

Watch Your Step: Optimal Policy in a Tipping Climate[†]

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We investigate the optimal policy response to the possibility of abrupt, irreversible shifts in system dynamics. The welfare cost of a tipping point emerges from the policymaker's response to altered system dynamics. Our policymaker also learns about a threshold's location by observing the system's response in each period. Simulations with a recursive, numerical climate-economy model show that tipping possibilities raise the optimal carbon tax more strongly over time. The resulting policy paths ultimately lower optimal peak warming by up to 0.5°C. Different types of posttipping shifts in dynamics generate qualitatively different optimal pretipping policy paths. (JEL D78, H23, Q54, Q58)

The possibility of irreversible climate tipping points has become one of the most prominent arguments in favor of stringent reductions in greenhouse gas emissions. Several lines of evidence suggest that components of the earth system can shift abruptly in response to the ongoing ramp-up of greenhouse gases (Alley et al. 2003; Overpeck and Cole 2006; Lenton et al. 2008). Integrated climate-economy models provide formal frameworks for analyzing the diverse intertemporal trade-offs at the heart of climate policy. For instance, the US government used these models to estimate the social cost of carbon for cost-benefit analyses of legislation. Yet by assuming that the climate system evolves smoothly, these models have been unable to connect tipping point concerns to policy. Indeed, the US government working group's report notes that the threat of thresholds is probably a crucial factor for policy, but existing models' limitations nonetheless excluded tipping points from the quantitative estimates (Greenstone, Kopits, and Wolverton 2011).

We address this policy need by integrating the possibility of climate tipping points into a benchmark integrated assessment model. For a fully integrated assessment of tipping points, we have to account for two important features novel to climate-economy modeling. These features ensure that we capture the full interaction between an abrupt shift in climate dynamics, the potential welfare loss, and the

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optimal policy response to both the possibility of a tipping point and the crossing of a tipping point. First, our tipping points directly shift the climate system.¹ Earlier work implemented tipping points as an exogenous loss of welfare or output. This exogenous loss is difficult to calibrate, making it hard to provide a good quantitative estimate of the effect of tipping points. In contrast, we endogenize the welfare loss from tipping into a new climate regime. This endogenous welfare loss reflects that today's optimal policy should anticipate the future policy response to an irreversible shift in the climate system. The welfare loss depends on the difference between welfare pre and posttipping, both of which depend on the current state of the economy and the climate. The optimal policy determines not only the probability of tipping but also the economic consequences of tipping, via the characteristics of the system at every given point in time. These characteristics include the stock of capital and also the stock of carbon dioxide that will drive future warming.

Our second novel feature is that the policymaker learns about the tipping point's location as the earth warms. While scientists are uncertain about the precise temperatures that trigger shifts in the climate system, these shifts do not occur randomly. Once we live through a period of warming, we learn whether its temperatures triggered a regime shift. If they did not, we can live in that world without incurring further damages from a shift in climate dynamics. Ignoring this informational value would overestimate the cost of a temperature increase and thereby bias optimal policies. Moreover, there is great scope for learning because climate tipping points are rife with uncertainty (Kriegler et al. 2009), as they are beyond both historical observations and the scope of large-scale climate models. Previous work with full integrated assessment models either made information exogenous or excluded learning altogether. To capture optimal learning in a full integrated assessment, we employ a recursive dynamic programming version of the DICE-2007 model, run with a one-year time step under an infinite planning horizon. Several papers have developed recursive implementations of DICE in order to analyze uncertain factors other than tipping points (Kelly and Kolstad 1999; Leach 2007; Crost and Traeger 2010; Jensen and Traeger 2011; Cai, Judd, and Lontzek 2012). Our recursive implementation extends the reduced-state version of Traeger (2012).

We derive optimal policies for two different types of tipping points from the scientific literature. The first tipping point increases the strength of temperature feedbacks in the climate system, and the second decreases the ability of the earth system to remove carbon dioxide (CO_2) from the atmosphere. These two tipping point possibilities generate qualitatively different policy paths because we explicitly model their geophysical effects and thereby endogenize the welfare loss. Specifically, as the CO_2 stock rises, posttipping policy has reduced scope to offset an increase in

¹ A different notion of tipping sometimes used in the economics literature refers to shifts between equilibria due to small changes in parameters. These changes can occur due to preferences in residential sorting models (Schelling 1971), network externalities in technology adoption models (Katz and Shapiro 1994), increasing returns in agglomeration models (Ellison and Fudenberg 2003), and human capital accumulation in growth models (Azariadis and Drazen 1990). The nonconvex control literature directly models the possibility of tipping a system into a new type of equilibrium (e.g., Skiba 1978; Brock and Starrett 2003; Mäler, Xepapadeas, and de Zeeuw 2003; Wagener 2003). In these deterministic models, it is always known whether a given policy path will or will not tip the system, so tipping along the optimal path depends entirely on the initial conditions. Skiba points divide the space of initial conditions into regions with and without optimal tipping.

temperature feedbacks and the welfare loss from crossing that threshold therefore increases. In contrast, the welfare loss from the carbon sink tipping point is relatively constant over time because the tipping point's effects on the earth system do not interact as strongly with the current CO₂ concentration. The importance of learning appears in how the tipping points affect the optimal carbon tax over time. In our base case specifications, tipping points raise the current optimal carbon tax by approximately 25 percent and raise the optimal prethreshold tax in 2050 by approximately 25–40 percent, depending on how the tipping point affects the climate system. With Bayesian learning, the chance of crossing a threshold in the next increment of warming rises as the world safely reaches higher temperatures. Optimal policy therefore does more to avert a marginal temperature increase when more warming has already occurred.

Other integrated models of climate thresholds represent a tipping point as a permanent shock to economic output or utility rather than a permanent change in the climate system. Gjerde, Grepperud, and Kverndokk (1999) simulate catastrophes that drop utility either to zero or to its level in 1990. They compare exogenous and endogenous hazards but do not consider a case with learning about the hazard. Keller, Bolker, and Bradford (2004) model a tipping point that alters ocean circulation. This tipping reduces production by an exogenous fraction. While their threshold is deterministic and known, its crossing is determined by atmospheric CO₂ and the uncertain sensitivity of the climate to CO₂. The model runs closest to our analysis assume three possible climate sensitivity states with time-constant probabilities and perfect learning in the year 2085 or the year 2185. Subsequent to our analysis, Lontzek, Cai, and Judd (2012) also model a collapse in ocean circulation using a permanent exogenous shock to production. The probability of their shock is a known function of temperature, with no scope for learning. By explicitly linking tipping points from the scientific literature to a standard representation of climate dynamics, we directly relate each physical change to the resulting welfare and policy impact.

A more stylized literature studies the implications of exogenous welfare losses, of exogenous tipping probabilities, or of endogenous tipping points without learning. The first strand of literature analyzes "climate catastrophes" with exogenous welfare consequences. It derives optimal emissions when the probability of a catastrophe increases with pollution (Clarke and Reed 1994; Tsur and Zemel 1996; Nævdal 2006; Nævdal and Oppenheimer 2007). The second strand of literature endogenizes welfare changes, but not the probability of the tipping point: the macroeconomic literature studies the potential for monetary policy to stabilize the economy when nominal interest rate rules might shift discontinuously in the future (Davig and Leeper 2007), and the real options literature studies optimal investment when demand dynamics might shift discontinuously in the future (Guo, Miao, and Morellec 2005). Because exogenously fixed transition probabilities control these regime shifts, the decision maker can change the welfare impact of tipping (self-insure) but not the likelihood of tipping (self-protect). We allow policy to adjust along both dimensions and compare the importance of each in the climate context. The third strand of literature analyzes endogenous shifts in the dynamics of natural resource systems (Heal 1984; Brock and Starrett 2003; Dasgupta and Mäler 2003; Mäler, Xepapadeas, and de Zeeuw 2003; Brozović and Schlenker 2011; Polasky, de Zeeuw, and Wagener

2011; de Zeeuw and Zemel 2012). Our model adds learning to these important contributions. The long time horizon of climate change and the slow evolution of the climate system together mean that future policymakers will have more information while potentially still making decisions that affect the probability of tipping. Our optimal policy recognizes the endogenous information gain available to future policymakers.

Sections I and II introduce the general model and analytically describe how anticipating tipping points affects optimal policy. Sections III and IV present our numerical integrated assessment model of climate change and show how possible tipping points alter the evolution of the optimal carbon tax and global temperature. Section V compares tipping points' policy importance to factors previously analyzed in the literature. It also discusses the implications of alternate ways of modeling tipping points. We conclude in Section VI. The Appendix provides the complete model description and additional results.

I. Modeling Tipping Points

Our tipping points are irreversible shifts in system dynamics that occur upon crossing a threshold in the state space. The policymaker does not know the precise location of the threshold. The probability of a tipping point occurring (i.e., the hazard rate) is endogenous. It depends on the evolution of the state variables, which in turn depends on policy choices as well as on the stochastics governing system dynamics. The policymaker learns that regions in the state space she has already visited are free of tipping points.² Crossing the threshold shifts the world from the "prethreshold" regime to a "postthreshold" regime with permanently altered system dynamics. Optimal pre and postthreshold policies together determine the welfare loss triggered by the tipping point.

The policymaker solves an infinite-horizon dynamic optimization problem. Optimal policy at time t depends on the vector \mathbf{S}_t of state variables. We denote the value of the optimal policy program by $V_\psi(\mathbf{S}_t)$.³ The parameter ψ indicates whether $V_\psi(\cdot)$ is the value function for the prethreshold regime ($\psi = 0$) or for the postthreshold regime ($\psi = 1$). In general, the threshold is an unknown function of the state variables. In the case of climate change, the threshold is the temperature level. Once the threshold is crossed, system dynamics change irreversibly. Returning state variables to earlier values does not restore the original dynamics. In our climate application, the new dynamics include melted ice sheets, large methane releases, or disrupted forest ecosystems; lowering temperature would not undo any of these changes over policy-relevant timescales. Similarly, macroeconomic changes can permanently alter expectations and institutions. Optimal policy in the prethreshold regime must consider its effect on both the pre and postthreshold value functions,

²The more of the relevant state space that is already explored without crossing the threshold, the more likely that the threshold is in the remaining unexplored state space. Some region of the relevant state space might not be explored even under policies optimized without considering tipping point possibilities. The probability mass on the permanently unexplored region can be interpreted as the chance that there is no tipping point.

³In our numerical application, each value function $V_\psi(\cdot)$ will be nonstationary. We absorb this nonstationarity into the state vector by making time t a component of \mathbf{S}_t .

but once the state variables cross the threshold, optimal policy depends only on postthreshold dynamics. Therefore, we solve the model recursively, starting with the postthreshold problem and then substituting the solution into the prethreshold problem.

