COMP3721 Question Bank 1 Suggested Solution

Sets, Relations, and Functions

Question 1

(a) True

(b) False

(c) True

(d) True

(e) True

(f) True

(g) False

(h) True

(i) False

Question 2

(a) $2^{\{7,8,9\}}$
$= \{\emptyset, \{7\}, \{8\}, \{9\}, \{7,8\}, \{7,9\}, \{8,9\}, \{7,8,9\}\}$
$2^{\{7,9\}}$
$= \{\emptyset, \{7\}, \{9\}, \{7,9\}\}$

Hence $2^{\{7,8,9\}} - 2^{\{7,9\}} = \{\{8\}, \{7,8\}, \{8,9\}, \{7,8,9\}\}$
Note that each element in the answer is a set containing the digit 8.

(b) $2^{\emptyset} = \{\emptyset\}$
Note that $\emptyset$ is a subset of every set (including the set $\emptyset$).

Question 3

First, we prove $A \cap (A \cup B) \subseteq A$:

$$\begin{aligned}
&x \in A \cap (A \cup B) \\
\Rightarrow &x \in A \text{ and } x \in (A \cup B) \\
\Rightarrow &x \in A
\end{aligned}$$

Second, we prove $A \subseteq A \cap (A \cup B)$:

$$\begin{aligned}
&x \in A && \text{(i)} \\
\Rightarrow &x \in A \text{ or } x \in B \\
\Rightarrow &x \in (A \cup B) && \text{(ii)} \\
\Rightarrow &x \in A \text{ and } x \in (A \cup B) && \text{Combining (i) and (ii)} \\
\Rightarrow &x \in A \cap (A \cup B)
\end{aligned}$$

Thus, $A \subseteq A \cap (A \cup B)$.

Question 4

(a) $\{1\} \times \{1,2\} \times \{1,2,3\}$
$= \{(1,1,1),(1,1,2),(1,1,3),(1,2,1),(1,2,2),(1,2,3)\}$

(b) $\emptyset \times \{1,2\}$
$= \emptyset$
Note: as there is no element in $\emptyset$, there is no element in the Cartesian product.

Question 5

(a) Let $A = \{1,2,3,4\}$, $B = \{v,w,x,y\}$, $f = \{(1,v),(2,v),(3,w),(4,x)\}$.

(b) Let $A = \{1,2,3,4\}$, $B = \{v,w,x,y,z\}$, $f = \{(1,v),(2,x),(3,y),(4,z)\}$.

(c) Let $A = \{1,2,3,4,5\}$, $B = \{w,x,y,z\}$, $f = \{(1,w),(2,w),(3,x),(4,y),(5,z)\}$.

(d) Let $A = \{1,2,3,4\}$, $B = \{w,x,y,z\}$, $f = \{(1,w),(2,x),(3,y),(4,z)\}$.

Question 6

(a) R is not reflexive, not symmetric and not transitive.
S is neither reflexive nor transitive but S is symmetric.

(b) $R \cup S$ is not reflexive, not symmetric and not transitive.

Question 7

To prove that R is an equivalence relation on A, we have to show that it has the following properties:

i) Reflexivity
$\forall a \in A, f(a) = f(a)$. Therefore $(a,a) \in R$.

ii) Symmetry
$\forall a,b \in A$, $(a,b) \in R$ implies $f(a) = f(b)$. This in turn implies $f(b) = f(a)$, and therefore $(b,a) \in R$.

iii) Transitivity
$\forall a,b,c \in A$, $(a,b) \in R$ implies $f(a) = f(b)$ and $(b,c) \in R$ implies $f(b) = f(c)$. Therefore $f(a) = f(c)$. Therefore $(a,c) \in R$

In conclusion, R is an equivalence relation on A.

Question 8

(a) i) Reflexivity
$\forall a \in A, a - a = 3 \cdot 0$. Therefore $(a, a) \in R$.

ii) Symmetry
$\forall a, b \in A$, $(a, b) \in R$ implies $a - b = 3c$ for some integer c. This in turn implies $(b - a) = 3(-c)$ for some integer $-c$. Therefore, $(b, a) \in R$.

iii) Transitivity
$\forall a, b, c \in A$, $(a, b) \in R$ and $(b, c) \in R$ imply $a - b = 3m$ and $b - c = 3n$, for some integers m and n. This in turn implies $(a - b) + (b - c) = 3m + 3n \Rightarrow a - c = 3(m + n)$. Therefore, $(a, c) \in R$.

