Mesh Parametrization

An Invitation to Geometry Processing

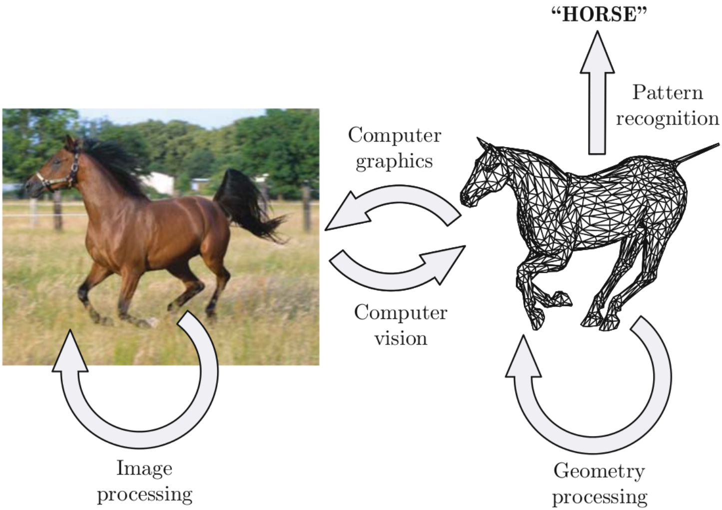
Tingran Gao

Department of Mathematics
Duke University

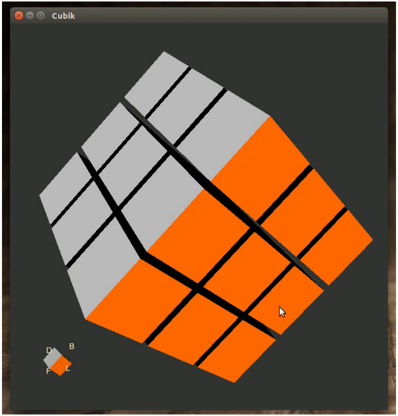
MATH 290 Guest Lecture
April 21, 2016

Table of Contents

What is Geometry Processing?



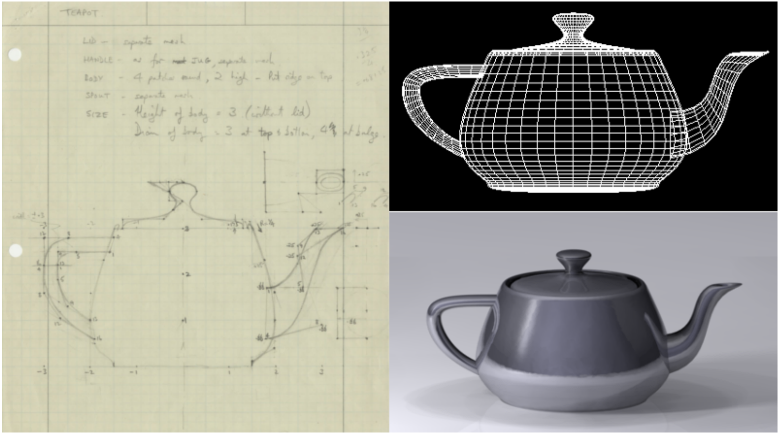
The Digital Virtual World...



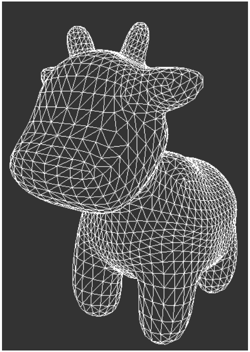
...of Dreams

Instantiated with Computer Graphics

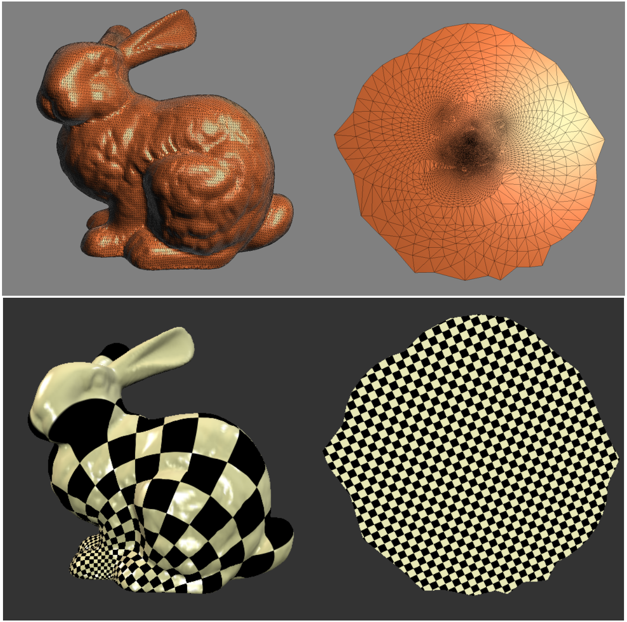
Utah Teapot: *Hello World* of Computer Graphics