In the postthreshold world, we obtain our value (and policy) functions from solving the following Bellman equation:

$$(1) \quad V_1(\mathbf{S}_t) = \max_{\mathbf{x}_t} \left\{ u(\mathbf{x}_t, \mathbf{S}_t) + \beta_t \int V_1(\mathbf{S}_{t+1}) d\mathbb{P} \right\}$$

$$\text{s.t.} \quad \mathbf{S}_{t+1} = g_1(\mathbf{x}_t, \epsilon_t, \mathbf{S}_t)$$

$$\mathbf{x}_t \in \Gamma(\mathbf{S}_t).$$

Here, \mathbf{x}_t is the vector of time t control variables, $u(\cdot)$ is the utility derived from time t consumption, and β_t is the discount factor.⁴ Constraints on the controls are captured by the set $\Gamma(\mathbf{S}_t)$. The transition function $g_1(\cdot)$ characterizes postthreshold dynamics. At time t , the next period's state vector is \mathbf{S}_{t+1} . It depends on the vector ϵ_t of independently and identically distributed stochastic shocks whose distribution is characterized by the probability measure \mathbb{P} . The decision maker maximizes the sum of immediate utility and discounted expected future welfare. The value function $V_1(\cdot)$ is defined as the fixed point of equation (1) and determines welfare after crossing the threshold. The welfare change from crossing the threshold depends on the state variables at the time of crossing and, thus, on the policy path chosen prior to crossing.

Prior to crossing a threshold, the value of the optimal policy program is given by the prethreshold value function:

$$(2) \quad V_0(\mathbf{S}_t) = \max_{\mathbf{x}_t} \left\{ u(\mathbf{x}_t, \mathbf{S}_t) + \beta_t \int \left[[1 - h(\mathbf{S}_t, \mathbf{S}_{t+1})] V_0(\mathbf{S}_{t+1}) \right. \right. \\ \left. \left. + h(\mathbf{S}_t, \mathbf{S}_{t+1}) V_1(\mathbf{S}_{t+1}) \right] d\mathbb{P} \right\}$$

$$\text{s.t.} \quad \mathbf{S}_{t+1} = g_0(\mathbf{x}_t, \epsilon_t, \mathbf{S}_t)$$

$$\mathbf{x}_t \in \Gamma(\mathbf{S}_t).$$

The prethreshold value function captures the possibility of crossing the threshold. The endogenous hazard $h(\mathbf{S}_t, \mathbf{S}_{t+1})$ determines the risk of crossing the threshold between the current and the next period. This probability generally depends on the current state variables and on how they change from one period to the next.⁵

⁴The dependence of the discount factor β_t on time can arise as a consequence of reformulating utility and normalizing consumption and state variables. In our climate change application, we use effective labor units for consumption and capital and transform the population-weighted utility function of per capita consumption into the form stated above (Traeger 2012).

⁵In general, the state space contains informational variables that tell the decision maker which part of the state space has already been explored. In our climate application, the decision maker keeps track of the greatest historic temperature.

With probability $1 - h$, the system dynamics stay unaltered and V_0 characterizes future welfare. With probability h , the system tips and V_1 determines future welfare from period $t + 1$ on. Because of the stochasticity in the equations of motion, we take expectations over the next period's value functions and over the hazard rate (via the integral). Once we have solved equation (1) for V_1 , we find V_0 as the fixed point of equation (2).

II. The Effects of Tipping Points on Optimal Policy

We now identify the channels by which tipping points affect optimal policy. The possible existence of a tipping point introduces two new terms into the marginal welfare impact of changing a control. For ease of exposition, we analyze the case where a single state variable determines the chance of crossing the threshold. The right-hand side of equation (2) characterizes welfare for an optimal choice of the controls (with optimality denoted by $*$). We evaluate the marginal welfare impact of varying a generic entry e_t of the control vector in the neighborhood of the optimum. In our climate change application, the temperature state variable determines the hazard, and the welfare impact of varying emissions determines the optimal carbon tax. Suppressing all arguments independent of e_t , the value of the optimal policy program is:

$$u(e_t^*) + \beta_t \int \left[[1 - h(\mathbf{S}_{t+1}(e_t^*))] V_0(\mathbf{S}_{t+1}(e_t^*)) + h(\mathbf{S}_{t+1}(e_t^*)) V_1(\mathbf{S}_{t+1}(e_t^*)) \right] d\mathbf{P}.$$

Varying e_t gives us the following trade-off characterizing optimal policies:

$$(3) \quad \frac{\partial u(e_t^*)}{\partial e_t} = -\beta \int \left\{ [1 - h(\mathbf{S}_{t+1}(e_t^*))] \frac{\partial V_0(\mathbf{S}_{t+1}(e_t^*))}{\partial \mathbf{S}_{t+1}} \frac{\partial \mathbf{S}_{t+1}(e_t^*)}{\partial e_t} \right. \\ \left. + h(\mathbf{S}_{t+1}(e_t^*)) \frac{\partial V_1(\mathbf{S}_{t+1}(e_t^*))}{\partial \mathbf{S}_{t+1}} \frac{\partial \mathbf{S}_{t+1}(e_t^*)}{\partial e_t} \right. \\ \left. - \underbrace{\frac{\partial h(\mathbf{S}_{t+1}(e_t^*))}{\partial \mathbf{S}_{t+1}}}_{(i)} \underbrace{\frac{\partial \mathbf{S}_{t+1}(e_t^*)}{\partial e_t}}_{(ii)} \underbrace{[V_0(\mathbf{S}_{t+1}(e_t^*)) - V_1(\mathbf{S}_{t+1}(e_t^*))]}_{(iii)} \right\} d\mathbf{P},$$

where primes ($'$) indicate derivatives.⁶ We interpret this equation for the case where an increase in e_t raises current utility but decreases expected future welfare. For instance, additional carbon dioxide emissions increase current utility (via higher consumption) but decrease future welfare by generating higher carbon stocks and temperatures.

⁶For a multidimensional state space, $\partial V/\partial \mathbf{S}_{t+1}$ and $\partial h/\partial \mathbf{S}_{t+1}$ denote gradients and $\partial \mathbf{S}_{t+1}/\partial e_t$ denotes the vector of state changes caused by the marginal change in e_t . These derivatives are taken with respect to the prethreshold dynamics g_0 .

The left-hand side of equation (3) characterizes the (immediate) benefits from increasing the policy variable. At the optimum, these benefits must balance the expected future costs. The costs are represented by the right-hand side of equation (3) and are subject to uncertainty (captured by the integration). The integrand in the first line represents the impact of policy on time $t + 1$ welfare under the prethreshold regime (i.e., on the prethreshold continuation value). This impact is composed of the control's impact on the state vector and the effect of the altered state vector on prethreshold value V_0 , and it is weighted by the probability of staying in the prethreshold regime $(1 - h)$. In a world without tipping points (where the hazard rate h is zero), the first line characterizes the full trade-off between current and future welfare.

The possibility of imminent tipping points introduces the second and third lines into the optimal policy trade-off: altering e_t now also changes time $t + 1$ welfare in the postthreshold world (second line) and changes the probability of entering the postthreshold world (third line). The first two lines together give the expected marginal welfare effect of increasing e_t in situations where the immediate hazard rate h is exogenous. These lines adjust a model without tipping points to account for the different marginal effect of the control e_t on pre and postthreshold welfare. We call the adjustment with respect to the case without potential tipping the *differential welfare impact (DWI)*:

$$(4) \quad DWI \equiv h \left[\frac{\partial V_0}{\partial \mathbf{S}_{t+1}} - \frac{\partial V_1}{\partial \mathbf{S}_{t+1}} \right] \frac{\partial \mathbf{S}_{t+1}(e_t^*)}{\partial e_t}.$$

The DWI is proportional to the hazard rate and to the difference in the marginal impact of the control on the pre and postthreshold value functions. If increasing the control decreases welfare relatively more in the postthreshold regime, then the differential welfare impact makes raising the control more costly.

The third line in equation (3) only arises when the tipping point's probability is endogenous. In this case, a change in the control e_t affects the hazard rate. The optimal policy now has to account for this marginal change in the hazard rate in response to a change in the control. We call this contribution the *marginal hazard effect (MHE)*. The MHE is composed of the response of the hazard rate to a change in the state vector (term i), the response of the state vector to a change in the control (term ii), and the total welfare change from switching regimes (term iii). For the tipping points in our climate application, increasing emissions raises the hazard rate and the welfare difference $[V_0 - V_1]$ is always positive. Therefore, the MHE increases the cost of emissions relative to a case without tipping points. In general, a change in the current control could also increase hazard rates at future times. Possible future tipping points are included in the prethreshold continuation value V_0 . If the current control alters the probability of crossing a threshold in the future, then $\partial V_0 / \partial \mathbf{S}_{t+1}$ will include this effect.

In summary, anticipating possible tipping points adjusts the first-order conditions governing optimal policy for the differential impact of the control on pre and postthreshold welfare (DWI) and for the control's effect on the immediate hazard rate (MHE). In our climate change application, the effects of DWI and MHE together increase the optimal carbon tax.

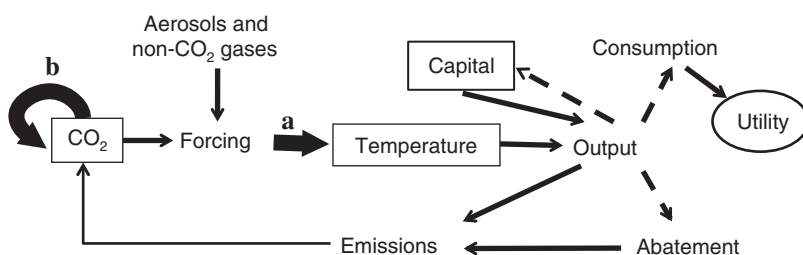


FIGURE 1. A SIMPLIFIED SCHEMATIC OF THE RELATION
BETWEEN THE ECONOMY AND THE CLIMATE

Notes: Boxes indicate stock variables, and dashed arrows indicate the decision variables of consumption, investment, and abatement. The climate feedback tipping point makes temperature more sensitive to forcing (a), and the carbon sink tipping point increases the persistence of CO_2 (b).

III. A Climate-Economy Model with Tipping Points and Learning

We now consider the effect of climate tipping points on the optimal carbon tax. The optimal carbon tax equals the social cost of carbon when evaluated along the optimal policy path in a welfare-maximizing integrated assessment model.