Therefore, R is an equivalence relation on A.

(b)

Languages and Regular Expressions

Question 1

a) $b^*ab^* \cup b^*ab^*ab^*$

b) $b^*(b^*ab^*ab^*ab^*)^*$

c) $b^*(ab \cup a)^*\, abb\, b^*(ab \cup a)^*$

Question 2

a) It is valid. Observe that $L(a^*b^*a^*b^*) = \{a\}^* \circ \{b\}^* \circ \{a\}^* \circ \{b\}^*$. Now, $e \in \{a\}^*$, $b \in \{b\}^*$, $aa \in \{a\}^*$ and $e \in \{b\}^*$. Therefore $baa \in a^*b^*a^*b^*$.

b) It is valid. Observe that b^*a^* represents a set of strings that have zero or more b's followed by zero or more a's while a^*b^* represents a set of strings that have zero or more a's followed by zero or more b's. Therefore, the intersection contains only the set of strings that are either the empty string e, all a's or all b's.

c) It is NOT valid. Since $b \in a^*b^*$ and $b \in b^*c^*$, so $b \in a^*b^* \cap b^*c^* \neq \emptyset$.

d) It is NOT valid. Observe that any substring cd in any element of the regular language, which is represented by $(a(cd)^*b)^*$, must be followed by one b. This is not possible for the string $abcd$.

e) It is NOT valid. Consider $a^2b^2 \in \{a^nb^n : n \geq 0\}$ and $bc \in \{b^nc^n : n \geq 0\}$, but $a^2b^3c \notin \{a^nb^{2n}c^n : n \geq 0\}$. Actually, $\{a^nb^n : n \geq 0\}\{b^nc^n : n \geq 0\} = \{a^nb^n : n \geq 0\}\{b^mc^m : m \geq 0\} = \{a^nb^{n+m}c^m : n, m \geq 0\}$.

Question 3

a) Proof: We will prove this by induction on the length of w. The basis case is that when $|w| = 0$. In this case, $w = e$, then $\{w\} = \{e\} = L(\emptyset^*)$, so $\{w\}$ is regular.

Now for the inductive hypothesis, assume that for $|w| \leq n$, $\{w\}$ is regular.

Next, for the inductive step, we wish to prove that for $|w| = n + 1$, $\{w\}$ is regular. Let $w = a_1a_2a_3 \ldots a_na_{n+1}$ where $a_i \in \Sigma$ for $\forall i$ and suppose $w' = a_1a_2a_3 \ldots a_n$. Then, by the inductive hypothesis, $\{w'\}$ is regular since $|w'| \leq n$. Also, by the axioms of regular expression, a_{n+1} is a regular expression and thus, $\{a_{n+1}\}$ is regular. Since the concatenation of two regular languages is regular, $\{w'\}\{a_{n+1}\} = \{w\}$ is regular.

By the principle of mathematical induction, the statement of 3a is proved.

b) Proof: We will also prove this by induction on the size of L. The basis case is that $|L| = 0$, that is, $L = \emptyset$. Then $L = L(\emptyset)$, thus L is regular.

As the inductive hypothesis, suppose that for any language $L \subset \Sigma^*$, if $|L| \leq n$ then L is regular.

Assume that $L = \{w_1, w_2, \ldots, w_{n+1}\}$ where $w_i \in \Sigma^*$, for $i = 1, ..., n+1$. Let $L' = \{w_1, w_2, \ldots, w_n\}$. Since $|L'| = n$, by the inductive hypothesis, L' is regular. Consider $\{w_{n+1}\}$, by the theorem proved in question 3a, $\{w_{n+1}\}$ is regular. Therefore, $L' \cup \{w_{n+1}\} = L$ is regular because it is known that the union of two regular languages is regular.

Therefore, every finite language is regular.

Question 4

$(ab \cup b)^*a^*$

Question 5

Proof: It is obvious that $L^* \subseteq (L^*)^*$, so we only need to show that $(L^*)^* \subseteq L^*$ holds.
For any $w \in (L^*)^*$, $w = w_0w_1 \ldots w_n$, where $w_i \in L^*$; and $w_i = u_{i0}u_{i1} \ldots u_{ik_i}$, where $u_{ij} \in L$. Then, we have $w = (u_{00} \ldots u_{0k_0})(u_{10} \ldots u_{1k_1}) \ldots (u_{n0} \ldots u_{nk_n}) \in L^*$. So we conclude that $(L^*)^* \subseteq L^*$, and $L^* = (L^*)^*$.

