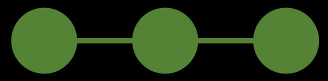
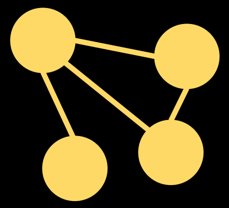
Balanced Trees

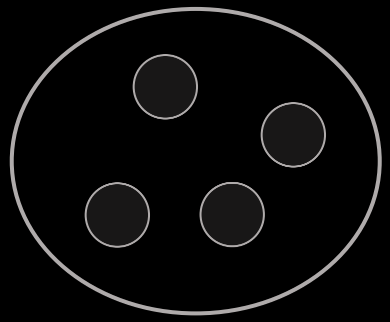
Data Structures & Algorithm

Categories of Data Structures

Linear Ordered	Non-linear Ordered	Not Ordered
Lists	Trees	Sets
Stacks	Graphs	Tables/Maps
Queues		





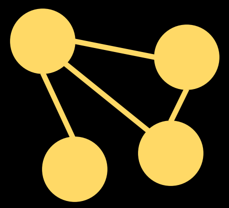


Recap

Non-linear Ordered

Trees

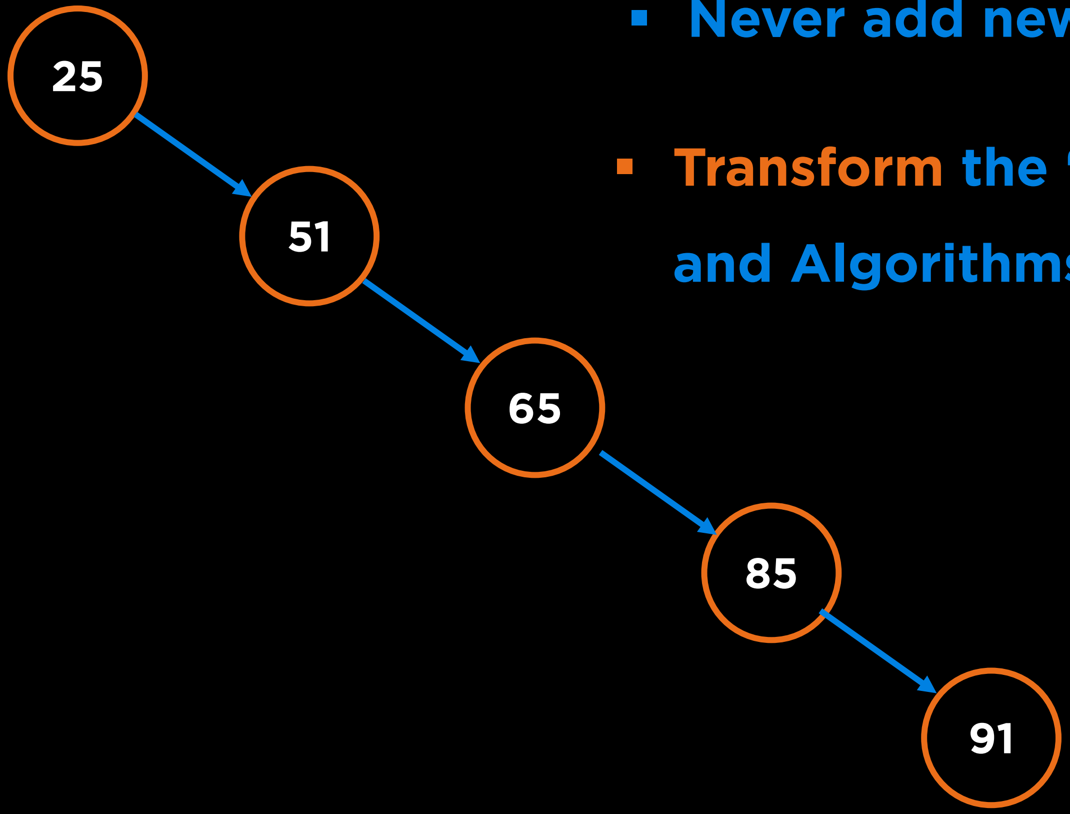
- **Binary Search Trees**
  - Operations
  - Traversals



Agenda

- Trees
    - More Properties Related to Height
- Binary Search Tree Performance
- Rotations
- Balanced Trees: AVL Trees
    - Properties
    - Insertion/Deletion
    - Performance

How do we fix the Worst Case?

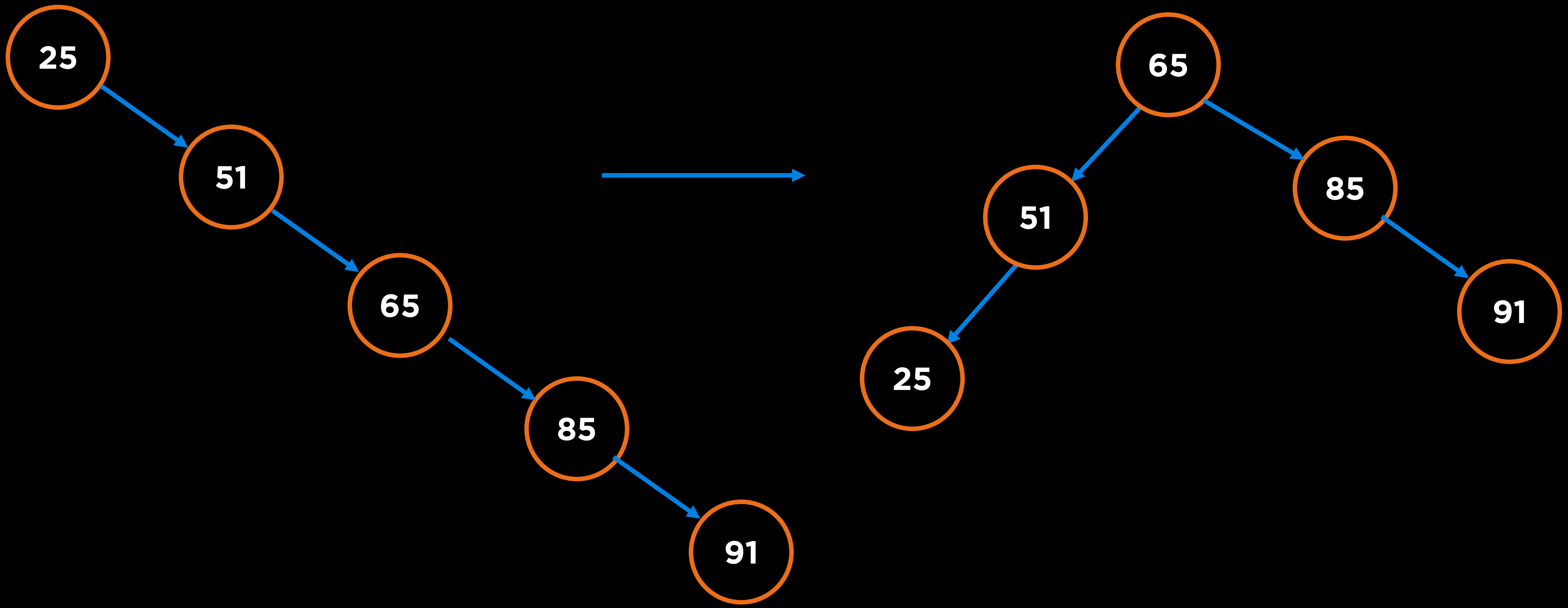


- Never add new leaves at the bottom: Increase size of node
- Transform the "Spindly" Tree to "Bushy Tree" using Tools and Algorithms

Rotations

Rotations

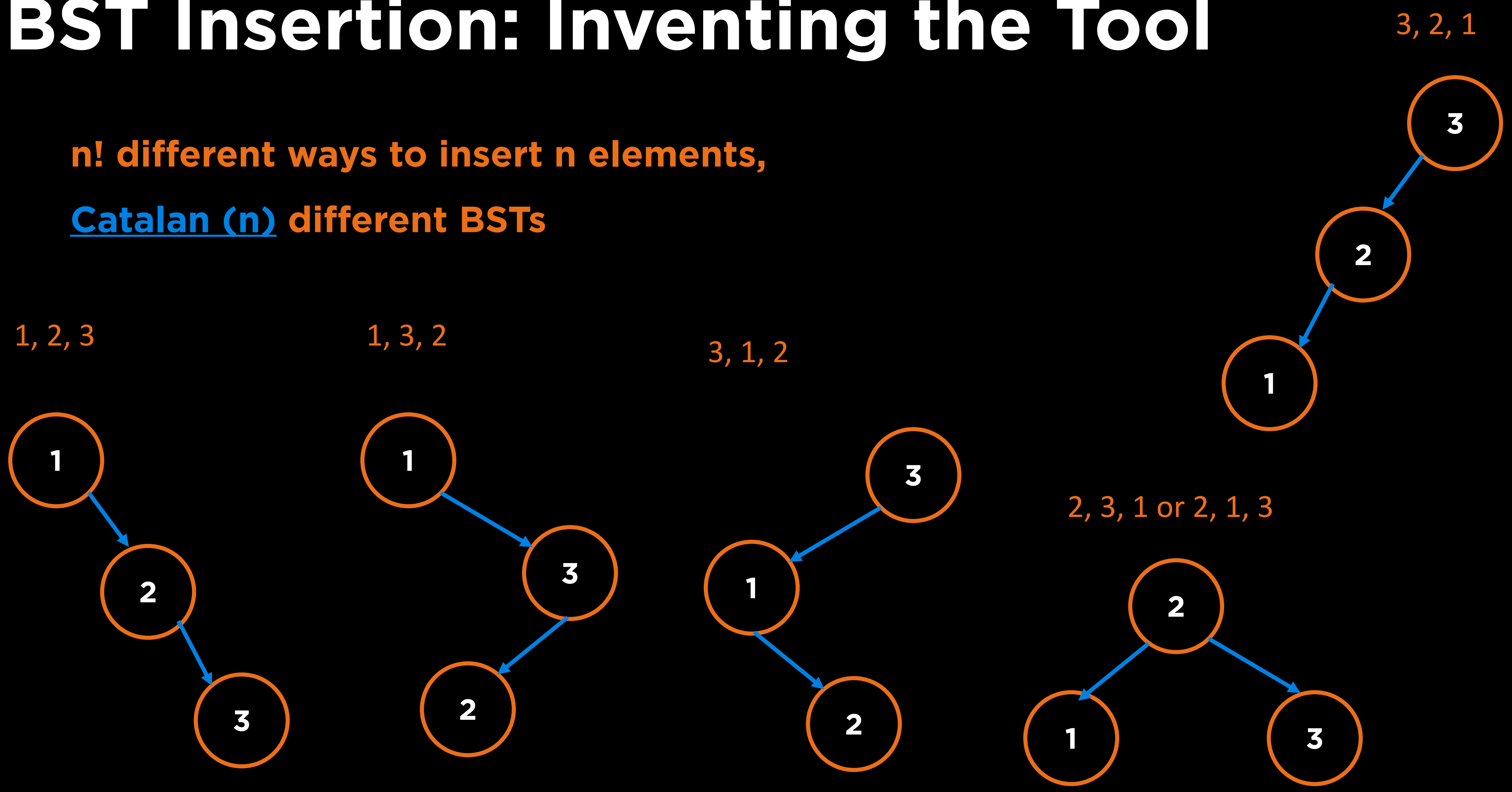
Tools to Rearrange the Tree Without affecting its Semantics