We reformulate the benchmark Dynamic Integrated model of Climate and the Economy (DICE) from Nordhaus (2008) as an infinite-horizon dynamic programming problem with a tipping point in the climate system and optimal learning about the threshold that triggers a tipping point.⁷ DICE is a Ramsey-Cass-Koopmans growth model that has an aggregate world economy interacting with a climate module (Figure 1). Gross economic output (or potential gross world product) is determined by an endogenous capital stock, an exogenously growing labor force, and exogenously improving production technology. Gross output produces carbon dioxide (CO_2) emissions. Nonabated CO_2 emissions accumulate in the atmosphere and ultimately translate into global warming, which causes damage proportional to world output. Cumulative temperature change reduces the total output available for allocation by the policymaker. The control variables are abatement and consumption, and residual output not allocated to these two options becomes capital investment. The state variables are capital per effective unit of labor, the stock of CO_2 in the atmosphere, the change in global mean surface temperature since 1900, and, to keep track of exogenously evolving variables, time. We integrate the tipping points into the infinite-horizon stochastic dynamic programming version of DICE by Traeger (2012). It runs at an annual (rather than decadal) time step, and it reduces the number of state variables by approximating the delay equation governing ocean cooling and the carbon cycle.⁸

⁷The Appendix provides the model equations. The standard DICE model is a nonlinear programming problem with constant system dynamics.

⁸For numeric efficiency, the model is normalized to effective labor units. Apart from tipping points, we also add annual stochastic weather shocks to the model, and we use a more elaborate approximation of the carbon cycle (see Appendix).

A tipping point irreversibly changes the climate system from its conventional representation in DICE to a new regime with altered dynamics. The tipping point occurs upon crossing some unknown temperature threshold. Emission decisions determine the future CO₂ stock, thereby affecting future temperatures and the probability that a tipping point occurs. The decision maker anticipates how he would choose emissions and consumption in the postthreshold world. The timing, probability, and welfare consequences of a regime switch are endogenous because they depend on the policies chosen before and after the threshold occurs.

To model tipping points, we specialize the recursive structure from Section I to DICE. We have one dynamic programming problem for the postthreshold world and another for the prethreshold world. The prethreshold world has standard DICE dynamics along with the tipping possibility and weather shocks calibrated to the historical record. A tipping point produces the postthreshold world by irreversibly changing the standard dynamics. We first solve the postthreshold problem and then use its solution in the prethreshold problem. We numerically solve each dynamic programming problem for the unknown value function using function iteration. Employing a projection method, we approximate the value functions by Chebychev polynomials and use collocation at the Chebychev nodes in the four-dimensional state space (Miranda and Fackler 2002).

We evaluate two tipping points of prominent concern in the climate science literature.⁹ In every model run, the policymaker faces a single tipping point and knows in advance what its effects would be. The first tipping point increases the climate feedbacks that amplify global warming (arrow a in Figure 1), and the second increases the atmospheric lifetime of CO₂ (arrow b in Figure 1). The first tipping point therefore increases the effect of emissions on temperature, and the second increases the time during which emissions affect the climate. The climate science literature has compiled a number of pathways by which tipping points could abruptly change the strength of feedbacks that determine surface temperature. As one example, warming could mobilize large methane stores locked in permafrost and in ice lattices (clathrates) in the shallow ocean (Hall and Behl 2006; Archer 2007; Schaefer et al. 2011). If warming mobilizes these methane stocks, they would cause further warming that could mobilize additional stocks. As another example, if land ice sheets begin to retreat on decadal timescales, the resulting loss of reflective ice could double the long-term warming predicted by models that hold land ice sheets fixed (Hansen et al. 2008). Temperature dynamics in DICE depend on a parameter known as climate sensitivity, which is the equilibrium warming from doubling CO₂. The value of 3°C used in DICE is inferred from climate models that hold land ice sheets and most methane stocks constant. We represent a climate feedback tipping point as increasing climate sensitivity to 4°C, 5°C, or 6°C.

The second tipping point reflects the possibility that carbon sinks weaken beyond the predictions of coupled climate-carbon cycle models. Warming-induced changes

⁹Each modeled tipping point is an extreme case: climate dynamics change severely, abruptly, and irreversibly. The scientific literature does not offer clear guidance on the best way to model a given tipping point, so we translate two common tipping stories into DICE's reduced climate system in order to gain intuition about policy implications.

in oceans (Le Quéré et al. 2007), soil carbon dynamics (Eglin et al. 2010), and standing biomass (Huntingford et al. 2008) could affect the uptake of CO₂ from the atmosphere. We represent these weakened sinks by decreasing the transfer of CO₂ out of the atmosphere by 25 percent, 50 percent, or 75 percent. The reader may think of this tipping point as reducing the long-term “decay rate” of atmospheric CO₂. If the threshold triggers a strong form of this tipping point, then the flow of carbon from land and ocean sinks back into the atmosphere can temporarily outweigh the flow of carbon out of the atmosphere. These flows can result in a temporarily negative decay rate as the earth system comes to a new equilibrium with more CO₂ in the atmosphere.

The system passes from the prethreshold regime ($\psi_t = 0$) into the postthreshold regime ($\psi_{t+1} = 1$) when cumulative temperature change T_{t+1} crosses an unknown threshold \tilde{T} . Climate science has identified potential tipping mechanisms, but scientists cannot yet clearly define their triggers or probabilities. In order to implement tipping points in a formal model, we make three assumptions. First, we assume that the trigger is a function of global mean surface temperature. Surface temperature is the most natural index in DICE for representing a threshold. Second, we assume that a given temperature either immediately triggers the tipping point or does not trigger it at all. This assumption seems reasonable in the absence of better scientific information.¹⁰ Third, we assume that every possible threshold temperature has a priori an equal chance of being the threshold. The closest the scientific literature comes to a probability distribution for thresholds are color-coded diagrams indicating the risks posed by different temperatures (Smith et al. 2009) and sets of probability intervals elicited from experts (Kriegler et al. 2009). Absent more guidance from the scientific literature, we decided to use a uniform prior distribution. This distribution recognizes that more warming imposes more threshold risk, and learning implies that a given increment of warming carries greater threshold risk when the world is already warmer.

The uniform distribution for \tilde{T} means that every temperature between the maximum temperature previously reached and an upper bound \bar{T} has an equal chance of being the threshold.¹¹ In our base case model runs, we use $\bar{T} = 4.27^\circ\text{C}$ so that the year 2005 expected value for the threshold is 2.5°C .¹² Sensitivity analyses vary \bar{T} between 3°C and 9°C , implying year 2005 expected values of about 1.9°C to 4.9°C .

¹⁰ Alternative scenarios could make the threshold a more complicated function of the temperature path, as, for instance, an average of recent years' temperatures. The computational model would then be significantly more complicated because of the additional lagged temperature states in the value function. The trigger mechanism and consequences, however, would be qualitatively similar.

¹¹ The optimal policy path in the absence of tipping points reaches a maximum temperature of 3.33°C . Our model with $\bar{T} > 3.33$ is therefore equivalent to one with the uniform distribution's upper bound at 3.33°C and probability $(\bar{T} - 3.33)/(\bar{T} - T_t)$ that there is no threshold.

¹² Using $E_{2005} \tilde{T} = 2.5^\circ\text{C}$ is consistent with the political 2°C limits for avoiding dangerous anthropogenic interference. In the most recent version of the “burning embers” diagram (Smith et al. 2009), the yellow (medium-risk) shading for the “risk of large-scale discontinuities” begins around 1.6°C of warming relative to 1900, while the red (high-risk) shading begins around 3.1°C of warming relative to 1900.

The probability of crossing the threshold between periods t and $t + 1$ conditional on not having crossed the threshold by time t is¹³

$$(5) \quad h(T_t, T_{t+1}) = \max \left\{ 0, \frac{\min \{T_{t+1}, \bar{T}\} - T_t}{\bar{T} - T_t} \right\}.$$

This expression is the hazard of crossing the tipping point. As the world reaches higher temperatures without reaching a threshold, the decision maker learns that the threshold is above the current temperature and updates his beliefs by moving probability density from the newly safe region to the remaining unexplored temperatures. Therefore, as the world safely reaches higher temperatures, each unit of temperature increase creates a greater hazard than it did at lower temperatures. The state variable T_t that controls the threshold crossing is a climate variable whose equation of motion is determined by CO₂ concentrations and does not reflect annual stochastic fluctuations.

IV. The Optimal Carbon Tax when Facing Tipping Points

We compare several sets of model runs to assess how the optimal carbon tax responds to the type of tipping point considered, to the strength of a tipping point, and to prior beliefs about the temperature threshold's location. The optimal carbon tax is the policy trajectory that maximizes the present value of net benefits within our extension of the benchmark DICE integrated assessment model. All of our graphs present results conditional on not having crossed the threshold: we want to understand how optimal policy changes in the face of a potential tipping point. Each graph compares the baseline scenario without tipping point awareness to runs with tipping points of various strengths. The Appendix contains additional results.

Figure 2 gives the effect of tipping points on the optimal carbon tax (social cost of carbon), the optimal CO₂ concentration path, and the optimal temperature path. The figure assumes the base case prior over the threshold location. The year 2015 optimal carbon tax is near \$10/tCO₂ in the absence of tipping points, the strongest version of the feedback tipping point increases it to \$13.5/tCO₂, and the strongest version of the carbon sink tipping point increases it to \$14/tCO₂. While tipping point possibilities have only a modest effect on near-term abatement, they can nonetheless have a large effect on cumulative abatement because they increase the optimal tax later in the century.¹⁴ The optimal path without possible tipping points produces a peak temperature (CO₂ concentration) of 3.3°C (637 ppm), reached in the year 2187

¹³ Along DICE-2007's optimized paths, the CO₂ stock increases monotonically until the model reaches a sufficiently high level of abatement. From this point on, the "decay" of CO₂ outweighs the flow of emissions, making the CO₂ stock decrease monotonically. Temperature in DICE follows the same pattern. When temperature is increasing, the probability of crossing the threshold is proportional to the difference between the next period's temperature and the current temperature. When temperature is decreasing, the probability of crossing the threshold is 0. As long as temperature is a quasiconcave function of time, we do not need an additional state variable to keep track of the highest historic temperature.

¹⁴ The climate feedback tipping points have their greatest proportional effect on the optimal carbon tax shortly after 2100, while the proportional effect of the carbon sink tipping points peaks shortly after 2050. The Appendix plots how abatement and other variables respond to each tipping point possibility.

