

Steering the Climate System: Using Inertia to Lower the Cost of Policy*

Derek Lemoine[†] and Ivan Rudik[‡]

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Conventional wisdom holds that the efficient way to limit warming to a chosen level is to price carbon emissions at a rate that increases exponentially. We show that this “Hotelling” tax on carbon emissions is actually inefficient. The least-cost policy path takes advantage of the climate system’s inertia by growing more slowly than exponentially. Carbon dioxide temporarily overshoots the steady-state level consistent with the temperature limit, and the efficient carbon tax follows an inverse-U-shaped path. Economic models that assume exponentially increasing carbon taxes are overestimating the minimum cost of limiting warming, overestimating the efficient near-term carbon tax, and overvaluing technologies that mature sooner.

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[†]Department of Economics, University of Arizona, dlemoine@email.arizona.edu

[‡]Department of Economics, Iowa State University, irudik@iastate.edu

Many economic models of climate change focus on trading off the costs and benefits of emission reductions, but the political process is instead oriented towards hard limits on global warming (Jaeger and Jaeger, 2010). In 2009, both the G8 group of developed countries and the Major Economies Forum supported a 2 degrees Celsius limit. The 2009 Copenhagen Accord and 2010 Cancun Agreements subsequently enshrined this 2 degrees Celsius limit as the goal of the United Nations negotiations towards a new climate change treaty (Gillis, 2014). Most observers expect temperature limits to continue directing the scientific and political discourse.¹

In response, the economic modeling community has extensively analyzed the implications of temperature limits for the cost and structure of emission policy (Clarke et al., 2014). Its primary tools are multisector general equilibrium models that determine the cheapest policy route to achieve a prespecified environmental goal. These numerical “cost-effectiveness integrated assessment models” are typically too complex to directly optimize each period’s tax on greenhouse gas emissions. Instead, they often adopt analytic results about the shape of the least-cost tax trajectory from the theoretical economics literature and then solve for the market equilibrium consistent with that policy path (Bauer et al., 2015). Contrary to conventional wisdom (e.g., Tol, 2013), we show that these analytic results do not apply to the case of temperature limits. Previous theoretical literature recommends an exponentially increasing tax on greenhouse gas emissions to achieve a limit on total pollution, but we show that the least-cost policy path to achieve a temperature limit actually employs a nonmonotonic tax on emissions which increases more slowly than exponentially. By implementing inappropriate theoretical results, general equilibrium models’ results have overstated the minimum cost of achieving temperature limits, overestimated the level of the near-term emission tax consistent with these limits, and overvalued technologies that mature sooner rather than later.

The conventional theoretical result implemented by general equilibrium models is that when policymakers seek to limit the quantity of carbon dioxide (CO_2) in the atmosphere, the least-cost tax on CO_2 emissions increases at the rate of interest plus the rate at which CO_2 “decays” in the atmosphere (Nordhaus, 1980, 1982; Peck and Wan, 1996; Goulder and Mathai, 2000).² This least-cost trajectory is commonly called a Hotelling trajectory: if we consider the atmosphere’s CO_2 -holding capacity as an exhaustible resource whose quantity is fixed by the chosen CO_2 limit, then the least-cost policy depletes the resource (via emissions)

¹Many economists have criticized environmental targets as implying an unrealistic jump in the damages from climate change (e.g., Nordhaus, 2008; Tol, 2013), but other economists have argued that we know too little about the economic costs of climate change to undertake a meaningful cost-benefit assessment (e.g., Pindyck, 2013; Stern, 2013). The latter perspective has, in the spirit of Baumol (1972), led some economists to argue for limiting economics to cost-effectiveness analyses of scientifically-grounded limits on total warming (e.g., Richels et al., 2004; Ackerman et al., 2009). We take no stand on this debate. Instead, we analyze the economic implications of the dominant approach to long-run climate policy.

²A number of papers, including Goulder and Mathai (2000), also explore how induced technological change affects the carbon tax trajectory.

according to the analysis of Hotelling (1931). The intuition is as follows. Along a least-cost trajectory, the policymaker must be indifferent to small deviations in the trajectory. Imagine that the policymaker considers deviating by allowing an additional unit of emissions today. Instead of spending money on reducing emissions today, the policymaker would invest those savings and compensate by undertaking additional emission reductions t years in the future. In order to return to the original CO₂ trajectory, the policymaker will not need to reduce future emissions by a full unit because the additional unit of emissions will have decayed at rate δ . By deviating in this fashion, the policymaker has earned interest at rate r in the years prior to t and has also seen the required spending decline at the rate δ of CO₂ decay. In order for the policymaker to be indifferent to this deviation, the marginal cost of emission reductions (i.e., the tax on CO₂ emissions) must grow at rate $r + \delta$.

We show that this logic is incomplete when policymakers aim to limit total warming. The reason is that an increase in CO₂ neither immediately nor fully translates into an increase in warming. The climate system displays substantial inertia, warming only slowly in response to additional CO₂.³ A year's temperature is driven not just by the contemporary quantity of CO₂ in the atmosphere but also by the level of CO₂ in previous years. Additional warming incurred by temporarily raising CO₂ cannot be undone simply by returning to the original CO₂ trajectory. By allowing additional warming over the next t years, a policymaker sacrifices some of the braking services provided by the inertia in the climate system. In order to return to the original temperature trajectory, the policymaker must undertake a sufficiently large quantity of emission reductions to bring time t CO₂ some distance *below* its original trajectory. This additional spending offsets the policymaker's earnings from interest and from the natural decay of CO₂. If the tax on CO₂ emissions increases exponentially, the policymaker could profitably deviate by borrowing money from time t to finance additional early emission reductions and then repaying the loan out of the savings from allowing greater emissions at a future time when the CO₂ tax has grown larger. The policymaker's profits arise because, by retaining more of the climate system's braking services, the additional early emission reductions allow the policymaker to increase time t emissions by a greater amount before reaching the original temperature trajectory. In order for the policymaker to be indifferent to small deviations in the emission trajectory, the tax on CO₂ emissions must grow more slowly than exponentially.⁴

³For example, interactions with ocean heat sinks mean that the next decades' warming will represent only about 50–60% of the eventual equilibrium warming corresponding to their likely CO₂ concentrations (Solomon et al., 2009). Even if we were to freeze all greenhouse gases at their current concentrations, the climate system's inertia means that we could expect total warming to more than double from the current level (Wetherald et al., 2001).

⁴The effect of inertia on the efficient CO₂ tax has rough parallels in the theory of nonrenewable resource extraction. Levhari and Liviatan (1977) extend the Hotelling setting so that the marginal cost of extraction increases in cumulative extraction. They show that equilibrium marginal profit grows more slowly than the rate of interest. If marginal profit grew faster than the rate of interest, then the resource owner would delay extraction in order to reduce future costs. In our setting, if the marginal cost of emitting grew at the (decay-

The presence of inertia in the climate system is valuable for a policy aiming to limit total warming. This value manifests itself in two ways. First, the climate system's braking services allow the policymaker to delay emission reductions without immediately incurring the full temperature penalty. For any positive consumption discount rate, the temporary disconnect between CO₂ and temperature provides a valuable degree of freedom which the policymaker uses to lower the present cost of policy. Second, the climate system's braking services allow the policymaker to reduce the cumulative quantity of abatement undertaken over time. By delaying the temperature consequences of additional emissions, the climate system's inertia provides more time for emissions to decay naturally. Even if future abatement costs are not discounted, the policymaker reallocates abatement over time so as to reduce the cumulative quantity of abatement undertaken. In the presence of discounting or of natural decay of CO₂, the climate system's inertia allows for a lower initial tax and reduces the overall cost of the policy program.

We derive an intuitive analytic expression for the least-cost emission trajectory in the presence of inertia and analyze its implications for climate policy. The least-cost emission tax trajectory is composed of two terms. The first, positive component is the classic Hotelling term, which makes the emission tax rise at the decay-adjusted interest rate. This first component would be the only component if the environmental target were expressed in terms of CO₂ or if temperature responded immediately to CO₂. The second, negative component is novel. At time t , this term is the product of the shadow cost of temperature and the change in time t temperature due to a marginal increase in time 0 abatement. The positive shadow cost of temperature reflects that additional warming uses up more of the atmosphere's braking capacity, even holding CO₂ constant. As temperature approaches the exogenous limit, its shadow cost grows large and this second component of the efficient emission tax becomes large in magnitude. The efficient emission tax therefore tends to decrease as the policymaker steers the climate into a steady state at the temperature limit.

Three specific results stand out. First, the least-cost CO₂ trajectory always overshoots the steady-state CO₂ level associated with a given temperature target.⁵ The climate system's

adjusted) rate of interest, then the policymaker would emit less now in order to increase emissions in the future. Further, Heal (1976) considers a case in which marginal cost becomes constant at a high enough level of cumulative extraction. He shows that the equilibrium resource price falls towards the long-run marginal cost. This result is similar to our carbon price dynamics immediately prior to reaching the long-run carbon price consistent with maintaining temperature exactly at the limit.

⁵Some numerical analyses have discussed "overshoot" pathways in the context of very low CO₂ targets and very low temperature targets (e.g., van Vuuren et al., 2011), but they often frame overshoot as a last resort rather than as a least-cost pathway in its own right. Remarks in Wigley (2003), Huntingford and Lowe (2007), and Wigley et al. (2007) suggested that overshoot trajectories might in fact be cheaper ways of achieving climate goals, and simulations in, for instance, den Elzen and van Vuuren (2007) and Clarke et al. (2009) have supported this conjecture. The latest report from the Intergovernmental Panel on Climate Change included overshoot scenarios and discussed how they affect the probability of maintaining temperature below given limits (Clarke et al., 2014). These scenarios have not used the least-cost trajectory derived in our work.

inertia provides a benefit: it allows emission reductions to be postponed without overshooting the temperature target. The least-cost emission path takes advantage of this delayed climatic response. Our calibrated numerical example suggests that the least-cost path overshoots a temperature target's steady-state CO₂ level by 50–100 ppm. The climate system's inertia enables a policymaker to temporarily increase CO₂ by approximately twice as much as in a case without inertia.