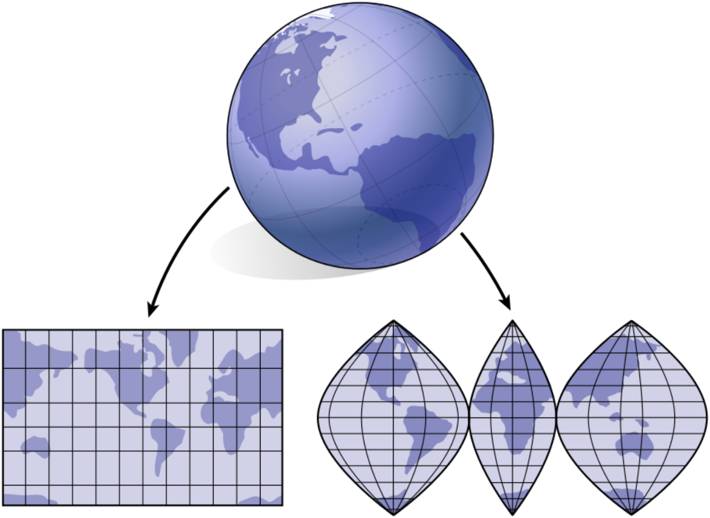
Texture on Triangular Meshes



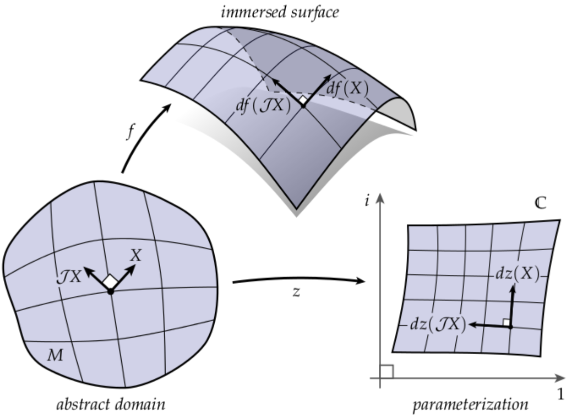
Mesh Parametrization for Texture Mapping



Cartography



Conformal Flattening: The Cauchy-Riemann Equation



$$\mathrm{d}z\,(\mathcal{J}X) = i\mathrm{d}z\,(X)$$

Conformal Flattening: The Cauchy-Riemann Equation

$(x, y) \mapsto (u(x, y), v(x, y))$ is a conformal map if and only if

$$\begin{cases} \dfrac{\partial u}{\partial x} & = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} & = -\dfrac{\partial v}{\partial x} \end{cases}$$

Conformal Flattening: The Cauchy-Riemann Equation

$(x, y) \mapsto (u(x, y), v(x, y))$ is a conformal map if and only if

$$\begin{cases} \dfrac{\partial u}{\partial x} &= \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} &= -\dfrac{\partial v}{\partial x} \end{cases}$$

Equivalently, write $z(x, y) = u(x, y) + iv(x, y)$, then

$$\mathrm{d}z(\mathcal{J}X) = i\mathrm{d}z(X)$$
$$\star\mathrm{d}z = i\mathrm{d}z$$

Conformal Flattening: The Cauchy-Riemann Equation

$(x, y) \mapsto (u(x, y), v(x, y))$ is a conformal map if and only if

$$\begin{cases} \dfrac{\partial u}{\partial x} &= \dfrac{\partial v}{\partial y} \\ \\ \dfrac{\partial u}{\partial y} &= -\dfrac{\partial v}{\partial x} \end{cases}$$

Equivalently, write $z(x, y) = u(x, y) + iv(x, y)$, then

$$\mathrm{d}z(\mathcal{J}X) = i\mathrm{d}z(X)$$
$$\star\mathrm{d}z = i\mathrm{d}z$$

Conclusion: In practice, it suffices to minimize the *conformal energy*

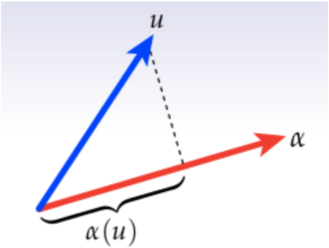
$$E_C(z) := \frac{1}{4}\|\star\mathrm{d}z - i\mathrm{d}z\|^2 .$$