Deterministic Finite Automata

Question 1

The corresponding DFA is $M = (K, \Sigma, \delta, s, F)$ where

$$
\begin{aligned}
K &= \{q_0, q_1, q_2, q_3, q_4, q_5\} \\
\Sigma &= \{0, 1\} \\
s &= q_0 \\
F &= \{q_5\}
\end{aligned}
$$

and δ is the function tabulated as follows:

q	σ	$\delta(q, \sigma)$
q_0	0	q_1
q_0	1	q_0
q_1	0	q_1
q_1	1	q_2
q_2	0	q_1
q_2	1	q_3
q_3	0	q_4
q_3	1	q_0
q_4	0	q_1
q_4	1	q_5
q_5	0	q_5
q_5	1	q_5

Question 2

Question 3

(a)

(b)

(c)

(d)

(e)

(f)

Nondeterministic Finite Automata

Question 1

a) a^*

b) $(a(ba \cup b))^*$

c) $(a \cup ba)^* bb$

Question 2

a)

b)

c)

Solution 1:

Solution 2: Food for thought: why is it necessary to intrdoduce the empty transition? Is it possible to eliminate q_0 and make q_1 the start and final state?

d)

Solution 1:

Solution 2:

Question 3

a) The NFA is drawn as follows:

b) First of all, we construct a useful lookup table:

State in the NFA q_i	$E(q_i)$	set of p s.t. $(q_i, a, p) \in \Delta$	set of p s.t. $(q_i, b, p) \in \Delta$
q_0	$\{q_0\}$	$\{q_1\}$	$\emptyset$
q_1	$\{q_1\}$	$\{q_3\}$	$\{q_2\}$
q_2	$\{q_0, q_2\}$	$\{q_0\}$	$\emptyset$
q_3	$\{q_3\}$	$\emptyset$	$\{q_0\}$

Then, the states in the corresponding DFA are found as follows:

States in the DFA Q_i	$\delta'(Q_i, a)$	$\delta'(Q_i, b)$
$Q_0 = E(q_0) = \{q_0\}$	$E(q_1) = \{q_1\}$	$\emptyset$
$Q_1 = \{q_1\}$	$E(q_3) = \{q_3\}$	$E(q_2) = \{q_0, q_2\}$
$Q_2 = \{q_3\}$	$\emptyset$	$E(q_0) = \{q_0\}$
$Q_3 = \{q_0, q_2\}$	$E(q_1) \cup E(q_0) = \{q_0, q_1\}$	$\emptyset$
$Q_4 = \{q_0, q_1\}$	$E(q_1) \cup E(q_3) = \{q_1, q_3\}$	$E(q_2) = \{q_0, q_2\}$
$Q_5 = \{q_1, q_3\}$	$E(q_3) = \{q_3\}$	$E(q_2) \cup E(q_0) = \{q_0, q_2\}$
$Q_6 = \emptyset$	$\emptyset$	$\emptyset$

The DFA is drawn as follows:

d) The NFA is drawn as follows:

First of all, we construct a useful lookup table:

State in the NFA q_i	$E(q_i)$	set of p s.t. $(q_i, a, p) \in \Delta$	set of p s.t. $(q_i, b, p) \in \Delta$
q_0	$\{q_0, q_1, q_2\}$	$\{q_3\}$	$\emptyset$
q_1	$\{q_1\}$	$\{q_0\}$	$\emptyset$
q_2	$\{q_2\}$	$\emptyset$	$\{q_0\}$
q_3	$\{q_3\}$	$\{q_4\}$	$\emptyset$
q_4	$\{q_4\}$	$\emptyset$	$\{q_5\}$
q_5	$\{q_5\}$	$\{q_6\}$	$\emptyset$
q_6	$\{q_6\}$	$\emptyset$	$\{q_7\}$
q_7	$\{q_7\}$	$\emptyset$	$\emptyset$

Then, the states in the corresponding DFA are found as follows:

States in the DFA Q_i	$\delta'(Q_i, a)$	$\delta'(Q_i, b)$
$Q_0 = E(q_0) = \{q_0, q_1, q_2\}$	$E(q_3) \cup E(q_0) = \{q_0, q_1, q_2, q_3\}$	$E(q_0) = \{q_0, q_1, q_2\}$
$Q_1 = \{q_0, q_1, q_2, q_3\}$	$E(q_3) \cup E(q_0) \cup E(q_4) = \{q_0, q_1, q_2, q_3, q_4\}$	$E(q_0) = \{q_0, q_1, q_2\}$
$Q_2 = \{q_0, q_1, q_2, q_3, q_4\}$	$E(q_3) \cup E(q_0) \cup E(q_4) = \{q_0, q_1, q_2, q_3, q_4\}$	$E(q_0) \cup E(q_5) = \{q_0, q_1, q_2, q_5\}$
$Q_3 = \{q_0, q_1, q_2, q_5\}$	$E(q_3) \cup E(q_0) \cup E(q_6) = \{q_0, q_1, q_2, q_3, q_6\}$	$E(q_0) = \{q_0, q_1, q_2\}$
$Q_4 = \{q_0, q_1, q_2, q_3, q_6\}$	$E(q_3) \cup E(q_0) \cup E(q_4) = \{q_0, q_1, q_2, q_3, q_4\}$	$E(q_0) \cup E(q_7) = \{q_0, q_1, q_2, q_7\}$
$Q_5 = \{q_0, q_1, q_2, q_7\}$	$E(q_3) \cup E(q_0) = \{q_0, q_1, q_2, q_3\}$	$E(q_0) = \{q_0, q_1, q_2\}$

The DFA is drawn as follows:

Question 4

(a) Let $L = L(M)$, $M = (\Sigma, K, \delta, s, F)$.
Let M' be a DFA such that, $M' = (\Sigma, K, \delta, s, F')$ where $q \in F'$ iff there exists a path in M from q to some state in F.
Given $w, w \in$Pref(L) iff w is a prefix of some string $s \in L$.
Thus $w \in$ Pref(L) iff w is a prefix of s and s has a path leading from the initial state in M to a final state in M.
Therefore, $w \in$ Pref(L) iff w stop at some point in the path that leads s from M to a final state. So, $w \in L$ iff w stop at some final state in M'. That means that Pref$(L) =$ L(M').
Pref(L) is regular since there exists a DFA that accept it.

(b) It is clear that Suf$(L) =$ (Pref$(L^R))^R$. If L is regular then L^R is regular. Since L^R is regular, Pref(L^R) is also regular and thus (Pref$(L^R))^R =$ Suf(L) is regular.

(c) Let M be a DFA such that $L(M) = L, M = (\Sigma, K, \delta, s, F)$. Let $M^{''} = (\Sigma, K, \delta, s, F^{''})$, where $q \in F^{''}$ iff $\exists x \in \Sigma^*$ such that $x \in L^{'}$ and x drives M from q to some $f \in F$ (one doesn't have to know how to exactly construct $M^{''}$, all we need to do is to show that there exists a DFA in this world that would accept the language). Obviously, $w \in L/L^{'}$ iff w leads a path to a final state in $M^{''}$. Thus $L/L^{'} = L(M^{''})$.

The Fundamental Theorem

Question 1

(a) T

(b) F

(c) T

(d) F

(e) T

(f) F

(g) F

(h) F

(i) T

Question 2

(a) Notation: $R(i, j, k)$ is denoted by R^k_{ij} in this part

$$\begin{aligned} R^4_{14} &= R^3_{14} \cup R^3_{14} {R^3_{44}}^* R^3_{44} = 1^*0(11^*0)^*00^*1(e \cup (e \cup 0 \cup 1)(e \cup 0 \cup 1)^*) \\ &= 1^*0(11^*0)^*00^*1(0 \cup 1)^* \end{aligned}$$

$$\begin{aligned} R^3_{14} &= R^2_{14} \cup R^2_{13} {R^2_{33}}^* R^2_{34} = \emptyset \cup 1^*0(11^*0)^*0(e \cup 0)^*1 \\ &= 1^*0(11^*0)^*00^*1 \end{aligned}$$

$$\begin{aligned} R^2_{14} &= R^1_{14} \cup R^1_{12} {R^1_{22}}^* R^1_{24} = \emptyset \cup (0 \cup 1^*0)(e \cup 11^*0)^*\emptyset = \emptyset \\ R^1_{14} &= R^0_{14} \cup R^0_{11} {R^0_{11}}^* R^0_{14} = \emptyset \\ R^1_{12} &= R^0_{12} \cup R^0_{11} {R^0_{11}}^* R^0_{12} = 0 \cup (e \cup 1)(e \cup 1)^*0 = 0 \cup 1^*0 = 1^*0 \\ R^1_{22} &= R^0_{22} \cup R^0_{21} {R^0_{11}}^* R^0_{12} = e \cup 1(e \cup 1)^*0 = e \cup 11^*0 \\ R^1_{24} &= R^0_{24} \cup R^0_{21} {R^0_{11}}^* R^0_{14} = e \cup 1(e \cup 1)^*\emptyset = \emptyset \end{aligned}$$

