

Nonparametric Detection of Gerrymandering in Multiparty Elections

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1 Preliminaries

Most of the traditional methods developed for detecting gerrymandering in first-past-the-post electoral systems assume that there are only political parties really contesting the election, or, at least, that the party system is regular in the sense that all parties field candidates in every district. This is certainly a very reasonable assumption in many cases: under a well-known empirical regularity known as *Durverger's law* FPTP tends to be correlated with the emergence of two-party systems. Moreover, many of the authors working on gerrymandering detection are motivated by U.S. legislative elections (state and federal), where the regular two-party pattern of competition prevails. However, in many other systems using FPTP we discover significant deviations from such patterns in the form of regional parties, strong independent candidates, minor parties that forgo campaigning in some districts, etc. In the face of such deviations, many of the traditional methods fail completely. Our objective, therefore, is to develop a method of detecting gerrymandering that can be applied to such partially-contested multiparty election.

1.1 Contribution

Our main contribution consists of the development of a *nonparametric methods for detecting gerrymandering in partially-contested multiparty elections*. By *nonparametric* we mean that, unlike most of the traditional statistical methods, the proposed method is free of assumptions about the probability distribution from which observed data points are drawn or the latent mechanism through which such data is generated. Instead we use statistical learning to identify regularities on the basis of the available empirical data.

By a *partially-contested multiparty elections* we mean any FPTP election where at least some candidates are affiliated into one or more *political parties* (after all, if every district-level election is completely independent and candidates cannot be affiliated into blocks, the very concept of gerrymandering as traditionally defined is meaningless), but *for every party there is at most one affiliated candidate in every district* (so there is no intra-party competition). For the sake of simplicity, we treat independent candidates as singleton parties.

In particular, we permit the following deviations from the two-party competition pattern:

- the number of parties can differ from two,
- the number of candidates within each district can differ from two,
- a party can run candidates in any number of electoral districts,
- the set of parties contesting the election varies from one district to another.

Another area in which our approach differs from traditional methods for detecting gerrymandering is that they have been tailored towards *testing a large ensemble of elections* (not necessarily from the same jurisdiction) rather than a single election. For instance, our original scenario was to test for evidence of gerrymandering in close to 2,500 Polish municipal elections. In particular, the proposed methods, like all statistical learning methods, require the researcher who wants to use them to have a large *training set of elections* that they believe to be sufficiently similar insofar as the translation of votes into seats is concerned. If there is a large ensemble of elections being tested, they might form such a training set itself. There is no requirement that the training set and the tested set be disjoint as long as we can assume that gerrymandering is not ubiquitous.

1.2 Prior Work

Among the methods of detecting gerrymandering that focus on the political characteristics of the districting plan (e.g., its impact on seats-votes translation or district-level vote distribution) the earliest focused on measuring how actual elections results deviate from a theoretically or empirically determined *seats-votes curve*. Such function, first introduced into political science by Butler [22] with his rediscovery of the *cube law*, have been intensely studied from the 1950-s to the 1990-s (see, e.g., 57, 16, 69, 99, 96, 101, 66, 44, 20, 26, 14, 37, 42). There is a consensus in the literature that a two-party seats-votes relation is usually described by a modified power law:

$$\frac{s_i}{1 - s_i} = \beta_i \left(\frac{v_i}{1 - v_i} \right)^\rho, \quad (1.1)$$

where s_i and v_i are, respectively, the seat- and vote-share of the i -th party, β_i is a party-dependent parameter, and ρ is a constant [101, 44]. However, only few authors have considered the case of multiparty elections [97, 59, 67], and their results are mostly heuristic.

The state-of-the-art approach to detecting gerrymandering is the *partisan symmetry* method. The general concept was first proposed by Niemi and Deegan [78], who noted that an election should not be regarded as gerrymandered if it deviates from a model seats-votes curve as long as the deviation is the same for each party, i.e., each party has the same seats-votes curve. The main challenge here lies in obtaining that curve from a single realization. The original idea has been to extrapolate therefrom by assuming a *uniform partisan swing*, i.e., that as the aggregate vote share of a party changes, its district-level vote shares increase or decrease uniformly and independently of their original levels. This assumption, first proposed in [21, 23], has been employed in, inter alia, [92, 17, 18, 101, 4, 45, 86, 87, 77, 79, 37, 2, 55]. However, in light of both theoretical and empirical criticism of the uniform partisan swing assumption [75, 6, 53, 10], a more sophisticated extrapolation method has been developed by Gelman and King [38, 39, 40], see also 58, 60, and 100. Yet neither of these two methods can account for multiple parties absent some unrealistic assumption that the relevant swing happens only between two parties identified as major.

The third approach is the *efficiency gap* method proposed by McGhee [74] and further developed in Stephanopoulos and McGhee [94]. It is based on the assumption that in an unbiased election all contending parties should waste the same number of votes. While *prima facie* attractive, this assumption is actually highly problematic because it requires the electoral system to match a very specific seats-votes curve [73, p. 296, 7]. In this respect it represents a methodological step backwards, making it again impossible to distinguish asymmetry from responsiveness. The McGhee-Stephanopoulos definition of wasted votes has also been criticized as counterintuitive [see, e.g., 29, pp. 1181-84, and 8, p. 5]. Most importantly for us, the efficiency gap method fails to account for multiple parties.