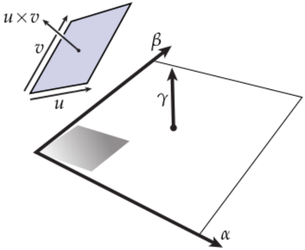
Discrete Exterior Calculus (DEC)

Differential forms "measure" the "volume" spanned by vectors:

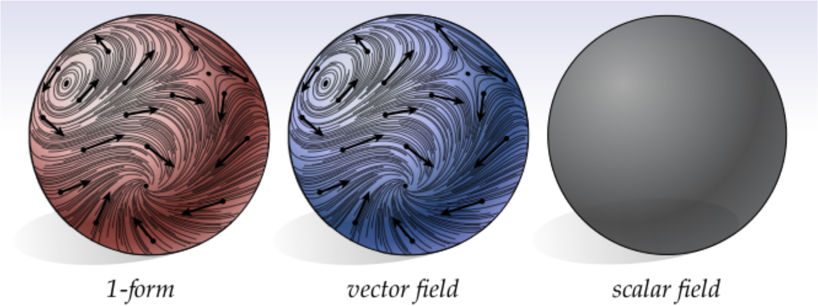
$$\alpha(u) = \langle \alpha, u \rangle$$

$$\alpha \wedge \beta(u, v) = \alpha(u)\beta(v) - \alpha(v)\beta(u)$$

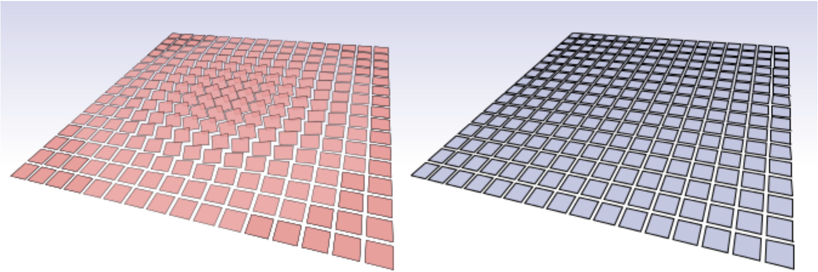




Differential 1-forms

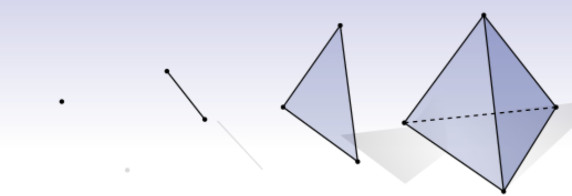


Differential 2-forms



Differential Forms on Triangular Meshes

On a triangular mesh, define $0, 1, 2$-forms by pairing with simplices

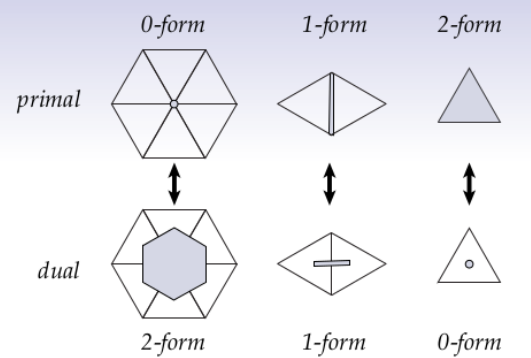


0-form (vertices)	1-form (edges)	2-form (triangles)	3-form (tetrahedra)
$\hat{\phi} \in \mathbb{R}^{\|V\|}$	$\hat{\alpha} \in \mathbb{R}^{\|E\|}$	$\hat{\beta} \in \mathbb{R}^{\|F\|}$	$\hat{\omega} \in \mathbb{R}^{\|T\|}$

Numbers assigned to each simplex!

Discrete Hodge Star

Hodge star $\star$ interpreted as "dual forms" defined on the dual mesh

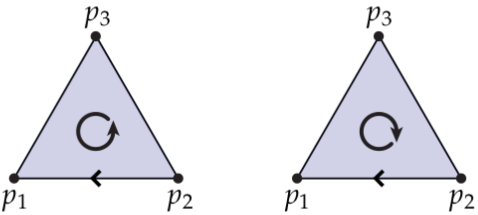


$$\hat{\star}\hat{\alpha} = \frac{|\sigma^*|}{|\sigma|}\hat{\alpha}$$

Discrete Exterior Derivative

Exterior derivative $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ defined by Stokes' Theorem

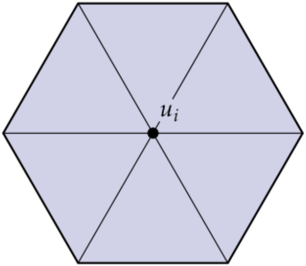
$$\int_{\Delta} \mathrm{d}\alpha = \int_{\partial\Delta} \alpha.$$



Oriented Sum of the numbers assigned on the boundary!