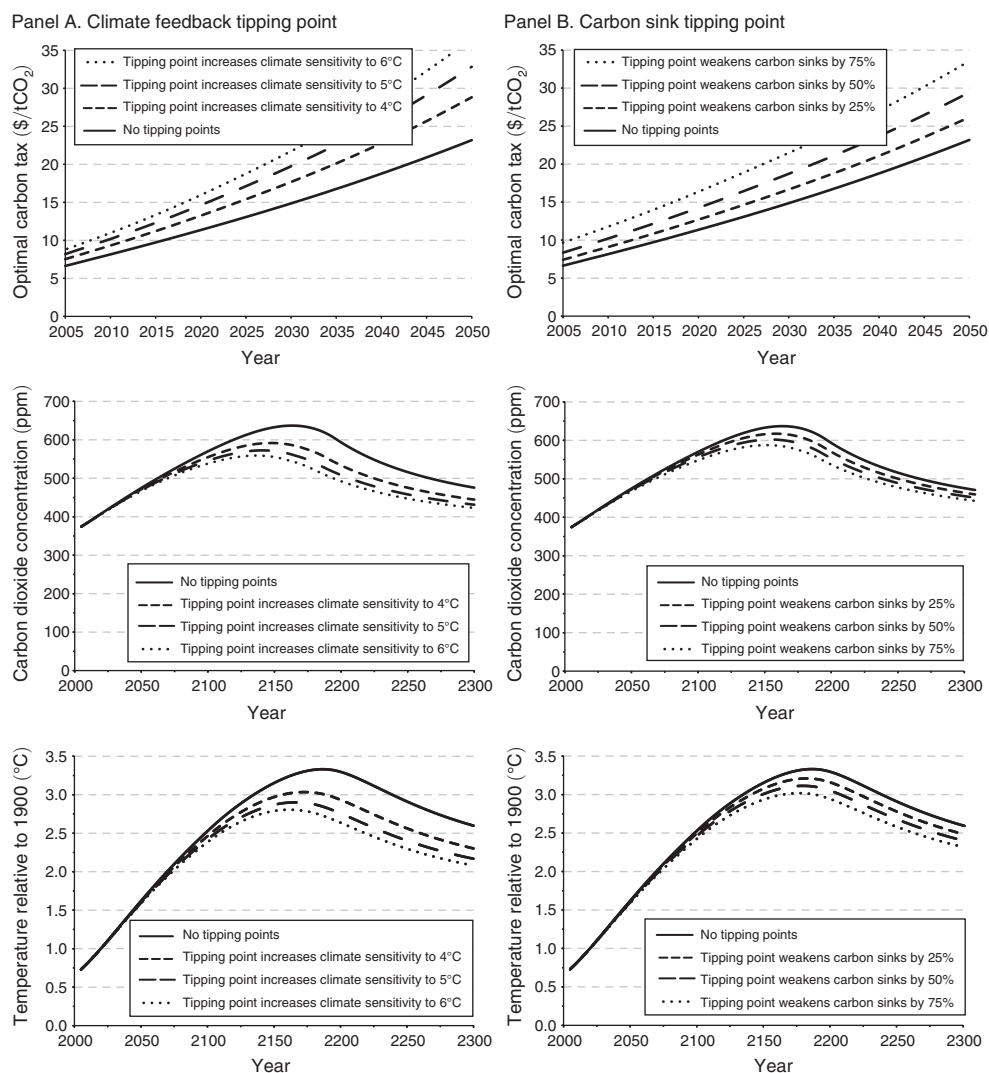


FIGURE 2

Notes: Time paths for the optimal carbon tax (current value), the CO₂ stock, and temperature under each type of tipping point possibility using expected draws. We simulate a path that happens to never cross a threshold in order to see how the modeled policymaker adjusts to the possibility over time. Results are for $\bar{T} = 4.27^{\circ}\text{C}$.

(2163). The optimal tax path in the presence of the weak climate feedback tipping point reduces this peak temperature to 3.0°C (592 ppm), while the higher taxes justified by the strong climate feedback tipping point further reduce peak temperature to 2.8°C (560 ppm). The tax path in the presence of the weak carbon sink tipping point reduces peak temperature only to 3.2°C (617 ppm), while the possibility of the strong carbon sink tipping point reduces peak temperature to 3.0°C (588 ppm). By reducing peak temperature and CO₂, the decision maker reduces the cumulative probability of crossing the temperature threshold. The stronger the anticipated tipping point, the more output the decision maker devotes to reducing this probability.

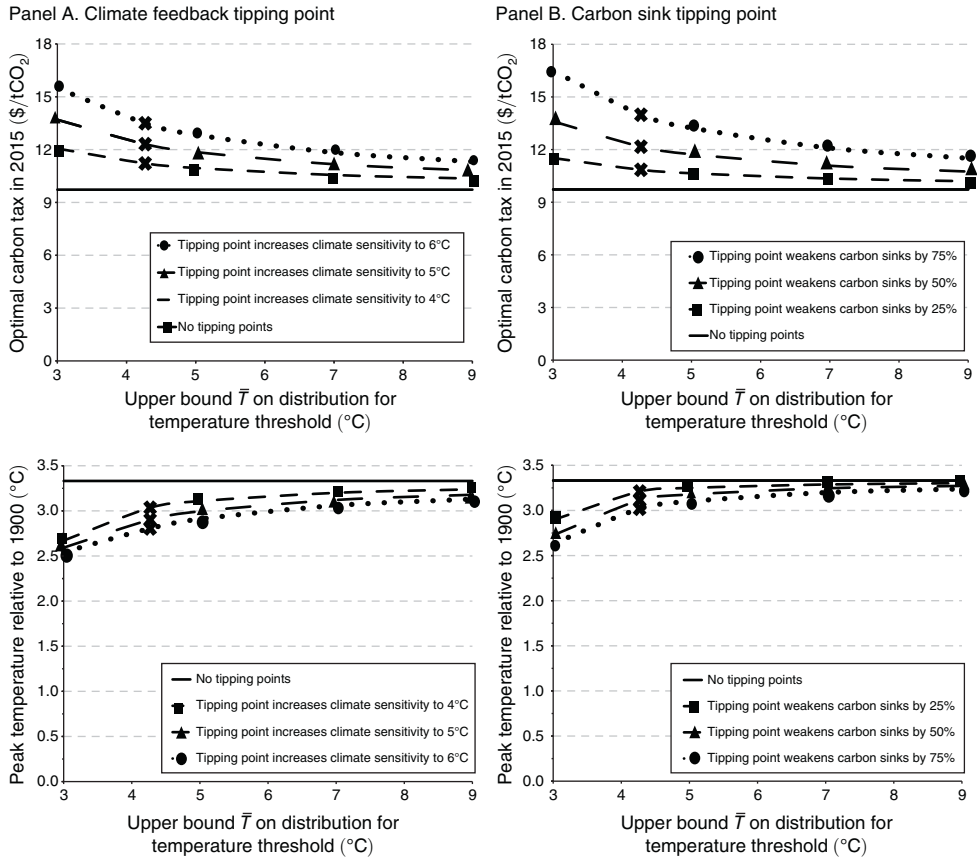


FIGURE 3

Notes: The optimal carbon tax in 2015 and the peak temperature reached for each upper bound \bar{T} for the temperature threshold's distribution. The plotted simulations assume expected draws of the weather shock and assume that the tipping point never occurs. Points marked by \times use the base case distribution.

Recognizing the present inability of climate science to provide a probability distribution for the temperature threshold, we consider the implications of more and less diffuse priors for the threshold location. Figure 3 plots the year 2015 optimal carbon tax and the peak temperature for values of \bar{T} between 3°C and 9°C, with all calculations still being for optimal policy paths conditional on not having crossed the threshold. As the upper bound \bar{T} increases, optimal policy converges asymptotically to the scenario without a tipping point. A more diffuse prior on the threshold location reduces the importance of the tipping point contributions in equation (3) by reducing both the hazard rate and its derivative. Lowering \bar{T} from its base case value has a stronger effect on optimal policy than does raising \bar{T} . For low values of \bar{T} , the hazard rate is a steeper function of emissions (raising the marginal hazard effect) and realized temperatures can approach regions with a high hazard rate (raising the differential welfare impact). When $\bar{T} = 3^\circ\text{C}$, the optimal carbon tax in 2015 rises as high as \$16.5/tCO₂ for the strong tipping points, with temperature peaking just above 2.5°C.

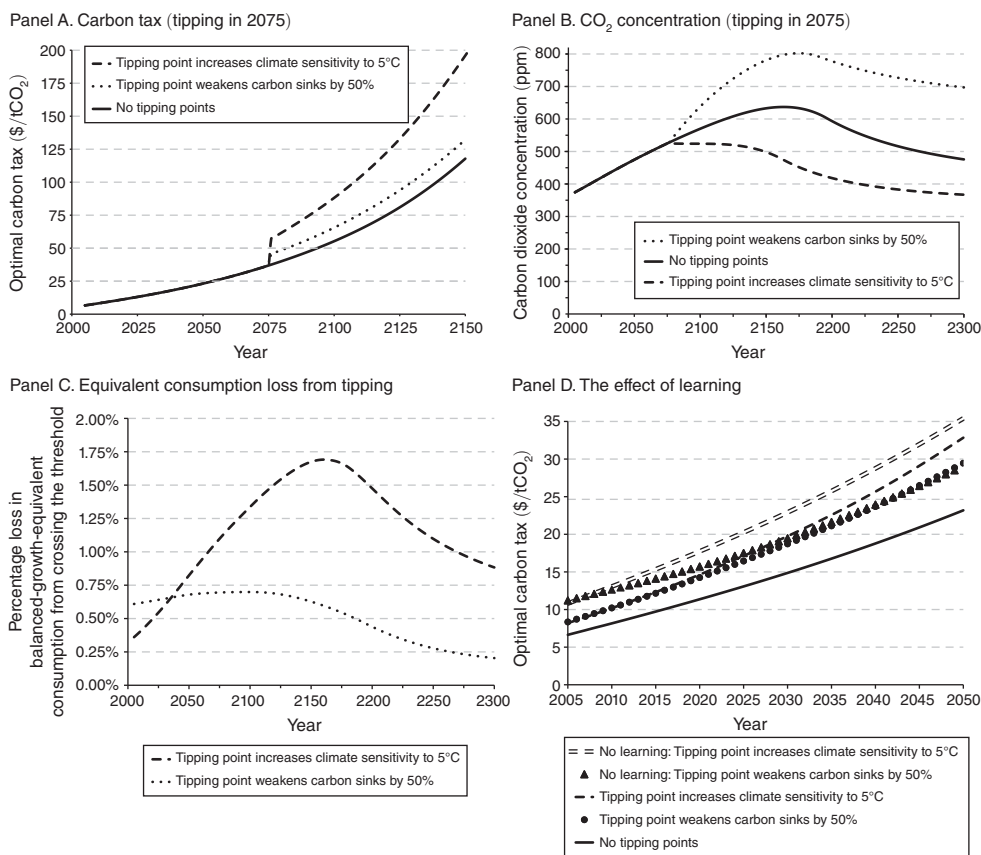


FIGURE 4

Notes: Some qualitative differences between tipping points, and also the effect of learning. Panels A and B show how the optimal policies react to a threshold crossing in the year 2075 when the policymaker was previously unaware of tipping point possibilities. Panel C depicts the (balanced growth equivalent) permanent consumption loss from tipping in a given year, using the optimal pretipping policy path. Panel D compares the optimal policies under Bayesian learning to the cases in which the initial distribution is never updated.

