncation, or

$$
\begin{aligned}
\mathbf{G} &= \mathbf{UWV}^T \\
\mathbf{G}^{-1} \approx \mathbf{G}^{-} &= \mathbf{VW}^{-}\mathbf{U}^T
\end{aligned} \tag{1.6}
$$

and

$$
W^{-} = \left[w_{ij}^{-} \right], w_{ij}^{-} = \begin{cases} 1/w_{ij} & w_{ij} > cutoff \\ 0 & otherwise \end{cases} \tag{1.7}
$$

The *cutoff* is determined by the smoothing parameter using the following rule

$$
cutoff = k \cdot 10^{-l} \tag{1.8}
$$

where $k = \begin{cases} 10^{-5} & Laplacian \\ 10^{-7} & voltage \end{cases}$, and λ is the smoothing parameter entered in the topographic mapping dialog box. Thus as smaller values of λ are entered (i.e. $l < 0$), fewer terms are truncated, and the solution is less smoothed.

Equation (1.1) holds for voltage interpolation. To obtain an interpolated Laplacian map, we must replace g(x) (equation (1.2)) with

$$
h_i\left(x_i\right) = \begin{cases} -\dfrac{1}{4p} \displaystyle\sum_{n=1}^{N} \dfrac{2n+1}{n^3\left(n+1\right)^3} P_n\left(x_i\right) & 1 \le i \le m \\ 1 & i = \left(m+1\right) \end{cases} \tag{1.9}
$$