Example: Laplace-Beltrami on $\Omega^0(M)$

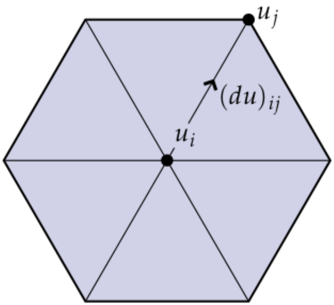
$$\Delta u = \star \mathrm{d} \star \mathrm{d} u, \qquad u \in \Omega^0(M).$$



Example: Laplace-Beltrami on $\Omega^0(M)$

$$\Delta u = \star \mathrm{d} \star \mathrm{d}u, \qquad u \in \Omega^0(M).$$

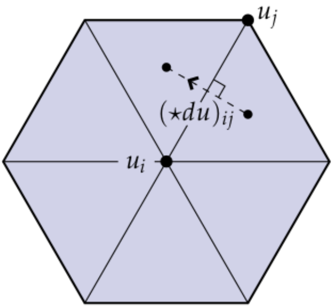
$$(du)_{ij} = \int_{e_{ij}} du = \int_{\partial e_{ij}} u = u_j - u_i$$



Example: Laplace-Beltrami on $\Omega^0(M)$

$$\Delta u = \star \mathrm{d} \star \mathrm{d} u, \qquad u \in \Omega^0(M).$$

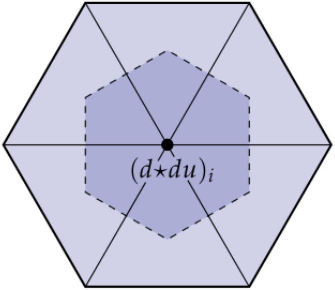
$$(\star du)_{ij} = \frac{|e_{ij}^\star|}{|e_{ij}|}(u_j - u_i)$$



Example: Laplace-Beltrami on $\Omega^0(M)$

$$\Delta u = \star \mathrm{d} \star \mathrm{d} u, \qquad u \in \Omega^0(M).$$

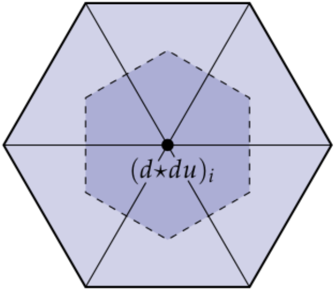
$$(d \star du)_i = \int_{C_i} d \star du = \int_{\partial C_i} \star du = \sum_j \frac{|e^\star_{ij}|}{|e_{ij}|}(u_j - u_i)$$



Example: Laplace-Beltrami on $\Omega^0(M)$

$$\Delta u = \star \mathrm{d} \star \mathrm{d} u, \qquad u \in \Omega^0(M).$$

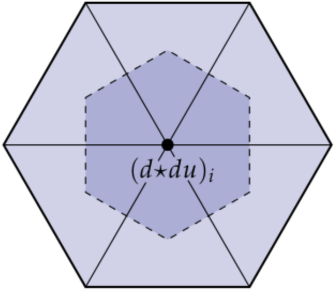
$$(\star d \star du)_i = \frac{1}{|C_i|} \sum_j \frac{|e_{ij}^\star|}{|e_{ij}|}(u_j - u_i)$$



Example: Laplace-Beltrami on $\Omega^0(M)$

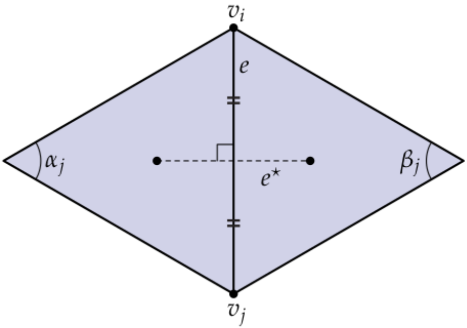
What is $\dfrac{|e_{ij}^*|}{|e_{ij}|}$?