Finally, there are several method designed to identify anomalies in the vote distribution indicative of standard gerrymandering techniques like packing and cracking. These include the *mean-median difference* test proposed by McDonald et al. [71], which measures the skewness of the vote distribution; the *multimodality* test put forward by Erikson [32]; the *declination coefficient* introduced by Warrington [104] and measuring the change in the shape of the cumulative distribution function of vote shares at 1/2; and the *lopsided wins* method of testing whether the difference between the winners' vote shares in districts won by the first and the second party is statistically significant [103]. Again, virtually of all those methods assume a two-party system. For instance, natural marginal vote share distributions in multiparty systems (such as the beta distribution or the log-normal distribution) are necessarily skewed. A similar assumptions underlies the declination ratio and the lopsided wins test. Finally, the multimodality test assumes a constant number of competitors.

1.3 Basic Concepts and Notation

Gerrymandering is usually defined as manipulation of electoral district boundaries aimed at achieving a political benefit. Hence, *intentionality* is inherent in the very concept. However, identical results can also arise non-intentionally, as geographic concentration of one party's electorate in small areas (major cities, regions) can produce similar effects to intentional packing. We use the term 'electoral bias' to refer to such 'natural gerrymandering'.

Our basic idea is to treat gerrymandering and electoral bias as *statistical anomalies* in the *translation of votes into seats*. Identification of such anomalies requires a reference point, either theoretical, such as a theoretical model of district-level vote distribution, or empirical, such as a large set of other elections that can be expected to have come from the same statistical population. As the former approach is burdened with the risk that the theoretical model deviates from the empirical reality, in this paper we focus on the latter.

There are three basic assumptions underlying our methodology. One is that we have a *training set* of elections that come from the same statistical population as the election we are studying. Another one is that gerrymandering (or any other form of electoral bias) is *an exception rather than a rule*. Thus, we assume that a substantial majority of the training set elections are free from bias. The third assumption is that while district-level results can be tainted by gerrymandering, *aggregate electoral results* (e.g., vote shares) never are.

One major limitation of our methodology lies in its *inability to distinguish gerrymandering from natural electoral bias*. This limitation is shared, however, with virtually all methods in which the evidence for gerrymandering is sought in analyzing voting patterns or any other variables which are ultimately a function of such patterns (e.g., seat shares, wasted votes, etc.). For many applications that may be enough, since for the end users it might not matter whether the bias in the electoral system is artificial or natural. For applications where that distinction matters, the proposed methods can still be useful to identify cases requiring more in-depth investigation.

Let us introduce some basic notation to be used throughout this paper:

set of districts We denote the set of districts by $D := \{1, \dots, c\}$.

set of parties We denote the set of parties by $P := \{1, \dots, n\}$.

set of contested districts For $i \in P$, we denote the set of districts in which the i -th party runs a candidate by D_i . Let $c_i := |P_i|$.

set of contesting parties For $k \in D$, we denote the set of parties that run a candidate in the k -th district by P_k . Let $n_k := |D_k|$.

district-level vote share For $i \in P$ and $k \in D$, we denote the district-level vote share of the i -th party's candidate in the k -th district by v_i^k . If there was none, we assume $v_i^k = 0$.

district-level seat share For $i \in P$ and $k \in D$, let s_i^k equal 1 if the i -th party's candidate in the k -th district won the seat, and 0 otherwise.

district size For $k \in D$, we denote the number of voters cast in the k -th district by w_k .

aggregate vote share For $i \in P$, let $v_i := (\sum_{k \in D_i} v_i^k w_k) / (\sum_{k \in D_i} w_k)$ be the aggregate vote share.

aggregate seat share For $i \in P$, we denote its aggregate seat share by $s_i := (\sum_{k \in D_i} s_i^k) / c_i$.

unit simplex For $n \in \mathbb{N}_+$, we denote the unit simplex by $\{\mathbf{x} \in \mathbb{R}_+^n : \|\mathbf{x}\|_1 = 1\}$ by Δ_n .

k -th largest / smallest coordinate For $n \in \mathbb{N}_+$, $\mathbf{x} \in \mathbb{R}^n$, and $k \in 1, \dots, n$, we denote the (k) -th largest coordinate of \mathbf{x} by x_k^\downarrow , and the (k) -the smallest one by x_k^\uparrow .

2 Seats-Votes Functions

Seats-votes curves are one of the fundamental concepts under the traditional approach to the quantitative study of electoral systems. It is a function that maps an aggregate vote share to

an aggregate seat share. Of course, it is easy to see that in reality even in two-party elections a seats-votes curve is not actually real-valued, but probability measure-valued, since the seat share depends on what we call ‘electoral geography’ – the distribution of district-level vote shares. We call this measure-valued function a *seats-votes function*, while reserving the name of a seats-votes curve to a function that maps a vote share to the expectation of its image under the seats-votes function. Note that both gerrymandering and electoral bias manifest themselves by deviation of the seats-votes function applicable to one or more parties from the ‘model’ seats-votes function caused by anomalies of the electoral geography.