By explicitly modeling the effects of tipping points on system dynamics and the ability to adapt policy to the new dynamics, we learn how different types of tipping points can have qualitatively different effects on the economy, on welfare, and on optimal policy. Figure 4, panel A compares how optimal policy reacts to triggering each type of tipping point. The presented scenario assumes that the policymaker is unaware of tipping possibilities until they occur in 2075.¹⁵ Upon discovering that a tipping point has increased climate sensitivity, the decision maker substantially increases carbon mitigation. However, upon discovering that a tipping point has weakened carbon sinks, the decision maker increases mitigation only by a small amount.

¹⁵ We assume that the policymaker is unaware of the tipping point to ensure that policy up to the year 2075 and the state variables in the year 2075 coincide for both tipping points. The represented differences between policy paths are therefore due to the different shifts in climate dynamics and not to different states of the economic or climate systems at the point of tipping.

Figure 4, panel B shows the corresponding paths of CO_2 . The increased mitigation after crossing the climate feedback tipping point sharply lowers CO_2 , though temperature and damages are still higher than if the tipping point had not occurred (see Figure A3 in Appendix A). In contrast, weakening carbon sinks drives CO_2 up sharply despite greater mitigation than in the case without tipping points. Mitigation increases only moderately because CO_2 stocks reach high levels regardless of the postthreshold policy, and additional units of CO_2 cause less warming once CO_2 stocks are already high.¹⁶

The optimal response to crossing each threshold endogenously determines the cost of crossing each threshold (Figure 4, panel C). We present this cost as the permanent (balanced growth equivalent) percentage loss of consumption.¹⁷ For the carbon sink tipping point, the permanent consumption loss is between 0.6 percent and 0.7 percent for the next 150 years. In contrast, the climate sensitivity tipping point induces a permanent consumption loss that increases from 0.35 percent to 1.7 percent over the next 150 years. If tipping happens early in time, at a low carbon stock, the consumption loss from the feedback tipping point is smaller because the policymaker can avoid some of the potential climate damages by sharply reducing emissions upon tipping. There is no similarly effective policy response for the carbon sink tipping point. If tipping happens later, at an already high level of atmospheric carbon, the consumption loss from the feedback tipping point is larger because the policymaker can no longer avoid its higher damages even by sharply increasing abatement. Therefore, we find that the welfare loss from crossing the climate feedback tipping point is initially smaller than the welfare loss from crossing the carbon sink tipping point, but it rises quickly to become over twice as great by 2150. These different tipping dynamics explain the results in Figure 2: as compared to the carbon sink tipping points, the climate feedback tipping points affect the optimal carbon tax relatively less in the near-term but relatively more later in the century. The contrasting postthreshold dynamics drive differences in policy responses and welfare losses, and these welfare losses explain the increasing effort spent avoiding the climate feedback tipping point.

Finally, Figure 4, panel D analyzes the consequences of learning for optimal policies. Our policymaker is a Bayesian learner who recognizes that temperature regimes that historically did not imply a threshold crossing will also not imply such a threshold crossing in the future. In particular, he realizes that if he did not cross the threshold when the world was warming, then he also will not cross the threshold as the world cools back down. Tipping points have twice as great an effect on the present optimal carbon tax when the policymaker does not anticipate learning about the location of the temperature threshold.¹⁸ Ignoring learning disregards the

¹⁶ The radiative forcing crucial to warming is logarithmic in CO_2 (see Appendix).

¹⁷ For each period, we derive two consumption paths growing at some constant rate g : the first path gives rise to the same welfare as the system's value function before tipping, and the second path gives rise to the same welfare as the system's value function after tipping. We then plot the (time-constant) percentage difference between these two (balanced growth equivalent) consumption paths.

¹⁸ In the initial period both policymakers have the same distribution over the tipping point location. Hence, the nonlearning policymaker is sure that there is no threshold below the year 2005 temperature. For this nonlearner, the marginal probability of tipping from an additional unit of warming then stays constant. However, the absolute tipping probabilities increase significantly over time (see discussion in next paragraph). This scenario is the correct

TABLE 1—THE CARBON TAX (*social cost of carbon*), LOSS FROM CROSSING A THRESHOLD, AND CHANCE OF CROSSING A THRESHOLD ALONG THE OPTIMAL POLICY PATHS

	With learning		Without learning		With learning	
	2015	2050	2015	2050	Peak	
<i>Panel A. Unaware of tipping possibility</i>						
Optimal carbon tax (\$/tCO ₂)			10	23	11.4	Emissions (Gt C y ⁻¹)
Tipping probability: i) that year (%)	0.57	0.76	5.5	26	637	CO ₂ concentration (ppm)
ii) from additional 0.01°C (%)	0.30	0.38	0.28	0.28	3.3	Warming since 1900 (°C)
<i>Panel B. Tipping raises climate sensitivity to 5°C</i>						
Optimal carbon tax (\$/tCO ₂)	12	33	15	35	10.1	Emissions (Gt C y ⁻¹)
Tipping probability: i) that year (%)	0.56	0.72	5.4	25	573	CO ₂ concentration (ppm)
ii) from additional 0.01°C (%)	0.30	0.37	0.28	0.28	2.9	Warming since 1900 (°C)
Bge consumption loss if tipping (%)	0.45	0.82	0.054	0.057	1.7	Bge loss (%; as at left)
<i>Panel C. Tipping weakens carbon sinks by 50%</i>						
Optimal carbon tax (\$/tCO ₂)	12	29	14	29	10.5	Emissions (Gt C y ⁻¹)
Tipping probability: i) that year (%)	0.56	0.73	5.4	25	602	CO ₂ concentration (ppm)
ii) from additional 0.01°C (%)	0.30	0.37	0.28	0.28	3.1	Warming since 1900 (°C)
Bge consumption loss if tipping (%)	0.62	0.68	0.16	0.073	0.70	Bge loss (%; as at left)

Notes: For the policymaker who is unaware of a tipping possibility, we calculate the hazard that would be faced along that same policy path in a model with tipping points. All calculations assume an initial expected threshold of 2.53°C. Bge stands for balanced growth equivalent.

value of the information obtained when the climate system does not tip. However, mitigation effort increases faster when there is learning. Eventually, the optimal carbon tax in the case with learning surpasses the optimal carbon tax in the case without learning. At a given information set, optimal mitigation is always higher if we neglect learning. As time passes, the optimal policy paths produce increasingly different information sets. The information set without learning always remains the same, while the information set with learning recognizes that the threshold is above the current temperature. Therefore, the Bayesian policymaker sees further temperature increases as more strongly increasing the probability of crossing the threshold, which increases the marginal hazard effect in the optimal carbon tax. Compared to the nonlearner, the Bayesian policymaker grows more cautious as he reaches higher temperatures.¹⁹

Table 1 summarizes the differences between our core scenarios and illustrates the relation between learning, tipping hazard, and balanced growth equivalent consumption loss. Anticipating the mid-strength climate sensitivity tipping point decreases peak emissions, CO₂, and warming by 10–13 percent, whereas anticipating the mid-strength carbon sink tipping point decreases these peak values by 5–7 percent. Both tipping points increase the optimal present-day carbon tax by approximately 25 percent. Removing learning would nearly double this increase. As the world continues warming, the Bayesian decision maker becomes

counterfactual to illustrate the effect of learning in our setting. Existing models that neglect learning usually directly define the annual hazard as an exogenous function of temperature, without explicitly mapping it to a distribution over thresholds. This formulation is useful for generating lower cumulative probabilities of tipping, but it does not generate a well-defined counterfactual for our setting.

¹⁹ Keller, Bolker, and Bradford (2004) find that anticipated one-time learning reduces optimal mitigation. Our analysis adds the dynamic effects.

increasingly concerned about the additional tipping risk from a further temperature increase, while the nonlearning decision maker does not change this judgment over time. As atmospheric temperature approaches its peak, further emissions primarily raise temperatures in a cooling world. Then the relation reverses, with the Bayesian decision maker becoming the one who is less worried about additional emissions tipping the climate system. Finally, the marginal hazard effect also depends on the welfare loss from crossing the threshold. Because the nonlearner judges the absolute tipping probability in future periods to be much higher, he also judges the probability of eventual tipping to be much higher and so judges the welfare loss from tipping now (rather than potentially tipping later) to be much smaller. We see this effect in Table 1, where the nonlearner's balanced growth equivalent consumption loss from crossing the threshold in a given period is significantly smaller than the Bayesian policymaker's loss.

V. Discussion

We have analytically decomposed how tipping points affect policy, and our numerical model demonstrated the importance of explicitly modeling how tipping points affect climate dynamics and how the policymaker can learn about them over time. We now compare the numerical importance of tipping points for the social cost of carbon to other factors analyzed in the literature. We then discuss our assumptions about the tipping point's trigger and about the distribution for the temperature threshold.

A. Comparing Tipping Points to Other Sources of Uncertainty

Anticipating tipping points increases the optimal carbon tax. We briefly discuss how the magnitude of the change compares to other extensions of the benchmark DICE model. The largest change in the optimal carbon tax follows from a change in the pure rate of time preference. In particular, Stern (2007) argues on ethical grounds for a pure rate of time preference of 0.1 percent, as opposed to the value of 1.5 percent used in DICE-2007 (and in the present study). Nordhaus (2007) shows that such a *ceteris paribus* reduction of impatience increases the optimal present-day carbon tax tenfold. However, Nordhaus also argues that an efficient climate policy has to rely on a model calibrating the consumption discount rate to the asset market. Crost and Traeger (2010) argue that a better calibration of preferences in DICE uses Epstein-Zin-Weil preferences that resolve the equity-premium and risk-free rate puzzles by disentangling risk aversion from intertemporal substitutability. This calibration increases the intertemporal elasticity of substitution to approximately 1.5 and doubles the optimal carbon tax. Sterner and Persson (2008) build limited substitutability between environmental and consumed goods into the DICE model and find an effect that grows to a similar magnitude during the second half of the century. All of these changes operate through a reduction of the risk-free consumption discount rate and have a significantly larger impact on the optimal carbon tax than does our incorporation of tipping points. In purely theoretic work, Weitzman (2009, 2010) shows that uncertainty over the damage function can potentially change the optimal carbon tax by a similar magnitude.

Nordhaus (2008) finds in a Monte Carlo simulation that the social cost of carbon under business-as-usual is hardly affected when drawing a set of eight crucial parameters from a distribution and averaging results. Similar Monte Carlo analyses have been carried out for the other two models used by the US government working group when estimating the social cost of carbon (Greenstone, Kopits, and Wolverton 2011). The PAGE model averages by default over different parameter draws, without generating major risk premia. Drawing from a large set of uncertain parameters and using more detail in the damage formulation, the FUND model's Monte Carlo simulation increases the business-as-usual social cost of carbon by \$3.50–\$16.00/tCO₂ (Anthoff, Tol, and Yohe 2009). The same paper shows that introducing equity weights into a regional analysis increases the social cost of carbon by \$17–\$32/tCO₂. A full stochastic programming analysis finds much smaller adjustments related to damage uncertainty, ranging from $-\$0.5$ to $+\$1.5$ /tCO₂ in the present and increasing tenfold by the end of the century (Croston and Traeger 2010). An explanation for the moderate nature of these findings is that technological progress in DICE-2007 outgrows smooth damages. Jensen and Traeger (2011) use a full stochastic programming framework to analyze the implications of growth uncertainty for optimal mitigation policy. They find that the optimal carbon tax increases by up to \$3/tCO₂ in the present when using Epstein-Zin-Weil preferences (with tenfold that increase by the end of the century), but the effect is much smaller when using standard entangled preferences.