$$(\star d \star du)_i = \frac{1}{|C_i|} \sum_j \frac{|e_{ij}^\star|}{|e_{ij}|}(u_j - u_i)$$



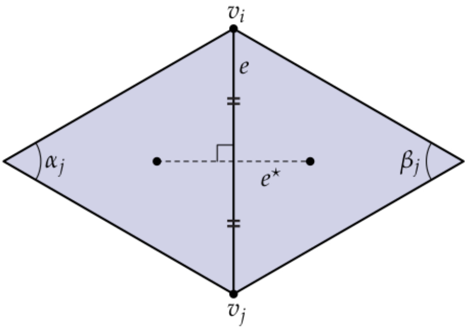
Example: Laplace-Beltrami on $\Omega^0(M)$

What is $\dfrac{\left|e_{ij}^*\right|}{\left|e_{ij}\right|}$?
– *the ratio of the length of the dual edge over the length of the original edge*



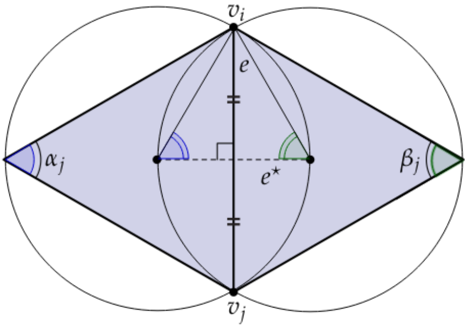
Example: Laplace-Beltrami on $\Omega^0(M)$

$$\frac{\left|e_{ij}^*\right|}{\left|e_{ij}\right|} = \frac{1}{2}\left(\cot\alpha_j + \cot\beta_j\right)$$



Example: Laplace-Beltrami on $\Omega^0(M)$

$$\frac{\left|e_{ij}^*\right|}{\left|e_{ij}\right|} = \frac{1}{2}\left(\cot\alpha_j + \cot\beta_j\right)$$



Cotangent Laplacian

$$\frac{\left|e_{ij}^*\right|}{\left|e_{ij}\right|} = \frac{1}{2}\left(\cot\alpha_j + \cot\beta_j\right)$$

$$\begin{aligned}(\star \mathrm{d} \star \mathrm{d} u)_i &= \frac{1}{|C_i|}\sum_j \frac{\left|e_{ij}^*\right|}{\left|e_{ij}\right|}\left(u_j - u_i\right) \\ &= \frac{1}{|C_i|}\underbrace{\sum_j \frac{1}{2}\left(\cot\alpha_j + \cot\beta_j\right)\left(u_j - u_i\right)}_{\text{Coincides with Finite Element Discretization}}\end{aligned}$$

Cotangent Laplacian

$$\frac{\left|e_{ij}^{*}\right|}{\left|e_{ij}\right|} = \frac{1}{2}\left(\cot\alpha_j + \cot\beta_j\right)$$

$$\begin{aligned}
(\star\mathrm{d}\star\mathrm{d}u)_i &= \frac{1}{|C_i|}\sum_j \frac{\left|e_{ij}^{*}\right|}{\left|e_{ij}\right|}\left(u_j - u_i\right) \\
&= \frac{1}{|C_i|}\underbrace{\sum_j \frac{1}{2}\left(\cot\alpha_j + \cot\beta_j\right)\left(u_j - u_i\right)}_{\text{Coincides with Finite Element Discretization}}
\end{aligned}$$

$$\Delta u\left(x_i\right) = \sum_j \frac{1}{2}\left(\cot\alpha_j + \cot\beta_j\right)\left(u\left(x_j\right) - u\left(x_i\right)\right)$$

Discrete Laplacians: No Free Lunch

	SYM	LOC	LIN	POS	PSD	CON
MEAN VALUE	○	●	●	●	○	○
INTRINSIC DEL	●	○	●	●	●	?
COMBINATORIAL	●	●	○	●	●	○
COTAN	●	●	●	○	●	●

Back to Cauchy-Riemann

$$\mathrm{d}z\left(\mathcal{J}X\right) = i\mathrm{d}z\left(X\right)$$
$$\star\mathrm{d}z = i\mathrm{d}z$$

Numerically, solve this equation in the *least-square sense*:

$$\min_z E_C\left(z\right) = \min_z \frac{1}{4}\left\|\star\mathrm{d}z - i\mathrm{d}z\right\|^2$$

Back to Cauchy-Riemann

$$\mathrm{d}z\left(\mathcal{J}X\right) = i\mathrm{d}z\left(X\right)$$
$$\star\mathrm{d}z = i\mathrm{d}z$$