Second, we show that recognizing the effect of inertia makes the efficient CO₂ tax path nonmonotonic and tends to decrease the efficient present-day CO₂ tax. By acting as a brake on warming, climatic inertia enables the policymaker to delay abatement and to reduce cumulative abatement, which reduces the initially efficient emission tax. Numerically, recognizing inertia can decrease the initial emission tax used to achieve a 2°C temperature target by 90%. The emission tax eventually reaches very high levels as the policymaker seeks to reduce the CO₂ concentration from its peak level, but as the CO₂ concentration then declines towards its required steady-state level, the emission tax also begins declining towards its corresponding steady-state level. Such a nonmonotonic pathway is not consistent with the standard Hotelling assumption.

Third, we show that a temperature target is always cheaper than the corresponding CO₂ target because it allows the policymaker to take advantage of the braking services provided by climatic inertia. If the primary benefits of either target arise from avoiding higher temperatures (as opposed to avoiding higher CO₂ or inducing a transition path with a particular damage profile), then this result argues strongly for refocusing policy and modeling efforts on temperature limits instead of CO₂ limits. Numerically, we find that a 2°C temperature target can be many times cheaper than its corresponding CO₂ target, trimming trillions of dollars from the present cost of the policy. The CO₂ target requires 20–25% more abatement over the next 200 years while also undertaking abatement earlier than necessary. Conventional economic models with Hotelling assumptions may be substantially overestimating the cost of policies to limit total warming.

Section 1 describes the model setting and shows that the least-cost CO₂ path overshoots the steady-state CO₂ level. Section 2 analyzes the least-cost policy trajectory, focusing on how inertia affects the standard Hotelling intuition. Section 3 develops a calibrated numerical example and calculates the excess cost and abatement incurred by a Hotelling trajectory. The final section concludes. The first appendix contains proofs and a derivation. The second appendix contains a phase portrait analysis of the least-cost policy, details of the numerical calibration, and analysis of a case in which the policymaker has access to a “geoengineering” technology for directly managing temperature.

1 Setting

A global planner seeks the least-cost emission path to limit global warming to an exogenous level \bar{T} . The setting is in continuous time, with an infinite-horizon planning period. Business-as-usual CO₂ emissions $E > 0$ arise exogenously. The policymaker chooses each instant's quantity of abatement $A(t)$, with the net emissions released to the atmosphere becoming $E - A(t)$. The cost of abatement is $C(A(t))$, where $C(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an increasing, twice-differentiable, continuous, and strictly convex function. Zero abatement costs nothing ($C(0) = 0$), and abatement cannot be negative ($A(t) \geq 0$).⁶

Atmospheric carbon dioxide $M(t)$ is increased by net emissions. CO₂ in excess of pre-industrial concentrations M_{pre} decays at rate $\delta \in (0, 1)$:

$$\dot{M}(t) = E - A(t) - \delta (M(t) - M_{pre}), \quad (1)$$

where dot notation indicates a time derivative. Atmospheric CO₂ generates forcing, which is a common measure of how much heat is trapped in a period. Following the scientific literature, forcing $F(M(t))$ is logarithmic in CO₂ (Kondratiev and Niilisk, 1960; Möller, 1963; Rasool and Schneider, 1971; Ramaswamy et al., 2001):

$$F(M(t)) = \alpha \ln (M(t)/M_{pre}), \quad (2)$$

where $\alpha > 0$ gives the additional forcing from a 1% increase in CO₂ relative to pre-industrial levels. If maintained forever, that additional forcing would generate $s > 0$ units of warming, where s is a transformation of the parameter commonly known as climate sensitivity. However, inertia in the climate system means that forcing does not immediately translate into temperature:

$$\dot{T}(t) = \phi [s F(M(t)) - T(t)]. \quad (3)$$

The parameter $\phi \in (0, 1)$ controls the degree of inertia in the system. Greater ϕ indicates less inertia. As $\phi \rightarrow 1$, there is no inertia: an instant's forcing completely determines that instant's temperature. As $\phi \rightarrow 0$, there is full inertia: temperature never changes irrespective of forcing. This temperature representation is similar to that used in more complex numerical models, except lacking an explicit ocean temperature state variable (Nordhaus, 1992, 1993, 2008).⁷

⁶We allow for net negative emissions ($A(t) > E$) in recognition of technologies for removing CO₂ from the atmosphere. These technologies have often been of primary interest in numerical literature discussing “overshoot” pathways. Constraining $A(t)$ to be less than E would not change the primary results.

⁷Nordhaus (1991) uses an identical formulation for the temperature transition, though ultimately linearizing the forcing relationship. When we later calibrate ϕ using his more recent and more complex discrete-time models, we obtain a value just below the low end of the studies he drew upon in 1991 and about half as large as the value he selected for his calibration. Our sensitivity analysis will include a value for ϕ close to his original calibration.

The initial time t_0 is given. The initial level of CO₂ is M_0 , and initial temperature is T_0 , which is strictly less than the policy target \bar{T} . The policymaker's objective is to select an abatement trajectory in order to minimize the present cost (using discount rate $r > 0$) of maintaining temperature below the policy target:⁸

$$\begin{aligned} V(M(t_0), T(t_0), t_0) &= \min_{A(t)} \int_{t_0}^{\infty} e^{-r(t-t_0)} C(A(t)) dt \\ &\text{subject to equations (1) and (3),} \\ &\quad T(t) \leq \bar{T}, \\ &\quad A(t) \geq 0, \\ &\quad M(t_0) = M_0, \quad T(t_0) = T_0. \end{aligned} \tag{4}$$

Consistent with international policy discussions and with the numerical cost-effectiveness models used to evaluate policy, all damages from climate change are reflected in the prior choice of \bar{T} , so that damages at lower temperatures do not affect the policy trajectory.⁹ The following assumption ensures that the policymaker faces an interesting problem:

Assumption 1. $E > \delta (M_{pre} e^{\bar{T}/(s\alpha)} - M_{pre})$.

The assumption guarantees two outcomes. First, the temperature limit would be violated if no abatement ever occurred, and second, maintaining temperature at \bar{T} requires strictly positive abatement. Given the assumption and the lack of any benefit from raising emissions beyond the business-as-usual level, we henceforth ignore the constraint that $A(t) \geq 0$.

It is clearly never optimal to reach \bar{T} and then reduce temperature: by Assumption 1, any such path has strictly greater cost than a path that reaches \bar{T} at the same point in time but then allows sufficient emissions to remain at \bar{T} . Further, it is clearly not optimal to maintain temperature strictly below \bar{T} at all times, as such a path is more costly than one that allows slightly more emissions yet still remains strictly below \bar{T} . We can therefore

⁸There exists a policy program that would satisfy the constraints and yield a convergent integral: hold abatement fixed at zero until CO₂ reaches \bar{M} (defined below), and then hold abatement equal to the level necessary to maintain \bar{M} . The cost of this program is zero until reaching \bar{M} and constant in current-value terms thereafter. The cost of this policy program is clearly finite. A least-cost policy program must therefore also have finite cost.

⁹A cost-benefit framework would explicitly model the possibility that warming imposes costs ("damages"), which would motivate emission reductions. In contrast, "cost-effectiveness" integrated assessment models avoid making assumptions about damages in order to focus on the costs imposed by policy-relevant environmental targets. These models implement analytic results derived for settings that lack explicit damages from warming and are the dominant approach to quantifying the cost of proposed policies and the relative value of new technologies. We demonstrate how to correct these influential models without taking a stand on the relative merits of the cost-effectiveness and cost-benefit approaches (see footnote 1). Our primary point about the implications of inertia is not sensitive to combining damages with a binding temperature limit.

reframe the problem as deciding on the optimal path towards \bar{T} and on the optimal time to reach \bar{T} .

Define \bar{M} as the unique CO₂ concentration compatible with the climate system remaining at \bar{T} :

$$\bar{M} \triangleq M_{pre} e^{\bar{T}/(s\alpha)}.$$

The climate dynamics themselves directly imply two important results that are typically overlooked in the literature.

Proposition 1.

1. *There exists a time q such that $\dot{M}(t) \leq 0$ for all times $t \geq q$ and $\dot{M}(t) < 0$ for some times $t \geq q$.*
2. *A path constrained by temperature limit \bar{T} can achieve strictly less cost than a path constrained by the corresponding CO₂ limit \bar{M} .*

Proof. See appendix. □

The first result says that a least-cost CO₂ trajectory overshoots the steady-state CO₂ level consistent with the temperature constraint. This occurs because of the inertia in the climate system. Any least-cost path must approach \bar{T} from below, and it must reach \bar{M} by the time it reaches \bar{T} . Climatic inertia means that the next instant's temperature is an average of the current temperature and of the steady-state (or “equilibrium”) temperature consistent with the current CO₂ concentration. In order to approach \bar{T} from below, the CO₂ concentration must approach \bar{M} from above. The drag in the climate system enables CO₂ concentrations to temporarily exceed their steady-state level without violating the temperature constraint. Any path that does not take advantage of this ability to overshoot the steady-state CO₂ level cannot be a least-cost path. The proposition's second result follows from the first: because a least-cost path must overshoot its steady-state CO₂ level, indirectly achieving a temperature constraint by directly constraining CO₂ must increase the cost of the efficient policy program.