In their analysis of tipping points, Keller, Bolker, and Bradford (2004) do not report the optimal carbon tax, but their abatement rate plots show that introducing the risk of a permanent shock to production increases the present-day carbon tax by only a minor amount. Moreover, anticipating one-time perfect learning in the year 2085 has essentially no effect on the present-day optimal carbon tax.²⁰ As in our model, the effect of the possible tipping point on optimal mitigation increases significantly over time. Lontzek, Cai, and Judd (2012) find that the possibility of a permanent productivity shock increases the abatement rate by a nearly constant amount over time. In contrast, we demonstrate how the specific channel by which tipping points affect the climate system determines how anti-tipping effort evolves over time.

In summary, we find that the effect of tipping points on the social cost of carbon is larger than for other uncertainties that have been modeled in full stochastic programming frameworks. However, model specifications that alter the consumption discount rate generally imply larger results. Monte Carlo assessments generate both larger and smaller effects. We find that the moderate effect on the optimal near-term carbon tax grows much larger within decades. We have also shown how characteristics of individual tipping points generate differences in optimal tax paths. A joint analysis of the comprehensive set of tipping possibilities described in the scientific literature would arguably imply a greater increase in the optimal carbon tax even in the near-term.

²⁰ Learning does raise the optimal carbon tax in the last two decades before learning occurs. They find a slightly larger increase in the present-day optimal carbon tax from introducing a deterministic tipping point (on the order of a few dollars).

B. Choices in Modeling Tipping Points

We have implemented versions of prominent tipping points into a full climate-economy model. In doing so, we made several assumptions that could be refined upon advances in scientific knowledge. First, we assumed that the consequences of tipping set in immediately after crossing a temperature threshold, and that the decision maker learns immediately about having crossed the threshold. A more realistic model might introduce a delay between the threshold crossing, its consequences, and possibly the decision maker's awareness of it. If the decision maker learned about the crossing only by observing the delayed consequences, then the delay would mostly correspond to additional discounting of the welfare loss from crossing the threshold. If the decision maker directly observed the threshold crossing, then she would have time to react to the changing system dynamics before they actually began causing harm. Both the effective discounting of the welfare loss and the ability to reduce the tipping point's severity would make the decision maker more willing to approach the threshold beforehand.

Our second simplifying assumption is the uniform distribution for the threshold's location. This distribution is the most natural choice based on current scientific knowledge, but another distribution will eventually become more realistic. The effect of a different distribution mostly depends on the probability mass it places on low temperatures and on the forms of learning available to the policymaker. In particular, our experiments with the upper bound of the uniform distribution suggest that distributions that place greater weight on low temperature thresholds could significantly increase the near-term social cost of carbon. In an earlier version of the present paper (Lemoine and Traeger 2012), we explicitly modeled the lack of confidence in the uniform distribution by allowing the policymaker to display ambiguity aversion. We analytically showed that increasing ambiguity aversion has an ambiguous effect on the social cost of carbon, but the numerical effect is to increase the social cost of carbon by a small amount in the near-term and by a larger amount over time.

Our third simplifying assumption is that tipping in our model is "nonstochastic." The decision maker expects the trigger to lie at some particular, though unknown, temperature. In consequence, the Bayesian learner recognizes a "safe region" encompassing all temperatures cooler than the current one. A concern might be that the clear delineation between safe and risky temperatures leads the policymaker to unrealistically freeze temperature at the edge of the safe region. We do not obtain such an unrealistic result because we integrate tipping point possibilities into a full climate-economy model that includes inertia in the climate system.²¹ While learning affects optimal policy, the qualitative shape of the optimal temperature trajectory remains similar to the benchmark model without tipping points. Observe the close connection between this third assumption and the first assumption. We would not actually expect tipping to be stochastic. Instead, any such stochasticity would serve to approximate a more complete model with uncertainty (and potentially learning) over the precise trigger *mechanism* underlying the tipping point.

²¹ See Lemoine and Traeger (2012) for a demonstration of how climatic inertia affects optimal policy paths.

VI. Conclusions

We have shown how to model economic decisions in the face of irreversible, policy-dependent tipping points that trigger endogenous damages from changes in the climate regime. We analytically demonstrated that tipping points affect optimal policy via two channels: the differential welfare impact (DWI) recognizes that today's policy choices affect welfare in the case that a tipping point occurs, and the marginal hazard effect (MHE) recognizes that today's policy choices affect the probability of crossing the threshold. Our numerical application developed a dynamic climate-economy model that includes the endogenous possibility of climatic tipping points, endogenous learning about the temperature threshold triggering tipping points, and endogenous welfare implications of tipping points. We find that the possibility of tipping points in the climate system raises the optimal carbon tax. Because of the small annual probability of crossing a climate threshold, these tipping points primarily affect the optimal carbon tax via the MHE. Further, the tipping point increment to the social cost of carbon is not merely a function of the current period's DWI and MHE but is also determined by how current emissions change "tipping lotteries" in all future periods. The climate system's warming delay can trigger future threshold crossings even if future CO₂ concentrations are stable.

Quantitatively, our base case tipping point possibilities increase the near-term optimal carbon tax by approximately 25 percent and our stronger tipping point possibilities increase the near-term optimal carbon tax by approximately 40 percent. Tipping points strongly increase our Bayesian policymaker's optimal tax over the next decades and, thus, have a strong effect on cumulative emissions and peak warming. The precise effects are sensitive to the type of tipping point, to the strength of the tipping point, and to the distribution for the threshold that triggers the tipping point. Carbon sink tipping points more strongly affect the near-term carbon tax, but climate feedback tipping points have an increasing effect over time and thereby more strongly reduce optimal peak temperature and CO₂. These two tipping points imply largely different postthreshold policy responses and very different time profiles for the equivalent consumption loss from tipping. Our results demonstrate the value of explicitly modeling tipping points' effects on system dynamics and endogenizing the welfare change. Moreover, the key role of the marginal hazard effect demonstrates the importance of endogenizing tipping probabilities. Finally, not allowing the policymaker to learn about thresholds would double a tipping point's effect on the present-day optimal carbon tax.

Our numerical conclusions have implications for economic modeling, climate science, and climate policy. First, economic models of climate change typically assume smooth changes in the climate system. More broadly, nearly all economic models dealing with growth and long-run dynamics assume smoothly changing systems. Models that do allow for discontinuous changes usually incorporate exogenous penalties. We have demonstrated the value of explicitly modeling the shifts in dynamics, explaining how feedback and carbon sink tipping points affect optimal abatement policies in different ways. Further, we have shown that the main effect of tipping points on optimal policy is often due to their endogeneity. It is important to model a tipping

point's structural effects rather than reducing it to a predetermined shock to utility, and it is important to capture the effect of policy decisions on the tipping point hazard.

Second, our work is a call to the climate sciences to improve knowledge about both the effects of tipping points on system dynamics and the types of temperature paths that trigger them. We have shown that different anticipated changes in dynamics can have quite different effects on the optimal carbon tax. We demonstrate the economic value of scientific information about tipping points and open the door to more comprehensive integrated assessments of abrupt climate transitions. Our sensitivity analysis has shown that these more comprehensive climate change assessments will benefit greatly from progress in the climate sciences that constrains the regions and probabilities of tipping point occurrence.

Third, our findings support the widespread supposition that the existence of tipping points in the climate system should have a strong influence on policy decisions. This influence would become stronger if we allowed the policymaker to face multiple tipping points at once. Numerical integrated assessment models are the main quantitative input into regulations that price carbon emissions. Yet past studies omitted climatic features we have shown to be highly relevant. We provide a quantitative basis for adjusting policy for the possibility of tipping points. Much work remains to make tipping representations more realistic, but we have demonstrated how to more comprehensively endogenize tipping possibilities and have provided a first assessment of their effect on policy.

APPENDIX

A. Additional Numerical Results for Possible Climate Tipping Points

Figure A1 gives additional results for optimal policy in the face of a possible tipping point. As indicated by the main text's results for the optimal tax, the tipping point possibility increases optimal abatement, and abatement jumps down upon first eliminating threshold risk by preventing temperature from increasing. Even though emissions jump up at this point, temperature inertia implies that these greater emissions still decrease temperature. Possible thresholds decrease the optimal investment rate very slightly, but they also eventually lead to slightly greater available output due to reduced climate damages. The differential welfare impact (DWI) has a very minor impact on the optimal tax as compared to the marginal hazard effect (MHE). Moreover, the current period's hazard explains (via MHE and DWI) only around 5 percent of the overall effect of tipping points on the optimal carbon tax. The remaining effects of tipping points on the optimal carbon tax are due to inertia in the climate system: increasing emissions today causes delayed warming that also poses a risk of tipping. We measure this effect of delayed tipping risk by comparing the marginal cost of emissions under the prethreshold value function to the marginal cost of emissions under a value function calculated for a model without any possibility of tipping.