Numerically, solve this equation in the *least-square sense*:

$$\min_z E_C\left(z\right) = \min_z \frac{1}{4}\left\|\star\mathrm{d}z - i\mathrm{d}z\right\|^2$$

Prevent *degeneracy* ($z = 0$), and "normalize" for *identifiability*:

$$\begin{aligned}\min_z \frac{1}{4}\left\|\star\mathrm{d}z - i\mathrm{d}z\right\|^2 \\ \text{s.t. } \|z\| = 1, \\ \langle z, \mathbf{1}\rangle = 0.\end{aligned}$$

In a Discrete World Everything Reduces to a Linear System

$$\begin{aligned}
\frac{1}{4}\left\|\star \mathrm{d}z - i\mathrm{d}z\right\|^2 &= \frac{1}{4}\left(\langle \star\mathrm{d}z, \star\mathrm{d}z\rangle + \langle i\mathrm{d}z, i\mathrm{d}z\rangle - 2\langle \star\mathrm{d}z, i\mathrm{d}z\rangle\right) \\
&= \frac{1}{2}\int_M z\Delta z \,\mathrm{dvol}_M(z) - \frac{i}{2}\int_M \mathrm{d}\bar{z}\wedge \mathrm{d}z \\
&=: (*) + (**)
\end{aligned}$$

In a Discrete World Everything Reduces to a Linear System

$$
\begin{aligned}
\frac{1}{4}\left\|\star\mathrm{d}z - i\mathrm{d}z\right\|^2 &= \frac{1}{4}\left(\langle\star\mathrm{d}z,\star\mathrm{d}z\rangle + \langle i\mathrm{d}z, i\mathrm{d}z\rangle - 2\langle\star\mathrm{d}z, i\mathrm{d}z\rangle\right)\\
&= \frac{1}{2}\int_M z\Delta z\,\mathrm{dvol}_M(z) - \frac{i}{2}\int_M \mathrm{d}\bar{z}\wedge\mathrm{d}z\\
&=: (*) + (**)
\end{aligned}
$$

- $(*)$ is the quadratic term $\bar{z}^\top Lz$, L is the contangent Laplacian
- $(**)$ is the *signed area* of $z(M)$ on $\mathbb{C}$, by Stokes' Theorem

$$
\int_M \mathrm{d}\bar{z}\wedge\mathrm{d}z = \int_{\partial M}\bar{z}\,\mathrm{d}z - z\,d\bar{z} = -\frac{1}{2}\sum_{e_{ij}\in\partial M}\bar{z}_i z_j - \bar{z}_j z_i
$$

Applied Mathematicians (Often) Have to Write Code

$$\frac{1}{4}\left\|\star\mathrm{d}z - i\mathrm{d}z\right\|^2 = \frac{1}{2}\bar{z}^\top (L - A)\, z$$

```
// L
SparseMatrix<Complex> L = d0^t * star1 * d0;

// Lc = L - A
foreach face f:
   foreach edge (vi,vj):
   Lc(vi,vj) = L(vi,vj) - 0.5 * i;
   Lc(vj,vi) = L(vi,vj) + 0.5 * i;
```

Constrained Quadratic Minimization: An Eigen-Problem

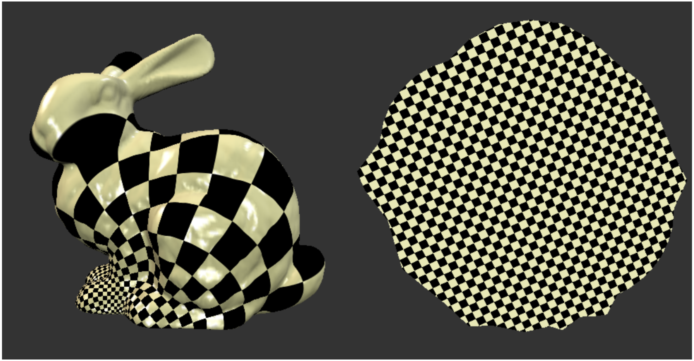
$$\min_z \frac{1}{2}\bar{z}^\top (L - A)\, z$$
$$\text{s.t.}\ \|z\| = 1,$$
$$\langle z, \mathbf{1}\rangle = 0.$$

