The Social Cost of Carbon When Damages are Unknown

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Integrated assessment models (IAMs) are economists' principal tool for analyzing the social cost of carbon (SCC). The "damage function" in IAMs, which links temperature to economic impacts, has come under fire from many economists because of its strong assumptions that may produce significant, and ex-ante unknowable misspecifications. I develop Bayesian learning and robust control frameworks for analyzing damage risk and uncertainty. I find that using robust control to guard against model misspecification raises the SCC, and anticipating future damage learning reduces the SCC. Interacting Bayesian learning with robust control actually results in the lowest SCC trajectory of all damage frameworks.

JEL: H23, Q54, Q58

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The social cost of carbon (SCC) quantifies the cost to society of emitting an additional unit of carbon into the atmosphere.[1] To estimate the SCC, economists have developed integrated assessment models (IAMs), which are macroeconomic models linked to a climate module. The link between the macroeconomy and the climate system in IAMs is the "damage function," a function that translates rising temperature into economic losses.[2] The world's true underlying damage function is highly complex and characterized by deep uncertainties stemming from a lack of knowledge of how warming will affect natural and economic systems. In order to develop tractable models in the face of these unknowns, integrated assessment modelers have been forced to pin down the damage function with strong assumptions. Alongside the discount rate, the damage function is one of the most contentious feature of IAMs. The damage function has drawn substantial criticism claiming that some of these assumptions, for example the benchmark DICE model's quadratic damage function (Nordhaus, 2008), have little scientific or economic grounding.[3] In fact, some economists have suggested abandoning quantitative integrated assessment because of the alleged arbitrariness of the damage assumptions underpinning it (Pindyck, 2013).

Rather than abandoning the quantitative integrated assessment agenda that has proved useful for analysis and policymaking, I integrate uncertainty and even skepticism about damages into a benchmark integrated assessment setting.[4] I advance quantitative IAM methodology by incorporating uncertainty and Bayesian learning for the damage function. I also implement robust control, a macroeconomic technique, to account for concerns from economists and scientists that the damage function in IAMs is misspecified. Using these methodological advances I aim to ascertain the policy impacts of (1) accounting for parametric uncertainty over the damage function calibration, (2) including endogenous damage learning in a manner that closely matches how real world modelers update damage functions, (3) allowing the policymaker to distrust her model, and (4) interacting learning with the policymaker's distrust of her own model.

Damage functions have garnered criticism from economists for a number of reasons. Some economists have noted that IAMs do not account for a variety of potential damage channels (Howard, 2014), that the studies used to calibrate the damage function underestimate the actual impact of warming (Hanemann, 2008; Howard and Sterner, 2014), and that integrated assessment modelers have imposed strong, arbitrary functional form assumptions (Pindyck, 2013). The mounting evidence of misspecification and underestimation of damages among the existing analyses has led some economists to suggest that low damage risk is baked into the models (Stern, 2013). Recent

[1]The SCC is typically in terms of either tons of carbon, or tons of CO_2. In addition, the optimal carbon tax is equivalent to the social cost of carbon along an optimal trajectory.

[2]This is specific to the benchmark DICE model. Other IAMs, such as FUND, may have explicit impacts of CO_2.

[3]DICE stands for the Dynamic Integrated model of Climate and the Economy.

[4]This is in line with recent calls for a new generation of models more capable of capturing uncertainties and risks amongst other features (Stern, 2013).

work analyzing the sensitivity of the optimal carbon tax to both the functional form and calibration of the damage function has demonstrated that errors in the damage function have non-trivial policy impacts (Stanton et al., 2009; Kopp et al., 2012; Weitzman, 2012).[5] Addressing these issues is critical since IAMs have become a key component of determining environmental regulation. For instance, a suite of IAMs is currently being used by the U.S. government to determine how to value greenhouse gas emissions in cost-benefit analyses of all federal policies with environmental impacts (Greenstone et al., 2013).

The weaknesses of current damage functions and a general lack of knowledge of damage impacts indicate that damage uncertainty should be explicitly captured in modeling. Yet it has been included only recently in IAMs in order to demonstrate the policy implications of including measurable uncertainty over damage parameters, and to show the inconsistent policy prescriptions of the conventional Monte Carlo approach to approximate true uncertainty (Crost and Traeger, 2013, 2014).[6] I build upon the existing literature by realistically capturing uncertainty, learning, and stochasticity in damages. To perform this analysis I integrate the benchmark DICE model into a dynamic programming framework that trades off near-term costs of emissions reductions with the future benefits of lower temperature via reduced damages.

The first of four policy frameworks I develop is the "uncertainty framework" where the policymaker treats the calibration of the damage function to observed data as uncertain. The policymaker assigns a distribution to the damage function's coefficient. The coefficient has historically been calibrated to damage estimates, while the functional form has been assumed to be quadratic. The policymaker utilizing the uncertainty framework does not update her beliefs over time.[7]

The second framework is the "learning framework" in which I develop a novel Bayesian learning mechanism for the calibrated damage coefficient. This weakens the usual assumption that policymakers do not update their damage beliefs. Under the learning framework the policymaker uses her annual observations of a noisy signal of the true damage coefficient to update her beliefs while

[5]Economists often analyze warming risk, captured by climate sensitivity uncertainty. Damage uncertainty is less studied, but there is evidence that it has greater policy implications than warming uncertainty (Lemoine and McJeon, 2013).

[6]The recursive IAM literature itself is relatively new and has mostly focused on policy and learning in the face of an uncertain climate sensitivity parameter (Kelly and Kolstad, 1999; Leach, 2007; Fitzpatrick and Kelly, 2014; Kelly and Tan, 2015), or potential tipping points (Lemoine and Traeger, 2014; Cai et al., 2015a,b; Lontzek et al., 2015; Lemoine and Traeger, 2016). The recursive IAM literature expands on an expansive set of previous works that attempt to approximate uncertainty using Monte Carlo methods (Hope, 2006; Stern, 2006; Nordhaus, 2008; Ackerman et al., 2010; Kopp et al., 2012). The Monte Carlo approach may lead to inconsistent policy prescriptions, i.e. simultaneously having more abatement effort but lower abatement cost, due to a Jensen's inequality argument (Crost and Traeger, 2013).

[7]This follows closely to Crost and Traeger (2013) however I parameterize damage uncertainty differently by differentiating between epistemic uncertainty over the damage function and true stochasticity in the realization of damages.

maintaining the conventional quadratic functional form. This closely represents how the DICE damage calibration has historically been updated and re-estimated in new vintages of the IAM, and effectively endogenizes the integrated assessment modeler's decision to update the damage function's calibration.[8]

The third framework is the "robust framework" which acknowledges the widespread belief that the damage functions used in IAMs are misspecified. The robust framework uses robust control techniques (Hansen and Sargent, 2007, 2008) in order to find policy rules that perform well even when the assumed damage function may not be correct.[9] Similar to some macroeconomic models, we have an extremely limited set of data to aid us in developing a damage function both in terms of the number of damage estimates, and also the range of historically observed temperatures. Our limited information on damages makes it difficult for a modeler or policymaker to statistically distinguish between alternative damage functions (Hansen and Sargent, 2014). A policymaker utilizing robust control considers her damage function to be only an approximation to reality and that there are unknown, and unlearnable misspecifications to her approximating damage function. When she optimizes her policy trajectory, the policymaker considers alternative damage models "close" to her approximating model as potential true models of the world (i.e. she weakens her damage functional form beliefs). She optimizes her policies while acknowledging these potential alternative damage functions in order to protect against unknown errors in her model.[10]

The final framework is the robust control and learning (RCL) framework where the robust framework and learning framework are combined so that the policymaker simultaneously updates her beliefs over the damage function and acknowledges that her model may be misspecified.[11]

I find that inclusion of Bayesian learning reduces the near term SCC. Under the learning framework, anticipated learning disincentivizes early abatement effort due to the option value of waiting to have better information before directing scarce economic resources toward abating emissions, but also due to the benefits of accelerated learning with additional emissions. Learning also increases the range of possible SCC realizations in the future. Since damages will stochastically reduce output and stochastically affect the policymaker's decisions, the SCC will also be stochastic. Introducing learning allows the policymaker to more flexibly adjust policy. Given the parameterization of the policymaker's beliefs and the annual damage shocks, the SCC in 2105 may be under $\$20/tCO_2$,

[8] For example, in the 1999 Regional Integrated Climate-Economy (RICE) model, low levels of warming increased output (Nordhaus, 2008), but in future model vintages, the damage function was updated so positive warming strictly reduces output. All models retained a quadratic functional form. See Tol (2009) for a survey of some previous impact analyses.

[9] See Roseta-Palma and Xepapadeas (2004), Athanassoglou and Xepapadeas (2012), Anderson et al. (2014), and Temzelides et al. (2014) for recent applications to integrated assessment and environmental models.

[10] Close is measured by relative entropy.

[11] See Cogley et al. (2007, 2008) for examples of interacting robust control with learning.

or over \$80/tCO_2. Without learning, the SCC outcomes are more tightly clustered, ranging from \$58/tCO_2 to \$65/tCO_2. Including concern for model misspecification by implementing robust control raises the entire SCC trajectory due to concerns that the damage function underestimates the costs of additional warming. The robust control SCC also increases more rapidly since rising temperature increases the potential costs of any model misspecifications. Conversely, when robust control and Bayesian learning are combined, the SCC is the lowest out of all frameworks. Layering deeper, Knightian uncertainty over the damage functional form on top of the conventional, measurable uncertainty over the damage calibration amplifies the policymaker's desire to resolve the measurable uncertainty via active learning. This intuitively unappealing finding highlights a problem with economists' decision frameworks for handling model uncertainty and learning in dynamic settings.

In order to develop these frameworks I must introduce additional state variables to hold time-varying sufficient statistics for the policymaker's beliefs and the stochastic realizations of damages. I demonstrate that sparse grid methods provide an accurate and computationally cheap way to substantially increase the number of states in dynamic stochastic IAMs. I use the methodology of Smolyak (1963) to construct a sparse grid for value function approximation such that computational complexity increases only *polynomially* in the number of states rather than exponentially (Winschel and Kratzig, 2010).[12] This affords major increases in computational efficiency for high dimensional problems. This work appears to be the first application of this powerful computational technique in the environmental literature, which opens up new avenues for research in climate modeling.

The paper is organized as follows. Section 1 describes the construction of the DICE damage function, an overview of the DICE model, and all four policy frameworks. Section 2 reports results on how the different assumptions on damages in each framework affect ex-post welfare and climate outcomes. Section 3 concludes. The appendix fully describe the dynamic stochastic DICE model, contains additional information on robust control, and details the computational methodology and error analysis.

1 Damages and the Integrated Assessment Modeling Framework

I begin by giving a brief overview of the DICE damage function, its calibration, and its major criticisms. Next, I give an overview of the DICE model as a whole, the novel framework that allows for learning over the damage function calibration, and how robust control techniques capture concerns about model misspecification. The end of the section describes how learning and robust

[12]This method has been applied recently in the macroeconomics literature to compute equilibria in overlapping generations models (Krueger and Kubler, 2004) and to solve stochastic growth models (Gonzalez and Rojas, 2009). See Judd et al. (2014) for a recent description of Smolyak's method.

control affect the trajectory of the optimal carbon tax.