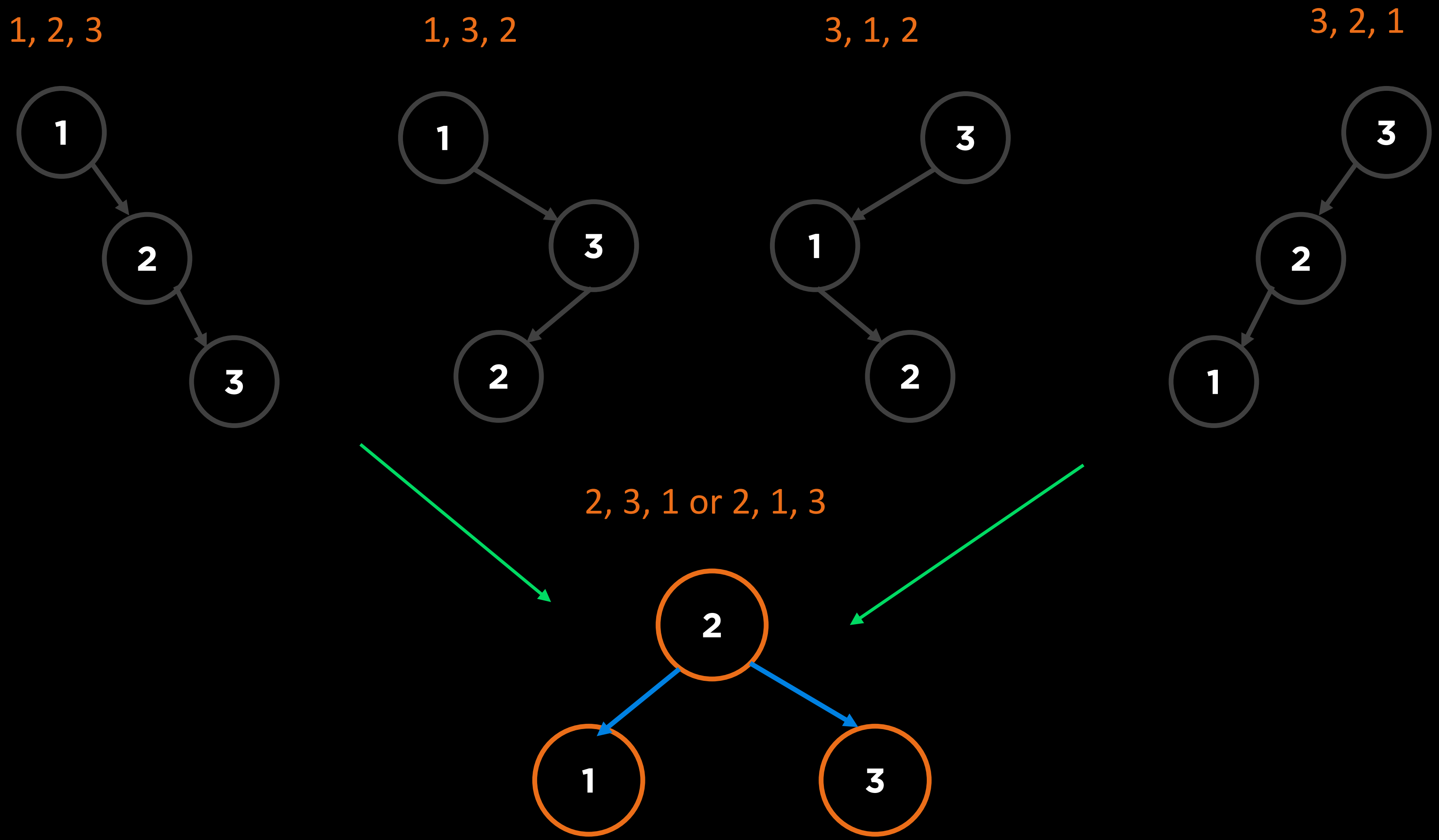
BST Insertion: Inventing the Tool

n! different ways to insert n elements,

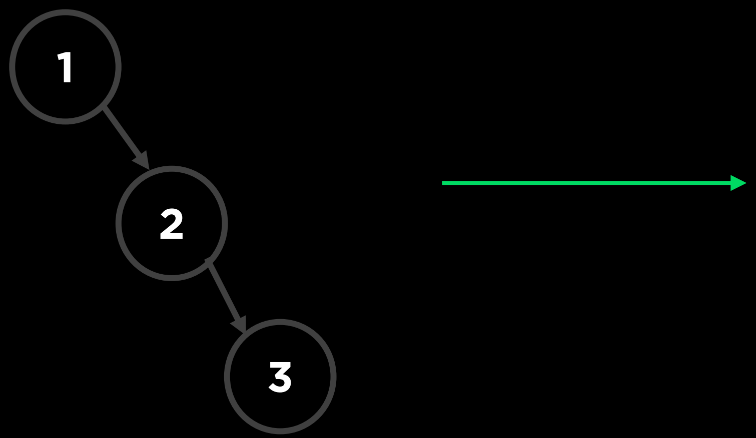
[Catalan (n)]() different BSTs



Goal



Left Rotation: Right Right Case

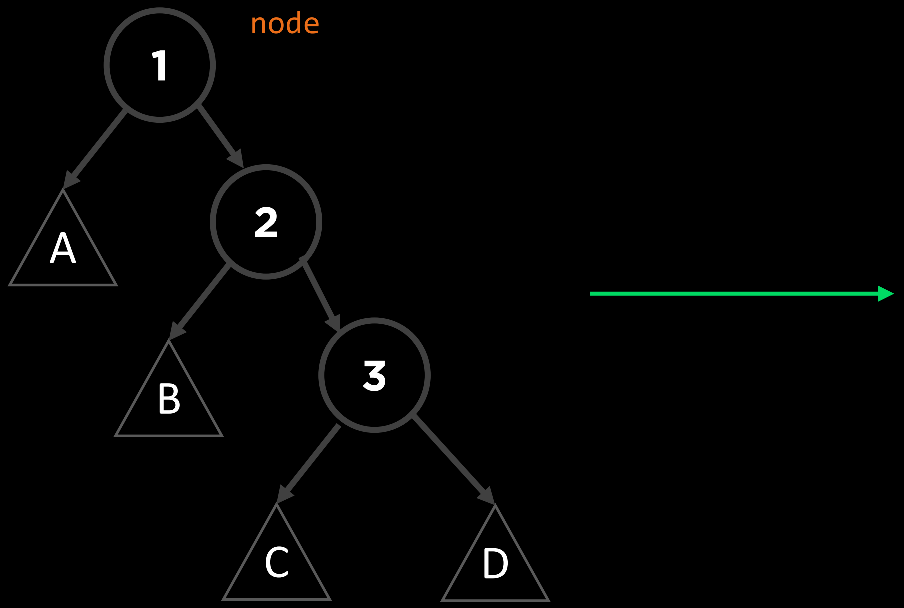


Left Rotation: Right Right Case

Single Rotation

Left Rotation: Right Right Case

```
rotateLeft (node)
{
        grandchild = node->right->left;
        newParent = node->right;
        newParent->left = node;
        node->right = grandchild;
        return newParent;
}
```



Left Rotation: Right Right Case

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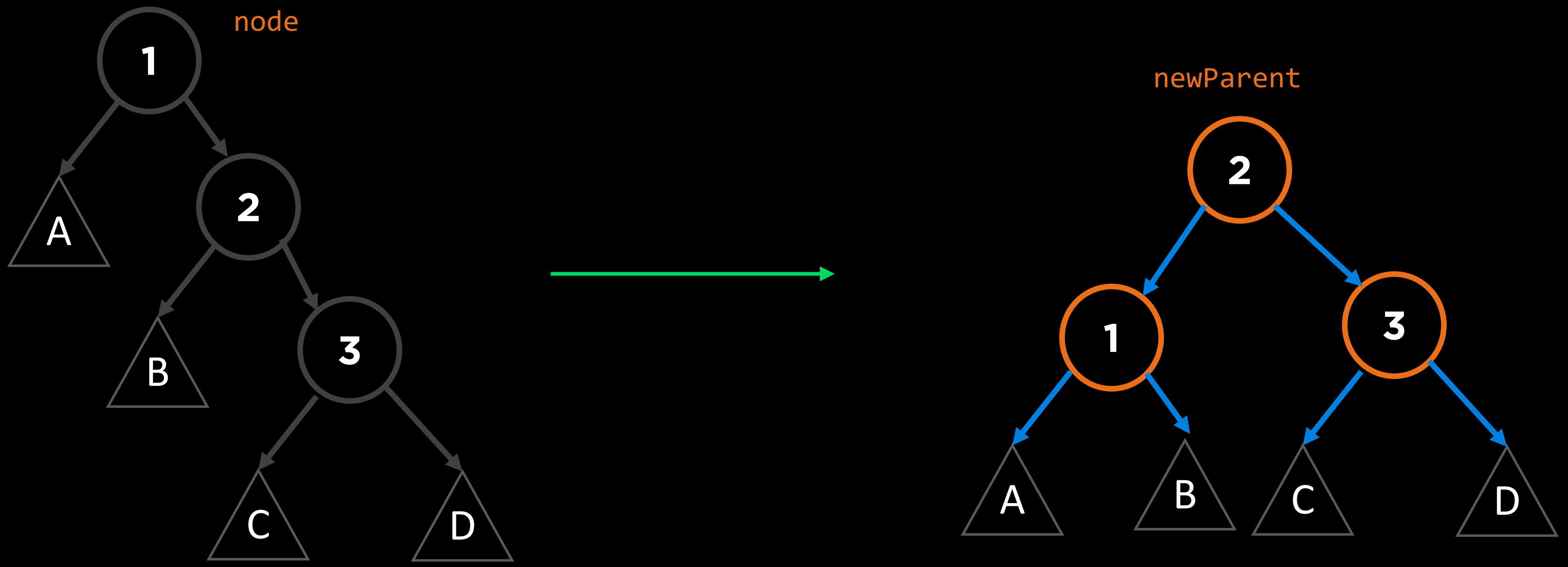