2 Least-cost policy

The policymaker's problem is an autonomous infinite-horizon control problem. Using the insights from the previous section, we rewrite the problem as a fixed endpoint, free terminal time problem with scrap value defined by the cost of maintaining temperature at \bar{T}

thereafter.¹⁰ The minimization problem therefore becomes¹¹

$$V(M(t_0), T(t_0), t_0) = \min_{A(t), \tau} \left\{ \int_{t_0}^{\tau} e^{-r(t-t_0)} C(A(t)) dt + e^{-r(\tau-t_0)} V(\bar{M}, \bar{T}, \tau) \right\}$$

subject to equations (1) and (3),

$$M(t_0) = M_0, \quad T(t_0) = T_0$$

$$\lim_{t \rightarrow \tau} M(t) = \bar{M}, \quad \lim_{t \rightarrow \tau} T(t) = \bar{T},$$

where τ is the chosen time at which temperature reaches \bar{T} and CO₂ reaches \bar{M} . We use limits to allow for the case where $\tau = \infty$. Because the least-cost policy will never decrease temperature once it has reached \bar{T} , the present formulation does not eliminate any trajectories that might solve (4). Form the current-value Hamiltonian:

$$H(M(t), T(t), A(t), \lambda_M(t), \lambda_T(t)) = C(A(t)) + \lambda_M(t) [E - A(t) - \delta (M(t) - M_{pre})] \\ + \lambda_T(t) \phi [s \alpha \ln(M(t)/M_{pre}) - T(t)].$$

In addition to the transition equations and the initial conditions, an optimal trajectory must satisfy the Maximum Principle and the costate conditions:

$$C'(A(t)) = \lambda_M(t), \tag{5}$$

$$\dot{\lambda}_M(t) = (r + \delta) \lambda_M(t) - \phi s \alpha \frac{\lambda_T(t)}{M(t)}, \tag{6}$$

$$\dot{\lambda}_T(t) = (r + \phi) \lambda_T(t), \tag{7}$$

where primes indicate derivatives. To pin down an optimal trajectory, we also require a transversality condition corresponding to the free terminal time τ . For $\tau < \infty$, this transversality condition is:

$$e^{-r(\tau-t_0)} H(M(\tau), T(\tau), A(\tau), \lambda_M(\tau), \lambda_T(\tau)) = - \frac{\partial [e^{-r(\tau-t_0)} V(\bar{M}, \bar{T}, \tau)]}{\partial \tau}. \tag{8}$$

If it is optimal to reach \bar{T} at some finite time τ , then the policymaker must be indifferent between bearing the time τ cost defined by the Hamiltonian and bearing the time τ portion

¹⁰One might attempt to set up the original constrained minimization problem using the necessary conditions of optimal control theory via a Lagrangian formulation. However, the rank constraint qualification is violated by the abatement control's failure to appear in the temperature constraint, which precludes the application of the standard necessary conditions for an optimal path, and the standard sufficiency conditions are violated by the concavity of the forcing function (Caputo, 2005).

¹¹The terminal condition on M follows from recognizing that $M(\tau) > \bar{M}$ means that temperature will rise above \bar{T} in the next instant and that $M(\tau) < \bar{M}$ implies that less abatement could have been undertaken while satisfying the constraint.

of the total “scrap cost” incurred by maintaining temperature at \bar{T} . If the right-hand side is greater, then the policymaker would find it cheaper to delay reaching \bar{T} , and if the left-hand side is greater, then the policymaker would find it cheaper to reach \bar{T} earlier. The formal analysis shows that this transversality condition implies that as t approaches τ , abatement approaches the level required to hold the system at \bar{M} and \bar{T} .

The least-cost abatement trajectory sets the marginal cost of abatement equal to the shadow cost of CO₂, as given by $\lambda_M(t)$ in equation (5). This is a familiar condition from economic analysis of climate change: the shadow cost of CO₂ defines the benefits of the next unit of emission reductions, which must equal the marginal cost of emission reductions along an optimal path. And the marginal cost of abatement is determined by a policy such as a carbon tax or a cap-and-trade program. However, the dynamics of the shadow cost of CO₂ are more interesting than commonly recognized. We begin by interpreting the shadow cost of temperature, and we then provide an explicit form for the shadow cost of CO₂ and rederive it from an intuitive no-arbitrage condition. We conclude this section with formal results about the optimal trajectories and the time when temperature reaches \bar{T} . The appendix develops a phase portrait analysis with further intuition.

First, note that all shadow costs are positive: another unit of temperature or CO₂ requires additional abatement, which raises the cost of the policy program. The shadow cost of temperature increases monotonically: because temperature itself monotonically approaches the constraint along a least-cost path, suddenly raising temperature by one unit leaves less room for adjustment the later it occurs. In fact, using equation (7), the shadow cost of temperature obeys a familiar Hotelling-like condition, adjusted for the effects of climatic inertia:

$$\lambda_T(t) = \lambda_T(t_0) e^{(r+\phi)(t-t_0)}. \quad (9)$$

Along an efficient policy path, the policymaker must be indifferent between accepting another unit of warming in any two instants. The benefit of delaying a unit of warming is composed of the time benefit $r \lambda_T(t)$ of delaying the cost by one more instant and also the inertial benefit $\phi \lambda_T(t)$ of beginning the following instant with a lower temperature. If there is extremely high inertia (ϕ small), then temperature would not have changed much between the two instants and the inertial benefit is small. But if there is extremely low inertia (ϕ large), then temperature would have changed a lot and the inertial benefit is high. Along an efficient path, these benefits must balance the additional cost imposed by delaying the temperature increase ($\dot{\lambda}_T(t) > 0$). Equating these benefits and costs yields the Hotelling-like condition.

The least-cost abatement policy is determined by the shadow cost of CO₂. From the costate equation (6), the evolution of the least-cost abatement policy is controlled by two terms. A first, positive term is the standard decay-adjusted Hotelling condition familiar from past literature. This term tends to increase abatement effort over time. The second, negative term is novel: it decreases abatement effort over time. This inertia adjustment works to slow the increase in abatement effort by taking advantage of the climate system’s braking services. As time passes and temperature begins approaching \bar{T} , $\lambda_T(t)$ becomes

arbitrarily large while $M(t)$ begins decreasing to \bar{M} and $\lambda_M(t)$ approaches the level required to hold CO₂ at \bar{M} . The growth of $\lambda_T(t)$ eventually dominates, which means that least-cost abatement effort tends to decrease as temperature approaches the constraint.

This second term exists because of the climate system's inertia.¹² The change in temperature is a fraction ϕ of the difference between “equilibrium” temperature $sF(M(t))$ and current temperature $T(t)$. If CO₂ were held constant, temperature would eventually reach its equilibrium level $sF(M(t))$. As $\phi \rightarrow 1$, inertia disappears and temperature immediately moves to its equilibrium level $sF(M(t))$. The change in temperature would therefore equal the change in forcing. Emissions directly pass through to temperature. However, in the presence of inertia, emissions no longer directly affect the contemporaneous change in temperature. Instead, net emissions affect how the CO₂ stock changes (i.e., $E - A(t)$ affects $\dot{M}(t)$), and the time t change in the CO₂ stock affects the second derivative of time t temperature (i.e., $\dot{M}(t)$ affects $\ddot{T}(t)$). Emissions (and abatement) therefore affect the contemporaneous *acceleration* in temperature, not the contemporaneous change in temperature. As a consequence, the effects of additional emissions are both delayed and persistent.

Using equations (6) and (9), the appendix shows that the marginal cost of abatement obeys the following relationship along the least-cost trajectory:

$$\lambda_M(t_0) = e^{-[r+\delta](t-t_0)}\lambda_M(t) + e^{-[r+\delta](t-t_0)}\lambda_T(t) \int_{t_0}^t e^{-(\phi-\delta)(t-i)} \frac{\phi s \alpha}{M(i)} di, \quad (10)$$

recalling that $C'(A(t)) = \lambda_M(t)$. The left-hand side is the present cost of abating an additional unit of CO₂ at time t_0 . The right-hand side is the present benefit of abating an additional unit of CO₂ at time t_0 . This benefit is determined by how the additional unit of abatement changes the state variables over time. First, abating an additional unit at time t_0 allows the policymaker to abate fewer units at time t . However, because CO₂ decays at rate δ , abating an additional unit at time t_0 does not enable the policymaker to reduce time t abatement by a full additional unit. If the target were expressed in units of CO₂ rather than temperature, then this would be the only term, and the shadow cost of abatement would grow at the decay-adjusted discount rate: $r + \delta$. This Hotelling-like condition recognizes that the policymaker should spend fewer dollars early because it discounts future spending and because additional CO₂ emissions have more chance to decay when emitted at an earlier time.

But the target is expressed in units of temperature, not CO₂. The second component of the present benefit of additional time t_0 abatement describes how it alters time t temperature by changing temperature (via forcing) between times t_0 and t . The total reduction in time

¹²The $M(t)$ in the denominator arises because forcing is logarithmic in CO₂, but the negative term would exist even under linear forcing.

t temperature from an additional unit of time t_0 abatement is:

$$\chi(t) \triangleq -\frac{dT(t)}{dA(t_0)} = -\int_{t_0}^t \frac{d\dot{T}(i)}{dA(t_0)} di = \int_{t_0}^t \left[e^{-\delta(i-t_0)} \frac{\phi s\alpha}{M(i)} + \underbrace{\phi \int_{t_0}^i \frac{d\dot{T}(j)}{dA(t_0)} dj}_{-\chi(i)} \right] di,$$

where the last equality uses equation (3) and recognizes that $dM(t)/dA(t_0) = -e^{-\delta(t-t_0)}$. Differentiate to obtain

$$\dot{\chi}(t) = e^{-\delta(t-t_0)} \frac{\phi s\alpha}{M(t)} - \phi \chi(t).$$

Integrating yields

$$\chi(t) = e^{-\phi(t-t_0)} \left[k + \int_{t_0}^t e^{(\phi-\delta)(i-t_0)} \frac{\phi s\alpha}{M(i)} di \right].$$

The constant k is $dT(t_0)/dA(t_0)$. However, by equation (3), time t_0 abatement does not affect time t_0 temperature. Therefore $k = 0$. Rearranging yields:

$$\chi(t) = e^{-\delta(t-t_0)} \int_{t_0}^t e^{-(\phi-\delta)(t-i)} \frac{\phi s\alpha}{M(i)} di > 0.$$

The exponential term inside the integral describes how additional time t_0 abatement changes time i forcing and how a change in time i forcing changes time t temperature. The present value of the effect of additional time t_0 abatement on time t temperature is $e^{-r(t-t_0)} \lambda_T(t) \chi(t)$, which is the second term on the right-hand side of equation (10).