1.1 The DICE Damage Function Its Criticisms

In the benchmark DICE model, damages, $D(T_t^{atm})$, multiplicatively reduce the level of time t output as a function of the time t atmospheric warming over pre-industrial levels, T_t^{atm}.[13] The damage function specific to the DICE model is given by:

$$D(T_t^{atm}) = d_1 \left(T_t^{atm}\right)^{d_2},$$

where d_2 is assumed to be 2 and d_1 is calibrated to data.[14] $D(T_t^{atm})$ is interpreted as the percent loss of economic output due to warming over pre-industrial levels.

Historically, the damage function has been calibrated by developing damage functions specific to different sectors such as health or agriculture, and then aggregating them up to a global damage function using a four-step procedure. This approach resulted in a calibration of $d_1 = 0.0028388$ (Nordhaus and Boyer, 2000).[15] Due to the tremendous ex-ante uncertainty in how warming will affect these sectors, the calibration procedure must assume functional forms on how warming affects each the individual sectors as well as impose the quadratic functional form assumption for the aggregate damage function. The most recent version of DICE takes a more simple calibration approach. The same quadratic damage function is used, but instead of devising sectoral damage functions, the quadratic damage function is directly calibrated to a set of monetized damage estimates contained in Tol (2009), with an upward adjustment of 25% to account for impacts that are difficult to estimate such as human conflict (Nordhaus and Sztorc, 2013). Even with this updated damage calibration procedure, the studies used to monetize the welfare impacts may be over 20 years old. Moreover the current damage calibration in DICE has not included impacts from a burgeoning strand of literature that has recently quantified a wide range of climate damages.[16]

Although the quadratic functional form is calibrated to data, the functional form itself is not. The quadratic functional form of damages is a critical assumption, and has been acknowledged to be arbitrary and lacking economic or scientific grounding (Stanton et al., 2009; Pindyck, 2013). Yet

[13] Although not investigated here, there are models that incorporate damages directly on capital (Kopp et al., 2012) or utility (Sterner and Persson, 2008; Barrage, 2014). Sterner and Persson (2008) explicitly includes non-market environmental services in the utility function with limited substitutability, which can be degraded by warming. There is also empirical evidence that climate damages affect the growth rate of output, not just the level (Dell et al., 2012).

[14] See the DICE documentation for a plot of the most recent damage function calibration (Nordhaus and Sztorc, 2013).

[15] The sectors included in the damage function are: agriculture, sea-level rise, health, non-market amenities, settlements and ecosystems, catastrophes, and other market sectors. See Nordhaus and Boyer (2000) for more details on the sectoral calibration approach.

[16] See Burke et al. (2015); Zivin et al. (2015) for several recent examples.

if we want to optimize climate policy, we must be able to quantify damages in order to know the benefits of mitigating emissions. Since there is a lack of theory and knowledge to tell us what the functional form should be (Pindyck, 2013), modelers have resorted to imposing their own choice of functional forms.

Economists have raised several issues with how the DICE damage function represents the relationship between warming and economic costs. In one example, a commonly cited adverse result of the quadratic damage functional form assumption is that it has been shown to not capture any meaningful risk of catastrophic impacts. Warming would need to reach nearly 20°C before half of economic output was lost to climate damages (Ackerman et al., 2010).[17] DICE accounts for the chance of catastrophe by adjusting d_1 upward using surveys of experts or judgment calls by the modeler.[18] Some scientists and economists have gone so far as to say that the way these kinds of catastrophic damages are implemented in IAMs is far removed from evidence in the natural sciences on catastrophes and impacts (Ackerman and Stanton, 2012; Lenton and Ciscar, 2012).[19]

1.2 The DICE Model

I introduce novel frameworks for Bayesian damage learning and robust control into a dynamic stochastic version of the DICE integrated assessment model. The model is a dynamic stochastic finite-horizon Ramsey growth model with an annual time step coupled to a climate module. A complete description of the model can be found in the appendix. The state space of the model is composed of capital (in effective labor units); the atmospheric, upper ocean and lower ocean CO_2 stocks; atmospheric and ocean temperatures; a state to capture stochastic damages; and for the frameworks that include learning, a state to capture a stochastically evolving sufficient statistic for the policymaker's beliefs.

Figure 1 displays a schematic of the climate-economy system of the DICE model. States are boxes, controls are ovals, and arrows denote relationships between the different quantities of interest. The policymaker's objective is to maximize the present value of her stream of expected discounted annual flow utility.

Each year, the policymaker begins with an existing level of capital in effective labor units. The effective capital stock is used in factor production to generate output, and as a byproduct, factor production releases industrial emissions into the atmosphere. The output generated from capital

[17] The damage function is calibrated to only calculate damages from warming under 3°C (Nordhaus and Sztorc, 2013), however any climate analysis extending beyond the next hundred years will likely exceed this bound.

[18] In recent work, others have incorporated explicit, stochastic tipping points into the model to capture catastrophe (Lemoine and Traeger, 2014; Cai et al., 2015a,b; Lontzek et al., 2015; Lemoine and Traeger, 2016).

[19] Howard and Sterner (2014) provide evidence that the calibration approach in the most recent version of DICE may still be underestimating damages by up to a factor of three.

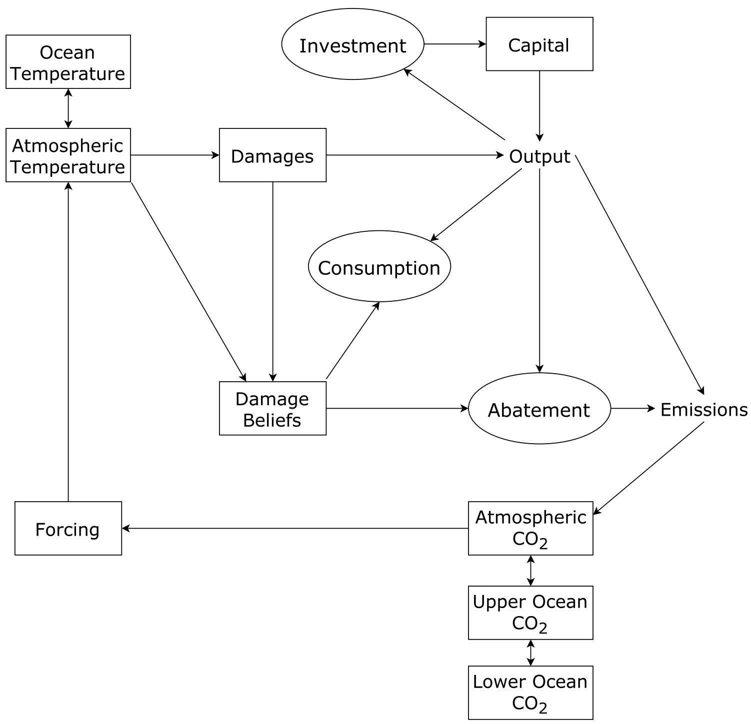

Figure 1: A schematic of the DICE model with learning. Squares are states, ovals are controls, and arrows denote relationships between states and controls.

can be used in three ways: consumption to increase her flow utility, abatement to reduce the level of industrial emissions in that year, or as investment to increase the future stock of capital. When optimizing abatement, consumption, and investment, the policymaker uses her current knowledge of damages, which is given by her distribution over the damage parameter d_1.

Emissions from exogenous land use changes and industrial emissions net of abatement directly enter the atmospheric stock of CO_2. CO_2 can flow in between the atmosphere and two other wells: the upper ocean and biosphere, and the lower ocean. Higher levels of atmospheric CO_2 increase radiative forcing, a measure of how much heat is trapped at the earth's surface by greenhouse gases, which in turn raises atmospheric temperature. The warmer atmosphere transfers thermal energy to the ocean which increases ocean temperature. Higher atmospheric temperature causes greater damages and reduces the amount of output produced from a given stock of capital via the damage function, subject to a stochastic annual shock to realized damages. The policymaker treats the calibrated damage coefficient, d_1, as uncertain and her beliefs about future damages affect her contemporaneous decisions. Thus far I have described the uncertainty framework. The policymaker's problem at some time t can be represented by a Bellman equation,

$$V_t^u(\boldsymbol{S_t}) = \max_{c_t, \alpha_t} \left\{ u(c_t) + \beta(t)\, E\left[\, V_{t+1}^u(\boldsymbol{S_{t+1}})\right]\right\} \tag{1}$$

subject to:

$$\boldsymbol{S_{t+1}} = f_u(\boldsymbol{S_t}, c_t, \alpha_t)$$

where V_t^u is the time t uncertainty framework value function, $f_u(\boldsymbol{S_t}, c_t, \alpha_t)$ is the set of transition equations without updating the distribution on d_1, $u(c_t)$ is the utility of consumption, α_t is abatement, $\beta(t)$ is a time-varying discount factor[20] and $\boldsymbol{S_t}$ is a vector of the capital, CO_2 stocks, temperature stocks, and damages. The policymaker's investment decision is just the residual output after consumption and abatement. The expectation is over the distribution of the damage state in time $t+1$, and the distribution of the unknown damage coefficient d_1. The distribution that characterizes beliefs over d_1 is not updated over time and is described below. In the terminal year, the policymaker maximizes utility subject to the terminal value function described in the appendix.

Building off this framework, the learning framework allows for observations of temperature and damages to be used to update the distribution of the damage calibration. This changes the Bellman

[20] The annual utility discount factor is constant. The modeled discount factor is a function of time because it adjusts for the non-constant growth rates of technology and labor via a Ramsey equation argument. This does not introduce time inconsistency issues.

equation to

$$V_t^l(\boldsymbol{S_t^+}) = \max_{c_t,\alpha_t}\left\{u(c_t) + \beta(t)\,E\Big[V_{t+1}^l(\boldsymbol{S_{t+1}^+})\Big]\right\} \tag{2}$$

subject to:

$$\boldsymbol{S_{t+1}^+} = f_l(\boldsymbol{S_t^+}, c_t, \alpha_t)$$

where $\boldsymbol{S_t^+}$ is the previous state vector along with a state variable for a sufficient statistic of the policymaker's time t beliefs, $V_t^l(\boldsymbol{S_t^+})$ is the time t value function for the learning framework policymaker, and $f_l(\boldsymbol{S_t^+}, c_t, \alpha_t)$ is the set of transition equations when beliefs can be updated. The expectation is over the distribution of time $t+1$ damages and also the distribution of her time $t+1$ beliefs of d_1 which will evolve stochastically as a function of her time t beliefs. The updating of solely the d_1 damage coefficient approximates the learning process of real world integrated assessment modelers who typically maintain the same damage function in updated versions of their models, but refine the calibration based on new information gathered over time.