Lagrange Multiplier: the optimal z_* is an eigenvector

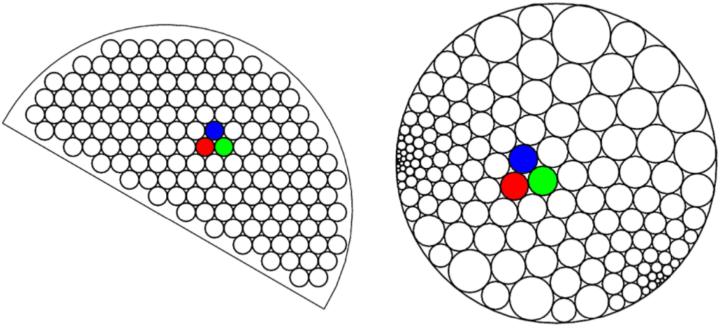
$$(L - A)\, z_* = \lambda_1 z_*,$$

where λ_1 is the smallest non-trivial eigenvlue of $(L - A)$.

Conformality: Angle-Preserving



Circle Packing

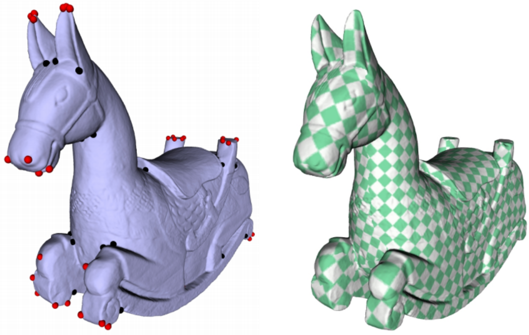


William Thurston (1946-2012)	Circle Packing Approximates Riemann Mapping
Oded Schramm (1961-2008)	Circle Packing and SLE

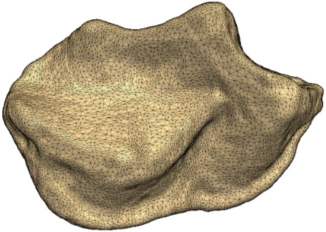
Curvature Prescription and Metric Scaling