It is instructive to examine how CO_2 decay and inertia affect $\chi(t)$. First, greater CO_2 decay (greater δ) reduces the time t temperature benefit from time t_0 abatement because early CO_2 has less of an effect on future forcing:

$$\frac{d\chi(t)}{d\delta} = e^{-\phi(t-t_0)} \int_{t_0}^t -i e^{(\phi-\delta)(i-t_0)} \frac{\phi s\alpha}{M(i)} di < 0.$$

If time t_0 CO_2 does not persist as long, then the forcing effect of time t_0 abatement also does not persist as long. In contrast, reduced inertia (greater ϕ) has conflicting effects: it increases the importance of late changes in forcing (increasing $\chi(t)$), but it reduces the importance of early changes in forcing (decreasing $\chi(t)$):

$$\begin{aligned} \frac{d\chi(t)}{d\phi} &= e^{-\phi(t-t_0)} \int_{t_0}^t [1 + (i-t)\phi] e^{(\phi-\delta)(i-t_0)} \frac{s\alpha}{M(i)} di \\ &= e^{-\phi(t-t_0)} \left[\int_{t_0}^{\hat{t}} [1 + (i-t)\phi] e^{(\phi-\delta)(i-t_0)} \frac{s\alpha}{M(i)} di + \int_{\hat{t}}^t [1 + (i-t)\phi] e^{(\phi-\delta)(i-t_0)} \frac{s\alpha}{M(i)} di \right], \end{aligned}$$

where $\hat{t} \triangleq \max\{t_0, t - \phi^{-1}\}$. The first integral is negative, describing the effect of reduced inertia operating through early changes in forcing, and the second integral is positive, describing the effect of reduced inertia operating through late changes in forcing. The net effect of reducing inertia depends on how the CO₂ concentration evolves over the interval, because high CO₂ concentrations make forcing less responsive to abatement. For times t sufficiently close to t_0 (i.e., for $t \leq t_0 + \phi^{-1}$), the first integral vanishes, in which case a reduction in inertia unambiguously increases $\chi(t)$. The initial portion of the carbon price trajectory is therefore flatter when there is more inertia in the system.

The following proposition establishes several characteristics of the least-cost trajectory:

Proposition 2. *Let τ be the first time at which $T(t) = \bar{T}$, and let x be the last time prior to τ at which $M(t)$ is nondecreasing. If $\dot{M}(t_0) > 0$, then $x > t_0$, $\dot{\lambda}_M(x) > 0$, and there exists a unique time $y \in (x, \tau)$ at which $\lambda_M(t)$ reaches a maximum. Further, for any least-cost trajectory, τ is finite.*

Proof. See appendix. □

The physical dynamics governing CO₂ accumulation require that abatement be increasing at the instant with the highest CO₂ concentration. The economic dynamics governing the evolution of abatement require that CO₂ either be declining or growing at a rate less than $r + \phi$ at the instant with the greatest abatement effort. If the CO₂ concentration is initially increasing along an optimal trajectory, then its peak occurs while abatement effort is still increasing, and abatement peaks during the later period in which CO₂ is declining towards \bar{M} . If the CO₂ concentration starts above \bar{M} and temperature starts sufficiently close to \bar{T} , then efficient abatement effort immediately begins reducing the level of CO₂. The efficient abatement path may be initially increasing or decreasing. In all cases, the policymaker steers the system so that it reaches \bar{T} in finite time. A policy that reduces abatement sufficiently to reach \bar{T} is cheaper than a similar policy that holds abatement just high enough to approach \bar{T} only asymptotically.

Finally, we note that a Hotelling-like trajectory can re-emerge in two types of policy environments. First, if the environmental constraint is expressed in terms of forcing rather than temperature, then the problem becomes equivalent to a constraint on CO₂. In this case, the least-cost carbon price grows at rate $r + \delta$. Second, if the policymaker has access to “geoengineering” technologies which allow her to directly reduce forcing by, for instance, shooting reflective particles into the atmosphere, then the appendix shows that the least-cost path for deploying these technologies has their marginal cost grow at rate $r + \phi$. Intuitively, the geoengineering control directly affects temperature, so an efficient policy pathway equates its marginal cost to the shadow cost of temperature. And we have already seen that the shadow cost of temperature grows at rate $r + \phi$, reflecting both the time benefit and the inertial benefit of delaying a unit of warming.

3 Numerical Example

We have seen that economic analyses of temperature limits have implemented policy programs that would be efficient for CO₂ limits but not for temperature limits. Is this distinction important? We next develop a calibrated numerical example in order to gain further qualitative intuition and to provide a first estimate of the gains from using the correct least-cost policy program.

As detailed in the appendix, we calibrate our theoretical setting's parameters to DICE-2007 (Nordhaus, 2008) via its implementation with an annual timestep in Lemoine and Traeger (2014). To solve the four-dimensional system of differential equations defined in Section 2, we begin with a triplet $(T(\tau), M(\tau), \lambda_M(\tau))$ such that $T(\tau) = \bar{T}$, $M(\tau) = \bar{M}$, and $\lambda_M(\tau)$ equals the marginal abatement cost that holds CO₂ constant at \bar{M} .¹³ We then seek the value of $\lambda_T(\tau)$ consistent with these conditions and with the initial conditions. For a given value of $\lambda_T(\tau)$, we solve the system of ordinary differential equations (1), (3), (6), and (7) from τ but with time flowing in reverse.¹⁴ In the resulting simulation, let x be the time t at which $M(t) = M_0$. At a solution to the system, it must also be the case that $T(x) = T_0$. An optimization routine searches for the value of $\lambda_T(\tau)$ such that $T(x) = T_0$. At a solution, the values $\lambda_M(x)$ and $\lambda_T(x)$ are the efficient $\lambda_M(t_0)$ and $\lambda_T(t_0)$.¹⁵ Using these initial values, we then simulate the model forward in actual time, setting $\lambda_M(t)$ to hold $M(t)$ constant at $M(\tau)$ for all times $t > \tau$.

Figure 1 shows the results for temperature limits of 2°C (left column) and 2.5°C (right column). The solid lines depict the least-cost pathways, and the dashed lines depict the standard Hotelling solution, which corresponds to the solution of a model constrained to limit CO₂ to \bar{M} . The climate system's inertia enables the least-cost policy to postpone abatement to later dates without overshooting \bar{T} . The Hotelling policy abates emissions too aggressively because it fails to take advantage of the climate system's inertia. Its resulting temperature trajectory is therefore lower than required by the temperature limit (top row), and the system's inertia in fact prevents temperature from ever reaching \bar{T} in finite time under the Hotelling policy. Whereas the least-cost policy overshoots \bar{M} by 50–100 ppm (middle row), the Hotelling trajectory never takes advantage of the breathing space afforded by the slowness with which the climate system reacts to overshooting \bar{M} . As a consequence, the carbon price starts out much higher under the Hotelling policy and rises more rapidly until abatement nears its steady-state level (bottom row). However, after the year 2100, the

¹³The condition on $\lambda_M(\tau)$ follows from the transversality condition, as shown in the proof of Proposition 2.

¹⁴In general, we cannot solve the model forward by searching for the initial shadow costs $\lambda_M(t_0)$ and $\lambda_T(t_0)$ that lead the system to obey the terminal conditions because, as is typical of saddle-path stable systems, values slightly off the desired trajectories lead the system to a wildly different outcome. Our solution method is closely related to the “reverse shooting” technique described in Judd (1998).

¹⁵When solving for the Hotelling trajectory, we begin with $M(\tau)$ equal to \bar{M} and $\lambda_M(\tau)$ equal to its corresponding steady-state value. No search is necessary as temperature can be effectively removed from the policymaker's problem.

least-cost policy does end up raising the carbon price to levels beyond any reached under the Hotelling trajectory. As CO₂ overshoots its steady-state level, the least-cost policy begins undertaking aggressive abatement so as to reduce CO₂ before temperature exceeds \bar{T} .¹⁶ Consistent with Proposition 2, the efficient carbon price peaks only after CO₂ has peaked, and the carbon price then declines swiftly towards its steady-state value.

The bottom row of Figure 1 also plots the Hotelling component (dotted lines) of the least-cost carbon price path, as given by the first term in equation (10). Recognizing inertia's braking services makes the least-cost trajectory differ from the Hotelling trajectory in two ways. First, recognizing inertia tends to bend the least-cost trajectory away from its Hotelling component. The gap between the Hotelling component and the least-cost path represents the trajectory adjustment for inertia, which we have seen slows the carbon price's rate of increase. Second, recognizing inertia also reduces the initial carbon price in order to delay abatement. This downward shift in the starting value flattens the Hotelling component of the least-cost trajectory relative to the full Hotelling path (compare the dotted and dashed lines). Near the initial time, the least-cost path differs from the Hotelling path primarily via the downward shift in the initial carbon price. The trajectory adjustment becomes more significant over time, beginning to strongly slow the carbon price's rate of increase near the end of this century, or around the same time that the least-cost CO₂ trajectory peaks.

Figure 2 shows how the strength of inertia (left column) and the choice of discount rate (right column) affect the least-cost trajectories for achieving a 2°C temperature limit. Reducing inertia (i.e., increasing ϕ) means that the least-cost policy has to reduce emissions faster in order to avoid \bar{T} : temperature increases faster than in the baseline case even as CO₂ follows a lower trajectory (dashed lines). In contrast, increasing inertia (i.e., reducing ϕ) means that the effect of current CO₂ on temperature is delayed even further. The initial portion of the emission price trajectory is therefore lower and, in line with our analytic results, flatter. CO₂ now peaks over 100 ppm above \bar{M} (dotted lines) even as temperature remains further from \bar{T} . However, even though increasing inertia lowers the initial carbon price, it does strongly raise the eventual peak carbon price (beyond the end of the plotted period) because the high degree of overshoot in CO₂ requires more aggressive abatement in order to return to \bar{M} .

The right column of Figure 2 shows the implications of reducing the annual consumption discount rate from the value of 5.5% used in DICE-2007 to the value of 1.4% used in Stern (2007). By raising the present cost of each unit of future abatement, the lower discount rate flattens the carbon price trajectory, which raises this century's carbon prices and lowers the next century's carbon prices. The initially higher carbon prices imply greater abatement early on, which lowers both the CO₂ and temperature trajectories. By increasing the present cost of future abatement, the lower discount rate reduces the economic importance of inertia. The more that CO₂ overshoots \bar{M} , the more abatement will eventually be needed to bring

¹⁶Even at its peak, abatement does not exceed business-as-usual emissions.