In multi-party elections there is another fundamental problem with seats-votes functions: the distribution of seats depends not only on the vote share and the electoral geography, but also on the competition patterns: the number of competitors and the distribution of their votes (or, to be more precise, on the distribution of the first order statistic of their votes) [24, 68, 25]. If we were to fit a single seats-votes for all parties without regard to competition patterns, the result would involve another source of randomness besides districting effects, namely the variation in such patterns. Hence, we would be unable to distinguish between a seats-votes function that deviates from the model because of electoral geography and a seats-votes function that also deviates from the model, but because of unusual competition patterns. Thus, we need to account for this effect by considering a seats-votes-competition pattern function rather than the usual seats-votes function.

Remark 2.1. Consider seats-votes curves in multi-party elections. If we assume that they are anonymous (i.e., identical for all parties), non-decreasing, and surjective, it turns out perfect proportionality ($s = v$) is the only seats-votes curve that does not depend on the distribution of competitors’ votes [12, Theorem 1].

It would be convenient if we were able to describe the competition pattern by a single numerical parameter. Our objective here is to find a measure of the ‘difficulty’ of winning a seat given the number of competitors and the distribution of their vote shares (renormalized so as to sum to 1). A natural choice would be the *seat threshold*:

Definition 2.1 (Seat Threshold). Fix $i = 1, \dots, n$, and assume that renormalized vote shares of the competitors of the i -th candidate equal some random variable \mathbf{Z} distributed according to some probability measure on $\Delta_{n,-i}$. A *seat threshold* of the i -th candidate is such $t_i \in [1/n, 1/2]$ that $Pr(S_i = 1|v_i) > 1/2$ for every $v_i > t_i$, i.e., the probability that the i -th candidate wins a seat with vote share equal v_i exceeds 1/2.

Proposition 2.1. It is easy to see that $Pr(S_i = 1|v) = 1 - F_{Z_i^{\downarrow}}(v/(1-v))$, where $F_{Z_i^{\downarrow}}$ is the cumulative distribution function of the renormalized vote share of the largest competitor.

Our next objective is to approximate the seat threshold in cases where we do not have any knowledge of the distribution of the competitors’ vote shares, but only a single realization thereof. We therefore need a statistic that is both a stable estimator of the distribution parameters and highly correlated with the value of the largest order statistic. We posit that the best candidates for such statistics are *measures of vote diversity* among competitors, and use a Monte Carlo simulation to test a number of such measures.

Observation 2.1. Let $\alpha \sim \text{Gamma}(1, 1)$, and let $\mathbf{V} \sim \text{Dir}(\{\alpha\}^n)$, where $n = 3, \dots, 12$ and $\text{Dir}(\alpha)$ is the Dirichlet distribution with parameter vector α . For a sample of 2^{16} realizations of \mathbf{V} we have calculated Spearman’s correlation coefficients [93] for:

1. α ,
2. $\mathbf{V}_1^{\downarrow}$, i.e., maximum of the coordinates,
3. \mathbf{V}_1^{\uparrow} , i.e., minimum of the coordinates,
4. median coordinate V_{med} ,
5. *Shannon entropy* [89], $H(\mathbf{V}) := -\sum_{i=1}^n V_i \log V_i$,

6. *Herfindahl–Hirschman–Simpson index* [48, 91, 46], $\sum_{i=1}^n V_i^2$,
7. *Gini coefficient* of the coordinates,
8. *Bhattacharyya angle* [9] between V and the barycenter of the simplex, $\arccos \sum_{i=1}^n \sqrt{V_i/n}$.

The results for $n = 3$ are given in Table 1, while those for $n = 6$ and 12 – in the Appendix.

	α	V_1^\downarrow	V_1^\uparrow	V_{med}	$H(V)$	$\sum_{i=1}^n V_i^2$	Gini	Bhatt.
α	1.000	-0.513	0.583	0.246	0.582	-0.565	-0.568	-0.588
V_1^\downarrow	-0.513	1.000	-0.806	-0.729	-0.938	0.965	0.965	0.917
V_1^\uparrow	0.583	-0.806	1.000	0.226	0.952	-0.918	-0.930	-0.967
V_{med}	0.246	-0.729	0.226	1.000	0.485	-0.564	-0.549	-0.436
$H(V)$	0.582	-0.938	0.952	0.485	1.000	-0.994	-0.993	-0.998
$\sum_{i=1}^n V_i^2$	-0.565	0.965	-0.918	-0.564	-0.994	1.000	0.997	0.986
Gini	-0.568	0.965	-0.930	-0.549	-0.993	0.997	1.000	0.986
Bhatt.	-0.588	0.917	-0.967	-0.436	-0.998	0.986	0.986	1.000

Table 1: Correlation Matrix for $n = 3$

We conclude that the *Herfindahl–Hirschman–Simpson index* is consistently the one that best correlates with the maximal coordinate while also being a reasonably good estimate of the distribution parameters. Accordingly, in our procedure for estimating the seat threshold we use its monotonic transform, i.e., the *effective number of competitors* [63, 98]:

Definition 2.2 (Effective Number of Competitors). The *effective number of competitors* of the i -th candidate, $i = 1, \dots, n$, is given by:

$$\varphi := \left(\sum_{j=1, j \neq i}^n z_j^2 \right)^{-1}, \quad (2.1)$$

where $\mathbf{z} \in \Delta_{n,-i}$ is a vector of the vote shares of that candidate's competitors multiplied by such constant in \mathbb{R}_+ that $\sum_{j=1, j \neq i}^n z_j = 1$.