1.2.1 The Mechanics of Updating the Damage Calibration

The policymaker under all four frameworks assigns the calibrated coefficient a lognormal prior distribution, $d_1 \sim log\mathcal{N}(\mu_t^B, \Sigma_t^B)$. In the frameworks with learning, the policymaker updates this distribution each year. I use a lognormal prior to for a combination of tractability and to ensure the policymaker believes that higher temperatures lead to higher damages.[21] Her immediate learning of the value of d_1 is hindered by a random IID noise process $w_t \sim log\mathcal{N}(\mu_w, \sigma_w^2)$ which captures random variation in how warming affects factor production. Examples of this are random shocks of droughts or cyclones. The noise enters the damage function multiplicatively,

$$D(T_t^{atm}, w_t) = d_1\,(T_t^{atm})^2\,w_t,$$

so that the damage function also has a lognormal distribution: $D(T_t^{atm}, w_t) \sim log\mathcal{N}\left((T_t^{atm})^2 + \mu_t^B, \Sigma_t + \sigma_w^2\right)$. Under the learning framework, the policymaker observes gross output from factor production before climate damages occur, Y_t^g, and the level of output net of damages, Y_t^n. Damages reduce the level of gross output as in the DICE model (Nordhaus, 2008),

$$Y_t^n = \frac{Y_t^g}{1 + d_1\,(T_t^{atm})^2\,w_t}.$$

[21] Nordhaus (2008) uses a normal distribution on d_1 but discards negative values in a Monte Carlo analysis. Crost and Traeger (2013) also use a lognormal distribution in a dynamic stochastic version of DICE.

Rearranging the expression and taking logs yields,

$$log\left(\frac{Y_t^g}{Y_t^n}-1\right)-log\left((T_t^{atm})^2\right)=log(d_1)+log(w_t).$$

$log(d_1)$ and $log(w_t)$ are normally distributed, and the policymaker observes every variable on the left hand side of the equality. Relabel the expression on the left hand side Q_t^B, where now $Q_t^B \sim \mathcal{N}(\mu_t^B+\mu_w, \Sigma_t^B+\sigma_w^2)$. With each observation of the random variable Q_{t+1}^B equal to some realized value q_{t+1}^B at time $t+1$, the policymaker updates her prior according to Bayes' Law,

$$\mu_{t+1}^B=\frac{\Sigma_t^B\left(q_{t+1}^B-\mu_\omega\right)+\sigma_w^2\mu_t^B}{\Sigma_t^B+\sigma_w^2} \tag{3}$$

$$\Sigma_{t+1}^B=\frac{\Sigma_t^B\sigma_w^2}{\Sigma_t^B+\sigma_w^2}. \tag{4}$$

Σ_{t+1}^B, the shape parameter of our lognormal posterior, monotonically declines over time.[22] μ_{t+1}^B, the location parameter, is a weighted average of its previous value, and the realization of $q_{t+1}^B-\mu_\omega$, a noisy signal of the underlying value of d_1. The prior location parameter is given more weight during the belief updating process when the damage shock has higher variance and the policymaker's signal of d_1 is more noisy. The prior location parameter is also assigned greater weight when the policymaker is more certain in her beliefs and her prior variance (and shape parameter) are small.

I assign the policymaker's expectation for d_1 in 2005 to be equal to the DICE calibration: $\mu_0^B=0.0028388$, and the variance is calibrated to be equal to $0.13^2\%$, a value used in previous analyses with the DICE model (Nordhaus, 2008). The probability density function of the coefficient is shown in left panel of Figure 2. The mean of the lognormal noise process is calibrated to 1 so that the average shock has no effect on damages. I calibrate the variance of the noise process so that the year 2005 variance of the composite random variable, $d_1\omega_t$ is approximately double the variance of the prior for d_1, a value used in recent work which acknowledges that the damage calibration and uncertainty over damages are likely too small (Crost and Traeger, 2013).[23] Given the noise calibration, the expected damage at 2°C using the DICE damage calibration is 1.14% of GDP. The right panel of Figure 2 displays the year 2005 prior distributions for damages at 1°C, 2°C, and 3°C. The 1°C distribution is highly peaked at near-zero damages, and the right tail decays rapidly. The 2°C and 3°C distributions peak at slightly higher damages, but are characterized by right tails that decay more slowly and assign greater weight to higher damage outcomes.

[22] This is the variance parameter of the underlying normal distribution, and μ_{t+1}^B is the corresponding mean.

[23] Due to the lack of information on damage variability, The value of σ_w^2 is a judgment call. Adjusting the noise variance does not meaningfully change the results.

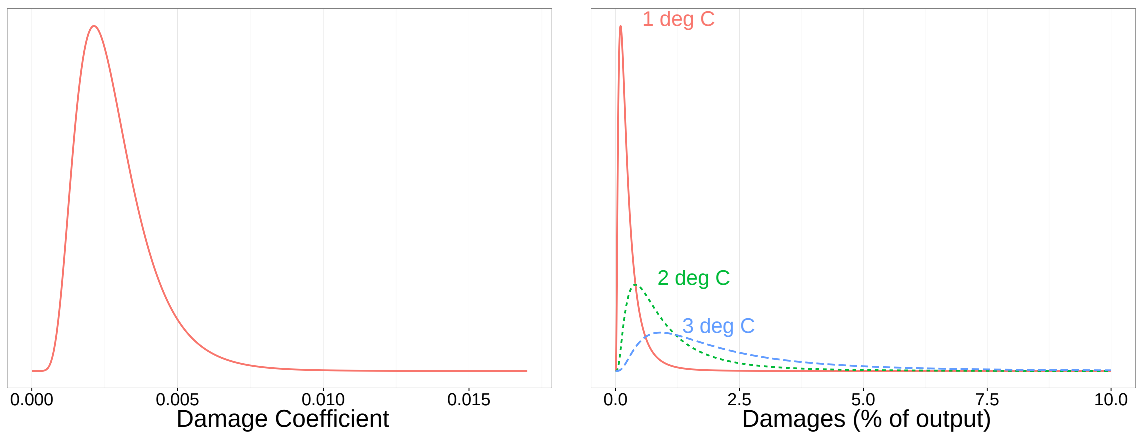

Figure 2: PDF of the coefficient's prior distribution in 2005 (left), and the PDF damages in 2005 at 1°C, 2°C, and 3°C of warming (right).

1.2.2 The Value of Emissions Reductions - Bayesian Learning

The potential to use observed data to update beliefs over uncertain model parameters changes the value of mitigating emissions. The marginal benefit of emissions reductions (marginal cost of additional emissions) is given by the SCC. The SCC in dollar terms is proportional to the change in the discounted stream of expected utility given an additional unit of emissions,[24]

$$SCC_t \propto -\beta(t)\, E\left[\frac{\partial V(\mathbf{S^+_{t+1}})}{\partial \mathbf{S^+_{t+1}}} \frac{\partial \mathbf{S^+_{t+1}}}{\partial e_t}\right],$$

where S^+_{t+1} is the state vector with the stochastically evolving belief state. After breaking up the gradient into its eight parts and recognizing that time t emissions do not affect time $t+1$ levels of capital, the shape parameter for beliefs over d_1, ocean temperature, nor ocean CO_2 stocks, we have that,

$$\begin{aligned} SCC_t \propto -\beta(t)\, E\Bigg[& \frac{\partial V(\mathbf{S^+_{t+1}})}{\partial M^{atm}_{t+1}} \frac{\partial M^{atm}_{t+1}}{\partial e_t} + \frac{\partial V(\mathbf{S^+_{t+1}})}{\partial T^{atm}_{t+1}} \frac{\partial T^{atm}_{t+1}}{\partial e_t} \\ & + \frac{\partial V(\mathbf{S^+_{t+1}})}{\partial \mathcal{L}_{t+1}} \frac{\partial \mathcal{L}_{t+1}}{\partial e_t} + \frac{\partial V(\mathbf{S^+_{t+1}})}{\partial \mu^B_{t+1}} \frac{\partial \mu^B_{t+1}}{\partial e_t}\Bigg], \end{aligned} \tag{5}$$

[24]The SCC is the marginal rate of substitution between emissions and capital (Cai et al., 2015b), where the division of equation (6) by the partial derivative of the value function with respect to capital, and an adjustment for labor and technology, puts the SCC into the correct dollar terms.

where in order, the partial derivatives are with respect to atmospheric CO_2, atmospheric temperature, percent of output remaining after damages, and the belief location parameter. Including Bayesian learning adds the last term of equation (6) to the SCC compared to a framework without learning.

Anticipating learning should yield two opposing option values. The first is the option value of waiting to take costly abatement action until the policymaker becomes better informed. This will reduce the SCC. Costly abatement action can be delayed into the future when the benefits of abatement, the avoided damages, are more certain. The first option value reduces the welfare cost of emissions through the location parameter. Equation (3) shows that additional time t emissions reduce μ_{t+1}^B by decreasing q_{t+1}^B. The variance of beliefs over d_1 is a function of both the location and scale parameters:

$$var_t(d_1) = \big(\exp(\Sigma_t^B) - 1\big)\exp(2\mu_t^B - \Sigma_t^B).$$

As the location parameter becomes smaller, all else constant, the variance of the distribution shrinks as well, affording the policymaker an active learning channel. One part of this option value will then be captured by emissions increasing future welfare through the location parameter and reducing the SCC, i.e.,

$$-\frac{\partial V_{t+1}^l(\mathbf{S_{t+1}^+})}{\partial \mu_{t+1}^B}\frac{\partial \mu_{t+1}^B}{\partial e_t} < 0.$$

But in addition, μ_{t+1} is decreasing and concave in time t emissions. The build up of atmospheric CO_2 actually allows active learning to occur faster. Therefore this option value may actually be increasing in magnitude over time as atmospheric CO_2 and atmospheric temperature increase.

The policymaker's second option value is the value of waiting before irreversibly emitting CO_2 into the atmosphere and causing irreversible warming. This raises the SCC. The policymaker reduces time t consumption and increases time t abatement to reduce emissions until she is more certain of the future damage costs of using output for consumption rather than abatement. The two option values push optimal policy in opposing directions. Others have found that the first option value tends to dominate and lead to a lower SCC in simple settings (Fisher and Narain, 2003). Ex-ante, it is unclear which one dominates in a dynamic stochastic version of DICE.

1.3 Robust Control

Many have noted that the damage functions used in IAMs are likely misspecified. Ideally, this skepticism about damage functions should be built into an IAM by modeling a policymaker who does not completely trust her model to be a precise representation of reality. The policymaker may instead believe that her model is only an approximation to the real world. In this case, the optimal policy she derives by solving her approximating model will almost surely not be optimal if used in real world policy. Instead of striving to develop an optimal policy using a model that is almost surely incorrect, the policymaker may instead prefer to find policy rules that are robust to unknown, and in the short-term unlearnable, errors in her model.[25] To capture concerns about damage misspecifications, I model a policymaker who uses robust control techniques in order to find policies that perform well even when the damage function inside her IAM may be incorrect (Hansen and Sargent, 2007, 2008).[26]

The policymaker utilizing robust control does not believe her model to be precisely correct, but she recognizes that she does have some information about the true damage function.[27] The policymaker begins with her approximating model, for example the quadratic damage function which has been calibrated to real world data, and then optimizes policy by finding decision rules that fare well over a set potential models that are statistically close to her approximating model. Statistically close is defined in terms of relative entropy. Alternative models more similar to her approximating model are given greater weight, while those that are much different get less weight during policy optimization. By taking this approach the policymaker is recognizing that her approximating damage function is probably "close" to the true damage function, but she wishes to guard against unknown errors by accounting for other possible damage functions, but with decreasing weights as they become more unlike the damage function in her own approximating model.