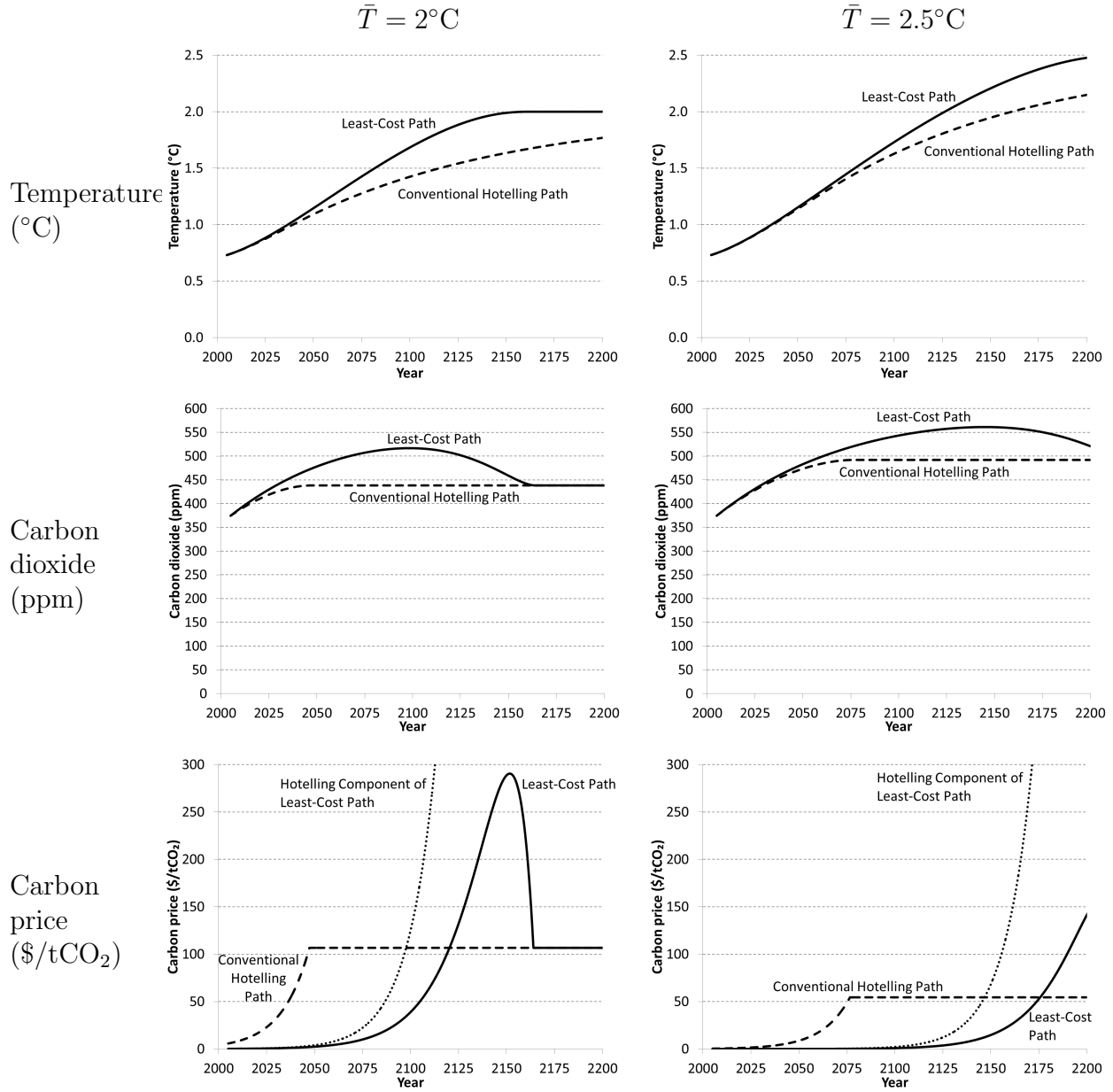


Figure 1: The least-cost trajectories (solid lines) for temperature, CO₂, and the carbon price for temperature limits of $\bar{T} = 2^\circ\text{C}$ (left) and $\bar{T} = 2.5^\circ\text{C}$ (right). Also, the conventional Hotelling-like paths (dashed lines), which are also the least-cost paths for the corresponding CO₂ constraint.

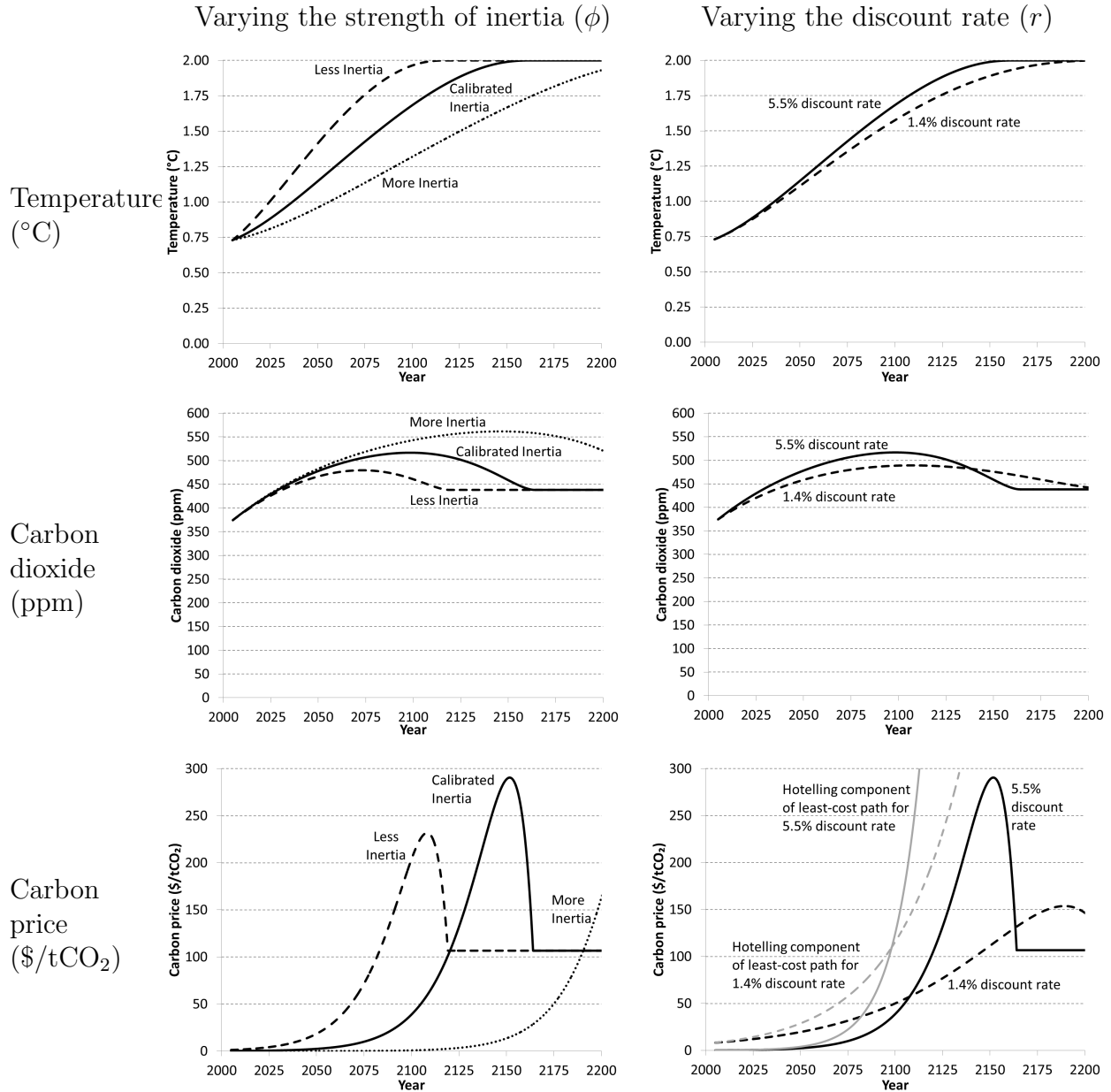


Figure 2: The least-cost trajectories for temperature, CO₂, and the carbon price for a temperature limit of $\bar{T} = 2^\circ\text{C}$. The solid lines show the paths under the baseline calibration. In the left column, dashed lines double ϕ to 0.0182 and dotted lines halve ϕ to 0.0046 (from the baseline value of 0.0091). In the right column, dashed lines lower r to 0.014 (from the baseline value of 0.055).

it back down to \bar{M} before temperature reaches \bar{T} (i.e., the higher the spike in the carbon price seen in the figures' bottom rows). Under the lower discount rate, the least-cost CO₂ trajectory overshoots \bar{M} by only around 50 ppm, less than two-thirds of the overshoot under the higher discount rate. Because CO₂ follows a flatter trajectory, temperature tracks CO₂ better than in the cases with a higher discount rate. The Hotelling component of the least-cost carbon price (gray lines) therefore plays a stronger role under the lower discount rate; inertia plays less of a role in dampening the increase in the carbon price when the policy path is doing less to take advantage of inertia. Because a primary advantage of inertia is the ability to postpone abatement, the policy implications of inertia depend strongly on the choice of discount rate.

Table 1 describes how the present cost of the policy program, the year 2005 carbon price, and cumulative abatement over the next 200 years vary with the temperature limit \bar{T} and with the recognition of climatic inertia. In all cases, the cost of the policy program and the initial carbon price both decline strongly in \bar{T} . The top panel reports results with the DICE-2007 consumption discount rate of 5.5%. By taking advantage of the climate system's inertia, the least-cost policy path can save over \$2 trillion as compared to the conventional Hotelling path. Recognizing inertia allows the policymaker to save money both by postponing abatement and by undertaking less cumulative abatement. The climate system's inertia allows for greater natural decay of CO₂ because it delays the temperature consequences of CO₂ emissions (granting more time for decay) and because it allows the CO₂ concentration to overshoot its steady-state level (decay is proportional to the quantity of CO₂).¹⁷ The interaction between inertia and CO₂ decay generates benefits even in the absence of discounting, and the interaction between inertia and discounting generates benefits even in the absence of decay. The ability to postpone emission reductions and to undertake fewer emission reductions in total lowers the initial carbon price by over 90%.

The bottom panel of Table 1 reports results with the Stern (2007) consumption discount rate of 1.4%. Adopting a CO₂ target instead of a temperature target (or failing to recognize inertia when designing the policy program) now increases costs by only a factor of 2–6 because inertia is more valuable under higher discount rates. However, lowering the discount rate also raises the overall present cost of each policy program, which means that the savings from adopting the temperature targets instead of their corresponding CO₂ targets nonetheless amount to between \$300 billion and \$15 trillion.