Left Rotation: Right Right Case

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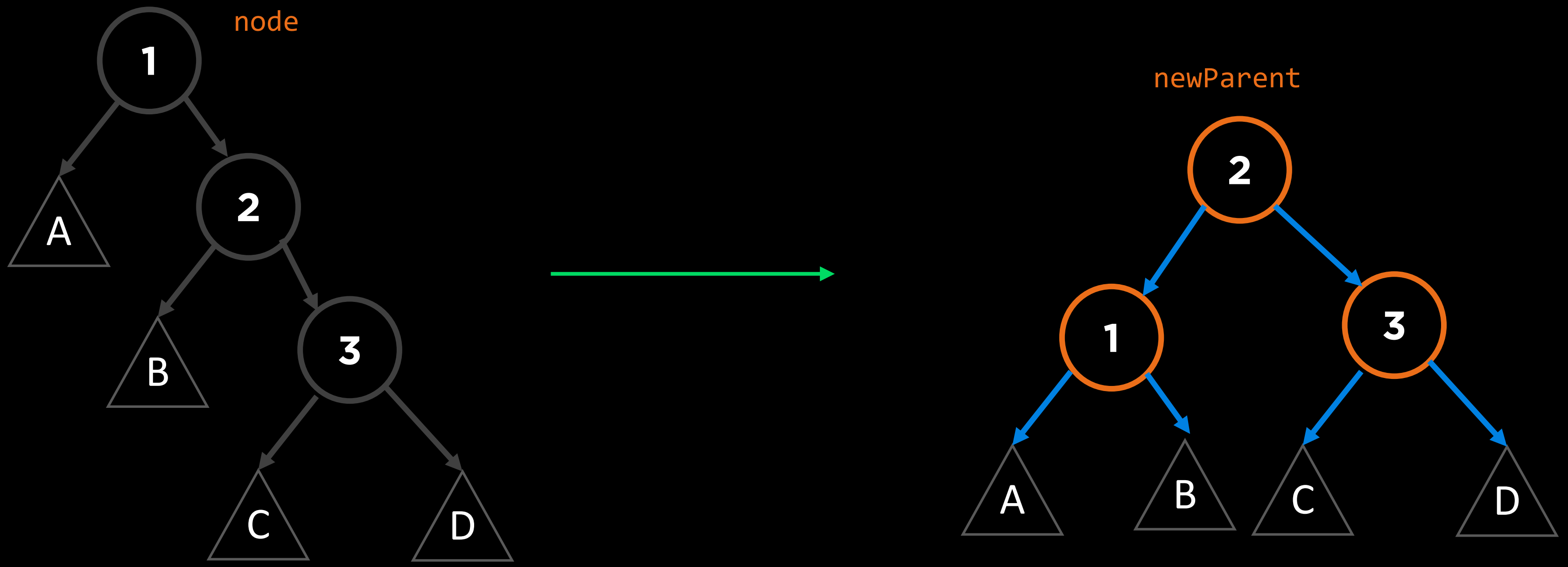
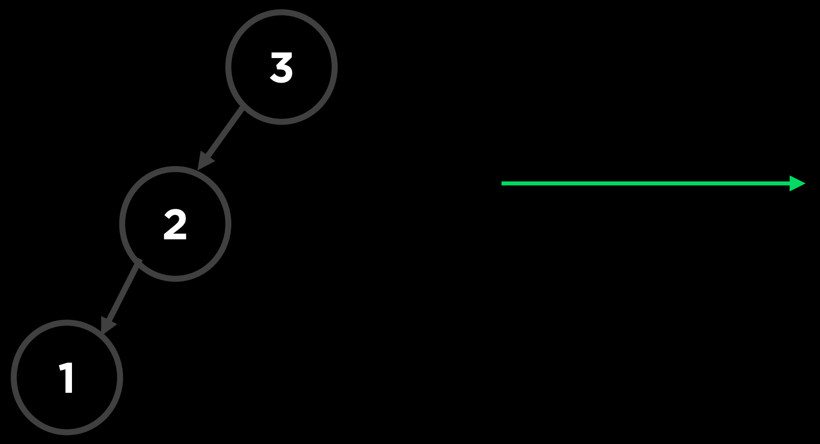
rotateLeft (node)
{
        grandchild = node->right->left;
        newParent = node->right;
        newParent->left = node;
        node->right = grandchild;
        return newParent;
}
```

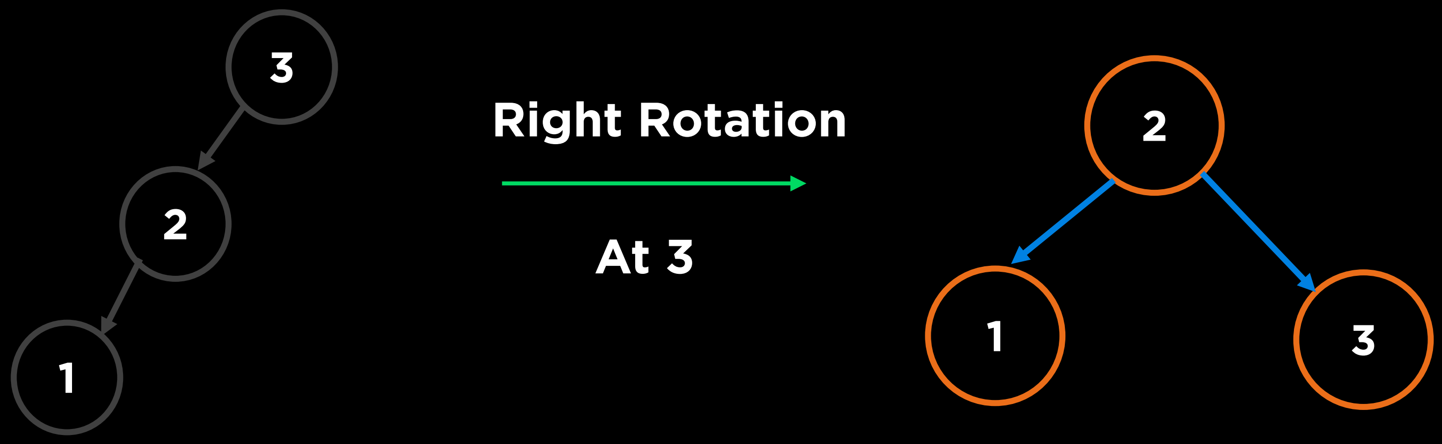
Take Constant Time, O(1)



Right Rotation: Left Left Case

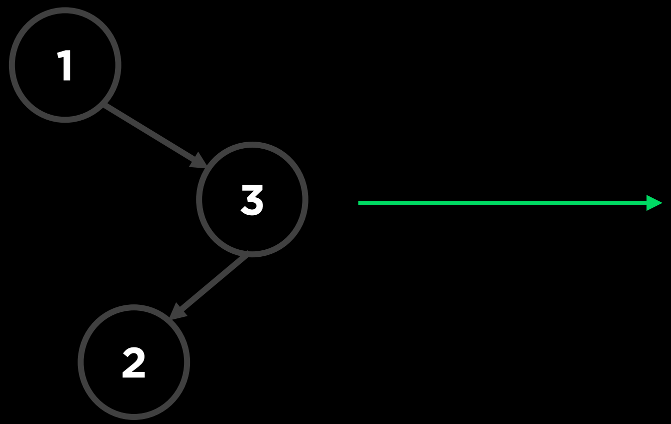


Right Rotation: Left Left Case

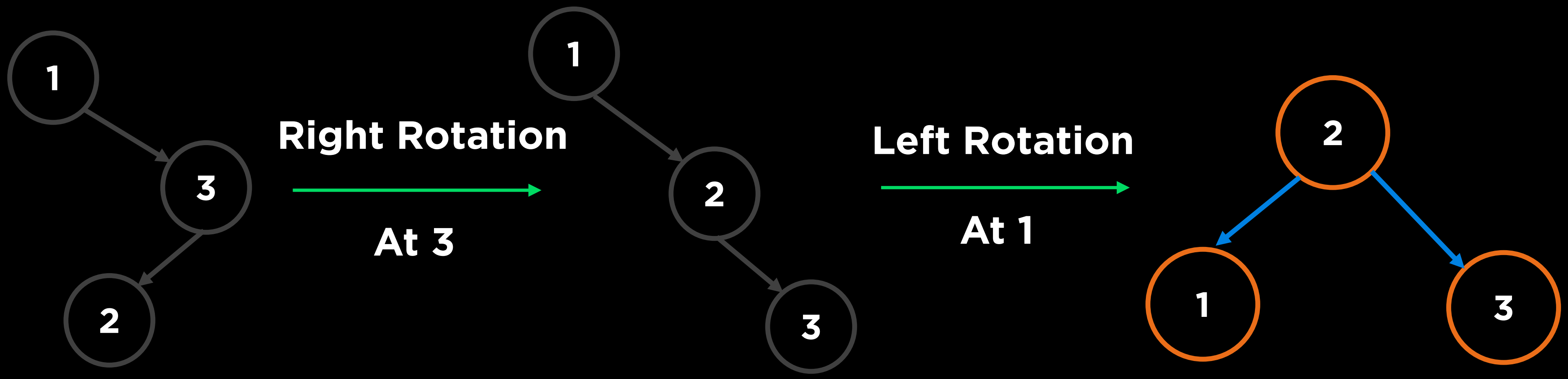


Single Rotation

Right Left Rotation: Right Left Case

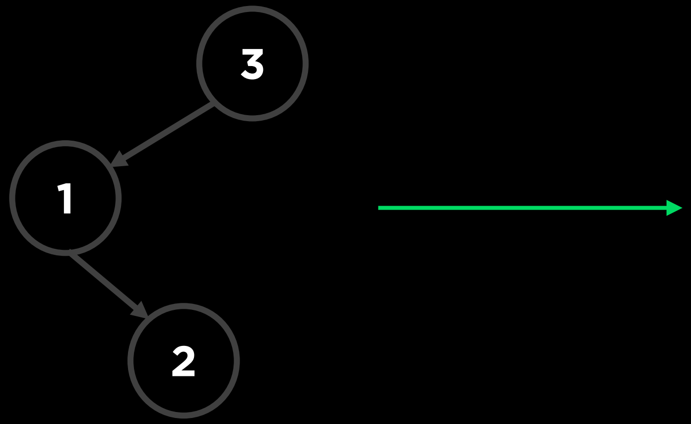


Right Left Rotation: Right Left Case

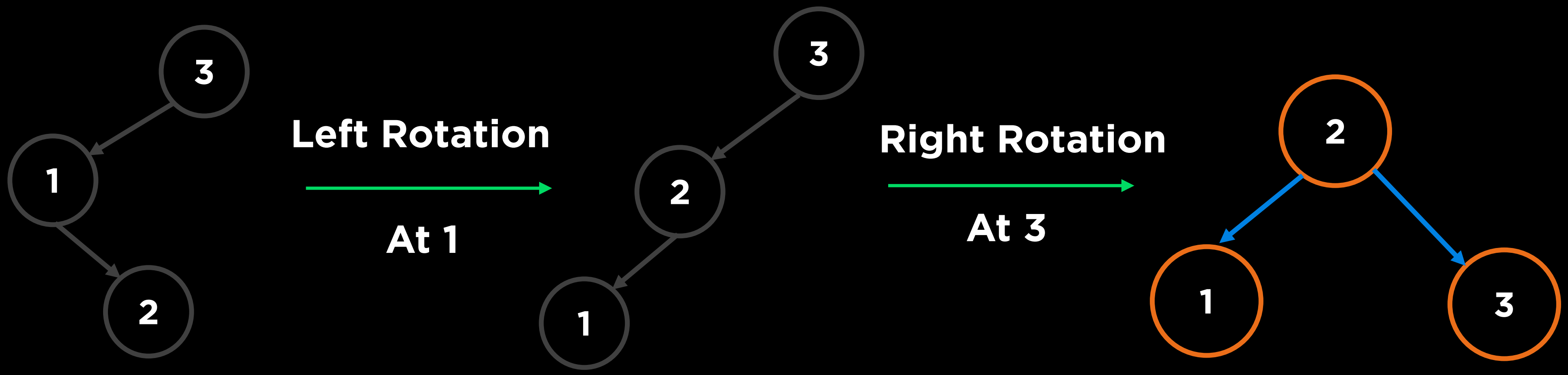


Double Rotation

Left Right Rotation: Left Right Case



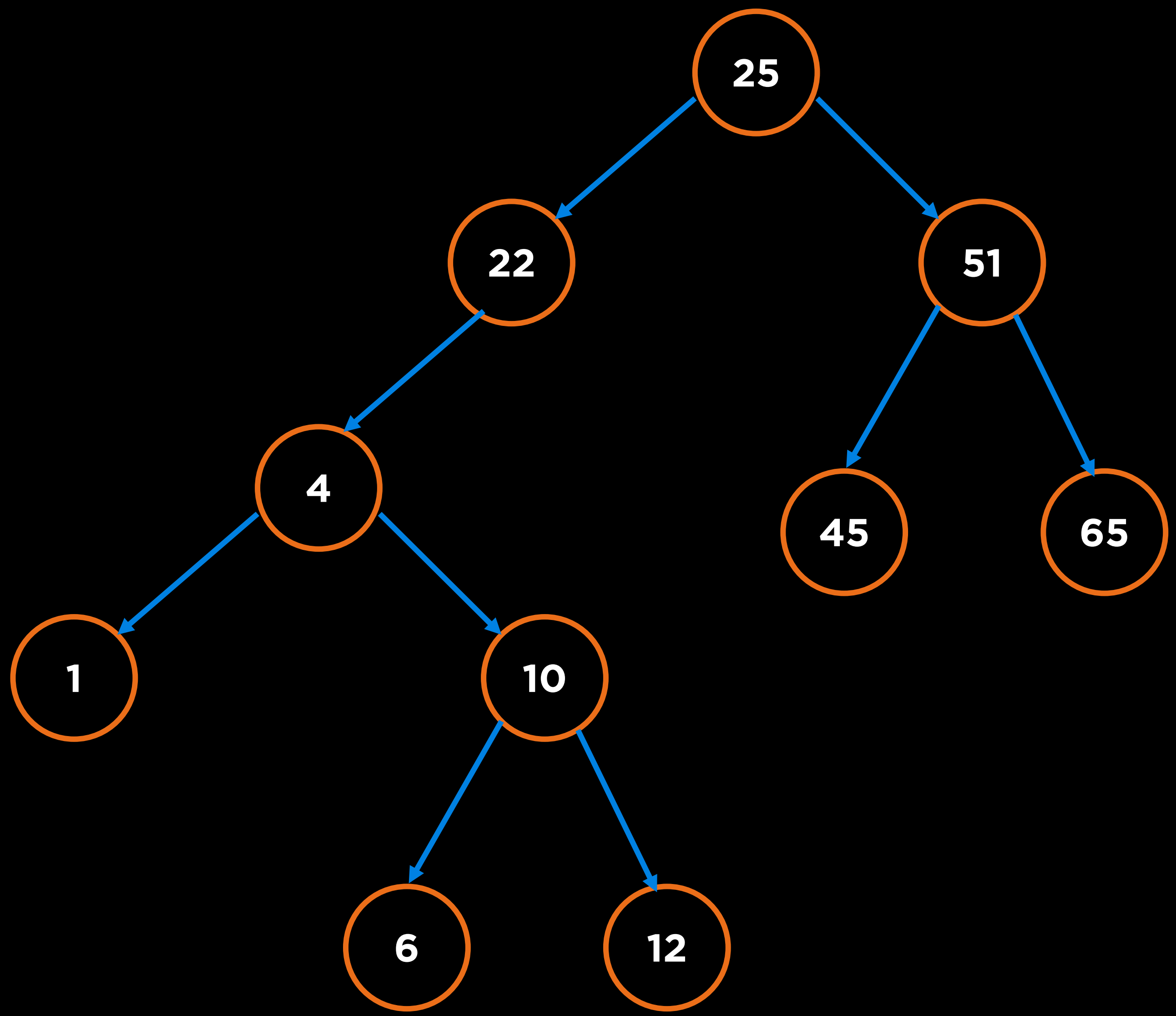
Left Right Rotation: Left Right Case



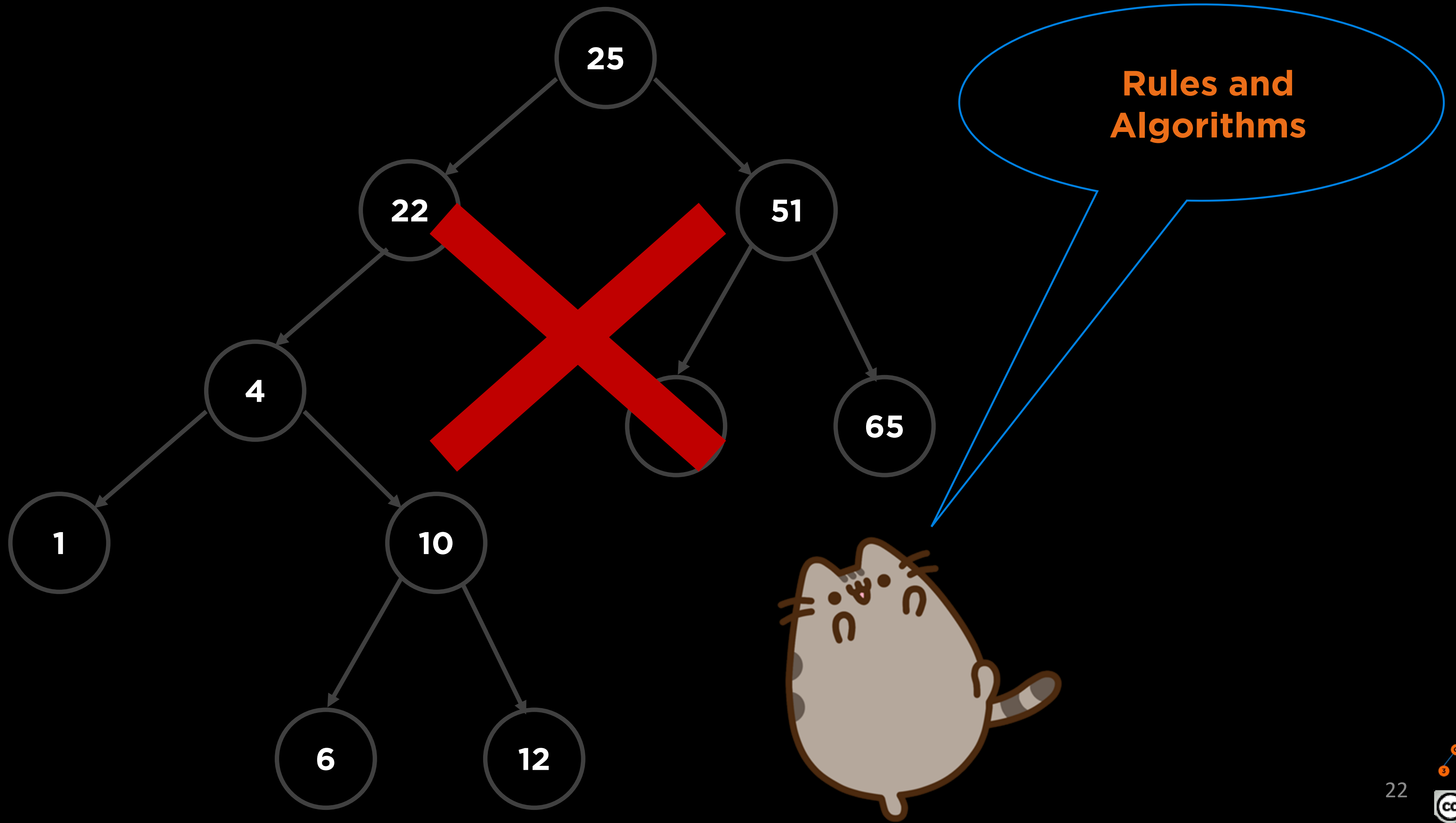
Double Rotation

Take Constant Time, O(1)

Tool Issue: Can Get Messy on a Prebuilt Tree



Fix for Messiness



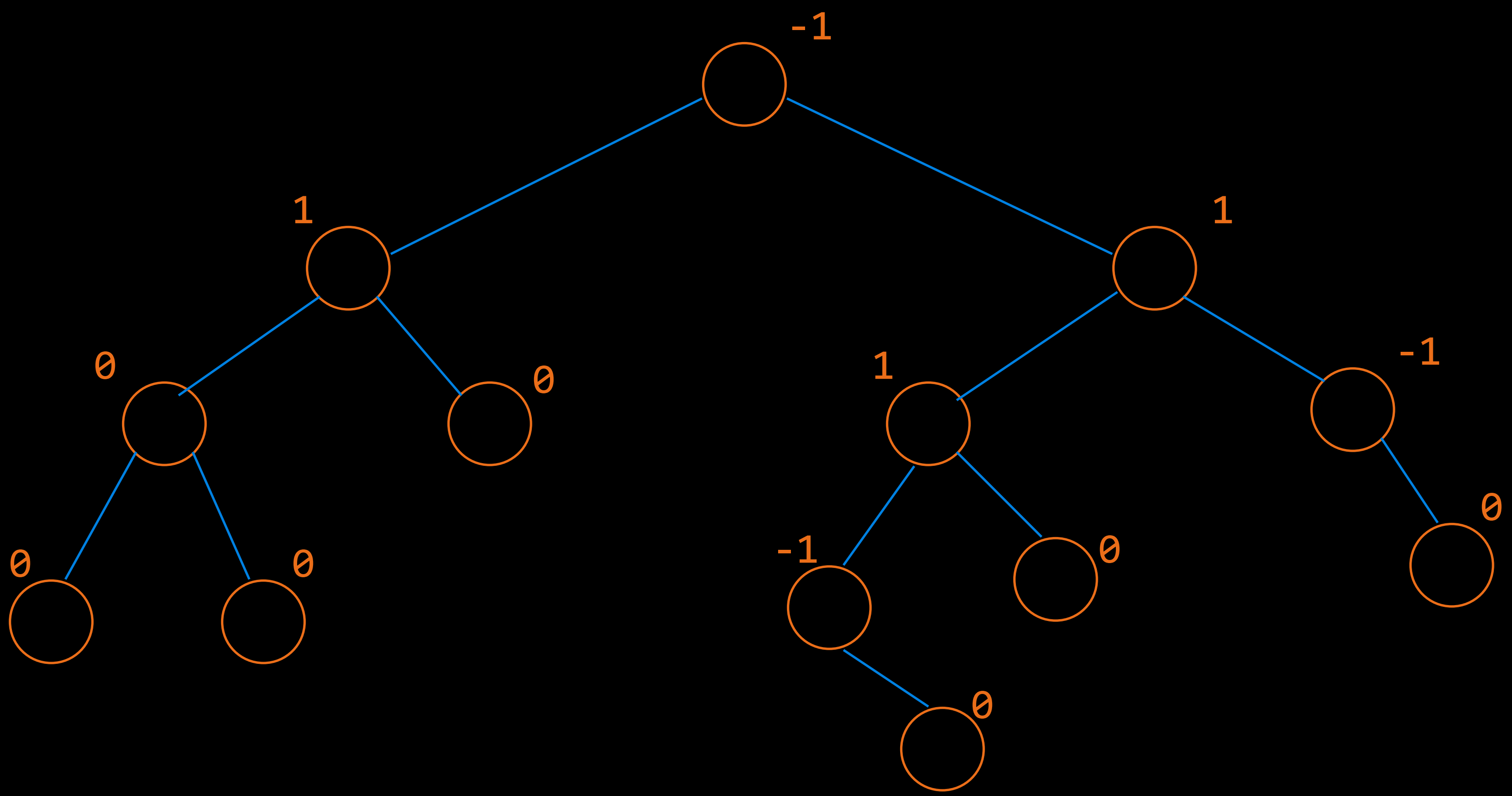
AVL Trees

Adelson-Velsky and Landis Trees

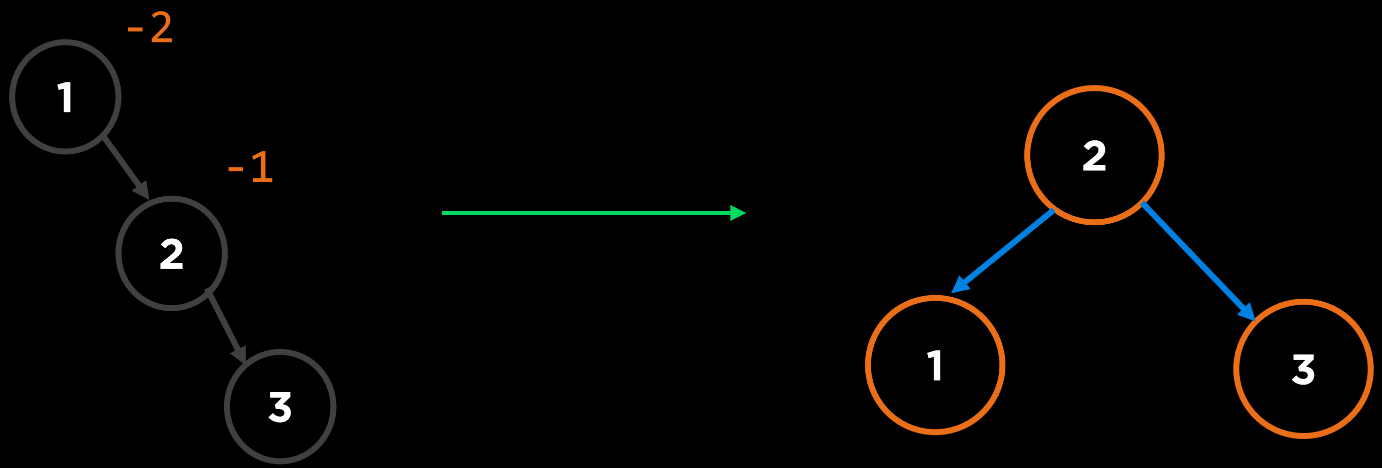
- Height Balanced Binary Search Trees
- Invariants:
  - Maintains BST invariants
    - Every node has 0, 1, or 2 children
    - Every element on the left is smaller and every element on the right is greater than a node.