Robust control captures potential misspecifications to a model as distortions to the state transition distributions, and in turn the policymaker's continuation value. These distortions reduce her perceived expected continuation values during policy optimization in order to induce her to use a

[25]Implicitly the four frameworks are taking different approaches to estimating an unknown data generating process for damages. In the uncertainty framework a quadratic damage function is estimated once before policymaking begins. The learning framework allows for periodic updates to the quadratic model, the robust framework considers the possibility of other approximations to the damage function, and the RCL framework combines the possibility of other damage functions with continual re-estimation of the calibrated coefficient on the approximating damage function.

[26]Robust control is one method of incorporating ambiguity aversion, which diverges from traditional subjective expected utility preferences. Robust control is axiomatized by multiplier preferences (Strzalecki, 2011), a special case of variational preferences (Maccheroni et al., 2006) where the ambiguity index is the relative entropy function (Kullback-Leibler divergence), an asymmetric measure of how close two distributions are to one another.

[27]In other generalizations of subjective expected utility, such as maxmin expected utility, the policymaker chooses a worst case model to optimize over given some set of models (Gilboa and Schmeidler, 1989). The fundamental difference between maxmin expected utility and multiplier preferences is in its axiomatization. The maxmin expected utility axioms further specialize the ambiguity index to a function equal to 0 for some closed and convex set of potential distributions for the damage model, and infinite otherwise. This draws a distinction between between the two preference orderings. Under multiplier preferences for robust control, the policymaker guards against a set of potential misspecifications to her approximating model weighted by how large the misspecification is, whereas under maxmin expected utility, the policymaker guards against a single worse-case misspecification within a set of possible misspecifications.

more robust policy, but the distortions do not affect the actual realized payoffs.[28] The policymaker maximizes her objective while accounting for potential distortions, weighted by their relative entropy. The distortions the policymaker faces are implemented via state-dependent shocks to the transition distributions, so that the distortions can be temporally linked and persist over time. This allows the distortions to represent real misspecifications to the policymaker's model or prior distribution. Hansen and Sargent (2007) demonstrate that the distortions capturing model misspecification are implemented in a non-linear dynamic programming framework by replacing expected continuation values in the policymaker's Bellman equation with a risk sensitivity operator,[29]

$$T^1(\beta_t V^r_{t+1}(\boldsymbol{S^+_{t+1}})|\theta_1) = -\theta_1 log\left(E_{S^+_{t+1}}\left[exp\left(-\frac{\beta_t V^r_{t+1}(\boldsymbol{S^+_{t+1}})}{\theta_1}\right)\right]\right),$$

where $V^r_{t+1}(\boldsymbol{S^+_{t+1}})$ is the time $t+1$ robust control value function, θ_1 is a penalty parameter (details below) that calibrates how strongly the policymaker guards against these misspecifications, and the expectation operator is over next year's state. This results in a new Bellman equation,

$$V^r_t(\boldsymbol{S^+_t}) = \max_{c_t,\alpha_t}\left\{U(c_t) + E_{d_1}\left[T^1(\beta_t V^r_{t+1}(\boldsymbol{S^+_{t+1}})|\theta_1)\right]\right\}. \tag{6}$$

subject to:

$$\boldsymbol{S^+_{t+1}} = f_{u/l}(\boldsymbol{S^+_t}, c_t, \alpha_t)$$

where the expectation acting outside the risk sensitivity operator is over the unknown damage coefficient d_1, and $f_{u/l}(\boldsymbol{S^+_t}, c_t, \alpha_t)$ may be the transition equations when learning or not learning depending on whether the policymaker is using the RCL framework or robust framework.

For any $\theta_1 \in (0,\infty)$, the risk sensitivity operator distorts the density of continuation values towards worse outcomes than if it was just the usual expected continuation value. As $\theta_1 \to \infty$, the risk sensitivity operator reduces to the usual subjective expected utility framework.[30] There typically exists some point $\bar{\theta}_1$, where the problem "breaks down" for $\theta_1 < \bar{\theta}_1$. I calibrate θ_1 by

[28] Hansen and Sargent (2008) demonstrate that the distortions can be represented as the optimal selection of a transition probability measure by an evil agent whose objective is to minimize the payoff of our policymaker. The evil agent's selection is subject to a penalty proportional to the relative entropy of its distorted transition probability measure relative to the approximating model's transition probability measure.

[29] Subscript 1 is to maintain the notation of Hansen and Sargent (2007) for the risk sensitivity operator that induces decision rules concerned with model misspecification.

[30] Consider a simple case where the continuation value, V^r_{t+1}, is distributed $N(1,1)$. The expected continuation value is just 1. However when we apply the risk sensitivity operator, the term inside the expectation is now lognormally distributed and we can use the mean and variance definitions for a lognormal distribution to show that the expected continuation value becomes $1-\frac{1}{2\,\theta_1}$. Clearly as $\theta_1 \to \infty$ we are back in the subjective expected utility world. As θ_1 decreases, the expected continuation value declines and the policymaker acts as if she is facing worse futures. This induces her to select policies that better guard against misspecifications which reduce her future welfare.

selecting values close to this breakdown point. This allows us to more clearly see the impact of robust control and more easily glean intuition from the results.[31]

1.3.1 The Value of Emissions Reductions - Robust Control

When applying the risk operator, the social cost of carbon for a policymaker utilizing robust control is now proportional to,

$$SCC_t \propto -\beta(t) \frac{E\left[exp\left(\frac{-\beta(t)V^r_{t+1}(\mathbf{S^+_{t+1}})}{\theta_1}\right)\frac{\partial V^r_{t+1}(\mathbf{S^+_{t+1}})}{\partial \mathbf{S^+_{t+1}}}\frac{\partial \mathbf{S^+_{t+1}}}{\partial e_t}\right]}{E\left[exp\left(\frac{-\beta(t)V^r_{t+1}(\mathbf{S^+_{t+1}})}{\theta_1}\right)\right]}.$$

Re-arranging the expression, we can recover the conventional SCC expression and an additively separable adjustment for robust control,

$$\begin{aligned} SCC_t \propto & -\beta(t)\, E\left[\frac{\partial V^r_{t+1}(\mathbf{S^+_{t+1}})}{\partial \mathbf{S^+_{t+1}}}\frac{\partial \mathbf{S^+_{t+1}}}{\partial e_t}\right] \\ & -\beta(t)\, cov\left(\frac{\partial V^r_{t+1}(\mathbf{S^+_{t+1}})}{\partial \mathbf{S^+_{t+1}}}\frac{\partial \mathbf{S^+_{t+1}}}{\partial e_t}, \frac{exp\left(\frac{-\beta(t)V^r_{t+1}(\mathbf{S^+_{t+1}})}{\theta_1}\right)}{E\left[exp\left(\frac{-\beta(t)V^r_{t+1}(\mathbf{S^+_{t+1}})}{\theta_1}\right)\right]}\right), \end{aligned} \tag{7}$$

where the first term is the conventional SCC expression, which is altered by robust control, and the second term is an additional robust control adjustment. The magnitude and direction of the robust control adjustment depends on the covariance of the marginal change in future welfare from an additional unit of emissions conditional on the distorted transition density (left argument), and how much an additional unit of emissions moves us into a lower value world, covaries with how much the transition density distortion moves us into a lower value world (right argument). This adjustment is several orders of magnitude smaller than the first component of the SCC so I ignore it in the quantitative analysis.

2 Results

I examine the SCC trajectories under the four frameworks, decompose the SCC into each channel outlined in Section 1 to determine how the different frameworks affect the SCC, and then analyze

[31] Calibrating θ_1 is typically done using detection error probabilities (Hansen and Sargent, 2008). I select specific values of θ_1 to simply show how a robustness-concerned policymaker acts, not to find the policy that corresponds to a policymaker desiring a specific amount of robustness.

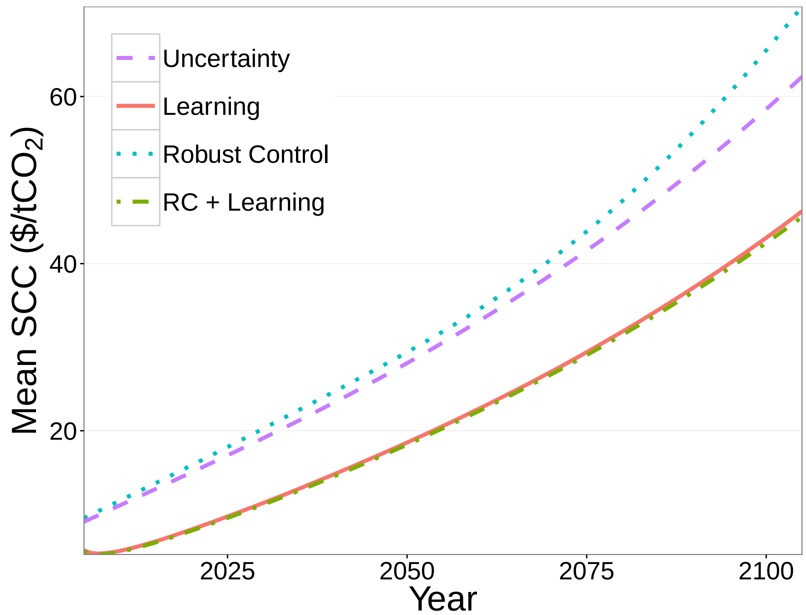

Figure 3: The mean SCC over 5000 simulations for each framework.

the distributions of SCC realizations. All results span years 2005-2105 and are averaged over 5000 simulations. In each simulation I draw a value from the year 2005 distribution for the damage coefficient and sample a vector of annual shocks from the distribution for the stochastic damage noise process. Each draw of a vector of damage shocks and each draw of the damage coefficient will have different impacts on each year's realization of damages. This will affect what the policymaker believes about damages under each of the learning frameworks, but it also randomly affects the policymaker's output budget, and her ability to consume, abate emissions, and invest for the future. This means that, even without learning, the SCC will follow a stochastic trajectory as has been noted by Cai et al. (2015b).

2.1 Social Cost of Carbon Trajectories

Figure 3 displays the mean SCC trajectory in $/t$CO_2$ over 5000 simulations for each framework when the true damage function follows the DICE assumptions. The uncertainty framework trajectory begins at $9/t$CO_2$ and rises to nearly $62/t$CO_2$ in 2105. The robust framework SCC trajectory follows very closely but begins rising higher than the uncertainty framework SCC as time progresses. The learning framework SCC starts lower than the SCC under the two frameworks without learning. The mean SCC trajectory begins $6/t$CO_2$ in 2005 and displays slight non-monotonicity in early years before increasing to $46/tCO$_2$.[32] Finally the RCL framework follows a similar SCC trajectory to the learning framework, however it actually begins slightly lower than the learning framework's SCC in 2005, and the difference between the two learning frameworks' SCCs grows over time.

2.2 Decomposing the SCC

To determine what drives these differences in the mean SCC trajectories, we can decompose each frameworks' SCC trajectory into four terms,

$$\frac{\partial V^{j}_{t+1}(\mathbf{S^{+}_{t+1}})}{\partial s^{i}_{t+1}}\frac{\partial s^{i}_{t+1}}{\partial e_t},$$

where s^i_{t+1} is either atmospheric CO_2, atmospheric temperature, damages or the prior location parameter, and j indexes each of the four policy frameworks.[33] All other states are unaffected by time t emissions and therefore do not enter the SCC. Finally, recall the covariance adjustment to the robust control SCC is dwarfed by the other components so I omit it from the analysis.[34]

Figure 4 displays the decomposition of the SCC into these four channels. Note that the y-axes are on different scales for each of the four panels. The upward SCC trajectory of all four frameworks is almost exclusively driven by the increasing welfare cost of additional atmospheric CO_2. The damage and temperature channels have a much smaller effect on the SCC because a marginal increase in emissions has a small effect on both temperature and damages. Moreover, due to inertia in the climate system, an additional unit of emissions will not fully translate into the corresponding amount of warming or damages for many years. Hence, this small increase in temperature and damages will be realized further in the discounted future and have a smaller present value impact compared to the increase in atmospheric CO_2 which realizes in the next year.