$$\begin{aligned} R^2_{13} &= R^1_{13} \cup R^1_{12} {R^1_{22}}^* R^1_{23} = \emptyset \cup 1^*0(e \cup 11^*0)^*0 = 1^*0(11^*0)^*0 \\ R^1_{13} &= R^0_{13} \cup R^0_{11} {R^0_{11}}^* R^0_{13} = \emptyset \\ R^1_{23} &= R^0_{23} \cup R^0_{21} {R^0_{11}}^* R^0_{13} = 0 \cup 1(e \cup 1)^*\emptyset = 0 \end{aligned}$$

$$\begin{aligned} R^2_{33} &= R^1_{33} \cup R^1_{32} {R^1_{22}}^* R^1_{23} = e \cup 0 \cup \emptyset(e \cup 11^*0)0 = e \cup 0 \\ R^1_{33} &= R^0_{33} \cup R^0_{31} {R^0_{11}}^* R^0_{13} = e \cup 0 \cup \emptyset(e \cup 1)^*\emptyset = e \cup 0 \\ R^1_{32} &= R^0_{32} \cup R^0_{31} {R^0_{11}}^* R^0_{12} = \emptyset \cup \emptyset = \emptyset \end{aligned}$$

$$\begin{aligned} R^2_{34} &= R^1_{34} \cup R^1_{32} {R^1_{22}}^* R^1_{24} = 1 \\ R^1_{34} &= R^0_{34} \cup R^0_{31} {R^0_{11}}^* R^0_{14} = 1 \cup \emptyset = 1 \end{aligned}$$

$$R^3_{44} = R^2_{44} \cup R^2_{43} {R^2_{33}}^* R^2_{34} = e \cup 0 \cup 1$$

$$\begin{aligned} R^2_{44} &= R^1_{44} \cup R^1_{42} {R^1_{22}}^* R^1_{24} = e \cup 0 \cup 1 \\ R^1_{44} &= R^0_{44} \cup R^0_{41} {R^0_{11}}^* R^0_{14} = e \cup 0 \cup 1 \cup \emptyset = e \cup 0 \cup 1 \\ R^1_{42} &= R^0_{42} \cup R^0_{41} {R^0_{11}}^* R^0_{12} = \emptyset \cup \emptyset = \emptyset \end{aligned}$$

$$R^2_{43} = R^1_{43} \cup R^1_{42} {R^1_{22}}^* R^1_{23} = \emptyset$$

$$R_{43}^1 = R_{43}^0 \cup R_{41}^0 {R_{11}^0}^* R_{13}^0 = \emptyset \cup \emptyset = \emptyset$$

(b) Our goal is to find $R(1,3,4)$. We first deduce terms needed in a backward way:

$$R(1,3,4) = R(1,3,3) \cup R(1,4,3)(R(4,4,3))^* R(4,3,3)$$

$$\begin{aligned}
R(1,3,3) &= R(1,3,2) \cup R(1,3,2)(R(3,3,2))^* R(3,3,2) \\
R(1,4,3) &= R(1,4,2) \cup R(1,3,2)(R(3,3,2))^* R(3,4,2) \\
R(4,4,3) &= R(4,4,2) \cup R(4,3,2)(R(3,3,2))^* R(3,4,2) \\
R(4,3,3) &= R(4,3,2) \cup R(4,3,2)(R(3,3,2))^* R(3,3,2)
\end{aligned}$$

$$\begin{aligned}
R(1,3,2) &= R(1,3,1) \cup R(1,2,1)(R(2,2,1))^* R(2,3,1) \\
R(3,3,2) &= R(3,3,1) \cup R(3,2,1)(R(2,2,1))^* R(2,3,1) \\
R(1,4,2) &= R(1,4,1) \cup R(1,2,1)(R(2,2,1))^* R(2,4,1) \\
R(3,4,2) &= R(3,4,1) \cup R(3,2,1)(R(2,2,1))^* R(2,4,1) \\
R(4,4,2) &= R(4,4,1) \cup R(4,2,1)(R(2,2,1))^* R(2,4,1) \\
R(4,3,2) &= R(4,3,1) \cup R(4,2,1)(R(2,2,1))^* R(2,3,1)
\end{aligned}$$