We shall see that the vote share and the number and effective number of competitors enable us to accurately classify candidates as winning and losing (see Figure 1 and Table 2).

Proposition 2.2. Clearly, with three candidates, i.e., two competitors, the classifier is exact (modulo ties), as the effective number of competitors uniquely determines the share of the larger one in their aggregate vote share:

$$\max\{z_{j_1}, z_{j_2}\} = \frac{1}{2} \left(1 + \sqrt{\frac{2}{\varphi_i} - 1} \right). \quad (2.2)$$

Then the decision boundary (i.e., the curve separating the space of candidates into winning and losing subspaces) is the set of points satisfying:

$$\varphi_i = \frac{1 - 2v_i + v_i^2}{1 - 4v_i + 5v_i^2}. \quad (2.3)$$

Model 2.1 (Decision Boundary for $n > 3$). For $n > 3$, the decision boundary is determined on the basis of the data using a support vector machine-based classifier [13, 28] with a third-order polynomial kernel, and then approximated by a strictly decreasing B-spline of degree 3, with boundary nodes at $1/n$ and $1/2$ and interior nodes fitted using cross-validation.

Definition 2.3 (Effective Seat Threshold). We refer to the value of the decision boundary for the candidate of the i -th party in the k -th district, ascertained for the given number and effective number of competitors, as the *effective seat threshold*, t_i^k .

Definition 2.4 (Effective Seat Threshold Classifier). An effective seat threshold classifier is a function $\mathfrak{s} : [0, 1] \times \mathbb{N} \times [1, \infty) \rightarrow \mathbb{B}$ that maps a triple (v, n, φ) to 0 iff the probability of winning with vote share v , $n - 1$ competitors, and φ effective competitors is below 1/2.

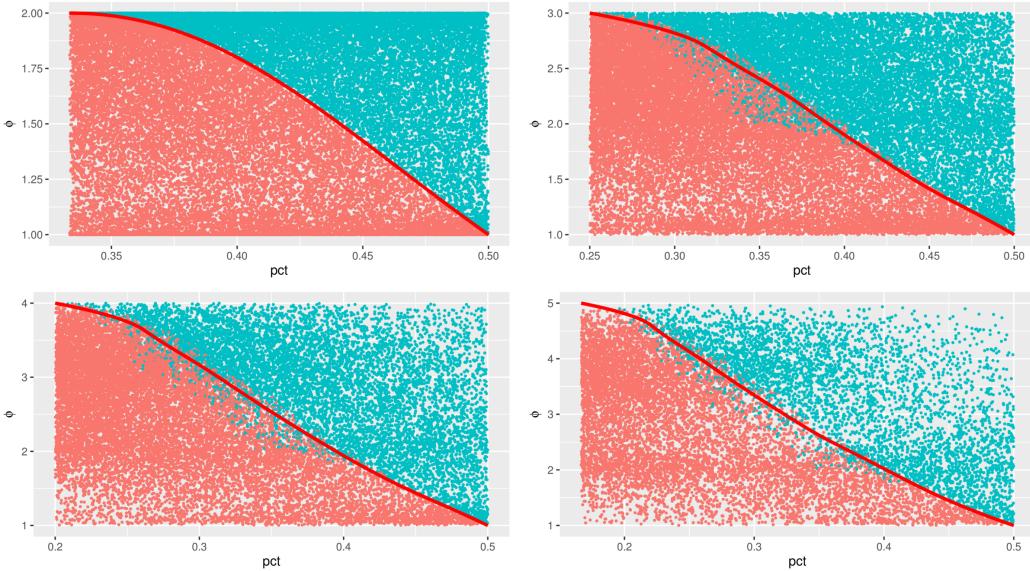


Figure 1: Effective seat thresholds for $n = 3, 4, 5, 6$. Blue points indicate successful candidates, while red points – unsuccessful candidates.

Definition 2.5 (Mean Effective Seat Threshold). *Mean effective seat threshold*, $t_i := \langle t_i^k \rangle_{k \in D_i}$ is our measure of the difficulty of winning a seat.

3 Nonparametric Seats-Votes Function Estimates

One possible approach to identifying the model seats-votes function is to construct one theoretically. We might start with some probabilistic model of intra-district vote distribution, then use it to calculate the seat threshold, and use a probabilistic model of inter-district vote distribution to calculate the probability of district vote share exceeding the seat threshold. Finally, either by convolving binomial distributions or by the central limit theorem we obtain the seat distribution.