We can determine the implications of learning and anticipated learning for the SCC by analyzing the difference between the dashed uncertainty framework trajectory and the solid learning framework trajectory in Figure 3, and observing the differences in the four SCC channels in Figure 4. Anticipating future learning decreases the SCC, as has been found in other settings analyzing tipping points and an uncertain climate sensitivity (Lemoine and Traeger, 2014; Kelly and Tan, 2015). Acknowledging her option value of waiting, the policymaker believes she will be able to

[32]The potential for a non-monotonic SCC trajectory has been noted previously by Fitzpatrick and Kelly (2014). This is due to several competing effects such as the resolution of uncertainty, growing marginal benefits of emissions in terms of active learning, and declining abatement costs.

[33]S^{+}_{t+1} contains the state for the location parameter of the policymaker's beliefs. For the uncertainty and robust frameworks this state is held constant at its initial value in 2005.

[34]The robust control covariance adjustment is typically less than $0.01/tCO$_2$.

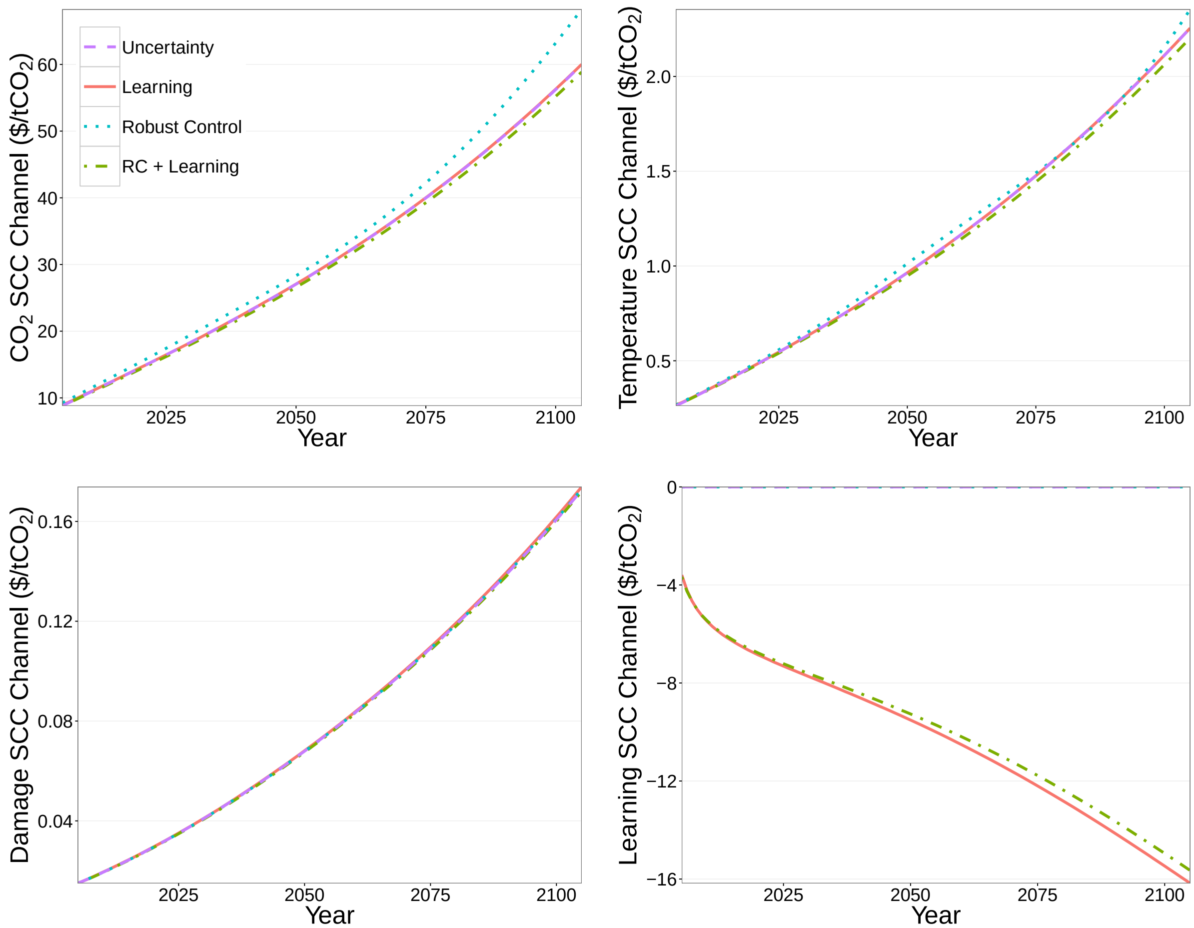

Figure 4: Decomposition of the mean social cost of carbon over 5000 simulations into the CO_2 channel (top left), temperature channel (top right), damage channel (bottom left), and learning channel (bottom right).

make more informed abatement decisions in the future and delays devoting scarce output resource towards abatement. Waiting to abate also provides benefits of accelerating learning and allowing the policymaker to make more informed decisions sooner. Both of these effects reduce the cost of additional emissions. This reduces the social cost of emitting an additional ton of CO_2.

Figure 4 shows that the primary driver of the lower SCC under the learning framework is the effect of emissions on the prior location parameter (bottom right panel), suggesting that active learning plays a large role in differences between the optimal policies. As more emissions are released into the atmosphere, the active learning channel becomes stronger, exercising a larger downward effect on the SCC over time.

Comparing the dashed uncertainty framework trajectory and the dotted robust control trajectory in Figure 3 shows the effects of concern for model misspecification. Guarding against potential model errors yields an SCC that is slightly higher up front, and also grows more quickly over time as temperature rises. Figure 4 demonstrates that the higher SCC under the robust framework comes about through the CO_2 channel, and that the CO_2 channel is also where the effect of robust control grows over time. The effect of robust control on the initial SCC is small since low temperatures early in the century keep the majority of the mass of the distribution for next year's damages on low values. This limits the size of the distortions to the damage transition caused by the risk operator. Since the potential distortions are small, this results in only a small increase in the SCC to guard against a potential underestimation of damages. Over time, damages rise quadratically in temperature, shifting more mass of the distribution of future damages onto larger values as demonstrated in Figure 2. The same distortion to the damage distribution now results in higher damages since the same misspecifications are becoming more costly. This raises the SCC faster than if the policymaker did not desire a robust policy. In general, the effect of concern about model misspecification is to assign more probability weight to lower continuation values and higher damage futures, which raises the SCC.

Lastly, comparing the dash-dotted RCL framework trajectory to the other three frameworks' trajectories in Figure 3 shows the impact of concern for model misspecification as it is layered on top of the ability to learn the true damage calibration over time. When learning is included, the effect of robust control flips. The combination of robust control and learning actually reduces the SCC compared to learning alone, robust control alone, or not learning at all. The reduction in the SCC from the combination of robust control and learning also grows over time. This counter-intuitive result occurs because misspecification concerns induce additional policy experimentation, lowering the cost of additional emissions. This outcome has appeared in other applications of robust control, namely by Cogley et al. (2007), who find that a policymaker has a similar desire to learn and experiment in a monetary policy setting. The stronger negative learning channel under the combination of robust control and learning shows that the policymaker has greater potential gains to resolving the damage uncertainty, which drives a lower SCC early since there are benefits to accelerating learning and reducing uncertainty. When facing an unknown model misspecification in addition to a measurable calibration uncertainty, reducing uncertainty becomes more valuable. The magnitude of potential distortions to the policymaker's model under robust control crucially depends on the mean and variance of damage beliefs. Learning provides a means to mitigate costly distortions to the model.

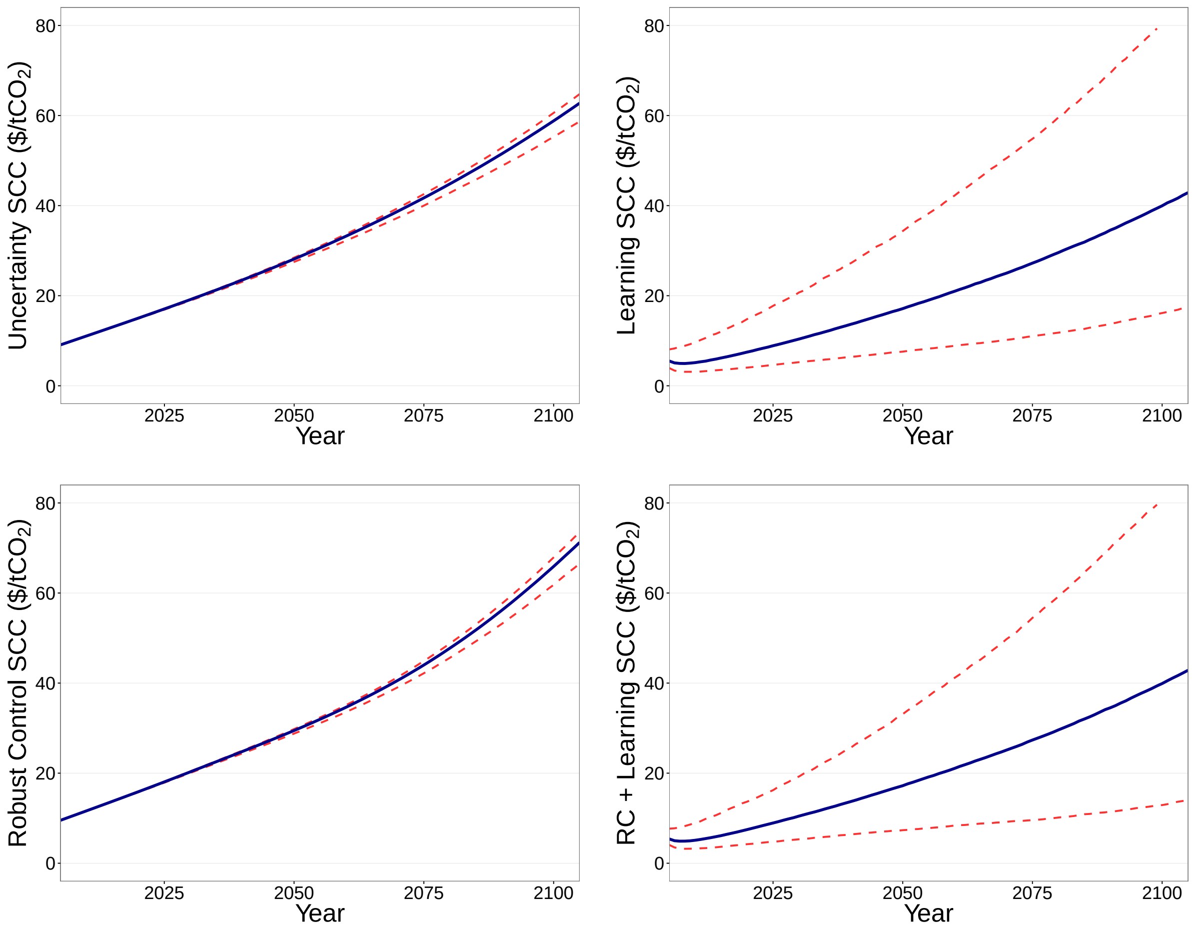

Figure 5: Median (solid), and the 5th and 95th percentiles (dashed) of the realized SCCs over 5000 simulations.