$$\begin{aligned}
R(1,3,1) &= R(1,3,0) \cup R(1,1,0)(R(1,1,0))^* R(1,3,0) \\
R(1,2,1) &= R(1,2,0) \cup R(1,1,0)(R(1,1,0))^* R(1,2,0) \\
R(2,2,1) &= R(2,2,0) \cup R(2,1,0)(R(1,1,0))^* R(1,2,0) \\
R(2,3,1) &= R(2,3,0) \cup R(2,1,0)(R(1,1,0))^* R(1,3,0) \\
R(3,3,1) &= R(3,3,0) \cup R(3,1,0)(R(1,1,0))^* R(1,3,0) \\
R(3,2,1) &= R(3,2,0) \cup R(3,1,0)(R(1,1,0))^* R(1,2,0) \\
R(1,4,1) &= R(1,4,0) \cup R(1,1,0)(R(1,1,0))^* R(1,4,0) \\
R(2,4,1) &= R(2,4,0) \cup R(2,1,0)(R(1,1,0))^* R(1,4,0) \\
R(3,4,1) &= R(3,4,0) \cup R(3,1,0)(R(1,1,0))^* R(1,4,0) \\
R(4,4,1) &= R(4,4,0) \cup R(4,1,0)(R(1,1,0))^* R(1,4,0) \\
R(4,2,1) &= R(4,2,0) \cup R(4,1,0)(R(1,1,0))^* R(1,2,0) \\
R(4,3,1) &= R(4,3,0) \cup R(4,1,0)(R(1,1,0))^* R(1,3,0)
\end{aligned}$$

We can then construct the regular expression as follows:

	$k=0$	$k=1$	$k=2$	$k=3$
$R(1,1,k)$	e			
$R(1,2,k)$	a	$a \cup ee^*a$ $= a$		
$R(1,3,k)$	$\emptyset$	$\emptyset \cup ee^*\emptyset$ $= \emptyset$	$\emptyset \cup ae^*b$ $= ab$	$ab \cup ab(a \cup b \cup e)^*(a \cup b \cup e)$ $= ab(a \cup b)^*$
$R(1,4,k)$	b	$b \cup ee^*b$ $= b$	$b \cup ae^*b$ $= ab \cup b$	$ab \cup b \cup ab(a \cup b \cup e)^*\emptyset$ $= ab \cup b$
$R(2,1,k)$	$\emptyset$			
$R(2,2,k)$	e	$e \cup \emptyset e^*a$ $= e$		
$R(2,3,k)$	b	$b \cup \emptyset e^*\emptyset$ $= b$		
$R(2,4,k)$	b	$b \cup \emptyset e^*b$ $= b$		
$R(3,1,k)$	$\emptyset$			
$R(3,2,k)$	$\emptyset$	$\emptyset \cup \emptyset e^*a$ $= \emptyset$		
$R(3,3,k)$	$a \cup b \cup e$	$a \cup b \cup e \cup \emptyset e^*\emptyset$ $= a \cup b \cup e$	$a \cup b \cup e \cup \emptyset e^*b$ $= a \cup b \cup e$	
$R(3,4,k)$	$\emptyset$	$\emptyset \cup \emptyset e^*b$ $= \emptyset$	$\emptyset \cup \emptyset e^*b$ $= \emptyset$	
$R(4,1,k)$	$\emptyset$			
$R(4,2,k)$	a	$a \cup \emptyset e^*a$ $= a$		
$R(4,3,k)$	$\emptyset$	$\emptyset \cup \emptyset e^*\emptyset$ $= \emptyset$	$\emptyset \cup ae^*b$ $= ab$	$ab \cup ab(a \cup b \cup e)^*(a \cup b \cup e)$ $= ab(a \cup b)^*$
$R(4,4,k)$	$a \cup e$	$a \cup e \cup \emptyset e^*b$ $= a \cup e$	$a \cup e \cup ae^*b$ $= a \cup ab \cup e$	$a \cup ab \cup e \cup ab(a \cup b \cup e)^*\emptyset$ $= a \cup ab \cup e$

Hence, the required regular expression $= R(1,3,4)$
$= ab(a \cup b)^* \cup (ab \cup b)(a \cup ab \cup e)^*(ab(a \cup b)^*)$
$= ab(a \cup b)^* \cup (ab \cup b)(a \cup ab)^*(ab(a \cup b)^*)$