Yamabe's Equation

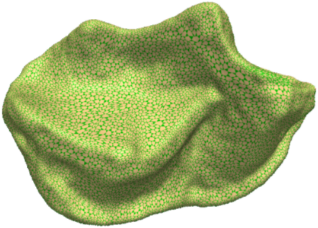
$$\Delta u = K - e^{2u}\widetilde{K}$$



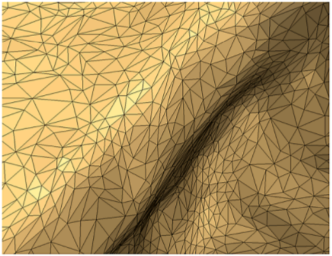
Mid-Edge Uniformization



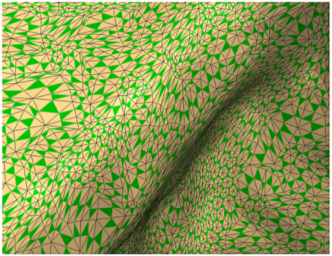
Discrete mesh



Mid-edge mesh

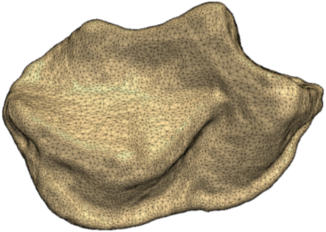


Surface mesh zoom-in

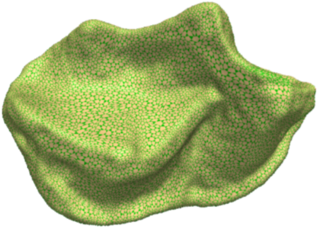


Mid-edge mesh zoom-in

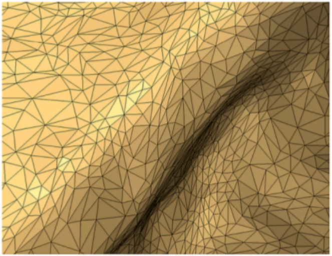
Mid-Edge Uniformization



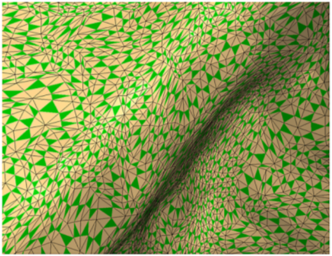
Discrete mesh



Mid-edge mesh

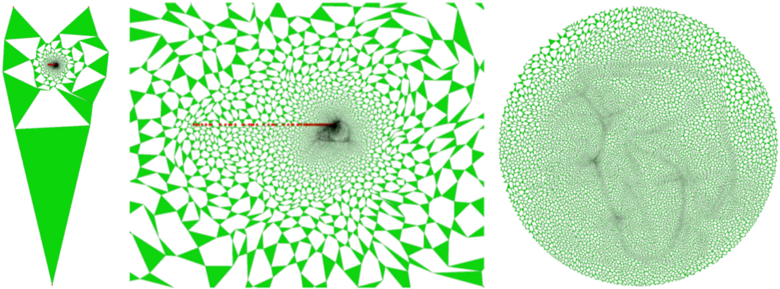


Surface mesh zoom-in

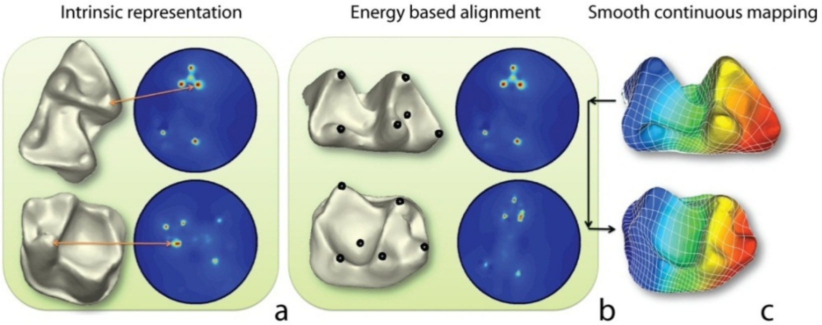


Mid-edge mesh zoom-in

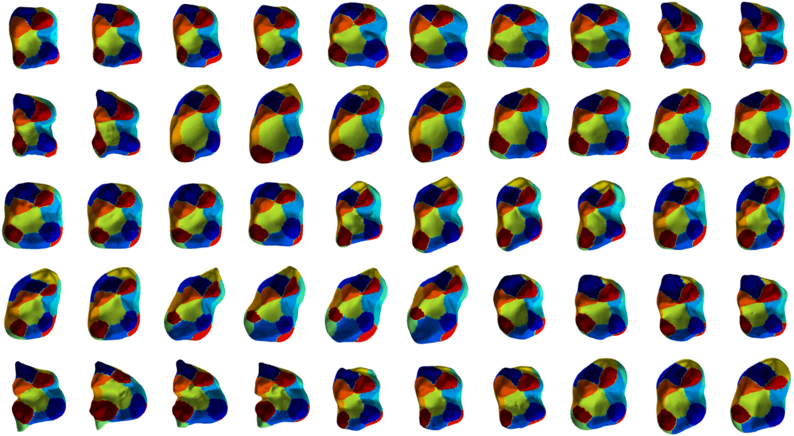
Mid-Edge Uniformization



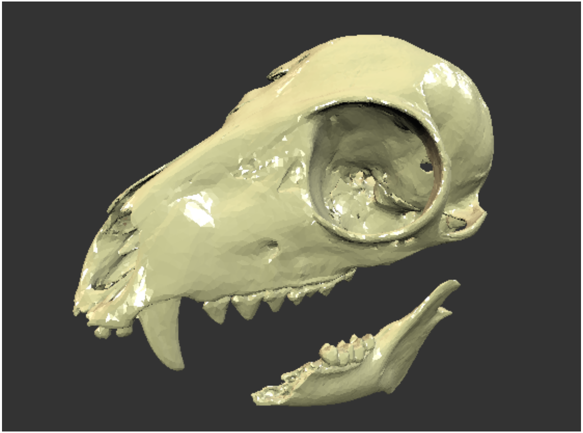
Evolutionary Anthropology

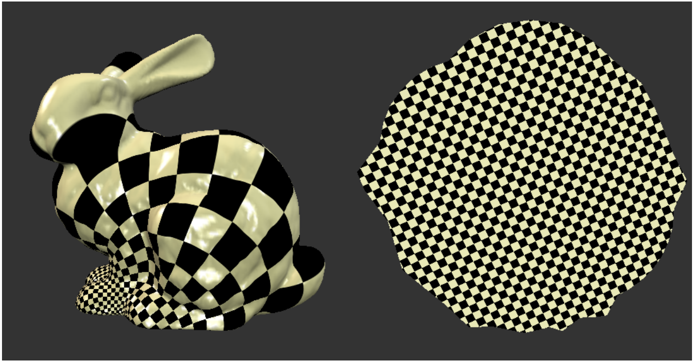


Evolutionary Anthropology



Evolutionary Anthropology: Many More Open Directions





Thank You!