2.3 The Distribution of SCC Realizations

Due to the stochastic nature of damages and the distribution of potential damage coefficients, the SCC may take many different paths. Figure 5 displays the median SCC trajectory and the 5$^{\text{th}}$ and 95$^{\text{th}}$ percentile SCC realizations. Each panel of Figure 5 is for a different framework. The frameworks that do not learn and update beliefs are in the left two panels. The non-learning frameworks are characterized by a tight distribution of SCC outcomes over the 5000 simulations. Indeed, decades into the 21$^{\text{st}}$ century, 90% of SCCs fall within in a \$1/tCO_2 range. However near the end of the first century, potential SCC trajectories begin to diverge, such that the optimal trajectory may be \$62/tCO_2 as in Figure 3 above, but also several dollars per ton of CO_2 higher or lower depending on the draws of the damage coefficient and the sequences of damage shocks.

This divergence occurs because different draws of the damage coefficient and different draws of the vector of damage noise shocks will reduce the output available to the policymaker in different ways over the simulation. If these non-learning policymakers face a long sequence of high damage realizations, the policymaker is left with less capital in the future relative to a sequence of low damage realizations. Her lower capital stock produces less output, reducing the benefit of abating an additional ton of emissions. The multiplicative damage function causes less absolute damage in dollar terms when the capital stock, and thus output, are low. Therefore abating an additional unit of emissions has less of a benefit in terms of preserving the level of future output. Moreover, since output is already low, abatement also has a higher opportunity cost since potential consumption will also be low, and marginal utility will be higher. Without learning, a sequence of high damage outcomes actually reduces the SCC.

The frameworks that do learn and update beliefs are shown in the right two panels of Figure 5. These frameworks display a much wider range of possible SCC outcomes, ranging from under \$20/tCO_2 to over \$80/tCO_2 at the end of the century. Even in the first few decades, the learning frameworks display a much large range of possible SCC realizations compared to the non-learning frameworks at the end of the century. The range of possible SCCs widens because learning allows the policymaker to more flexibly adjust policy. If the policymaker faces a sequence of high damage realizations, she revises her beliefs and assigns more posterior mass to high values for the damage coefficient. This change in her beliefs increases the expected benefits of abating another ton of emissions, allowing the SCC to rise higher than without learning. Conversely, if she faces a sequence of low damage realizations, she revises her beliefs downward which reduces the expected marginal benefit of abating emissions, and therefore the SCC.

3 Conclusions

Most economists believe that the damage functions in IAMs are misspecified. Yet analyses of optimal climate policy have not investigated the potential for learning damages, or for adapting policy to take account of the widely noted concerns of damage function misspecification. I fill this gap in the literature by comparing the performance of four policy frameworks with different degrees of damage assumptions. The uncertainty framework accounts for uncertainty in the calibration of the damage function while assuming a known functional form. The learning framework acknowledges that integrated assessment modelers do learn over time and allows the policymaker to update her distribution over the uncertain damage function calibration within the model, assuming the same functional form. The robust framework borrows techniques from the macroeconomic literature to incorporate concerns that the damage function is misspecified in unknown ways. The robust policymaker constructs policies that guard against model misspecification. Last, the RCL framework combines concern for model misspecification with updating of beliefs over the damage coefficient.

I find that the pure effect of endogenously updating beliefs over the damage coefficient lowers the near-term SCC. Concern for robustness tends increase the SCC in order to guard against any harmful model misspecifications. Combining learning and robust control leads to the lowest up-front SCC out of all the frameworks. The learning and non-learning frameworks also display stark differences in the potential trajectories of the SCC. The learning frameworks allow for more flexible policy that correctly adapts to damage information.[35] However, the non-learning frameworks actually lead to lower SCCs when damages are high in a given simulation.

The results of this analysis have broader implications for modeling and policy. The combination of robust control and learning results in a peculiar outcome where the SCC is actually the lowest out of the four frameworks. This goes in the face of conventional intuition that suggests when faced with unknown model misspecifications, essentially a form of ambiguity, the policymaker would insure against bad futures with a higher SCC today. This raises questions about how our conventional decision frameworks actually perform in capture attitudes towards uncertainty and learning in complex dynamic settings.

However, recent work acknowledging tipping points in the climate system or effects on growth rates (Lemoine and Traeger, 2014; Cai et al., 2015a,b; Moore and Diaz, 2015; Lemoine and Traeger, 2016) have estimated SCCs significantly higher than the one used in US regulatory analysis. Although the regulatory SCC does not account for damage uncertainty and learning, this lower estimate is consistent acknowledging the potential for future damage learning. If we take the recent proliferation of climate impact estimates (that have yet to be incorporated in the damage function) as evidence of our future potential to update damage functions, it is possible that the current regulatory SCC may not be as far off from an SCC that accounts for our ability to refine our model parameters.

[35] Conditional on the rest of the model being specified correctly.

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Appendix

The appendix gives full details on the dynamic stochastic version of the DICE model, then formally describes Robust Control, and finally details how to construct a grid and Chebyshev interpolant using Smolyak's method and analyzes the accuracy of this approximation scheme.

A DICE Model

The policymaker's objective is to maximize the net present value of her expected stream of welfare, subject to the climate-economy transition equations and constraints. Here, I describe the problem for the learning framework. The uncertainty framework is equivalent but without updating the prior location parameter. Using either of the robust control frameworks replaces the expected continuation value with the risk sensitivity operator. The model is:

$$V_t(k_t, T_t^{atm}, T_t^{ocean}, M_t^{atm}, M_t^{up}, M_t^{lo}, \mu_t, \mathcal{L}_t) =$$
$$\max_{c_t,\alpha_t} \left\{ u(c_t) + \beta(t)\, E\left[V_{t+1}(k_{t+1}, T_{t+1}^{atm}, T_{t+1}^{ocean}, M_{t+1}^{atm}, M_{t+1}^{up}, M_{t+1}^{lo}, \mu_{t+1}, \mathcal{L}_{t+1}) \right] \right\}$$

subject to transitions:

$$k_{t+1} = e^{-(g_{L,t}+g_{A,t})}\left[(1-\delta_k)k_t + (1-\psi_t\alpha_t^{a_2})\mathcal{L}_t Y_t - c_t\right]$$

$$\begin{bmatrix} M_{t+1}^{atm} \\ M_{t+1}^{up} \\ M_{t+1}^{lo} \end{bmatrix} = \begin{bmatrix} 1-\phi_{12} & \phi_{21} & 0 \\ \phi_{12} & 1-\phi_{21}-\phi_{23} & \phi_{32} \\ 0 & \phi_{23} & 1-\phi_{32} \end{bmatrix} \begin{bmatrix} M_t^{atm} \\ M_t^{up} \\ M_t^{lo} \end{bmatrix} + \begin{bmatrix} e_t \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} T_{t+1}^{atm} \\ T_{t+1}^{ocean} \end{bmatrix} = \begin{bmatrix} 1-\varphi_{21}-\xi_2 & \varphi_{21} \\ \varphi_{12} & 1-\varphi_{21} \end{bmatrix} \begin{bmatrix} T_t^{atm} \\ T_t^{ocean} \end{bmatrix} + \begin{bmatrix} \xi_1 F_{t+1}(M_{t+1}^{atm}) \\ 0 \end{bmatrix}$$

$$\mu_{t+1} = \frac{\Sigma_t\left(log\left(\frac{1}{\mathcal{L}_{t+1}} - 1\right) - log({T_{t+1}^{atm}}^2) - \mu_\omega\right) + \sigma_w^2\mu_t}{\Sigma_t + \sigma_w^2}$$

$$\Sigma_{t+1} = \frac{\Sigma_t\sigma_w^2}{\Sigma_t + \sigma_w^2}$$

$$\mathcal{L}_{t+1} = \frac{1}{1 + \mathbf{d_1}\,{T_{t+1}^{atm}}^{d_2}\, w_{t+1}}$$

and subject to constraints:

$$\alpha_t \leq 1$$
$$c_t + (\psi_t\alpha_t^{a_2})\mathcal{L}_t Y_t \leq \mathcal{L}_t Y_t$$

$V_t(\cdot)$ is one of the four frameworks' value functions. Let K_t be capital, A_t be production technology, and L_t be labor. The eight states of the model are: effective capital $k_t = \frac{K_t}{A_t L_t}$; atmospheric carbon dioxide M_t^{atm}; upper ocean carbon dioxide M_t^{up}; lower ocean carbon dioxide M_t^{lo}; atmospheric temperature T_t^{atm}; ocean temperature T_t^{ocean}; prior location parameter μ_t; and the percentage of output remaining after damages $\mathcal{L}_t$. The prior shape parameter is not included as a state in the approximation since it evolves exogenously, but I include it above to show how it changes over time. The controls available to the policymaker are effective consumption $c_t = \frac{C_t}{A_t L_t}$ and the fraction of industrial emissions abated α_t, where C_t is total consumption. Investment at time t is calculated as a residual of the output remaining after the policymaker selects her policy vector.

The economy produces gross output Y_t from effective capital k_t according to the production function:

$$Y_t = k_t^{\kappa},$$

where κ is the Cobb-Douglas capital elasticity. Damages reduce output multiplicatively such that output net of damages is given by $\mathcal{L}_t Y_t$. Each year, the remaining output can be used towards two areas: consumption and abatement. The consumer has the common iso-elastic utility function, $u(C_t) = \frac{c_t^{1-\eta}}{1-\eta}$, where $\eta = 2$. The cost of abatement as a percentage of output is $C(\alpha_t) = \psi_t \alpha_t^{a_2}$, where ψ_t declines over time to capture improving abatement technology, and consumption additively reduces output. Any residual output after consumption and abatement is invested towards the capital stock. Each year, the capital also depreciates at a rate δ_k. Next year's capital stock is kept in effective terms by $g_{L,t}$ and $g_{A,t}$, which adjust the transition equation to account for population growth and technology improvements. The percentage of output remaining after damages, $\mathcal{L}_t$, evolves stochastically due to an IID lognormal noise process $w_t \sim log\mathcal{N}(\mu_\omega, \Sigma_\omega)$. The parameter $\mathbf{d_1}$ in bold is unknown, and due to the noise, cannot be learned immediately. However, the policymaker observes $\mathcal{L}_t, Y_t$ and T_t^{atm} each year, which allows her to update her prior distribution over $\boldsymbol{d_1}$ each year. Her prior at time t is that $\boldsymbol{d_1} \sim log\mathcal{N}(\mu_t, \Sigma_t)$. The policymaker uses her beliefs each year in order to predict future losses and make consumption and abatement decisions. Under the uncertainty framework, the prior distribution remains constant over time. The distributions of $\boldsymbol{d_1}$ and w_t are calibrated as described in the main text.