  - For every node x, Balance Factor = 0, -1 or 1

AVL Tree

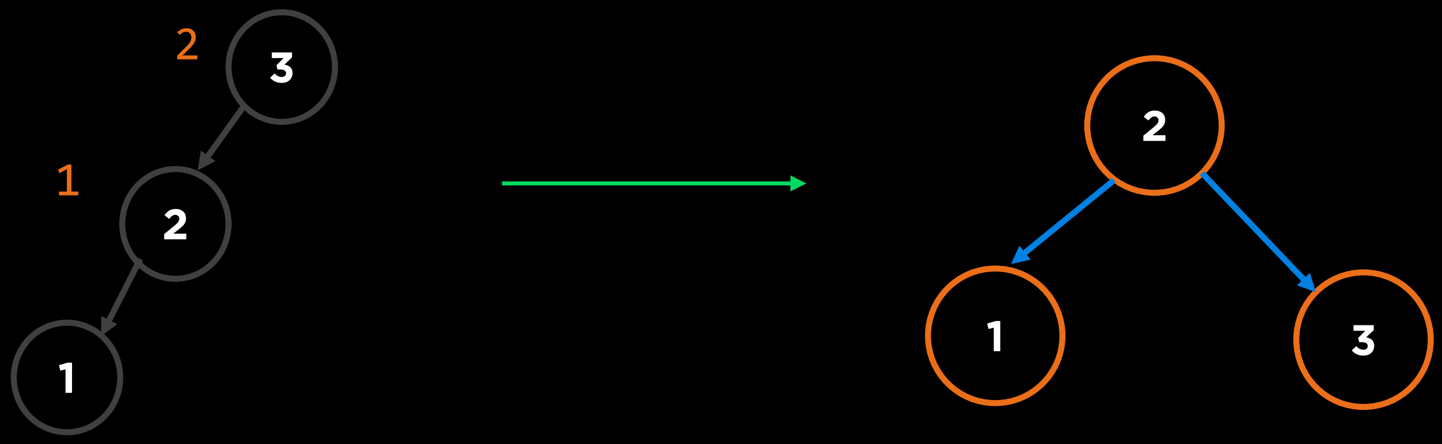
Balance Factor of x = Height (left subtree of x) - Height (right subtree of x)



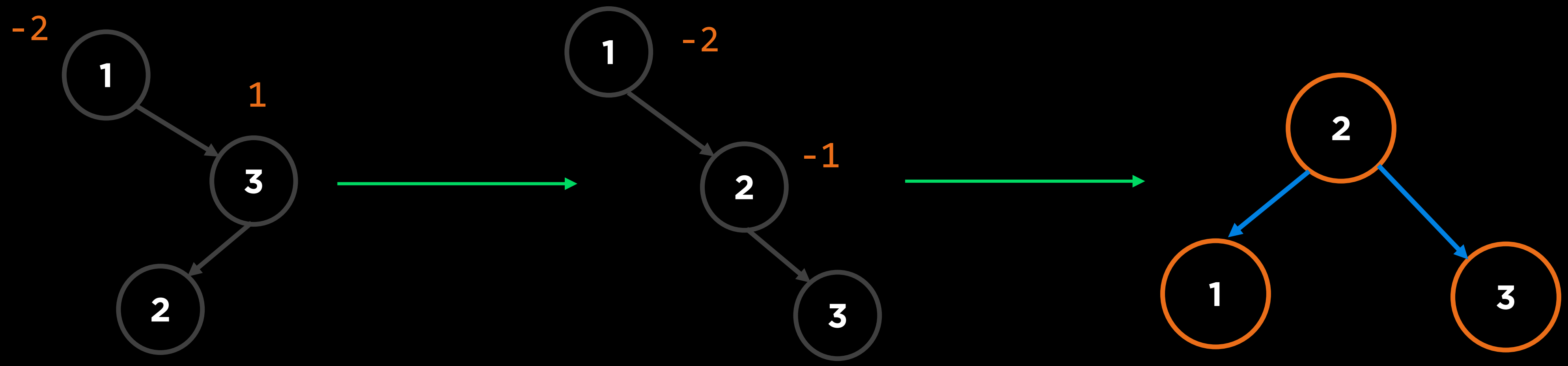
Left Rotation: Right Right Case



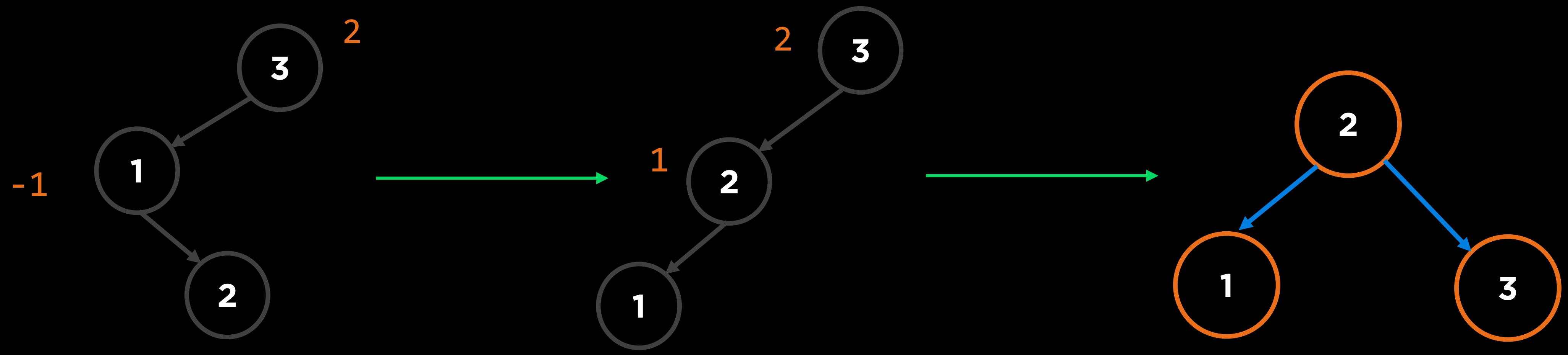
Right Rotation: Left Left Case



Right Left Rotation: Right Left Case



Left Right Rotation: Left Right Case



AVL Tree Rotations

Balance Factor of x = Height (left subtree of x) - Height (right subtree of x)

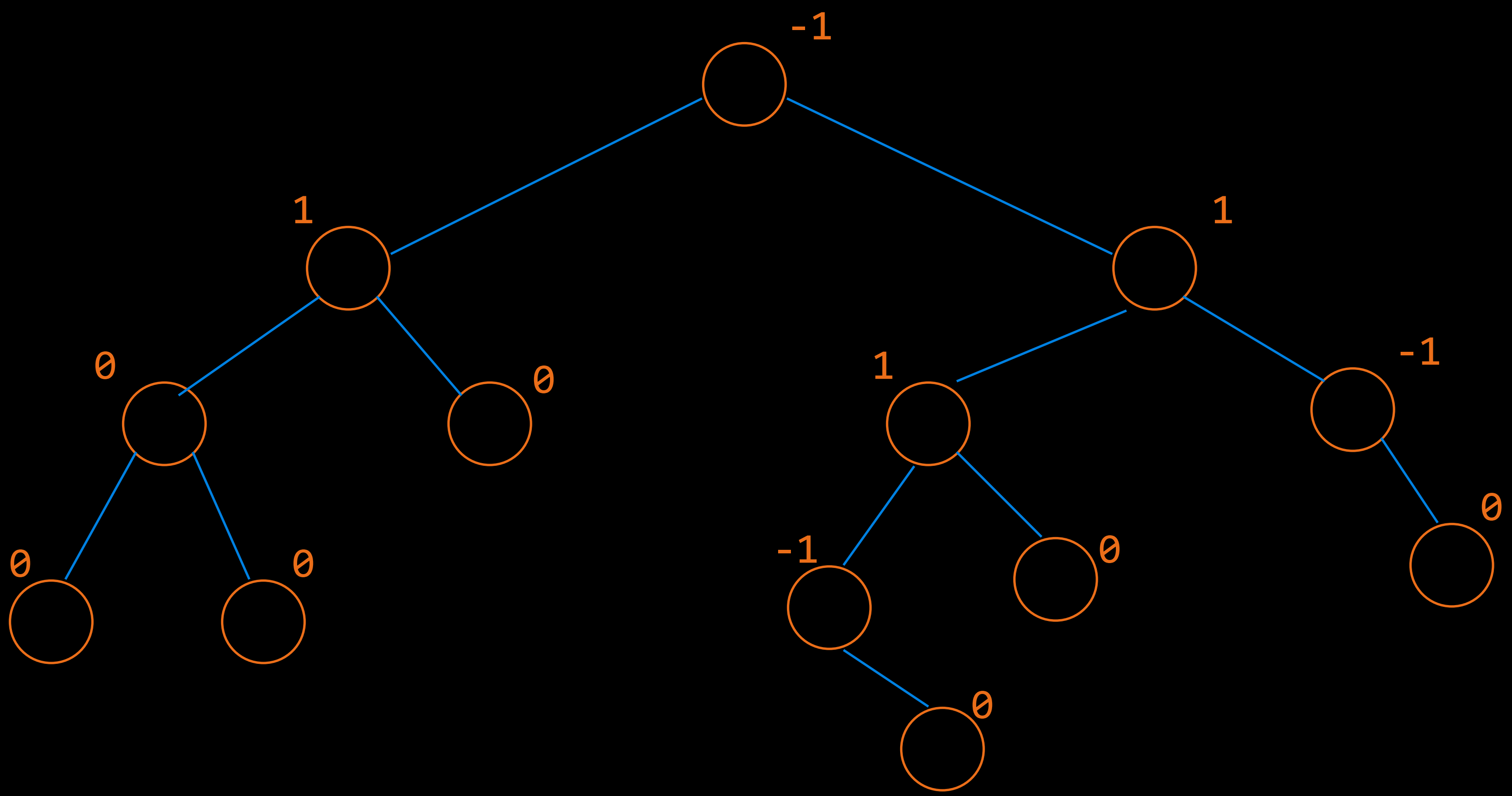
Case (Alignment)	Balance Factor		Rotation
	Parent	Child	
Left Left	+2	+1	Right
Right Right	-2	-1	Left
Left Right	+2	-1	Left Right
Right Left	-2	+1	Right Left

If Balance Factor of x = Height (right subtree of x) - Height (left subtree of x),

Reverse all signs in the table!

AVL Tree

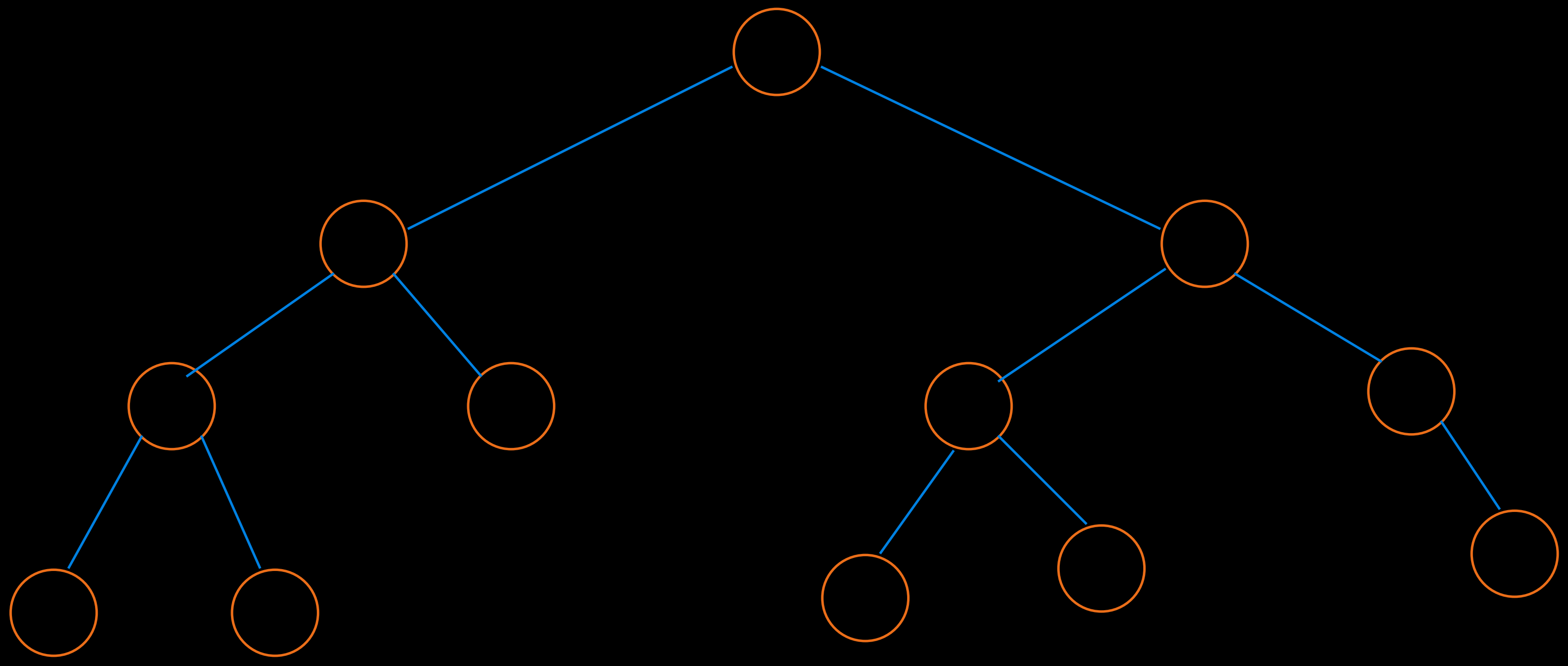
Height of an AVL Tree with n Nodes = $1.44 \log_2 (n+2)$



AVL Insert, Delete and Search

Worst Case ~ Height = log n

And Common Operations will be O(log n)



AVL Trees: Insertion/Deletion

- Same as Binary Search Trees
- Identify deepest node that breaks the Balance Factor rule; Start rotating and move further up the search path
- After Insertion/Deletion height of all nodes in Search Path may change

AVL Insert

Insert 25