Learning enters the model by expanding the set of safe temperatures and concentrating probability mass on temperatures yet to be explored (Figure A2, panel A). Therefore, as the world reaches higher temperatures, a contemplated temperature increase poses a greater hazard because it cuts through more probability mass

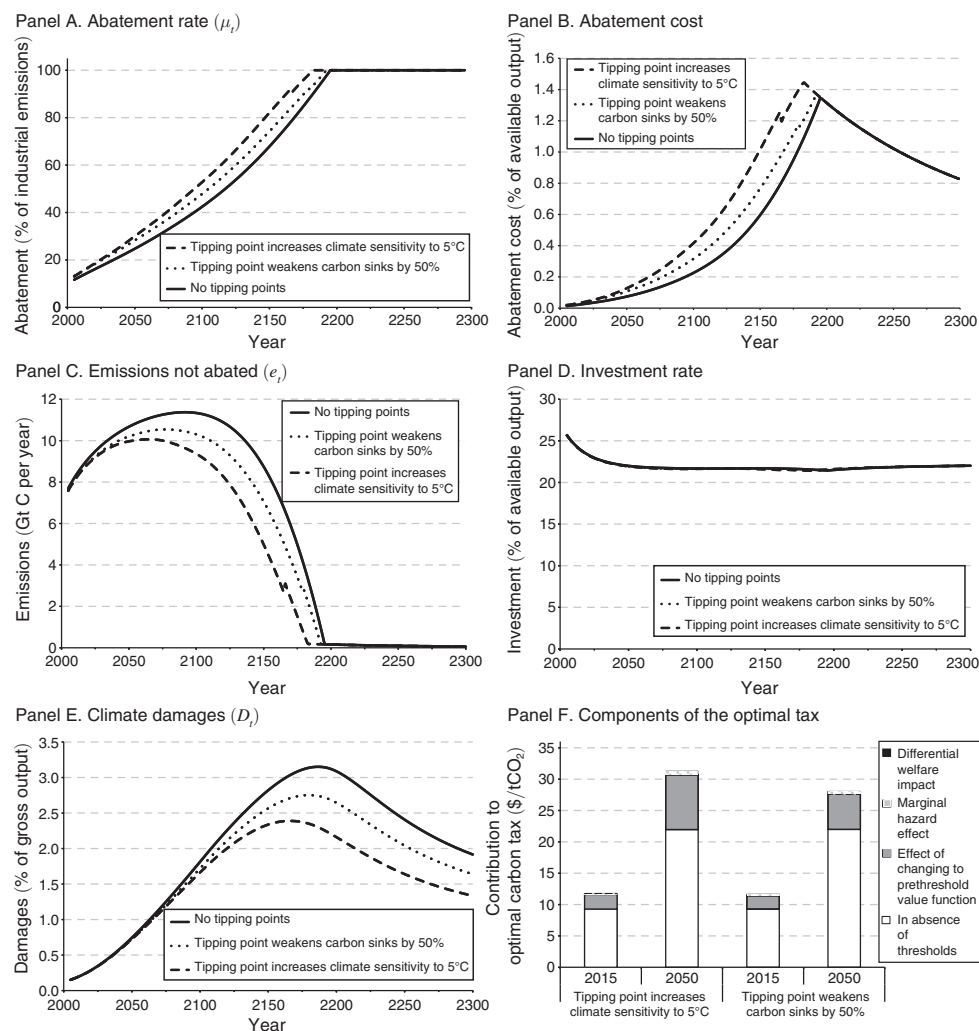


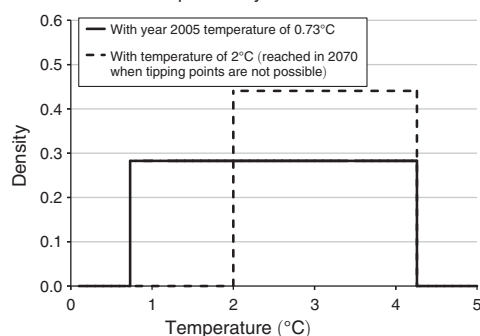
FIGURE A1. ADDITIONAL RESULTS ALONG OPTIMAL POLICY PATHS FOR THE BASE CASE THRESHOLD DISTRIBUTION

Notes: The plotted simulations assume expected draws of the weather shock and also assume that the tipping point never occurs. The lines in the investment plot almost completely overlap, and the differential welfare impact (DWI) is almost negligible in the optimal tax decomposition.

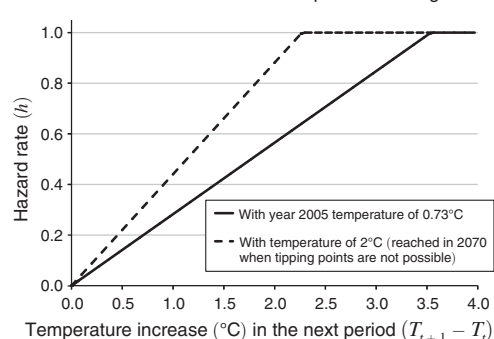
(Figure A2, panel B). Lower temperature profiles imply both less learning and flatter hazard functions for each temperature reached (Figure A2, panel C). Optimal policy in the face of possible tipping points lowers the hazard rate by reducing temperature change over time (Figure A2, panel D).²²

Figure A3 complements Figure 4 by further illustrating the policy response to crossing a threshold. It assumes that the policymaker is ignorant of the tipping point possibility until 2075, when the tipping point actually occurs. The effect of crossing a threshold depends on how it affects system dynamics and on how policy can compensate for the altered dynamics. The main text showed that the carbon tax

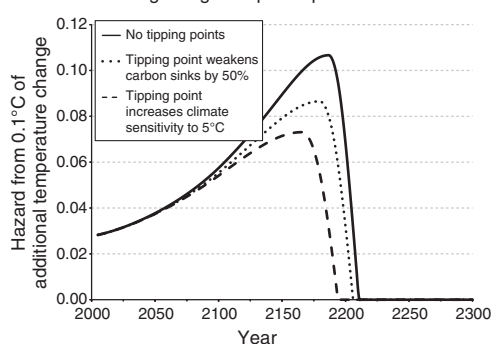
²² Each hazard path has a kink at the year 2100 when the exogenous non-CO₂ forcing ceases to increase.

Panel A. Base case probability distribution for \tilde{T} 

Panel B. Hazard as a function of temperature change



Panel C. Learning along the optimal path



Panel D. Hazard along the optimal path

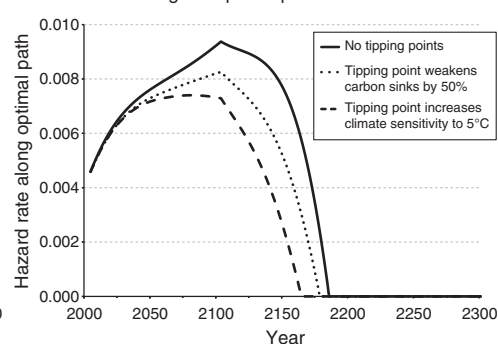


FIGURE A2

Notes: As the time t expected temperature increases without crossing a threshold, the probability distribution for the threshold level \tilde{T} places more mass on temperatures yet to be reached (panel A). Each additional increase in temperature therefore also produces a greater risk of crossing the threshold (panel B). Learning increases the hazard posed by a fixed increment of temperature change as long as temperatures are increasing (panel C). The actual hazard along the optimal path (panel D) also depends on the chosen emission policy. The plotted simulations assume expected draws of the weather shock and also assume that the tipping point never occurs. The hazard in the cases without tipping points is calculated as if they were in fact possible but the decision maker is unaware of the tipping possibility.

responds more strongly to the threshold crossing when it increases climate feedbacks. It also showed that the CO_2 concentration in the regime with strengthened climate feedbacks follows a lower path than in a case without thresholds, but degraded carbon sinks produce a much higher carbon concentration by increasing the persistence of CO_2 in the atmosphere. Temperature increases under either tipping point, but the more aggressive abatement under the climate feedback tipping point keeps the peak temperature below levels reached under the carbon sink tipping point.

B. The Dynamic Climate-Economy Model

Our analysis builds on the stochastic, dynamic programming version of DICE-2007 introduced in Traeger (2012). This Appendix gathers the complete set of equations, discusses changes, and explains how we extend the model to incorporate tipping points. The Matlab code is available on the journal's website (<http://dx.doi.org/10.1257/pol.6.1.137>). Table B1 provides the parameterization. For numeric efficiency, this reformulated DICE-2007 model uses effective labor units for capital

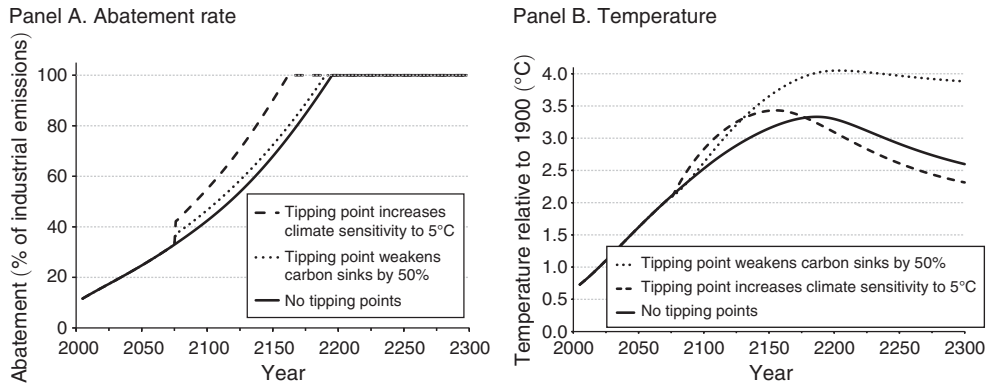


FIGURE A3. THE EFFECT OF TIPPING POINTS ON POSTTHRESHOLD POLICY AND STATE VARIABLES

Notes: Simulations assume the decision maker is ignorant of threshold possibilities until 2075, when the specified tipping point occurs. The plotted simulations assume expected draws of the weather shock.

$k_t = \frac{K_t}{A_t L_t}$ and consumption $c_t = \frac{C_t}{A_t L_t}$.²³ Moreover, the model reformulates the exogenous processes of DICE in continuous time, which eases an accurate downscaling of DICE's originally decadal time step to our annual step size. Finally, the model reduces the state space by approximating the ocean cooling dynamics and the carbon cycle.

As discussed in Section II, we have to solve recursively for two different value functions in the presence of tipping points. The prethreshold value function is:

$$\begin{aligned}
 V_0(k_t, M_t, T_t, t) \\
 = \max_{c_t, \mu_t} \frac{c_t^{1-\eta}}{1-\eta} + \beta_t \int [[1 - h(T_t, T_{t+1})] V_0(k_{t+1}, M_{t+1}, T_{t+1}, t+1) \\
 + h(T_t, T_{t+1}) V_1(k_{t+1}, M_{t+1}, T_{t+1}, t+1)] d\mathbf{IP}
 \end{aligned}$$

subject to

(Effective capital)

$$k_{t+1} = e^{-(g_{L,t} + g_{A,t})} \left[(1 - \delta_k) k_t + (1 - \Psi_t \mu_t^{a_2}) \frac{Y_t}{1 + D_t} - c_t \right]$$

(CO₂)

$$M_{t+1} = e_t + M_t [\mathbf{b}_{11} + b_{21} [\mathbf{b}_{12} + (b_{22} + b_{32} b_{23}) \alpha_B(M_t, t) + b_{32} b_{33} \alpha_O(M_t, t)]]$$

(Temperature)

$$T_{t+1} = T_t + C_T \left[F(M_{t+1}, t+1) - \frac{f}{s} T_t - [1 - \alpha_T(T_t, t)] C_O T_t \right]$$

²³The model employs the more convenient, equivalent formulation of technological progress as augmenting labor productivity. The value function itself is now a function of effective capital, and it is normalized by the factor $(A_t^{1-\eta} L_t)^{-1}$.