In DICE, CO_2 is captured by a three box system where the stock of CO_2 can flow between the atmosphere, biosphere and upper ocean, and the lower ocean. I calibrate the parameters governing carbon dynamics, ϕ_{ij} for $i, j = 1, 2, 3$, to that of Cai et al. (2015). Each year, emissions e_t are released and enter the atmospheric CO_2 stock:

$$e_t = \sigma_t(1 - a_t)Y_t + B_t,$$

where σ_t gives the emissions intensity of gross output, and B_t are emissions from exogenous sources. Atmospheric emissions generate radiative forcing, which is the energy per surface area due to increases in atmospheric CO_2. Forcing at time t is given by:

$$F(M_t^{atm}, t) = f\,\frac{log(M_{t+1}^{atm}/M_{pre})}{log(2)} + EF_t.$$

f is the forcing from a doubling of the atmospheric CO_2 stock, M_{pre} is the preindustrial level of CO_2 and EF_t is forcing from exogenous, non-CO_2 sources.

Time $t+1$ atmospheric temperature is determined by time t's atmospheric and ocean temperatures, and time $t+1$ forcing. Time $t+1$ ocean temperature is a function of time t atmospheric and ocean temperatures. The parameters governing temperature dynamics φ_{kl} for $k, l = 1, 2$ and ξ_1, ξ_2 are also calibrated to Cai et al. (2015).

Exogenous economic processes of the model are governed by the following equations:

$$L_t = L_0 + (L_\infty - L_0)(1 - exp(-t\delta_L)) \qquad \text{(Labor population)}$$

$$g_{L,t} = \delta_L \left(\frac{L_\infty}{L_\infty - L_0}\,exp(t\delta_L) - 1\right)^{-1} \qquad \text{(Labor population growth rate)}$$

$$A_t = A_0\,exp\left(\frac{g_{A,0}}{\delta_A}\left[1 - e^{-t\delta_A}\right]\right) \qquad \text{(Production technology)}$$

$$g_{A,t} = g_{A,0}\,exp(-t\delta_A) \qquad \text{(Production technology growth rate)}$$

$$\beta_t = exp(-\rho + (1-\eta)g_{A,t} + g_{L,t}) \qquad \text{(Labor and tech adjusted discount factor)}$$

and exogenous climate processes follow:

$$\psi_t = \frac{a_0\sigma_t}{a_2}\left(1 - \frac{1 - exp(tg_\psi)}{a_1}\right) \qquad \text{(Abatement cost technology)}$$

$$B_t = B_0\,exp(tg_B) \qquad \text{(Non-industrial } CO_2 \text{ emissions)}$$

$$\sigma_t = \sigma_0\,exp\left(\frac{g_{\sigma,0}}{\delta_\sigma}\left[1 - exp(-t\delta_\sigma)\right]\right) \qquad \text{(Gross emissions per unit of output)}$$

where index 0 indicates the year 2005. Table 1 contains the values of the model parameters.

The model is solved by value function iteration on a sparse grid generated by Smolyak's method with $\mu = 3$ on a 400 year finite horizon. The value function is approximated by a set of orthogonal Chebyshev polynomials. The terminal value function at year 2405 is the value function corresponding to the uncertainty framework in an infinite horizon with all exogenous processes held at their

2405 levels.[1] The selection of the year 2405 value function has shown little effect on the trajectories over the first 100 years.

B Robust Control

The policymaker faces objective randomness in how damages realize over time. However, she also faces subjective uncertainty due to her limited knowledge of how temperature actually manifests as damages. Her attitudes towards the first, objective source of uncertainty, may be different than her attitudes towards the second, subjective source of uncertainty, often called ambiguity. The disconnect in these two forms of uncertainty is introduced into the axiomatization of preference orderings as a weakening of the Axiom of Independence.

Robust control puts ambiguity into context as a concern about model misspecification. The ways her model is misspecified are unknown and unlikely to be learned in a reasonable timeframe, so she is unable to apply reasonable probability distributions over possible true models and stay in the usual subjective expected utility setting. For example, assuming the damage function is truly a polynomial, she may not know the exact degree of polynomial. To capture attitudes towards ambiguity stemming from model misspecification, robust control introduces a new set of preferences for the policymaker: multiplier preferences (Strzalecki, 2011). Multiplier preferences are a special case of variational preferences (Maccheroni et al., 2006) where a penalty parameter θ_1 determines the policymaker's attitude towards ambiguity and also gauges how robustly she designs policy to misspecifications in her model. Multiplier preferences are characterized by a relative entropy function which measures the "distance" between distributions.[2] The relative entropy function,

$$D_{t+1}(p_{t+1}||q_{t+1}) = \int log\left(\frac{dp_{t+1}}{dq_{t+1}}\right) dp_{t+1},$$

enters the policymaker's Bellman equation as a minimization problem:

$$V_t(\boldsymbol{S_t}) = \min_{q_{t+1}} \left\{ \max_{c_t,\alpha_t} \left[\int u(c_t)\, dq_{t+1} + \theta_1 D_{t+1}(p_{t+1}||q_{t+1}) + \beta V_{t+1}(\boldsymbol{S_{t+1}}) \right] \right\}$$

Where $u(c_t)$ is year t utility, p_{t+1} is her approximating model of the transition probabilities, and q_{t+1} is a model of transition probabilities selected by an "evil agent."[3] In the robust control context, the policymaker maximizes her expected discounted welfare stream subject to the distortion q_{t+1}

[1] In practice this terminal value function is obtained by solving a finite horizon model over 1000 years. The value function in the first year is a close approximation to a true infinite horizon value function.

[2] In variational preferences, θ_1 and the relative entropy function are consolidated into a general function called the ambiguity index.

[3] The optimal policy is independent of the order of the maximization and minimization.

Table 1: The parameterization of the dynamic stochastic DICE-2007 model. The carbon and temperature dynamics are calibrated to Cai et al. (2015). Similar to Lemoine and Traeger (2014), δ_κ varies over time to account for annual decisionmaking and to better match DICE results with the new annual timestep.

Parameter	Value	Description
A_0	0.027	Initial production technology
$g_{A,0}$	0.009	Initial annual growth rate of production technology
δ_A	0.001	Annual change in growth rate of production technology
L_0	6514	Year 2005 population (millions)
L_∞	8600	Asymptotic population (millions)
δ_L	0.035	Annual rate of approaching asymptotic population level
σ_0	0.13	Initial emissions intensity of output (Gigatons of carbon per unit output)
δ_σ	0.003	Annual change in growth rate of emissions intensity
a_0	1.17	Cost of backstop technology in 2005 ($1000 per ton of carbon)
a_1	2	Ratio of initial backstop technology cost to final backstop technology cost
a_2	2.8	Abatement cost function exponent
g_Ψ	-0.005	Annual growth rate of backstop technology cost
B_0	1.1	Initial annual non-industrial CO_2 emissions (Gigatons of carbon)
g_B	-0.01	Annual growth rate of non-industrial emissions
EF_0	-0.06	Year 2005 exogenous forcing W/m^2
EF_{100}	0.30	Year 2105 exogenous forcing W/m^2
κ	0.3	Capital elasticity in production
δ_κ	0.06	Annual depreciation rate of capital
M_{pre}	596.4	Pre-industrial atmospheric CO_2 (Gigatons of carbon)
ρ	0.015	Annual rate of pure time preference
η	2	Relative risk aversion
ϕ_{12}	0.019	Carbon transfer coefficient for atmosphere to upper ocean
ϕ_{23}	0.0054	Carbon transfer coefficient for upper ocean to lower ocean
ϕ_{21}	0.01	Carbon transfer coefficient for upper ocean to atmosphere
ϕ_{32}	0.00034	Carbon transfer coefficient for lower ocean to upper ocean
ξ_1	0.037	Coefficient translating forcing to temperature ($^\circ C/(W/m^2)$)
ξ_2	0.047	Coefficient governing outgoing long wave radiation to space
φ_{12}	0.01	Temperature transfer coefficient for atmosphere to ocean
φ_{21}	0.0048	Temperature transfer coefficient for ocean to atmosphere
f	3.8	Forcing from a doubling of CO_2 (W/m^2)
K_0	137	Initial absolute capital in 2005 (Trillions of USD)
M_0^{atm}	808.9	Year 2005 atmospheric CO_2 (Gigatons of carbon)
M_0^{up}	1255	Year 2005 biosphere and upper ocean CO_2 (Gigatons of carbon)
M_0^{lo}	18365	Year 2005 lower ocean CO_2 (Gigatons of carbon)
T_0^{atm}	0.7307	Year 2005 atmospheric temperature (Degrees Celsius)
T_0^{ocean}	0.0068	Year 2005 ocean temperature (Degrees Celsius)
μ_0	-5.95	Year 2005 prior location parameter (mean 0.0028388)
Σ_0	1.24×10^{-5}	Year 2005 prior shape parameter (variance $(0.013)^2$)

to her transition probability measure p_{t+1}. However, the size of the distortion is modulated by the size of the penalty parameter θ_1, which determines the size of the cloud of models around her approximating models she deems plausible. As shown in Hansen and Sargent (2007), the evil agents optimal selection can be represented by a risk sensitivity operator,

$$T^1(V_{t+1}(\boldsymbol{S_{t+1}})|\theta_1) = -\theta_1 log \left(E \left[exp \left(-\frac{V_{t+1}(\boldsymbol{S_{t+1}})}{\theta_1} \right) \right] \right),$$

The expectation is over the transition densities for the observable states. The risk sensitivity operator minimizes the expectation of the policymaker's value function by selecting a distortion to the transition probability, but subject to a penalty θ_1 for selecting distortions that are too large in terms of relative entropy. The role of θ_1 is to capture how big this distortion is, and how well the policymaker wants to guard against potential misspecifications. If $\theta_1 = \infty$ the operator reduces to the standard expected continuation value, but if $\theta_1 < \infty$, then the distribution underlying the future value is twisted by m^*_{t+1} which gives the worst possible distortion to the transition densities for the policymaker. The smaller θ_1 is, the larger the possible distortions she may face and the more concerned she is about having a robust decision rule.

C Smolyak's method

Suppose the true value function of dimension d is $v : [-1, 1]^d \to \mathbb{R}$.[4] A polynomial approximant to v,

$$\hat{v}(\boldsymbol{x}; \boldsymbol{\alpha}) = \sum_{n=1}^{N} \alpha_n \, f_n(\boldsymbol{x}),$$

can be constructed from a set of d-dimensional basis polynomials, $f_n(\boldsymbol{x}) : [-1, 1]^d \to \mathbb{R}, n = 1, ..., N$, and a vector of scalars $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_N)$. $\hat{v}$ is obtained through value function iteration. At a given year t, the value function approximant $\hat{v}_t(\boldsymbol{x}; \boldsymbol{\alpha})$ is maximized at each point of the grid, $\{\boldsymbol{x_1}, ..., \boldsymbol{x_N}\}$, where each $\boldsymbol{x_i} = (x_i^1, ..., x_i^d)$ is a point in d-dimensional space. The maximized values are used to update the approximant through the following relationship,

$$B \, \boldsymbol{\alpha} = v_t^g,$$

where B is an N x N matrix of the basis polynomials evaluated at the grid points, $\boldsymbol{\alpha}$ is the Nx1 vector of basis coefficients, and v_t^g is a Nx1 vector of the maximized values of $\hat{v}_t(\boldsymbol{x}; \boldsymbol{\alpha})$ at the grid points $\{\boldsymbol{x_1}, ..., \boldsymbol{x_N}\}$, after the maximization step in year t. Solving the system of linear equations

[4] Any value function outside of this domain can be normalized to be entirely within it.