```
IF tree is right heavy
{
        IF tree's right subtree is left heavy
                Perform Right Left rotation
        ELSE
                Perform Left rotation
}
ELSE IF tree is left heavy
{
        IF tree's left subtree is right heavy
                Perform Left Right rotation
        ELSE
                Perform Right rotation
}
```

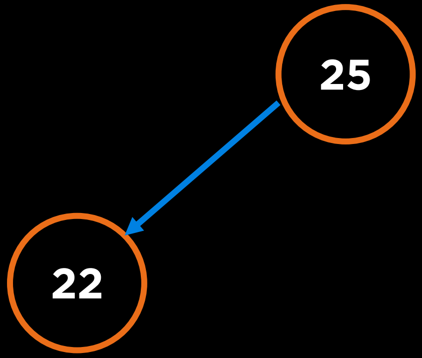
AVL Insert

Insert 22

```
IF tree is right heavy
{
            IF tree's right subtree is left heavy
                        Perform Right Left rotation
            ELSE
                        Perform Left rotation
}
ELSE IF tree is left heavy
{
            IF tree's left subtree is right heavy
                        Perform Left Right rotation
            ELSE
                        Perform Right rotation
}
```

AVL Insert

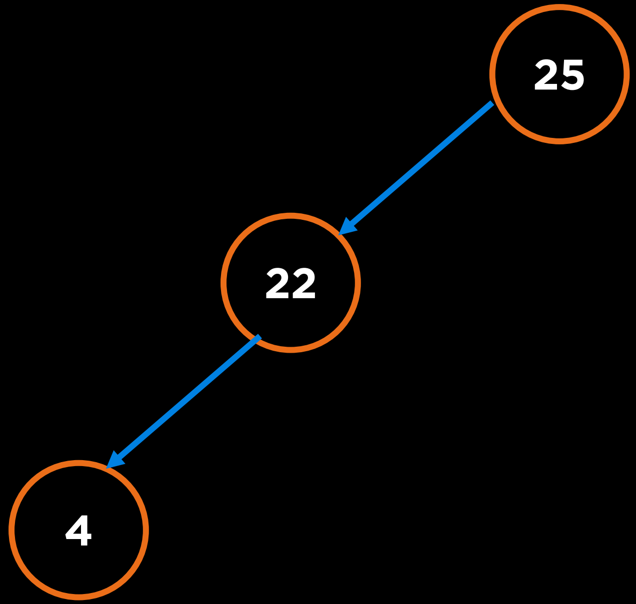
Insert 4




```
IF tree is right heavy
{
            IF tree's right subtree is left heavy
                        Perform Right Left rotation
            ELSE
                        Perform Left rotation
}
ELSE IF tree is left heavy
{
            IF tree's left subtree is right heavy
                        Perform Left Right rotation
            ELSE
                        Perform Right rotation
}
```

AVL Insert

Insert 4




```
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        ELSE
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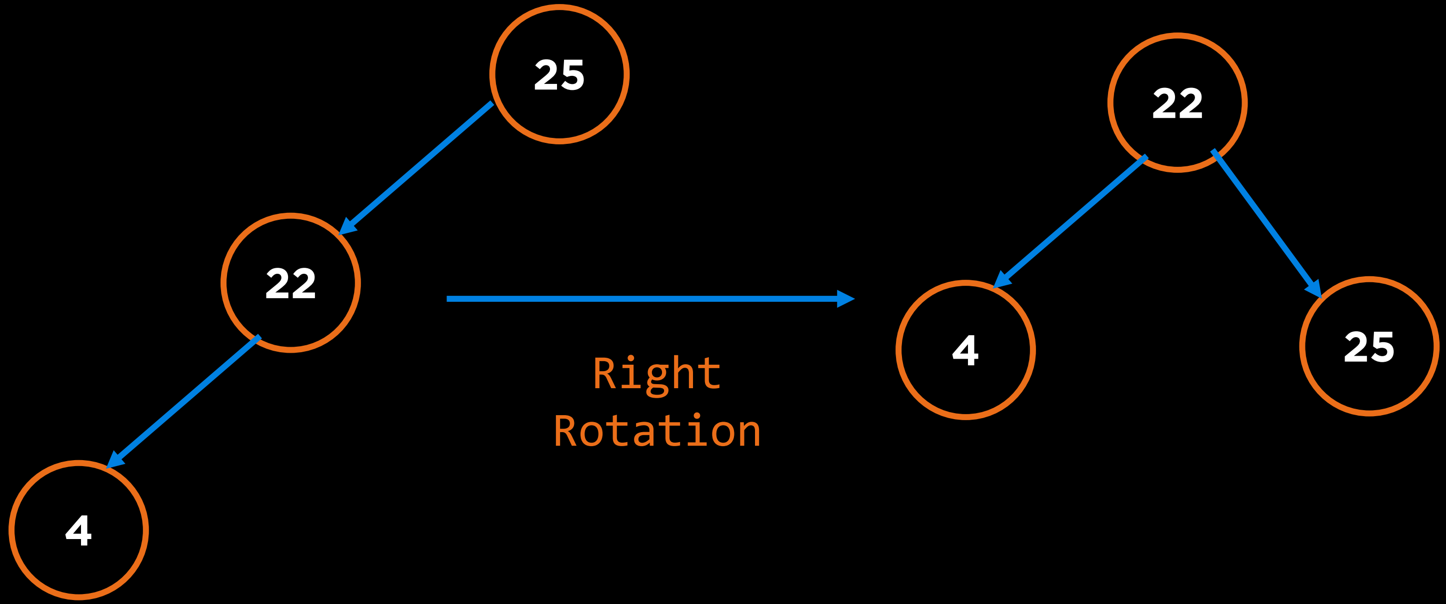
}
ELSE IF tree is left heavy
{
        IF tree's left subtree is right heavy
                Perform Left Right rotation
        ELSE
                Perform Right rotation
}
```

AVL Insert

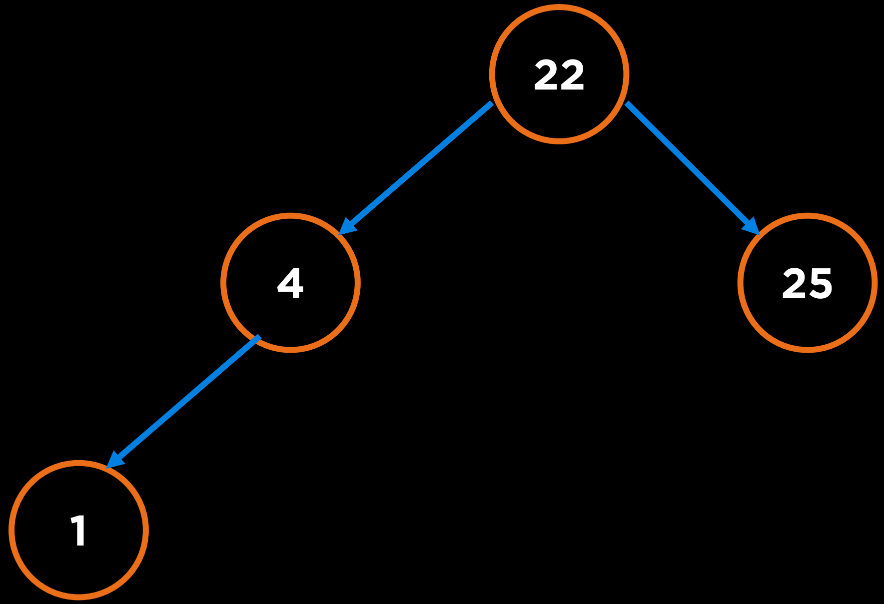
Insert 1