4 Conclusions

We have shown that the standard assumption of a decay-adjusted Hotelling path for the price of carbon is not the efficient path under a temperature target. Instead, the efficient path

¹⁷The concavity of forcing in the stock of CO₂ amplifies the benefits of inertia because it allows the policymaker to overshoot the steady-state CO₂ level by a greater amount and for a longer period of time.

Table 1: The present cost of each policy program, the initial efficient carbon price, and cumulative abatement over the next 200 years.

	Temperature limit (°C)		
	2	2.5	3
<i>5.5% discount rate</i>			
Cost of efficient path from 2005–2200 (\$billion)	98	1.5	0.0001
Cost of Hotelling path from 2005–2200 (\$billion) ^a	2,465	181	1.4
CO ₂ price along the efficient path in 2005 (\$/tCO ₂)	0.18	0.003	0.000003
CO ₂ price along the Hotelling path in 2005 (\$/tCO ₂) ^a	5.8	0.39	0.003
Abatement from 2005–2200 along the efficient path (Gt C)	682	235	6.6
Abatement from 2005–2200 along the Hotelling path (Gt C)	891	522	169
<i>1.4% discount rate</i>			
Cost of efficient path from 2005–2200 (\$billion)	12,366	1,125	9
Cost of Hotelling path from 2005–2200 (\$billion) ^a	28,464	6,094	315
CO ₂ price along the efficient path in 2005 (\$/tCO ₂)	8.3	1.3	0.05
CO ₂ price along the Hotelling path in 2005 (\$/tCO ₂) ^a	21	4.2	0.26
Abatement from 2005–2200 along the efficient path (Gt C)	716	296	52
Abatement from 2005–2200 along the Hotelling path (Gt C)	899	537	184

^a The Hotelling paths are also the least-cost paths for the corresponding CO₂ constraint.

combines the standard Hotelling term with a second term that reflects the climate system's inertia. Recognizing inertia means that the efficient CO₂ trajectory tends to overshoot the steady-state level corresponding to the temperature target. Further, recognizing inertia reduces the cost of the policy program and also tends to reduce the next decades' carbon prices in exchange for increasing the next century's carbon prices. Numerically, these effects are substantial, with the conventional Hotelling assumption inflating the estimated cost of a 2°C target by up to a factor of 10 and inflating the initially efficient carbon dioxide price by over \$5 per ton.

Our results have implications for both policy and economic modeling. In terms of policy, some scientists have advocated for adopting temperature targets instead of CO₂ targets because they allow greater flexibility to incorporate new information about how CO₂ affects temperature (e.g., Allen and Frame, 2007; Roe, 2010). And some economists have argued for temperature targets as a means of directly limiting climate damages rather than trusting the impact models embedded in cost-benefit assessments (e.g., Richels et al., 2004; Ackerman et al., 2009). We identify a new advantage that is independent of these concerns. Framing policy in terms of a limit on temperature rather than on CO₂ allows the efficient policy trajectory to take advantage of the climate system's inertia. This efficient trajectory temporarily exceeds the corresponding CO₂ limit. The flexibility granted by the climate system's inertia substantially reduces the present cost of the policy program, though we assume that the temporary "overshoot" in CO₂ does not impose additional environmental costs through, for instance, ocean acidification.

Second, numerical general equilibrium models are the primary tool for estimating the cost of proposed climate policies. However, we have shown that common implementations of these models typically overstate the cost of temperature targets. These implementations tend to assume Hotelling price paths (e.g., Thomson et al., 2011; Bauer et al., 2015) and/or represent a temperature constraint via a constraint on forcing or CO₂ (e.g., Azar et al., 2010; Edenhofer et al., 2010; Kriegler et al., 2014). Given the high degree of inertia in the climate system, the savings from taking advantage of inertia can be a large fraction of the estimated costs. Furthermore, these technology-rich integrated assessment models are used to learn about the relative values of prospective low-carbon technologies, but this relative value likely depends on whether the carbon price follows a Hotelling path or instead follows the inverse-U-shaped trajectory described in the present paper. Implementing the more complex price path is likely to complicate the models' solution algorithms, but our results suggest high payoffs since the simpler implementations appear likely to produce highly misleading results. Given that international policy discussions are focusing on temperature limits, it should be a high priority to reassess these models' conclusions using frameworks that take advantage of the braking services provided by the climate system's inertia.

First Appendix: Formal Analysis

This appendix contains proofs and the derivation of equation (10). The second appendix contains a phase portrait analysis of the least-cost policy, details of the numerical calibration, and analysis of a case in which the policymaker has access to a “geoengineering” technology.

Proof of Proposition 1

Assumption 1 implies that temperature along a least-cost path must either reach \bar{T} in finite time or approach it asymptotically from below. By equation (3), there exists $\epsilon > 0$ such that if $T(t) \in (\bar{T} - \epsilon, \bar{T})$ and $\dot{T}(t) > 0$, then $\dot{M}(t) > 0$. And once temperature attains \bar{T} , CO_2 must remain no larger than \bar{M} in order to prevent temperature from rising past the constraint. Further, Assumption 1 also implies that along a least-cost trajectory, CO_2 must remain no less than \bar{M} once temperature has attained \bar{T} . CO_2 must be strictly above \bar{M} at some instant before temperature attains \bar{T} , and CO_2 must remain fixed at \bar{M} once temperature attains \bar{T} . Therefore, along any least-cost trajectory, there exists some time q such that $\dot{M}(t) \leq 0$ for all times $t \geq q$ and such that $\dot{M}(t) < 0$ for some time $t \geq q$. This establishes the first part of the proposition.

The second part follows immediately from observing that a policymaker constrained to keep CO_2 no greater than the steady-state level \bar{M} corresponding to \bar{T} never lets temperature reach \bar{T} . Any path that satisfies the constrained CO_2 problem therefore also satisfies the corresponding constrained temperature problem. However, we have seen that the least-cost CO_2 trajectory must exceed \bar{M} in the constrained temperature problem. The least-cost path that satisfies the temperature constraint therefore does not satisfy the corresponding CO_2 constraint. Constraining CO_2 introduces an additional binding constraint that strictly increases the cost of the least-cost policy pathway.

□

Proof of Proposition 2

First consider the CO_2 trajectory. We know by Proposition 1 that it is nonincreasing after some time prior to τ . Combined with the assumption that $\dot{M}(t_0) > 0$, we have that there exists a last time $x \in (t_0, \tau)$ at which $\dot{M}(t)$ is nondecreasing. At this interior maximum, it must be the case that $\dot{M}(t) = 0$ and $\ddot{M}(t) < 0$. Differentiating equation (1), we have

$$\ddot{M}(t) = -\dot{A}(t) - \delta \dot{M}(t).$$

At a point where $\dot{M}(t) = 0$, $\ddot{M}(t) < 0$ if and only if $\dot{A}(t) > 0$. We know by equation (5) that marginal abatement cost equals the shadow cost of CO_2 . This establishes that $\dot{\lambda}_M(x) > 0$.

At an interior maximum of $\lambda_M(t)$, it must be the case that $\dot{\lambda}_M(t) = 0$ and $\ddot{\lambda}_M(t) \leq 0$. Differentiating equation (6), we have:

$$\ddot{\lambda}_M(t) = (r + \delta) \dot{\lambda}_M(t) + \left[\frac{\dot{M}(t)}{M(t)} - (r + \phi) \right] \phi s \alpha \frac{\lambda_T(t)}{M(t)}.$$

At a point where $\dot{\lambda}_M(t) = 0$, $\ddot{\lambda}_M(t) \leq 0$ if and only if $\dot{M}(t)/M(t) \leq r + \phi$. Recognizing that $\dot{M}(t) < 0$ at all times $t \in (x, \tau)$, we have that $\ddot{\lambda}_M(t) < 0$ at any $t \in (x, \tau)$ for which $\dot{\lambda}_M(t) = 0$.

We have already seen that $\dot{\lambda}_M(x) > 0$. Now consider the first time τ when $T(t) = \bar{T}$. Proposition 1 shows that CO_2 must exceed \bar{M} before returning to \bar{M} , so CO_2 must be higher the instant before τ : $M(\tau - \epsilon) = \bar{M} + \gamma$ for ϵ small and $\epsilon, \gamma > 0$. In order to achieve the temperature limit in the next moment, abatement must be such that $\dot{M}(\tau - \epsilon) = -\gamma$. This implies that:

$$\dot{M}(\tau - \epsilon) = E - A(\tau - \epsilon) - \delta(\bar{M} - M_{pre} + \gamma) = -\gamma,$$

which holds if and only if:

$$A(\tau - \epsilon) = E - \delta(\bar{M} - M_{pre}) + (1 - \delta)\gamma.$$

To maintain temperature at \bar{T} at time τ , abatement must satisfy $A(\tau) = E - \delta(\bar{M} - M_{pre})$. Therefore abatement is higher the instant before time τ and $\dot{\lambda}_M(\tau - \epsilon) < 0$. By the Intermediate Value Theorem, there exists some time $y \in (x, \tau - \epsilon)$ such that $\dot{\lambda}_M(y) = 0$. We have already established that $\ddot{\lambda}_M(y) < 0$ for all such y , so there is a unique maximum of $\lambda_M(t)$ between times x and τ .

Finally, consider the time τ at which the system achieves \bar{T} . Begin by assuming τ is finite. Assumption 1 guarantees that abatement will be positive from that time onward and completely determined by the level necessary to maintain temperature at \bar{T} . Recognizing that $\dot{M}(\tau) = 0$ and $\dot{T}(\tau) = 0$, the transversality condition becomes

$$e^{-r(\tau-t_0)} C(A(\tau)) = -e^{-r(\tau-t_0)} \frac{\partial V(\bar{M}, \bar{T}, \tau)}{\partial \tau} + r e^{-r(\tau-t_0)} V(\bar{M}, \bar{T}, \tau).$$

The current value of the policy program at time τ is:

$$\begin{aligned} V(\bar{M}, \bar{T}, \tau) &= \int_{\tau}^{\infty} e^{-r(t-\tau)} C \left(E - \delta \left(M_{pre} e^{\bar{T}/s\alpha} - M_{pre} \right) \right) dt \\ &= \frac{1}{r} C \left(E - \delta \left(M_{pre} e^{\bar{T}/s\alpha} - M_{pre} \right) \right). \end{aligned}$$

Therefore,

$$\frac{\partial V(\bar{M}, \bar{T}, \tau)}{\partial \tau} = 0.$$

The transversality condition becomes:

$$C(A(\tau)) = C \left(E - \delta \left(M_{pre} e^{\bar{T}/s\alpha} - M_{pre} \right) \right),$$

which holds if and only if $A(\tau) = E - \delta \left(M_{pre} e^{\bar{T}/s\alpha} - M_{pre} \right)$, where the right-hand side is the abatement level required to hold CO₂ constant at \bar{M} . A trajectory with a finite time τ at which the system reaches (and remains at) \bar{M} and \bar{T} does satisfy the necessary transversality condition.