Table 2: First sets of grid point locations.

i	Nested: E_i	Disjoint: E_i^d
1	0	0
2	$(0, -1, 1)$	$(-1, 1)$
3	$\left(0, -1, 1, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

for $\boldsymbol{\alpha}$ at each year t yields that year's value function approximant $\hat{v}_t(\boldsymbol{x}; \boldsymbol{\alpha})$.

In order to reduce the number of optimization problems that must be solved to obtain each year's value function, the Smolyak method constructs a sparse grid.[5] First, a unidimensional set of grid points is constructed on the extrema of the Chebyshev polynomials. Smolyak's method does not use Chebyshev zeros as in other collocation approaches. However, not all extrema are used. Consider sets of extrema that form a sequence $E_1, E_2, E_3, ..$ and satisfy the following two conditions:

1. There are $m(i) = 2^{i-1} + 1$ points in a set E_i for $i \geq 2$ and $m(1) = 1$
2. $E_i \subset E_{i+1}\ \forall i$

The rule to construct the unidimensional sets of grid points that satisfy these conditions are:

$$E_1 = 0$$
$$E_2 = \left(-cos\left(\frac{(1-1)\pi}{(m(2)-1)}\right), -cos\left(\frac{(2-1)\pi}{(m(2)-1)}\right)\right)$$
$$E_i = \left(-cos\left(\frac{(1-1)\pi}{(m(i)-1)}\right), -cos\left(\frac{(2-1)\pi}{(m(i)-1)}\right), ..., -cos\left(\frac{(i-1)\pi}{(m(i)-1)}\right)\right) \text{ for } i = 3, 4, ..$$

The second condition above implies that the sets E_i are nested, forcing many unnecessary repetitions in the computation, increasing the cost. To avoid double repetitions, consider a sequence of disjoint sets, $E_1^d, E_2^d, E_3^d, ..$ such that their union has the same elements of the E sequences: $E_1^d = E_1$ and $E_i^d = E_i \backslash E_{i-1}$. This avoids unnecessary repetition of grid points and calculations during the computation. Table 2 shows the first three sequences of grid points for the nested and disjoint sets.

Next, the unidimensional sets of grid points are used to create a higher dimensional grid through tensor products. In order to select the appropriate tensor products to construct the grid, the

[5] This description of Smolyak's method follows that of Judd et al. (2014).

Smolyak rule is applied:

$$d \leq \sum_{j=1}^{d} i_j \leq d + \mu.$$

d is the dimension of the state space, i_j are the indices for E_{i_j} sequences of points along each dimension, and μ is an integer that detremines the level of approximation where larger μ implies a higher quality approximation. The Smolyak rule says to form a d dimensional grid by constructing tensor products of unidimensional grids where the sum of sequence indices i_j, for each dimension j, are weakly greater than the dimension of the space d, but weakly less than the dimension of the space plus the approximation level μ. For larger μ, more tensor products satisfy Smolyak's rule, so more points are placed in the grid. For example, suppose that $d = 2$ and $\mu = 2$. Tensor products are formed from sequences E_i^d that satsify: $2 \leq \sum_{j=1}^{d} i_j \leq 4$. E_{i_j} pairs that satisfy this are: $\{E_1, E_1\}, \{E_1, E_2\}, \{E_1, E_3\}, \{E_2, E_1\}, \{E_2, E_2\}, \{E_3, E_1\}$. The tensor product of each pair creates points in d dimensional space that will construct the final 2-dimensional grid. The first several points are constructed as follows: $\{1, 1\}$ generates the 2 dimensional point $(0, 0)$, $\{1, 2\}$ generates $(0, 1)$ and $(0, -1)$. Continuing this process will construct the entire 13 point grid.

The final step is to construct the Smolyak interpolant, which follows almost identically to the process of constructing the grid. The basis polynomials for the interpolant are Chebyshev polynomials. Order the polynomials in a set Ω according to their their recurrence relation, $\Omega = \{T_0(\boldsymbol{x}), T_1(\boldsymbol{x}), T_2(\boldsymbol{x}), ...\}$ where $T_0(\boldsymbol{x}) = 1, T_1(\boldsymbol{x}) = x$, and $T_{n+1}(\boldsymbol{x}) = 2\,x\,T_n(\boldsymbol{x}) - T_{n-1}(\boldsymbol{x})$ for $n = 2, 3, ...$ Sets $P_1, P_2, P_3, ..$ are selected from Ω sequentially starting from the first element and must satisfy the following two conditions:

1. There are $m(i) = 2^{i-1} + 1$ points in a set P_i for $i \geq 2$ and $m(1) = 1$

2. $E_i \subset E_{i+1}\ \forall\, i$,

The nested nature of the sets creates many unnecessary repetitions of elements. To avoid this, we construct disjoint sets $P_i^d = P_i \backslash P_{i-1}$. The Smolyak rule, $d \leq \sum_{j=1}^{d} i_j \leq d + \mu$ is applied to the disjoint sets to construct the required tensor products of the sets of Chebyshev polynomials. The linear combination of the basis functions created by the tensor products will be the final Smolyak interpolant.[6] For example, consider the case where d=2 and $\mu = 1$. The Smolyak rule states to select tensor products of sets P_{i_j} where the sum of the indices across dimensions is weakly greater than 2 and weakly less than 3. Three combinations of sets satisfy this: $\{P_1, P_1\}, \{P_1, P_2\}, \{P_2, P_1\}$. Along the x dimension, the polynomial in set P_1 is 1, and the polynomials in set P_2 are x and $2x^2 - 1$. The polynomials for the y dimension are the same. The final basis functions that comprise

[6] For more examples of construction of the grid and interpolant see Judd et al. (2014).

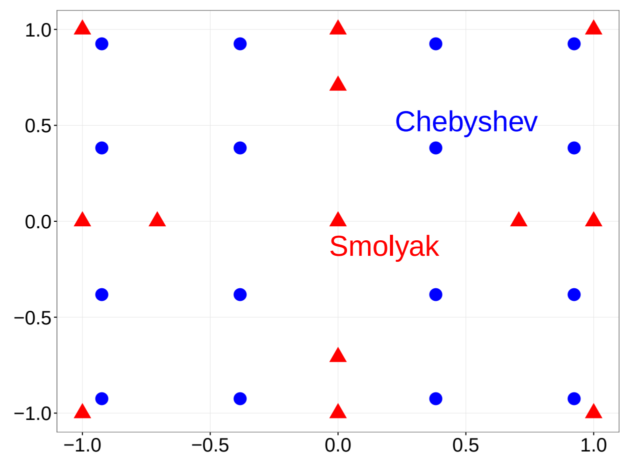

Figure 1: The 2-dimensional Smolyak grid at Chebyshev extrema with 13 grid points (red triangles), and a 2-dimensional conventional grid at Chebyshev zeros with 4^2 grid points (blue circles). Both grids are on $[-1,1] \times [-1,1]$.

the interpolant are: $1, x, y, 2x^2 - 1, 2y^2 - 1$. Figure 1 displays the construction of a conventional tensor product grid and a Smolyak grid on $[-1,1] \times [-1,1]$. The Smolyak grid, indicated by circles, consists of 13 points whereas the conventional, Chebyshev grid, indicated by x's, consists of 16. As the dimension of the space increases, the difference in the density of grid points between the two methods will increase, as will the relative efficiency of the Smolyak method.

C.1 Error Analysis

Smolyak's method affords us the ability to circumvent exponential growth in the number of grid points on a collocation grid. However to have trust in the results, an analysis of the error in the approximation is required. The convention in the literature is to analyze the Euler equation error, e.g. see Winschel and Kratzig (2010); Judd et al. (2014) for Euler equation error analysis for Smolyak grids. Along an equilibrium trajectory, the Euler equation must be precisely satisfied. Any deviation from the equilibrium trajectory, due to approximation error, incurs a welfare loss. With this knowledge we can reformulate the Euler equation to be in unitless per-consumption terms which allows us to quantify the error as dollars lost by following the approximated policy instead of the true policy. Santos (2000) demonstrates that the Euler equation error is the same order of magnitude as the error in the optimal control rates. However, the DICE model does not admit an Euler equation since the number states exceeds the number of controls. Instead of analyzing the Euler error, I opt for another equation that must be satisfied in equilibrium: the Bellman equation.

Table 3: Maximum and average log10 Bellman errors for the deterministic DICE model solved using Smolyak's method with $\mu = 3$ on the tightest and widest domains used in this paper.

	Tightest Domain		Widest Domain	
	2005-2105	2005	2005-2105	2005
$\mathcal{L}^\infty$	-4.58	-4.58	-4.15	-4.57
$\mathcal{L}^1$	-5.93	—	-4.64	—

The Bellman equation for the deterministic DICE model is,

$$V_t(\mathbf{S_t}) = \max_{c_t,\alpha_t} \{u(c_t) + \beta V_{t+1}(\mathbf{S_{t+1}})\}$$

We can rearrange the Bellman equation and divide by two additional terms, assuming they are non-zero, to arrive at,

$$\frac{1}{\partial u(c_t)/\partial c} \frac{1}{c_t} \left[V_t(\mathbf{S_t}) - \max_{c_t,\alpha_t} \{u(c_t) + \beta V_{t+1}(\mathbf{S_{t+1}})\} \right] = 0 \tag{1}$$

Equation 1 is formed by moving the right hand side of the Bellman to the left, and then dividing by the marginal utility of consumption, and also by consumption. The Bellman equation is originally in unit-free terms, dividing by marginal utility puts the Bellman into consumption terms, similar to how the social cost of carbon is converted to dollar terms through the marginal value of capital. The division by consumption again makes the equation unit-free, but in per-consumption terms which allows for an economic interpretation. If (1) is not precisely satisfied, the error can be interpreted as the loss in consumption using the approximated policy rule (Judd and Guu, 1997; Fernandez-Villaverde et al., 2016). An error of 10^{-3} implies a loss of \$1 for every \$1000 spent, while an error of 10^{-6} implies a loss of \$1 for every \$1,000,000 spent.

Table 3 displays the $\mathcal{L}^\infty$ (maximum) and $\mathcal{L}^1$ (average) Bellman equation errors in log10 units for a simulation of the deterministic DICE model solved on the widest domain used in this paper, and also a tighter domain that only allows for a solution because damages are no longer stochastic.[7] Both domains display excellent accuracy, with the worst case errors both incurring mistakes of less than \$1 per \$10,000 spent. For the tightest domain, this error actually occurs in the first year of the simulation, while accuracy improves as the policymaker moves forward in time. On average over a 100 year simulation, the tightest domain displays errors over an order of magnitude smaller than the widest domain, although the accuracy on the widest domain is still excellent.

[7] Technically the model is the uncertainty model with damage shocks of variance 10^{-10} and a variance of beliefs of 10^{-10}. This allows me to test the approximation quality on the 8 state grid. Small but non-zero variances are required to take expectations of damages.

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