```
IF tree is right heavy
{
        IF tree's right subtree is left heavy
                Perform Right Left rotation
        ELSE
                Perform Left rotation
}
ELSE IF tree is left heavy
{
        IF tree's left subtree is right heavy
                Perform Left Right rotation
        ELSE
                Perform Right rotation
}
```

AVL Insert

Insert 2




```
IF tree is right heavy
{
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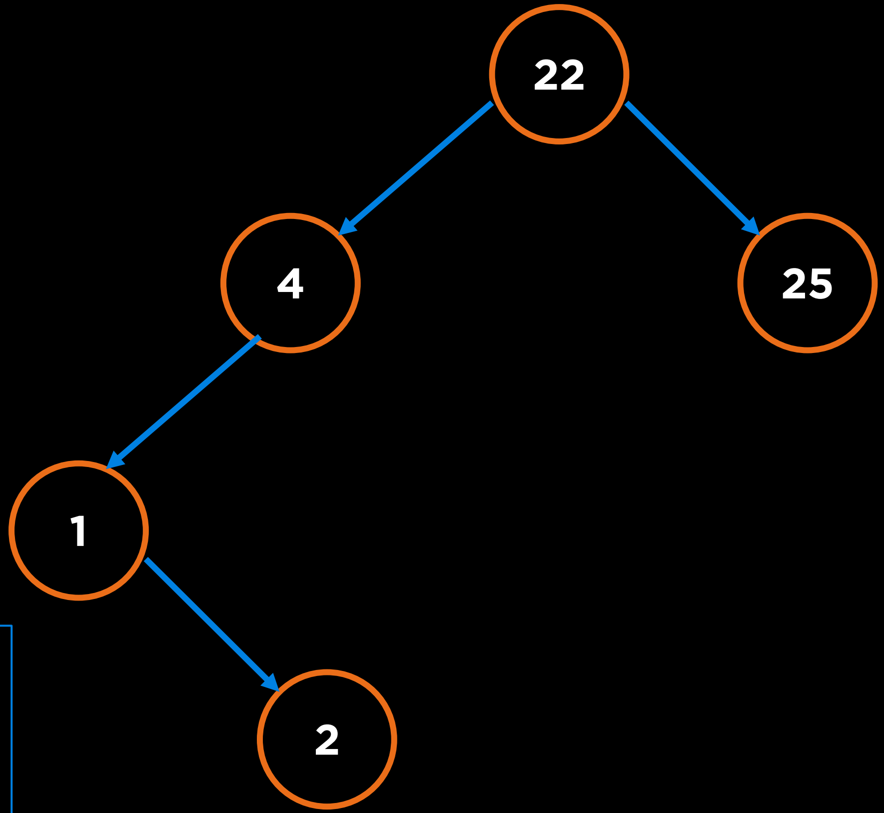
                Perform Right Left rotation
        ELSE
                Perform Left rotation
}
ELSE IF tree is left heavy
{
        IF tree's left subtree is right heavy
                Perform Left Right rotation
        ELSE
                Perform Right rotation
}
```

AVL Insert

Insert 2




```
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{
        IF tree's left subtree is right heavy
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        ELSE
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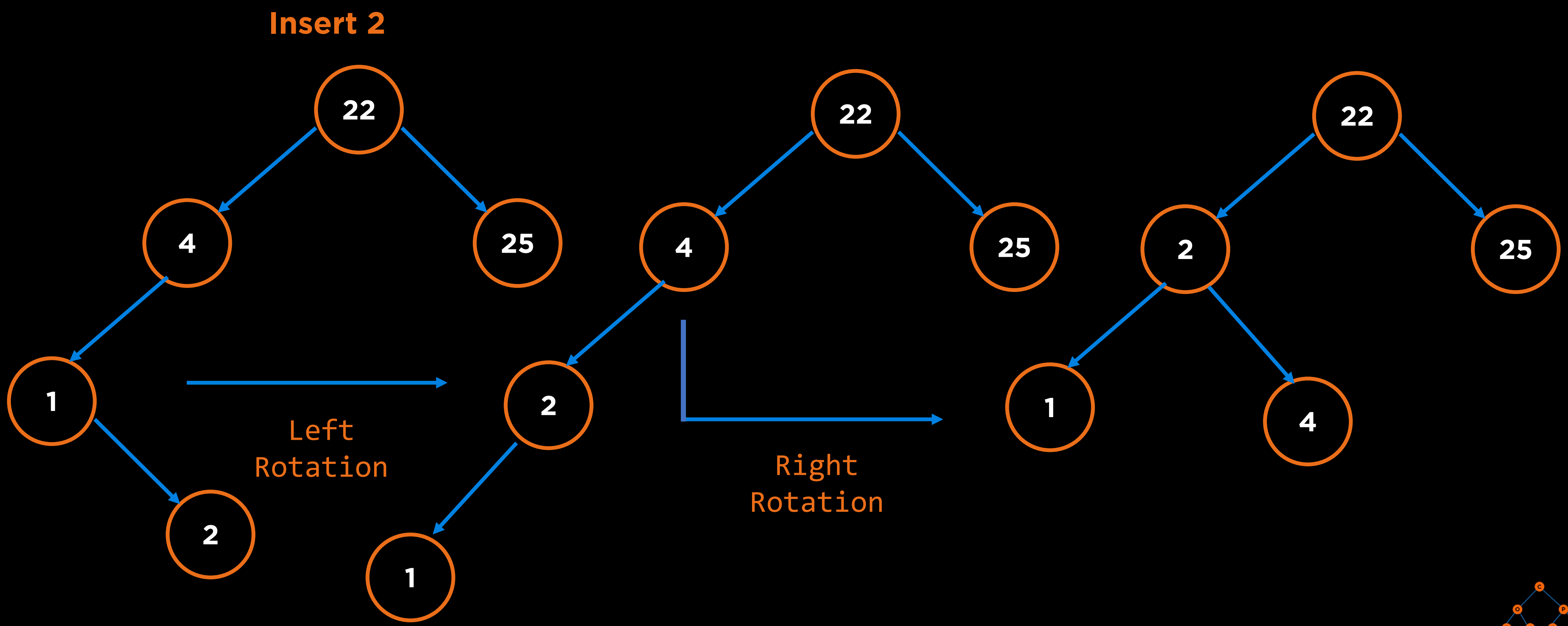
}
```

AVL Insert


Insert 2
22
4
25
1
2
Left
Rotation
22
4
25
2
1
Right
Rotation
22
2
25
1
4

AVL Tree : C++ Node Class

```cpp
 class TreeNode {
    public:
      int val;
      int height; // Or Balance Factor
      TreeNode *left;
      TreeNode *right;
      TreeNode(int x) : val(x), left(nullptr), right(nullptr) {}
};
```

AVL Tree : C++ Insert

```cpp
TreeNode* insert(TreeNode* root, int key)
{
    if (root == nullptr)
        return new TreeNode(key);

    else if (key < root->val)
        root->left = insert(root->left, key);

    else
        root->right = insert(root->right, key);

    return root;
}
```

```
IF tree is right heavy
{
        IF tree's right subtree is left heavy
                Perform Right Left rotation
        ELSE
                Perform Left rotation
}
ELSE IF tree is left heavy
{
        IF tree's left subtree is right heavy
                Perform Left Right rotation
        ELSE
                Perform Right rotation
}
```

Questions

AVL Tree : C++ Insert

```cpp
TreeNode* insert(TreeNode* root, int key)
{
    if (root == nullptr)
        return new TreeNode(key);

    else if (key < root->val)
        root->left = insert(root->left, key);

    else
        root->right = insert(root->right, key);