One unavoidable weakness of any theoretical seats-votes curve lies in the fact that a systematic deviation therefrom might just as easily arise from electoral bias as from incongruities between the theoretical distributional assumptions and the empirical reality. To

n	R	n	R	n	R
3	.0035	9	.0073	15	.0144
4	.0137	10	.0067	16	.0148
5	.0152	11	.0142	17	.0176
6	.0137	12	.0186	18	.0182
7	.0136	13	.0152	19	.0133
8	.0068	14	.0171	20	.0168

Table 2: Classification error R for the effective seat threshold classifier.

avoid this issue we derive our model seats-votes function solely from the reference election dataset with minimal theoretical assumptions¹ by using the kernel regression method [76, 105]. Its general idea is to estimate the conditional expectation of a random variable at a point in the condition space by averaging the values of its realizations at neighboring points with distance-decreasing weights. As the method can be sensitive to the choices of hyperparameters, we discuss them in some detail.

Model 3.1 (Locally-Constant Kernel Regression). Let $S \in \mathbb{R}$ be a random *response variable*, and let $\mathbf{X} \in \mathfrak{F}$, where \mathfrak{F} is some linear *feature space* and $D := \dim \mathfrak{F}$, be a vector of *predictor variables*. Assume we have a vector of N realizations of S , \mathbf{s} , and an $N \times \dim \mathfrak{F}$ matrix of realizations of \mathbf{X} , \mathbf{x} . We denote its j -th row by \mathbf{x}_j . Then the *locally-linear kernel regression estimate* of the conditional expectation of S given a vector of predictors $\mathbf{x}_0 \in \mathfrak{F}$ is given by:

$$\mathbb{E}(S|\mathbf{x}_0) = \frac{\sum_{j=1}^N s_j K((\mathbf{x}_j - \mathbf{x}_0)\mathbf{h}_{\mathbf{x}_0, \mathbf{x}_j})}{\sum_{j=1}^N K((\mathbf{x}_j - \mathbf{x}_0)\mathbf{h}_{\mathbf{x}_0, \mathbf{x}_j})}, \quad (3.1)$$

where N is the number of observations (in our case – sum of the number of parties over all elections in our set of elections), K is a second-order *kernel*, and $\mathbf{h}_{\mathbf{x}_0, \mathbf{x}_j} \in \mathbb{R}_+^D$ is a *bandwidth* parameter for the pair $(\mathbf{x}_0, \mathbf{x}_j)$. In other words, we average the values of s over all parties with weights determined by the value of the kernel at $(\mathbf{x}_j - \mathbf{x}_0)\mathbf{h}_{\mathbf{x}_0, \mathbf{x}_j}$.

Definition 3.1 (Kernel). We use the standard *D-variate Gaussian kernel*:

$$K(\mathbf{x}) := (2\pi)^{-k/2} \exp\left(-\frac{1}{2}\|\mathbf{x}\|_2\right) \quad (3.2)$$

Definition 3.2 (Multivariate Adaptive Nearest-Neighbor Bandwidth [15, 1, 90, 88]). For $\mathbf{x}_0, \mathbf{x}_j \in \mathfrak{F}$, the i -th coordinate of the *multivariate adaptive nearest-neighbor bandwidth*, $i = 1, \dots, D$, is given by:

$$(h_{\mathbf{x}_0, \mathbf{x}_j})_i = h_{0,i} |x_{0,i} - x_{N_k^i(\mathbf{x}_j),i}|, \quad (3.3)$$

where $N_k^i(\mathbf{x})$ is the index of the k -th nearest neighbor of \mathbf{x} along the i -th dimension of the feature space under the absolute difference metric, and $\mathbf{h}_0 \in \mathbb{R}_+^D$ is a scaling vector.

We still need to choose two hyperparameters of the model: the scaling vector \mathbf{h}_0 and the nearest-neighbor parameter k . This is usually done by leave-one-out cross-validation [65, 51] with the objective function defined either as an L_1 or L_2 distance between the predicted and actual value vectors [30], or as the Kullback-Leibler 1951 divergence between the former and the latter. We use the latter variant together with an optimization algorithm by Hurvich et al. [50] which penalizes high-variance bandwidths (with variance measured as the trace of the parameter matrix) in a manner similar to the Akaike information criterion [3].

4 Measuring Deviation from the Seats-Votes Function

By this point, we have estimated a party’s expected seat share given its aggregate vote share and the competition patterns in the districts it contests. But what we actually need is a measure of how much the actual seat share deviates from that expectation. A natural choice would be the difference of the two. It is, however, inappropriate, as seat shares only assume values within a bounded interval $[0, 1]$ and there is no reason to expect seat share distributions to be even approximately symmetric around the mean.

We therefore use another measure of deviation: the probability that a seat share deviating from the median more than the empirical seat share could have occurred randomly. Note how this quantity is analogous to the p -value used in statistical hypothesis testing.

¹In particular, we assume that the seat shares are distributed according to some absolutely continuous probability measure supported on $[0, 1]$

Definition 4.1. Electoral Bias *p*-Value Let s_i be an empirical seat share and let μ be the conditional distribution of the aggregate seat share given the empirical aggregate vote share and the empirical mean effective seat threshold, i.e., the value of the seats-votes function. Then the *electoral bias p-value* is given by:

$$\pi_i = \min\{\mu((0, s_i)), \mu((s_i, 1))\} = \min\{F(s_i), 1 - F(s_i)\}, \quad (4.1)$$

where F is the cumulative distribution function of μ .