Now assume that there is no finite τ at which the system attains \bar{T} . The system asymptotically approaches \bar{M} , which means that abatement asymptotically approaches $E - \delta \left(M_{pre} e^{\bar{T}/s\alpha} - M_{pre} \right)$. Therefore

$$\lim_{\tau \rightarrow \infty} \lambda_M(\tau) = C' \left(E - \delta \left(M_{pre} e^{\bar{T}/s\alpha} - M_{pre} \right) \right),$$

which implies

$$\lim_{\tau \rightarrow \infty} \dot{\lambda}_M(\tau) = 0.$$

Using equation (6), we have

$$\lim_{\tau \rightarrow \infty} \dot{\lambda}_M(\tau) = (r + \delta) C' \left(E - \delta \left(M_{pre} e^{\bar{T}/s\alpha} - M_{pre} \right) \right) - \phi s \alpha \lambda_T(t_0) e^{(r+\phi)(\tau-t_0)} \bar{M}^{-1} = -\infty.$$

But $\dot{\lambda}_M(t)$ cannot asymptote to both zero and negative infinity. We have a contradiction. The time τ must be finite.

□

Derivation of equation (10)

Substitute $\lambda_T(t)$ into equation (6):

$$(r + \delta)\lambda_M(t) - \dot{\lambda}_M(t) = \frac{\phi s \alpha}{M(t)} \lambda_T(t_0) e^{(r+\phi)(t-t_0)}.$$

Integrate and rearrange:

$$\begin{aligned} & \int_{t_0}^t \left[-(r + \delta)\lambda_M(i) + \dot{\lambda}_M(i) \right] di = \int_{t_0}^t -\frac{\phi s \alpha}{M(i)} \lambda_T(t_0) e^{(r+\phi)(i-t_0)} di \\ \Leftrightarrow & \int_{t_0}^t \left[-(r + \delta)e^{-(r+\delta)(i-t_0)} \lambda_M(i) + e^{-(r+\delta)(i-t_0)} \dot{\lambda}_M(i) \right] di = \int_{t_0}^t -e^{-(r+\delta)(i-t_0)} \frac{\phi s \alpha}{M(i)} \lambda_T(t_0) e^{(r+\phi)(i-t_0)} di \\ \Leftrightarrow & e^{-(r+\delta)(t-t_0)} \lambda_M(t) - \lambda_M(t_0) = -\phi s \alpha \lambda_T(t_0) \int_{t_0}^t \frac{e^{(\phi-\delta)(i-t_0)}}{M(i)} di. \end{aligned}$$

Substitute in $\lambda_T(t_0) = e^{-(r+\phi)(t-t_0)} \lambda_T(t)$ and rearrange:

$$\lambda_M(t_0) = e^{-[r+\delta](t-t_0)} \lambda_M(t) + e^{-[r+\delta](t-t_0)} \lambda_T(t) \int_{t_0}^t e^{-(\phi-\delta)(t-i)} \frac{\phi s \alpha}{M(i)} di.$$

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Second Appendix

The first section contains the phase portrait analysis. The second derives the least-cost trajectory for a geoengineering control. The third describes the numerical example's calibration.

A Phase portrait analysis

We construct conditional phase portraits in order to better understand the evolution of abatement and CO₂ along a least-cost trajectory. Figure A1 depicts conditional phase portraits for a period with low temperature (top panel) and for a period with high temperature (bottom panel). These two snapshots correspond, respectively, to the early part of this century and to sometime late in this century or early in the next. The emission price (λ_M) is on the vertical axes, and CO₂ (M) is on the horizontal axes. Let $a(\cdot)$ denote the inverse of marginal abatement cost, so that $A(t) = a(\lambda_M(t))$. By the properties of $C(\cdot)$, we have that $a(0) = 0$ and $a'(\cdot) > 0$.

In each panel, the downward-sloping solid curve depicts, from equation (1), the M -nullcline:

$$M(t)|_{\dot{M}(t)=0} = \frac{1}{\delta} [E - a(\lambda_M(t))] + M_{pre}.$$

At these combinations of CO₂ and abatement, the CO₂ concentration is stationary. Decay increases in CO₂, so higher levels of CO₂ become stationary at lower levels of abatement. This curve is linear if abatement cost is quadratic. The downward-sloping dashed curve in each panel depicts, from equation (6), the λ_M -nullcline:

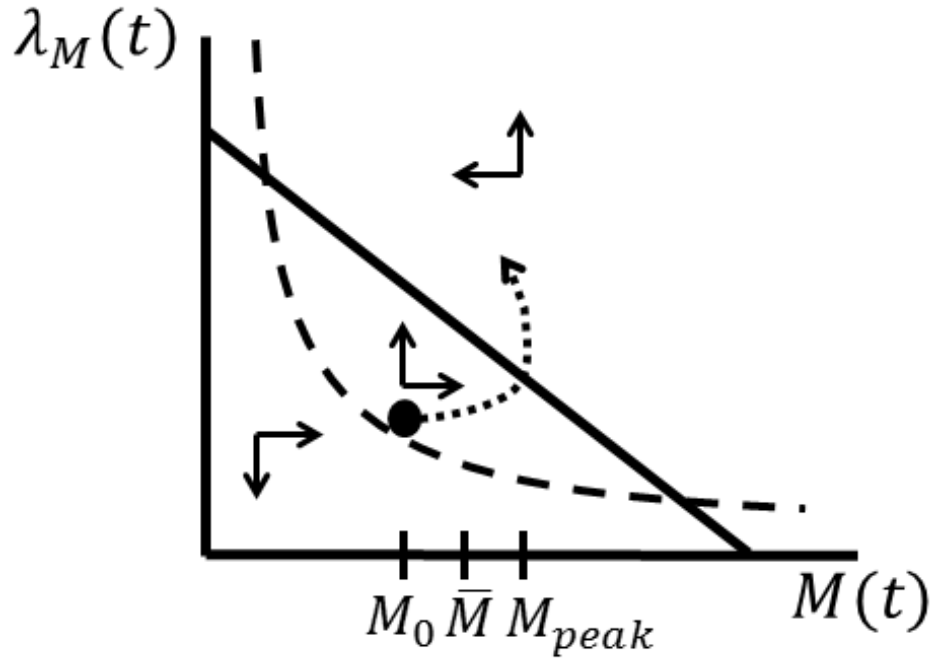
$$\lambda_M(t)|_{\dot{\lambda}_M(t)=0} = \frac{\phi s \alpha}{r + \delta} \frac{\lambda_T(t)}{M(t)} = e^{(r+\phi)(t-t_0)} \frac{\phi s \alpha}{r + \delta} \frac{\lambda_T(t_0)}{M(t)}.$$

At these combinations of CO₂ and abatement, a least-cost trajectory holds abatement constant. The nullcline's convexity arises because forcing is logarithmic in CO₂, and the nullcline shifts out as the shadow cost of temperature increases. The arrows describe the direction of motion in each sector. They follow from recognizing that

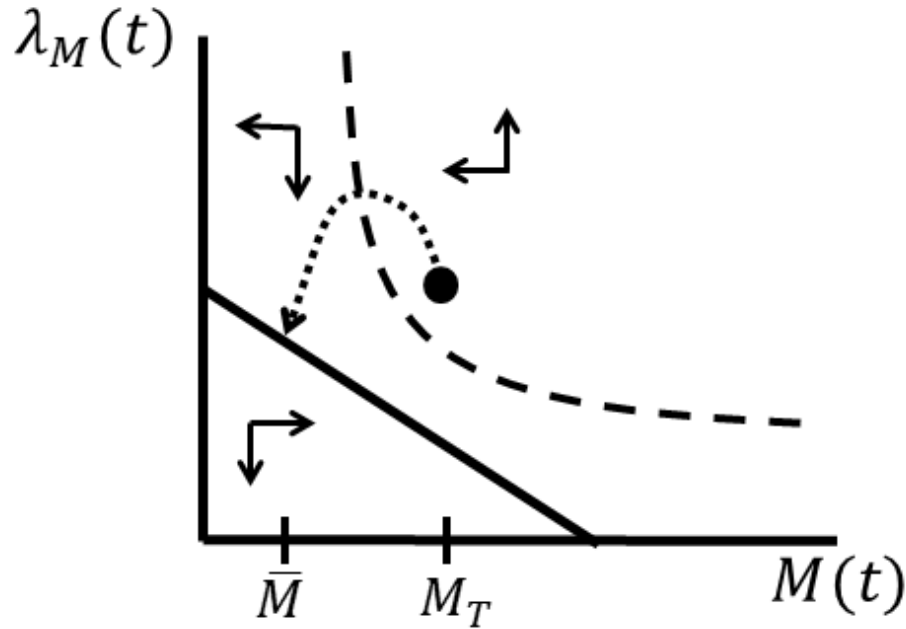
$$\frac{\partial \dot{M}(t)}{\partial \lambda_M(t)} < 0, \quad \frac{\partial \dot{\lambda}_M(t)}{\partial M(t)} > 0.$$

In sectors above (below) the M -nullcline, the direction of motion is to the left (right). In sectors to the right (left) of the λ_M -nullcline, the direction of motion is upward (downward).