    return root;
}
```

```
IF tree is right heavy
{
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        ELSE
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}
ELSE IF tree is left heavy
{
        IF tree's left subtree is right heavy
                Perform Left Right rotation
        ELSE
                Perform Right rotation
}
```
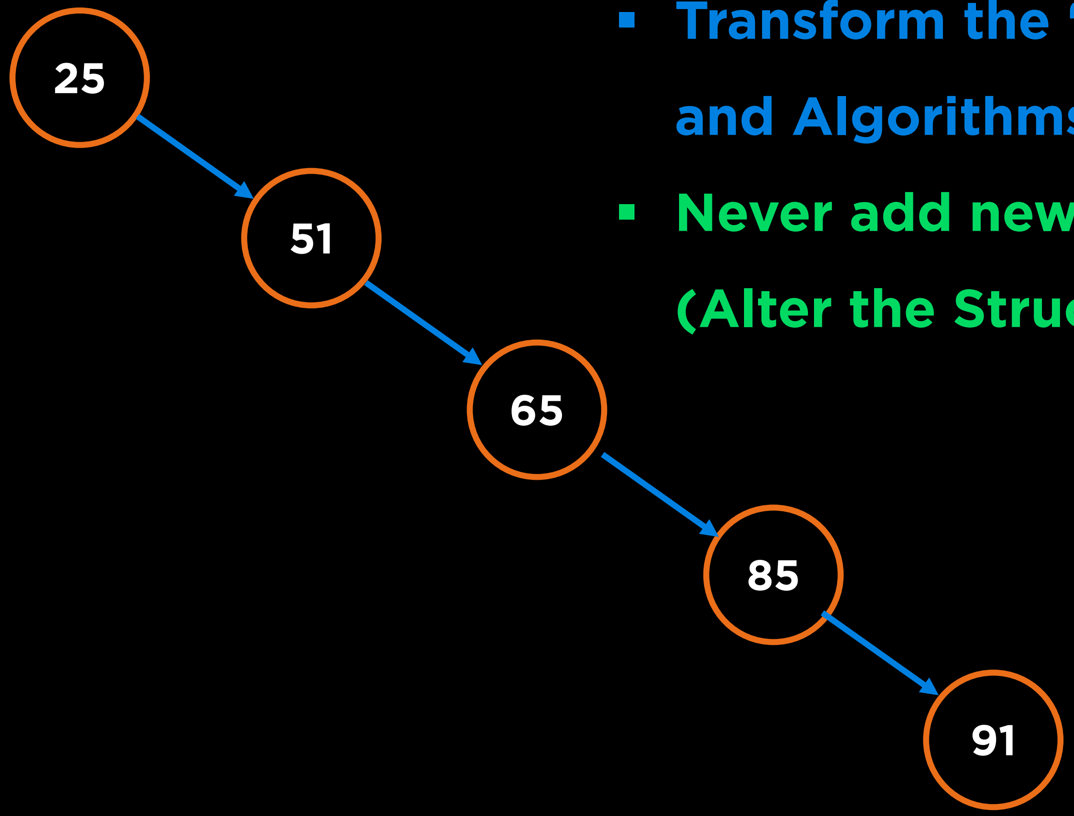

Agenda

- B Trees
  - Properties
  - Insertion
  - Use Cases
  - B Trees vs B+ Trees
- Splay Trees
  - Properties

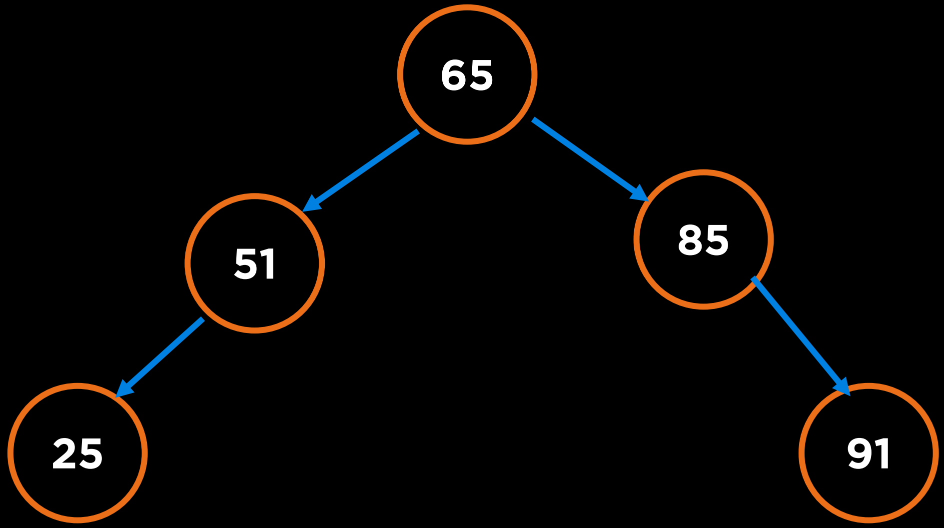
How do we fix the Worst Case in a BST?



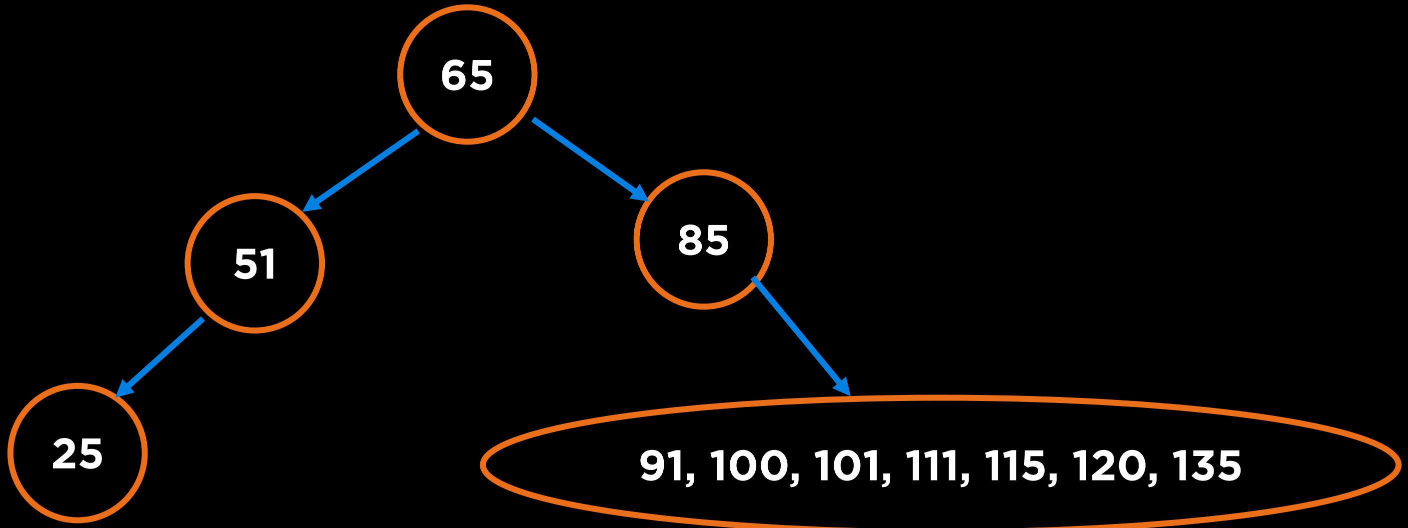
- Transform the "Spindly" Tree to "Bushy Tree" using Tools and Algorithms (Transformation)
- Never add new leaves at the bottom: Increase size of node (Alter the Structure by Design)

Insertion in Non-Random Order

Insert 100, 101, 111,
115, 120, 135

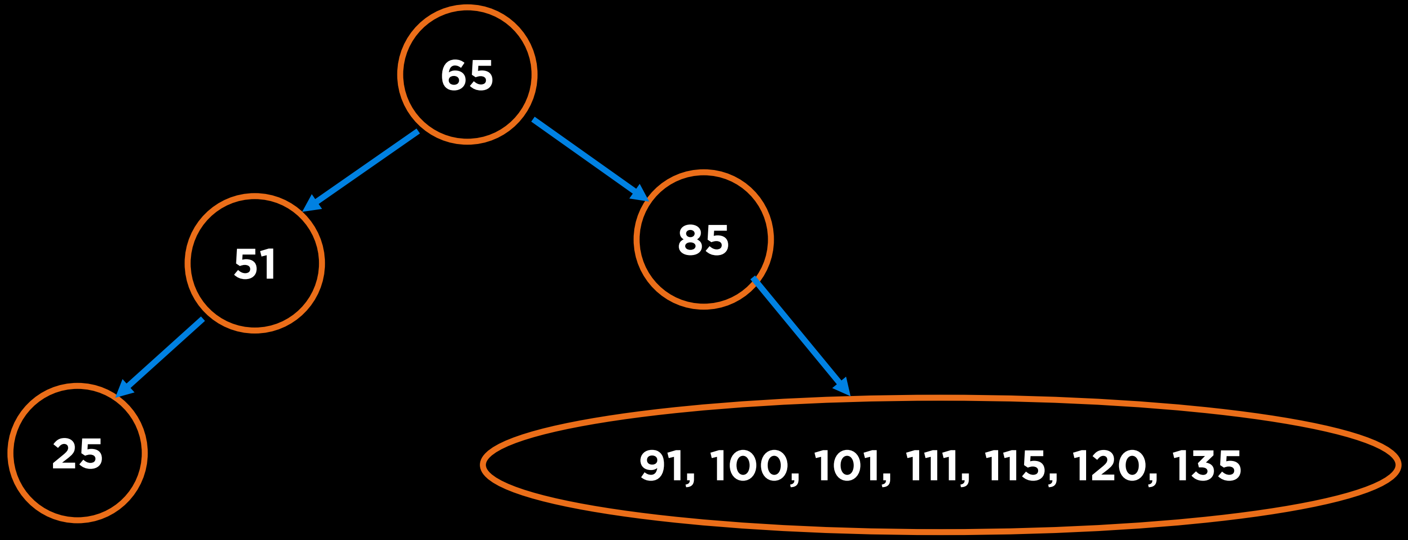


Insertion in Non-Random Order



Insertion in Non-Random Order

- Can lead to overstuffing
- Overstuffed trees have better balanced height
- Consistent BST



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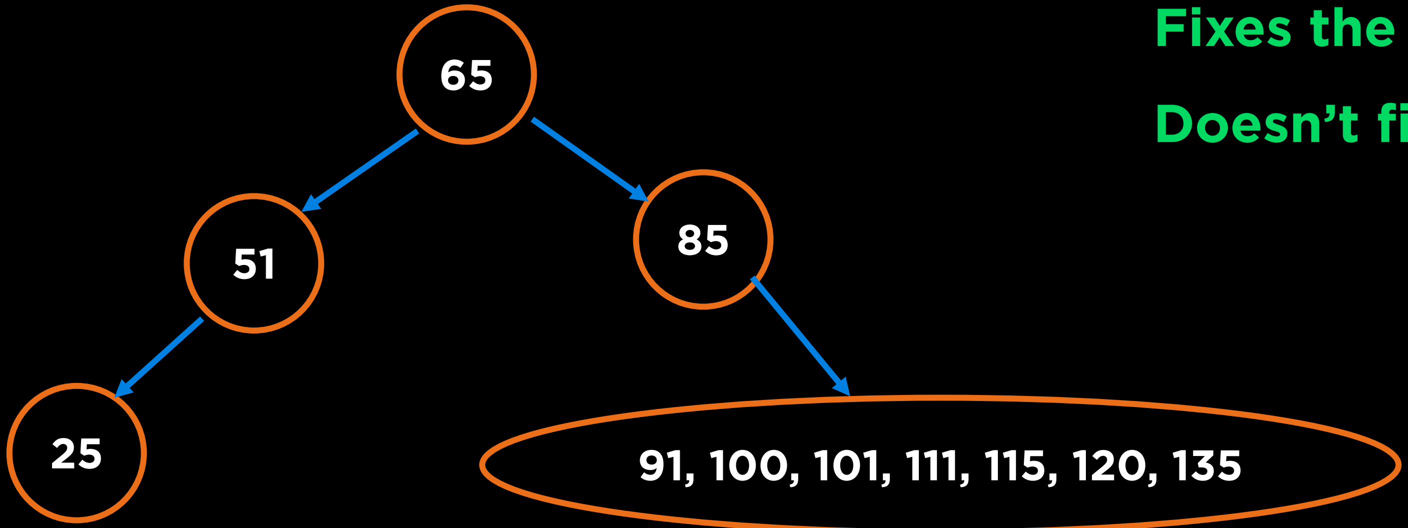


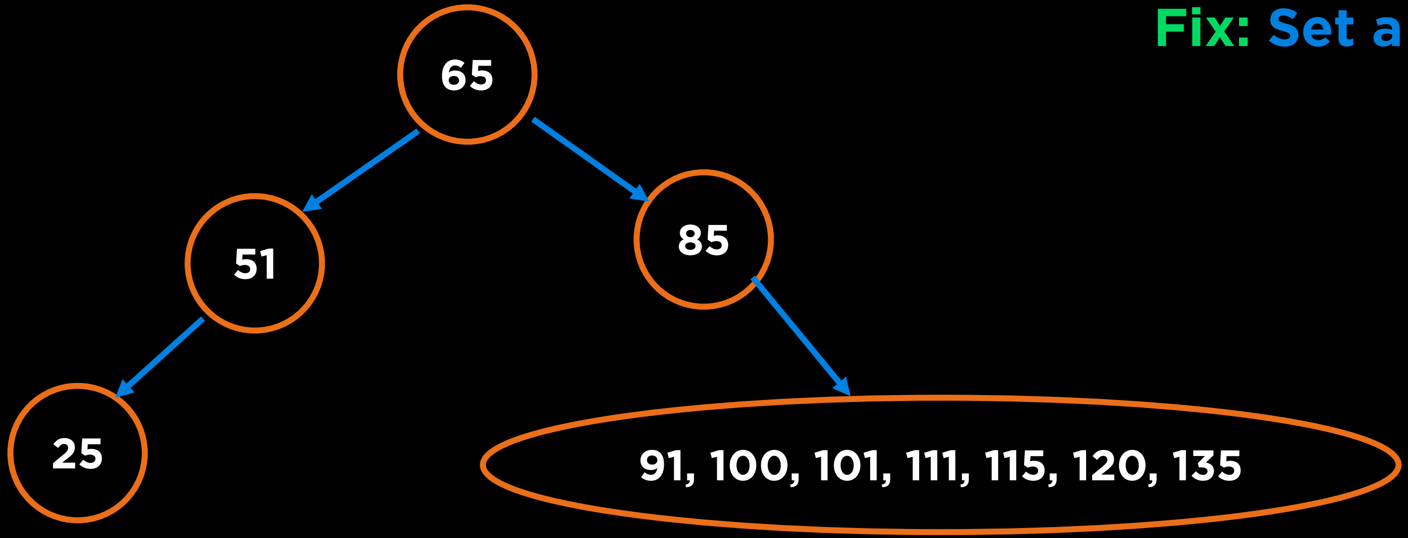
Fixes the height imbalance problem

Doesn't fix Non-random insertion

Insertion in Non-Random Order