We thus need not a regression estimator, but a conditional cumulative distribution function estimator. One approach would be to estimate the conditional density of S_i [82, 81] and integrate it numerically. This method, however, is prone to potential numerical errors. We therefore use another approach, relying on the fact that a conditional cumulative distribution function is defined in terms of the conditional expectation, and therefore the problem of estimating it can be treated as a special case of the kernel regression problem.

There remains one final problem: when comparing parties contesting different number of districts, we need an adjustment for the fact that the probability of getting an extreme value depends on that number (decreasing exponentially as the number of contested districts increases). In particular, except for very rare electoral ties, single-district parties always obtain extreme results. Thus, if for a party contesting k districts we include parties contesting fewer districts in the training set, we overestimate the probability of obtaining an extreme seat share. To avoid that problem, the kernel model for parties with exactly k districts, $k \in \mathbb{N}$, is trained only on parties with as many or fewer contested districts. If the distribution of the number of contested districts has a tail, it is optimal to adopt a cutoff point k_0 such that for the set $P_{c \geq k_0}$ of parties contesting k_0 or more districts each party is compared with a model trained on all parties in $P_{c \geq k_0}$.

5 Aggregation

The final step is the aggregation of party-level indices into a single election-level index of electoral bias. We would like our aggregation function to: (1) assign greater weight to major parties than to minor parties; (2) be sensitive to very low *p*-values and less sensitive to even substantial differences in large *p*-values; and (3) be comparable among elections, i.e., independent of the number of parties and districts. An easy example of such a function is the *weighted geometric mean* given by:

$$\pi := \exp\left(\sum_{i=1}^n w_i \log \pi_i\right), \quad (5.1)$$

where w_i is the number of votes cast for the i -th party divided by the number of all valid votes cast in the election (including in districts not contested by i).

6 Experimental Test

Before applying our proposed method to empirical data, we wanted to sure that it really works – both in terms of high precision and of high recall. But one fundamental problem in testing any method for the detection of gerrymandering, especially outside the familiar two-party pattern, lies in the fact that we have very few certain instances thereof. Therefore we first tested our method on a set of simulated elections, consisting both of ‘fair’ districting plans, drawn at random with a distribution intended to approximate the uniform distribution on the set of all admissible plans, and of ‘unfair’ plans generated algorithmically. We used real-life precinct-level data from the 2014 municipal election in our home city of Kraków, where the two leading parties were nearly tied in terms of votes. That allowed us to generate ‘gerrymandered’ plans for both of them.

6.1 Experimental Setup

Our baseline dataset consisted of a neighborhood graph of 452 electoral precincts, each of which was assigned three parameters: precinct population, $w_k \in \mathbb{N}$, varying between 398 and 2926 (but with 90% of the population taking values between 780 and 2420); party p 's vote share p_k , varying between 9.1% and 66.7% (but with 90% of the population taking values between 20.8% and 48.5%); and party q 's vote share q_k , varying between 11.5% and 56.7% (but with 90% of the population taking values between 21.0% and 44.1%). On the aggregate, party p won the election with 33.15% of the vote, but party q was a close runner-up with 32.64% of the vote. There have been seven third parties, but none of them had any chance of winning any seats (in particular, none has come first in any precinct). In drawing up plans, we fixed the number of districts at 43 (the real-life number of seats in the municipal council) and the permissible population deviation at 25%.

6.2 Algorithm for Generating Fair Plans

Our sample of fair districting plans consisted of 128 partitions of the precinct graph generated using the Markov Chain Monte Carlo algorithm proposed by Fifield et al. [33]. It used the Swendsen-Wang algorithm [95], as modified by Barbu and Zhu [5], to randomly walk the graph of solutions. In each iteration, we randomly ‘disable’ some of the edges within each district of the starting districting plan (independently and with a fixed probability); identify connected components adjoining district boundaries; randomly choose R such components (where R is chosen from some fixed discrete distribution on \mathbb{R}_+) in such manner that they do not adjoin one another; identify admissible exchanges; and randomly accept or reject each such exchange using the Metropolis-Hastings criterion. Barbu and Zhu [5] have shown that if $\Pr(R = 0) > 0$, the algorithm is ergodic, and Fifield et al. [33, 34] – that in such a case its stationary distribution is the uniform distribution on the set of admissible districtings. This algorithm has a better rate of convergence than classical Metropolis-Hastings, but obtaining satisfactory performance still required additional heuristics like simulated annealing [70, 41].

6.3 Algorithm for Generating Unfair Plans

To generate unfair districting plans we used an algorithm by Szufa et al. [35, Ch. 3.7] based on integer linear programming. The idea is to consider all feasible districts (connected components of the precinct graph with aggregate population within the admissible district population range), K_1, \dots, K_d , and to solve the following optimization problem for $x = p, q$:

Problem 6.1. *For*

$$\xi \in \mathbb{B}^d$$

maximize

$$\sum_{j=1}^d \xi_j \operatorname{sgn} \left(\sum_{k \in K_j} x_k w_k - \sum_{k \in K_j} y_k w_k \right) \quad (6.1)$$

subject to

$$\sum_{j=1}^d \xi_j = s, \quad (6.2)$$

$$\sum_{j=1}^d \mathbf{1}_{K_j}(k) = 1 \text{ for every } k = 1, \dots, c, \quad (6.3)$$

where $y = q$ if $x = p$ and $y = p$ if $x = q$.