TABLE B1—PARAMETERIZATION OF THE NUMERICAL MODEL FOLLOWING DICE-2007

Parameter	Value	Description
A_0	0.027	Initial production technology
$g_{A,0}$	0.009	Initial annual growth rate of production technology
δ_A	0.001	Annual rate of decline in growth rate of production technology
L_0	6514	Population in 2005 (millions)
L_∞	8600	Asymptotic population (millions)
δ_L	0.035	Annual rate of convergence of population to asymptotic value
σ_0	0.13	Initial emission intensity before emission reductions (Gt C per unit output)
$g_{\sigma,0}$	−0.0073	Initial annual growth rate of emission intensity
δ_σ	0.003	Annual change in growth rate of emission intensity
a_0	1.17	Cost of backstop technology in 2005 (\$1,000 per t C)
a_1	2	Ratio of initial backstop cost to final backstop cost
a_2	2.8	Abatement cost exponent
g_ψ	−0.005	Annual growth rate of backstop cost
B_0	1.1	Initial annual nonindustrial CO ₂ emissions (Gt C)
g_B	−0.01	Annual growth rate of nonindustrial emissions
EF_0	−0.06	Forcing in 2005 from non-CO ₂ agents (W m ^{−2})
EF_{100}	0.30	Forcing in 2105 from non-CO ₂ agents (W m ^{−2})
κ	0.3	Capital elasticity in Cobb-Douglas production function
δ_κ	0.06	Annual depreciation rate of capital
d_1	0.0028	Coefficient on temperature in the damage function
d_2	2	Exponent on temperature in the damage function
s	3	Climate sensitivity (°C)
f	3.8	Forcing from doubled CO ₂ (W m ^{−2})
M_{pre}	596.4	Preindustrial atmospheric CO ₂ (Gt C)
C_T	0.03	Translation of forcing into temperature change
C_O	0.3	Translation of surface-ocean temperature gradient into forcing
b_{11}, b_{12}, b_{13}	0.978, 0.023, 0	Transfer coefficients for carbon from the atmosphere
b_{21}, b_{22}, b_{23}	0.011, 0.983, 0.005	Transfer coefficients for carbon from the combined biosphere and shallow ocean stock
b_{31}, b_{32}, b_{33}	0, 0.0003, 0.9997	Transfer coefficients for carbon from the deep ocean
ρ	0.015	Annual pure rate of time preference
η	2	Relative risk aversion (also aversion to intertemporal substitution)
k_0	137/($A_0 L_0$)	Effective capital in 2005, with US \$137 trillion of capital
M_0	808.9	Atmospheric carbon dioxide (Gt C) in 2005
T_0	0.73	Surface temperature (°C) in 2005, relative to 1900

Note: Several values are rounded, and C_T and δ_κ vary slightly over time in order to reproduce the DICE results with an annual timestep.

(Output constraint)

$$c_t + \Psi_t \mu_t^{a_2} \frac{Y_t}{1 + D_t} \leq \frac{Y_t}{1 + D_t}$$

(Nonnegativity constraint for emissions)

$$\mu_t \leq 1.$$

The state variables are effective capital k_t , atmospheric CO₂ M_t , cumulative temperature change T_t , and time t . Tipping points change the bold parameters. The controls are effective consumption c_t , abatement μ_t , and, as a residual, investment. Capital depreciates at rate δ_k , and capital investment comes from any available output not allocated to the control variables of consumption c_t and abatement μ_t . The exogenous variable Ψ_t and parameter a_2 determine the cost of abating the chosen fraction

μ_t of emissions. The term outside the brackets in the capital transition equation adjusts for the growth of labor and technology to keep capital in effective terms. Gross output Y_t is a function of the capital stock:

$$(\text{Gross output}) \quad Y_t = k_t^\kappa.$$

The parameter κ gives the capital elasticity in a Cobb-Douglas production function. Climate damages D_t reduce gross output in accord with the total temperature change:

$$(\text{Damages}) \quad D_t = d_1(\epsilon_t T_t)^{d_2},$$

where the independent, normally distributed multiplicative shock ϵ_t has probability measure IP. Optimal policy adjusts current controls in anticipation of possible future shocks, and a given period's realized shock affects the residual output allocated to investment. We calibrate the mean-1 weather shock to the years 1881–2010 in the NASA Goddard Institute for Space Studies (GISS) Surface Temperature Analysis dataset.²⁴ We take expected temperature in each year to be the mean of the surrounding ten years' realized temperatures. The realized standard deviation of the resulting time series of multiplicative shocks is 0.0068.²⁵ This multiplicative noise captures period-to-period temperature variability that makes extreme outcomes more likely as CO₂ increases.

The carbon dynamics in DICE-2007 are determined by a transition matrix governing the flow between the atmospheric stock (stock 1), the combined biosphere and shallow ocean stock (stock 2), and the deep ocean stock (stock 3). Departing from Traeger (2012), we represent the combined biosphere and shallow ocean stock as a fraction $\alpha_B(M, t)$ of the atmospheric stock, and the deep ocean stock as a fraction $\alpha_O(M, t)$ of the atmospheric stock. We fit these functions for the pre and postthreshold regimes using a 16th-degree Chebychev polynomial and a set of seven paths calculated with full DICE dynamics.²⁶ The parameter b_{12} gives the fraction of atmospheric CO₂ absorbed by land and ocean sinks over a single timestep. The parameter $b_{11} = 1 - b_{12}$ determines the fraction of CO₂ that remains in the atmosphere from period to period, with the remaining terms in the CO₂ transition equation together governing the transfer of carbon from land and ocean sinks back into the atmosphere. As described in Section III, carbon sink tipping points reduce the parameter b_{12} by a fraction, which also changes b_{11} , α_B , and α_O . Time t emissions e_t are given by:

$$(\text{Emissions}) \quad e_t = \sigma_t(1 - \mu_t)Y_t + B_t.$$

The exogenous variable σ_t is the emission intensity of gross output and B_t gives exogenous CO₂ emissions from nonindustrial sources such as land use change.

²⁴ Available at <http://data.giss.nasa.gov/gistemp/>.

²⁵ We implement the continuous distribution numerically using a Gauss-Legendre quadrature rule with eight nodes.

²⁶ These seven paths are: the no-policy DICE path, the optimized DICE path, two paths with abatement rates similar to those optimal in the presence of tipping points, a path with emissions twice as great as along DICE's optimized path, a path with emissions half as great as along DICE's optimized path, and a path without emissions.

DICE-2007 determines time t surface temperature from the stock of CO_2 , from surface temperature in the previous period, and from the previous period's difference between the surface temperature and the deep ocean temperature. We represent the deep ocean temperature as a fraction $\alpha_T(T, t)$ of surface temperature, where α_T is an interpolated function of temperature and time. We calibrate this function to the same DICE-2007 paths used in calibrating the carbon stock approximations and again use a 16th-degree Chebychev polynomial. Forcing $F(M_t, t)$ measures the additional energy (W m^{-2}) trapped at the earth's surface by greenhouse gases and other atmospheric agents. Forcing is concave in CO_2 :

$$(\text{Forcing}) \quad F(M_t, t) = f \ln(M_t/M_{pre}) + EF_t,$$

where f is the forcing from doubled CO_2 and EF_t gives the time t exogenous (non- CO_2) forcing. The parameter s in the temperature transition equation is climate sensitivity, or the equilibrium temperature change from doubling CO_2 concentrations. This parameter is altered by climate feedback tipping points as described in Section III. Finally, the parameter C_O determines how a temperature gradient between the surface and the deep ocean affects forcing at the surface, and the parameter C_T controls the speed with which aggregate forcing changes temperature.

The transition equations for the exogenous variables are

$$(\text{Production technology}) \quad A_t = A_0 \exp \left[\frac{g_{A,0}}{\delta_A} (1 - e^{-t\delta_A}) \right]$$

(Growth rate of production technology)

$$g_{A,t} = g_{A,0} e^{-t\delta_A}$$

$$(\text{Labor}) \quad L_t = L_0 + (L_\infty - L_0)(1 - e^{-t\delta_L})$$

$$(\text{Growth rate of labor}) \quad g_{L,t} = \delta_L \left[\frac{L_\infty}{L_\infty - L_0} e^{t\delta_L} - 1 \right]^{-1}$$

$$(\text{Effective discount factor}) \quad \beta_t = \exp(-\rho + (1 - \eta) g_{A,t} + g_{L,t})$$

(Uncontrolled emissions per output)

$$\sigma_t = \sigma_0 \exp \left[\frac{g_{\sigma,0}}{\delta_\sigma} (1 - e^{-t\delta_\sigma}) \right]$$

$$(\text{Abatement cost factor}) \quad \Psi_t = \frac{a_0 \sigma_t}{a_2} \left(1 - \frac{1 - e^{t g_\Psi}}{a_1} \right)$$

$$(\text{Nonindustrial CO}_2 \text{ emissions}) \quad B_t = B_0 e^{t g_B}$$

$$(\text{Non-CO}_2 \text{ forcing}) \quad EF_t = EF_0 + 0.01(EF_{100} - EF_0) \min\{t, 100\},$$

where the subindex 0 denotes year 2005 values. Traeger (2012) shows that these continuous-time interpolations of the exogenous processes match their

discrete-time and recursively-defined counterparts in the original DICE model and ease an accurate downscaling of the time step. Our model replicates DICE's results when run with DICE's decadal policy path, or when we optimize with a ten-year timestep. When optimizing with an annual timestep (and without tipping point possibilities), the policymaker's ability to smooth emissions within a decade leads to the peak CO₂ level being about 30 ppm lower than in DICE, the maximum temperature being about 0.15°C lower, and the year 2015 social cost of carbon being about \$4/tCO₂ lower.

The primary computational challenge in solving the model lies not in finding the optimal actions for a given value function but in determining the value functions that satisfy the relations in equations (1) and (2). We solve the problem by value function iteration using a 10⁴ basis of Chebychev polynomials to approximate the value functions over a set of 10⁴ Chebychev nodes (see Kelly and Kolstad 1999, 2001; Traeger 2012). Our break criterion is a tolerance of 10⁻⁴ on the maximal coefficient change. The approximation intervals in the no-tipping runs cover effective capital values from 3 to 6, atmospheric carbon stocks from 700 to 1,800 Gt C, and temperature levels of 0.5 to 4°C above 1900 (plus the infinite time horizon mapped to the unit interval). We changed the approximation intervals for the pre and postthreshold value functions in order to accommodate altered dynamics due to tipping points.²⁷

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²⁷When approximating the postthreshold value functions with the climate feedback tipping point, our temperature approximation interval extends to 6°C. When approximating the postthreshold value functions with the carbon sink tipping point, our temperature approximation interval extends to 6.5°C for the weak and middle versions and to 7°C for the strong version, and our atmospheric carbon approximation interval extends to 2,800 Gt C for the weak and middle versions and to 3,200 Gt C for the strong version. Finally, when undertaking the sensitivity analysis for the threshold distribution's upper bound, we limit the prethreshold temperature approximation interval's maximum to the lower of the threshold distribution's upper bound and the standard approximation interval's maximum.

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