The top panel depicts a case in which the nullclines intersect: business-as-usual emissions are sufficiently great that the M -nullcline is pushed out, and temperature is sufficiently far below \bar{T} that its shadow cost is low and the λ_M -nullcline is pushed in. This case corresponds to the present day for a sufficiently lax temperature target. The point M_0 depicts a typical



(a) Near-term



(b) Long-term

Figure A1: Phase portraits conditional on λ_T . Solid curves give the M -nullclines, dashed curves give the λ_M -nullclines, dotted curves depict least-cost trajectories, and arrows give the direction of motion in each sector. The top panel corresponds to a case with $T(t)$ sufficiently far below \bar{T} , and the bottom panel corresponds to a case with $T(t)$ closer to \bar{T} .

starting point, and $\bar{M} > M_0$ indicates the steady-state level of CO₂ corresponding to \bar{T} . The optimal emission price begins by following the dotted curve. It starts at a relatively low level in the space between the two nullclines, and it increases along with CO₂. It eventually crosses the M -nullcline at M_{peak} , at which point CO₂ begins to fall even as abatement continues increasing. This crossing illustrates how the least-cost CO₂ trajectory temporarily overshoots the terminal level \bar{M} .

As time passes, the shadow cost of temperature increases and the λ_M -nullcline shifts out.¹ Eventually we reach a situation such as the bottom panel, where the two nullclines no longer intersect. This corresponds to a world like that in the next century, once temperatures are closer to the chosen limit and once technological change has potentially lowered business-as-usual emissions. It also corresponds to the present world under a sufficiently stringent temperature target. In this panel, CO₂ has already peaked. The story from the last panel finished at a point such as M_T , where we pick up in this panel. As already noted, abatement is increasing and CO₂ is decreasing. The terminal condition has the policymaker hitting the M -nullcline at \bar{M} . As CO₂ falls, the system crosses the λ_M -nullcline. At this point, abatement peaks. As the policymaker steers the system towards \bar{T} , it decreases abatement towards the level compatible with steady-state \bar{M} .

In sum, we have seen that the type of CO₂ trajectory depends on the stringency of the temperature limit. For a sufficiently lax limit, least-cost policy increases CO₂ past its terminal level, relying on the climate system's inertia to avoid crossing \bar{T} . It then decreases CO₂ back towards its terminal level, using both abatement and natural decay. For a sufficiently stringent target, CO₂ begins far enough past its terminal level that abatement policy immediately begins decreasing CO₂. In either case, least-cost abatement policy generally increases before decreasing. This least-cost abatement trajectory looks quite different from the conventionally assumed, monotonically increasing Hotelling-like trajectory, and the least-cost CO₂ trajectory looks quite different from the CO₂ trajectory implied by capping concentrations at the terminal level \bar{M} .

Finally, consider how the least-cost CO₂ trajectory changes with properties of the climate system. In the top panel, whether CO₂ initially increases or decreases depends on how M_0 corresponds to the gap between the nullclines. For sufficiently large M_0 , abatement begins at a sufficiently high level to decrease CO₂. This case is more likely the larger are ϕ , s , α , and $\lambda_T(t_0)$. For a given temperature, larger ϕ (i.e., lower inertia) increases the speed with which warming responds to any CO₂ in excess of \bar{M} . Larger s and α increase the effect of CO₂ on temperature, which decreases \bar{M} and so increases the degree to which M_0 is overshooting \bar{M} . Finally, greater $\lambda_T(t_0)$ corresponds to a more stringent temperature target, which also decreases \bar{M} and increases the degree of overshoot from M_0 .

¹And if business-as-usual emissions exogenously decrease, then the M -nullcline shifts in.

B Least-cost geoengineering trajectory

The only way to achieve a CO₂ target is to reduce emissions or, perhaps, to suck CO₂ directly out of the atmosphere, but a temperature target could be achieved by directly reducing forcing. Geoengineering methods for reducing forcing typically involve “solar radiation management”: if we reflect incoming solar radiation by injecting particles into the atmosphere, by placing mirrors in space, or by brightening the tops of clouds, then we can reduce forcing without reducing greenhouse gases. These methods are drawing increasing attention because they are potentially cheap but also potentially full of surprises and side-effects (Keith, 2000; Shepherd, 2012; Caldeira et al., 2013).

Extend the theoretical setting by allowing for a geoengineering control in the form of solar radiation management. The time t level of the control is $G(t) \geq 0$, and the cost of exercising the control is a strictly increasing, convex function $Z(G)$, where $Z(0) = 0$. The geoengineering control reduces contemporaneous forcing, which changes the temperature transition to

$$\dot{T}(t) = \phi [s \{F(t) - G(t)\} - T(t)]. \quad (\text{A-1})$$

The policymaker’s objective is to select abatement and geoengineering trajectories in order to minimize the present cost of maintaining temperature weakly below \bar{T} :

$$\begin{aligned} V(M(t_0), T(t_0), t_0) = & \min_{A(t), G(t)} \int_{t_0}^{\infty} e^{-r(t-t_0)} [C(A(t)) + Z(G(t))] dt \\ & \text{subject to equations (1) and (A-1),} \\ & T(t) \leq \bar{T}, \\ & A(t) \geq 0, \\ & G(t) \geq 0, \\ & M(t_0) = M_0, T(t_0) = T_0. \end{aligned}$$

The current-value Hamiltonian becomes:

$$\begin{aligned} H(M(t), T(t), A(t), G(t), \lambda_M(t), \lambda_T(t)) = & C(A(t)) + Z(G(t)) \\ & + \lambda_M(t) [E - A(t) - \delta (M(t) - M_{pre})] \\ & + \lambda_T(t) \phi [s \{\alpha \ln(M(t)/M_{pre}) - G(t)\} - T(t)]. \end{aligned}$$

The necessary conditions are unchanged, except that the new temperature transition equation must be obeyed and there is now an additional condition:

$$Z'(G(t)) = \lambda_T(t) \phi s.$$

Along a least-cost path, the marginal cost of geoengineering increases with the shadow cost of temperature, which we have seen increases exponentially at rate $r + \phi$.

C Numerical calibration

We calibrate the example to DICE-2007 (Nordhaus, 2008), as implemented with an annual timestep in Lemoine and Traeger (2014). All baseline runs use the 5.5% annual consumption discount rate ($r = 0.055$) generally consistent with this model.²

The full DICE model includes three carbon reservoirs. Lemoine and Traeger (2014) approximate DICE's full carbon dynamics by making the decay rate of CO₂ a function of the atmospheric CO₂ stock and time. Along the optimal path in DICE, the time-varying decay rate for CO₂ in excess of its pre-industrial level starts at 0.0141, declines to 0.0119 in 100 years, and declines to 0.0068 after 200 years. Using the average value over the first 100 years, we have $\delta = 0.0138$. We calibrate business-as-usual CO₂ emissions E to DICE's initial value. This yields $E = 9.97$ Gt C per year.

In the forcing relationship, we take $M_{pre} = 596.4$ Gt C and follow Ramaswamy et al. (2001, Table 6.2) in using $\alpha = 5.35$ W m⁻², which is approximately equivalent to the parameters used in DICE. The full DICE model includes two temperature reservoirs. Lemoine and Traeger (2014) simplify this setting by representing the deep ocean temperature as a function $\alpha_T(T, t)$ of surface temperature and time. In their discrete-time setting, the temperature transition equation becomes

$$T_{t+1} - T_t = C_T \left[F_{t+1} - \frac{\alpha \ln(2)}{cs} T_t - [1 - \alpha_T(T_t, t)] C_O T_t \right],$$

where we have used cs for climate sensitivity so as to avoid confusion with the present paper's notation. The present paper's parameter s gives equilibrium warming per unit of forcing, whereas DICE's $cs = 3$ gives equilibrium warming from doubled CO₂. Relating the two parameters, we have:

$$s = \frac{cs}{\alpha \ln(2)} = 0.809 \text{ } ^\circ\text{C} [\text{W m}^{-2}]^{-1}.$$

Using explicit Euler difference methods, we find:

$$\phi = \frac{C_T \left[F_{t+1} - \frac{\alpha \ln(2)}{cs} T_t - [1 - \alpha_T(T_t, t)] C_O T_t \right]}{s F_t - T_t}.$$

Along DICE's optimal trajectory, the inferred value of ϕ starts at 0.0129, falls to 0.0056 after 100 years, and falls to -0.0030 after 200 years (reflecting that the ocean begins transferring heat to the atmosphere as the CO₂ concentration declines). Using the average value over the first 100 years, we have $\phi = 0.0091$.

²Technically, this setting with stationary output should use a discount rate no greater than 1.5% to be consistent with DICE-2007: consumption growth in the Ramsey equation is negative once we subtract the cost of abatement.

In DICE, the cost (as a fraction of time t output) of abating a fraction μ_t of business-as-usual emissions is $\Psi_t \mu_t^{a_2}$, where $a_2 = 2.8$ and

$$\Psi_t = \frac{a_0 \sigma_t}{a_2} \left(1 - \frac{1 - e^{(t-t_0)g_\Psi}}{a_1} \right), \text{ with } \sigma_t = \sigma_0 \exp \left[\frac{g_{\sigma,0}}{\delta_\sigma} (1 - e^{-(t-t_0)\delta_\sigma}) \right].$$

The parameters are $a_0 = 1.17$, $a_1 = 2$, $g_\Psi = -0.005$, $\sigma_0 = 0.13$, $g_{\sigma,0} = -0.0073$, and $\delta_\sigma = 0.003$. Initial output Y (without adjusting for climate damages) in DICE is approximately 85 trillion dollars. We represent the cost of abatement $A(t)$ as

$$C(A(t)) = \Psi_0 \left[\frac{A(t)}{E} \right]^{a_2} Y.$$

Finally, from DICE-2007, we have the initial CO₂ stock as $M_0 = 808.9$ Gt C, the initial global mean surface temperature as $T_0 = 0.7307$ °C, and the initial time as $t_0 = 2005$.

We solve the four-dimensional system of differential equations with a constrained optimization solver. The solver selects $\lambda_T(\tau)$ to minimize the distance between T_0 and $T(x)$, where the main text defines x as the time t at which $M(t) = M_0$. We use Matlab's ode23 solver with the finest tolerance that the solver allows. We search for a solution on a 700-year mesh and discretize time to every 0.01 years. When the ode solver returns a trajectory for a given guess of $\lambda_T(\tau)$, we select the time x by finding the latest mesh point where CO₂ is within 0.1 Gt C of M_0 .³ Finally, once the solver has converged to the optimal value of $\lambda_T(\tau)$, we use the trapezoidal method to approximate the integral of abatement cost over the mesh points.

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³In practice, only one mesh point ever satisfies the condition of having CO₂ within 0.1 Gt C of M_0 .

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