In practice, it is infeasible to enumerate all possible districts with hundreds of precincts. We therefore first artificially combine leaf nodes, small precincts, and similar precincts until the number of precincts is reduced below 200. Only then we run the ILP solver and recover the full solution by replacing combined precincts with their original elements. We then run a local neighborhood search to find a local maximum. To obtain less extreme districting plans, we include an additional constraint on the required margin of victory.

6.4 Results

Our sample of fair districting plans yielded a distribution of electoral results varying from 26 to 17 seats for party p , with the median at 21. Those results corresponded to aggregate p -values between .18 and .37. Accordingly, none of the fair plans was classified as gerrymandered at the significance level .05, giving us perfect precision 1.

Gerrymandered plans varied from a 37-seat to a 29-seat plan for party q , corresponding to aggregate p -values from .006 to .064, and from a 36-seat to a 29-seat plan for party p , corresponding to aggregate p -values from .016 to .096. In total, 24 out of 28 gerrymandered plans were classified as such at the significance level .05, yielding recall .857. Note, however, that all plans that we failed to recognize as gerrymandered were highly inefficient ones.

7 Empirical Test

We have tested our method on data from four training sets of elections:

1. \mathcal{D}_{14} , 2014 Polish municipal elections (2412 elections, 15,848 parties, 37,842 districts, 131,799 candidates),
2. \mathcal{D}_{18} , 2018 Polish municipal elections (2145 elections, 10,302 parties, 32,173 districts, 86,479 candidates),
3. \mathcal{D}_U , U.S. House of Representatives elections, 1900-2016 period (2848 elections, 13,188 parties, 23,390 districts, 71,314 candidates)²,
4. \mathcal{D}_N , national legislative elections from 15 countries (206 elections, 53,721 parties, 52,321 districts, 237,331 candidates).

The following countries were included in the \mathcal{D}_N dataset:

1. United Kingdom – all general elections from 1832 (47 cases), SMD-s only;
2. Canada – all general elections from 1867 (42 cases);
3. Denmark – all general elections held under FPTP, i.e., from 1849 to 1915 (32 cases);
4. New Zealand – all general elections from 1946 until the 1994 reform (17 cases);
5. India – all general elections from 1962 until 2014 (14 cases);
6. Malaysia – all general elections from 1959 until 2018 (12 cases);
7. Philippines – all general elections, 1987-1998, and SMD results, 1998-2013 (9 cases);
8. Japan – single-member-district results from elections from 1996 to 2014 (7 cases);
9. Ghana – 2000-2016 elections (4 cases);
10. South Africa – 1984-1989 elections (4 cases);
11. Poland – upper house elections from 2011 (3 cases);
12. Taiwan – single-member-district results from elections under parallel voting (3 cases);
13. Nigeria – 2003 and 2011 elections;
14. Kenya – 2002 and 2007 elections.

²Each state election is treated as distinct. We do not track party identity beyond any individual election, wherefore for instance the Republican party in, say, the 1994 House election in Pennsylvania and in the 1994 House election in New York (or the 1996 House election in Pennsylvania) is counted as two different parties.

Data has been obtained from the Constituency-Level Election Archive [61].

Those 15 countries were chosen as major countries using the FPTP system that have been categorized as at least partly free under the Freedom House Freedom in the World survey [36]. We have chosen elections that, even if not always democratic by modern standards, were at least minimally competitive. We note that the countries listed include cases with strong regional parties (UK, Canada) or very large number of small parties and independent candidates (India), as well as cases with party systems developing or otherwise fluid. They also include instances in which allegations of gerrymandering have already appeared in the literature [47, 49, 72, 56, 19, 85, 64, 80, 52, 102, 27].

dataset	c			n		φ			n_R		
	med	min	max	avg	max	avg	min	max	avg	min	max
PL2014	15	1	23	3.48	10.22	2.78	1.55	6.79	3.29	2.00	7.20
PL2018	15	1	15	2.81	6.60	2.37	1.62	4.72	2.74	1.91	5.60
Intl	197	8	659	3.78	21.21	2.24	1.65	3.62	2.59	1.91	4.32
US	6	1	53	2.96	9.00	1.87	1.00	3.82	2.06	1.00	4.33

Table 3: Basic characteristics of electoral datasets: the number of non-unopposed districts c and the number of candidates n , the effective number of candidates φ , and the number of candidates with at least 5% vote share, all averaged over such districts.

7.1 Results by Training Set

In this subsection, we report the raw results for our training sets and test whether the method agrees with classical measures of gerrymandering for two-party U.S. elections.

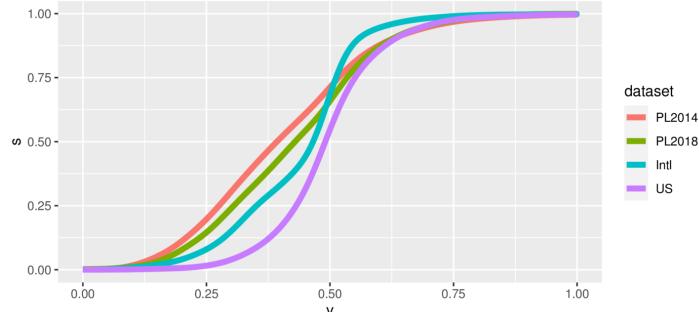


Figure 2: Nonparametric seats-votes curves for the four training sets – comparison.

The incidence of electoral bias in the four datasets under consideration was as follows:

dataset	avg p -value	$\Pr(\pi < .05)$
\mathcal{D}_{14} (Poland 2014)	.240	.0079
\mathcal{D}_{18} (Poland 2018)	.241	.0141
\mathcal{D}_N (non-U.S. national elections)	.311	.0049
\mathcal{D}_U (U.S. elections)	.307	.0021
\mathcal{D}_U restricted to post-1970 elections	.268	.0122

Table 4: Incidence of Electoral Bias in Four Election Datasets.

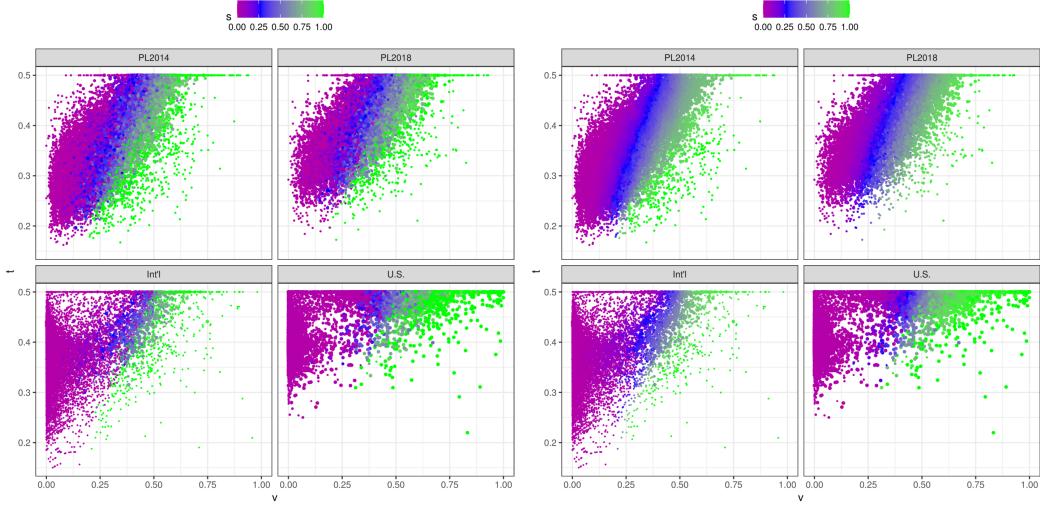


Figure 3: The empirical (left) and expected (right) seat shares as functions of the vote share and the effective seat threshold.

Finding an appropriate baseline for comparison, however, is quite difficult. Even if we were able to easily model the expected scale of random electoral bias, it would still be impossible to determine whether deviation from it is caused by the

deficiencies of our method or by actual instances of gerrymandering. Hence, a more appropriate test would be to analyze whether our measure agrees with other methods for detecting gerrymandering used in the literature. Of course, we can do this test only for two-party elections, such as those from the U.S. elections dataset. As most modern methods assume the absence of large-scale malapportionment, we have dropped the pre-1970 observations. We have compared our coefficient with absolute values of four classical indices: Gelman-King partisan bias, efficiency gap, mean-median difference, and declination coefficient. Scores for those methods were obtained from PlanScore [43]. Expected association is negative.

For other datasets, we are at present left with qualitative analysis of the most biased cases. For the full U.S. dataset D_U these were: several Missouri elections from the 1900s to 1920s (especially the 1926, 1916, 1906, and 1902), the 1934 Indiana and New Jersey elections, and the 1934 New Jersey elections. Missouri and Indiana were at the time highly malapportioned [31], but the case of New Jersey requires further study.

For the non-U.S. national elections dataset the most biased instances include the 1874 U.K. general election (a famous electoral inversion where the Liberal Party decisively lost in terms of seats – 242 to 350 – despite winning a plurality of the popular vote), the 1873 Danish Folketing election (held in highly malapportioned districts), the 1882 Canadian federal election, and the 1841 U.K. general election, while among modern elections – the 2013 Malaysian election, the 2014 Indian general election, the 2005 and 2009 Japanese elections, and the 1983 U.K. general elections (with geographical bias documented by [83, 84, 54, 11]).

Coefficient	Estimate	Std. Error	t Value	p-value
(Intercept)	.340	.011	29.785	< 2e-16 ***
abs(bias)	.278	.186	1.492	.136
abs(effGap)	-.895	.189	-4.725	3e-06 ***
abs(meanMed)	-.781	.278	-2.807	.005 **
abs(declin)	.116	.060	1.923	.055 .

Table 5: Correlation with Classical Measures of Gerrymandering.

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