- Can lead to overstuffing
- Overstuffed trees have better balanced height
- Consistent BST

Fix: Set a limit, L on the node filling

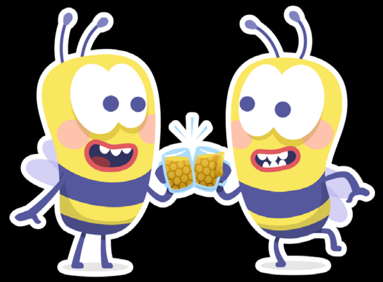


B Trees

Property #1

Each Node is a Block Containing Multiple "Keys", Keys at most l

5	7	25	44



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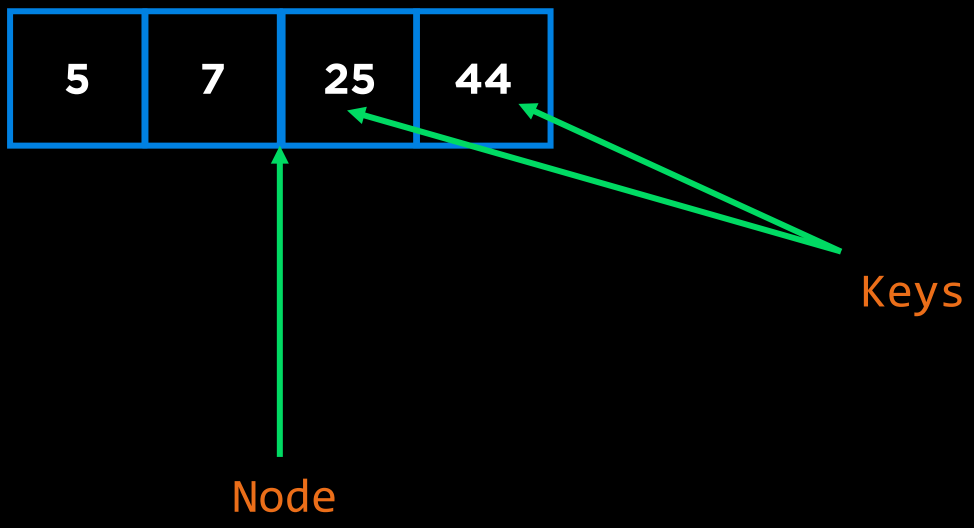
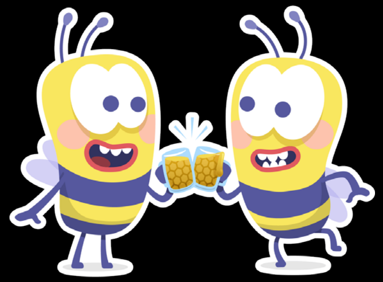
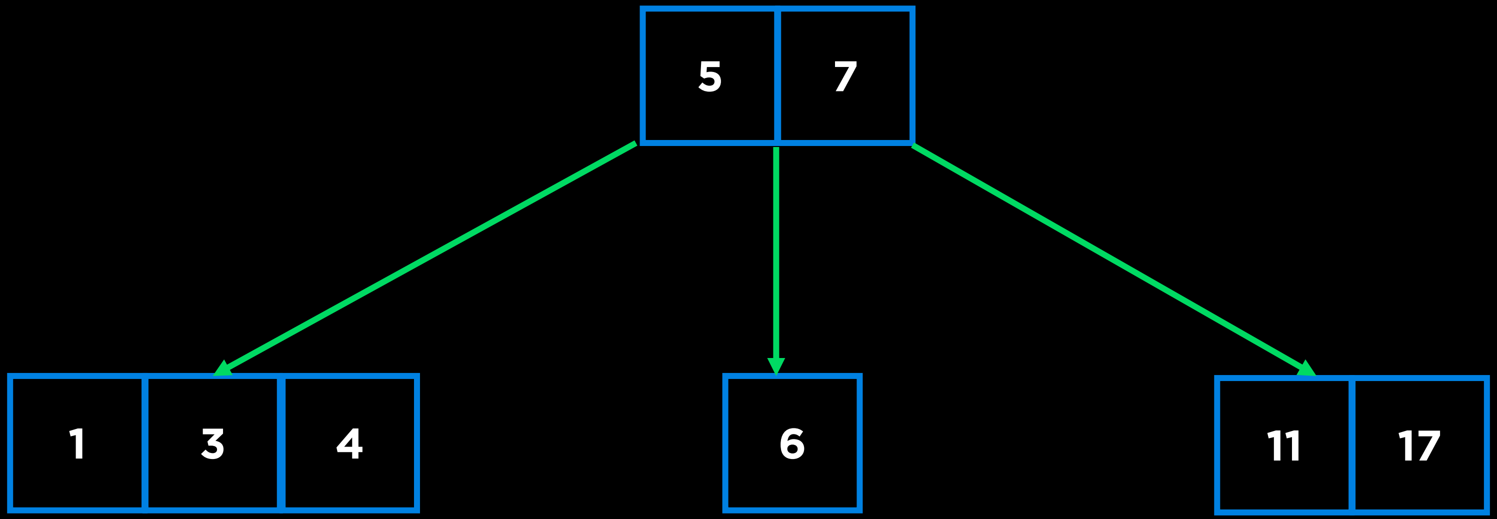




Property #2

B Trees are n-ary Trees, each node has up to n children. They follow BST property



Property #2

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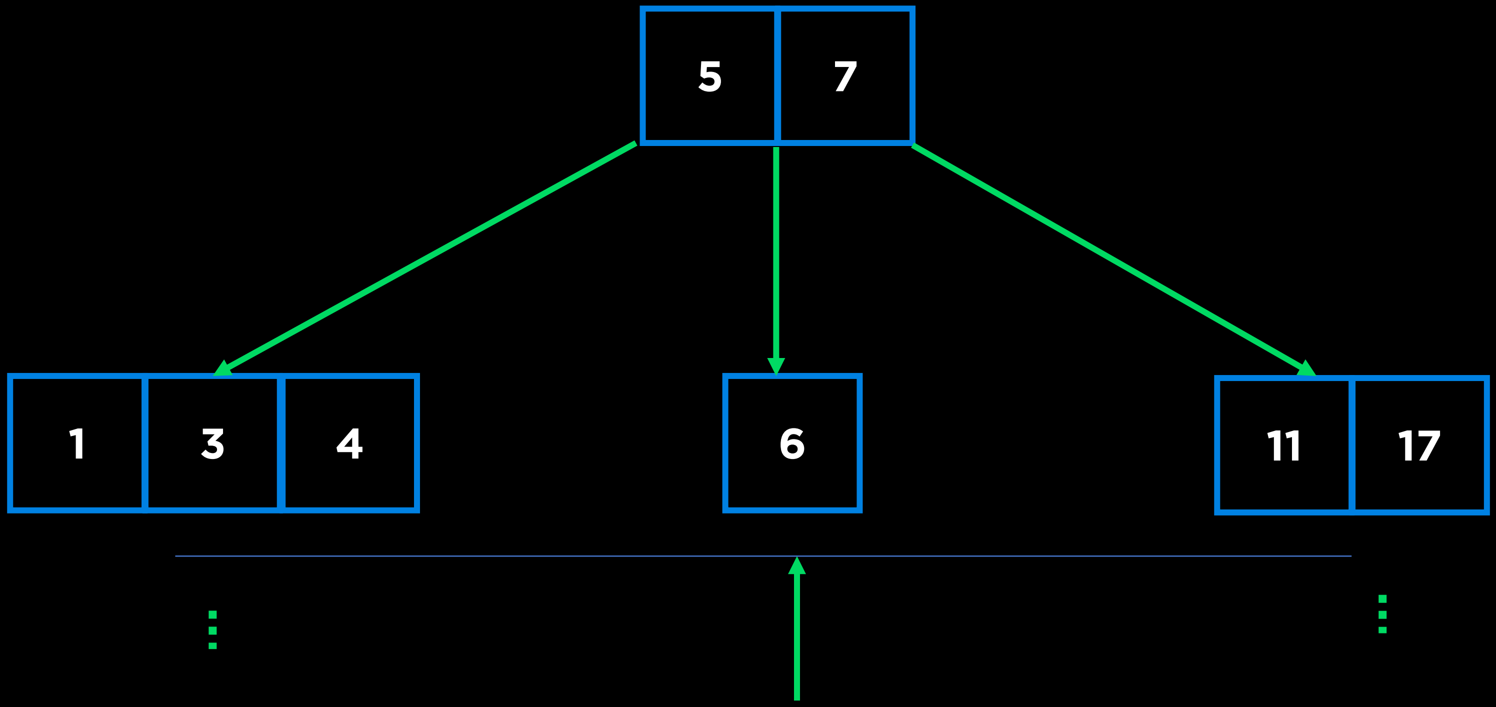
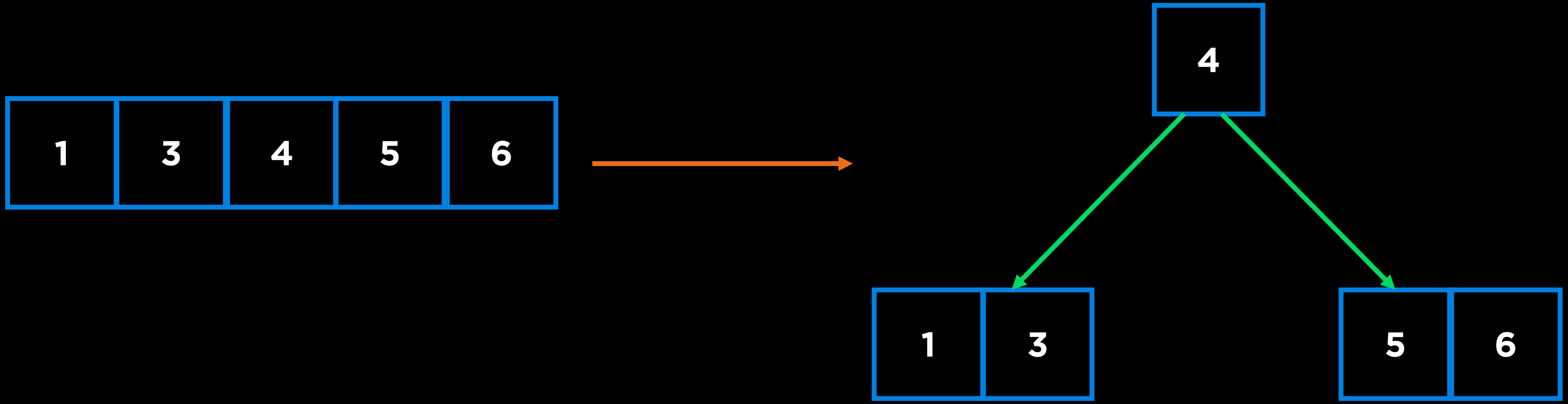


n Children, here n = 3

Not a Binary Tree!

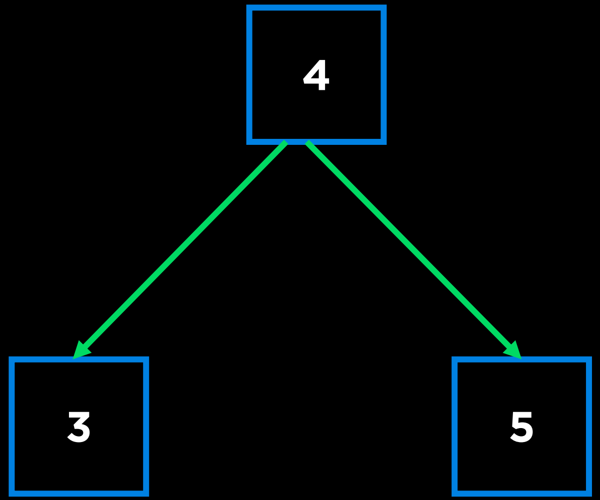
Property #3

Tree Building is Bottom-up



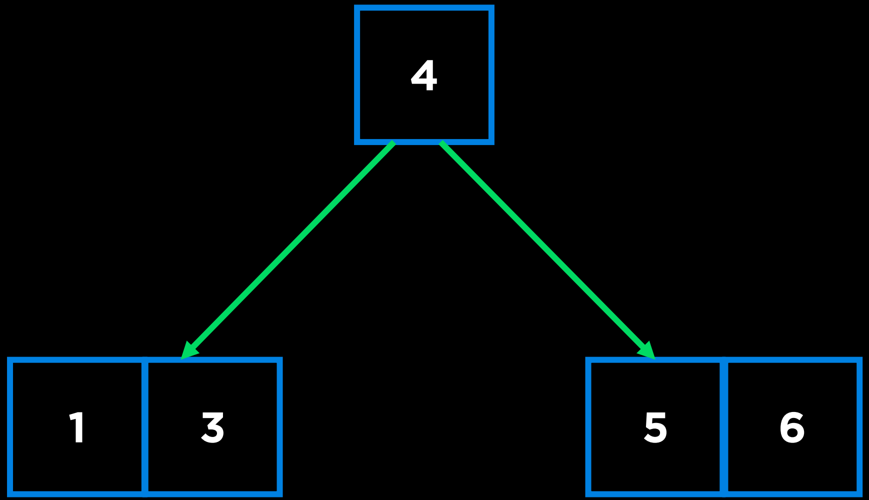
Property #4

Order “n” tree has at most n children (n=2, l=1 is a BST)



Property #5

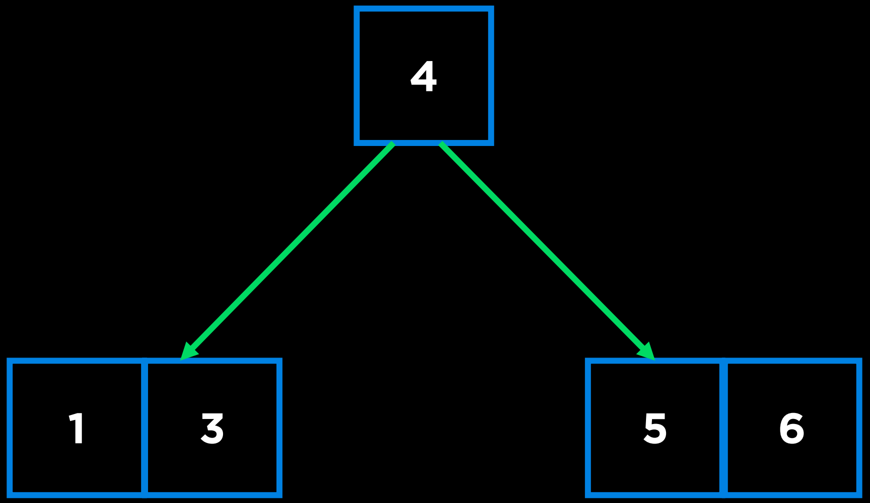
Leaves are at same depth



Property #6

Keys

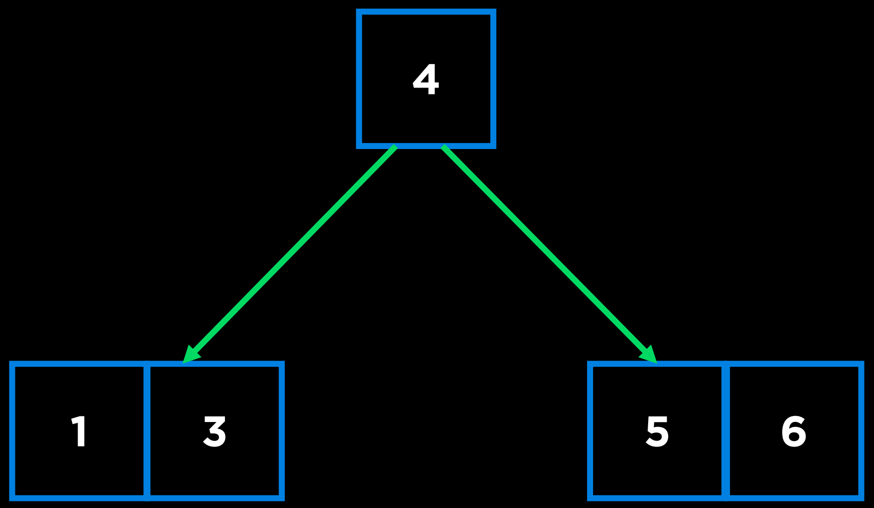
- n=3, l=2
- Non-leaf nodes store up to n-1 keys. Thus, l is atmost n-1 for internal nodes.
- All keys are in Sorted Order
- Leaf nodes have [ceil(l/2), l] keys except when tree has less than l/2 elements (Not strictly enforced)



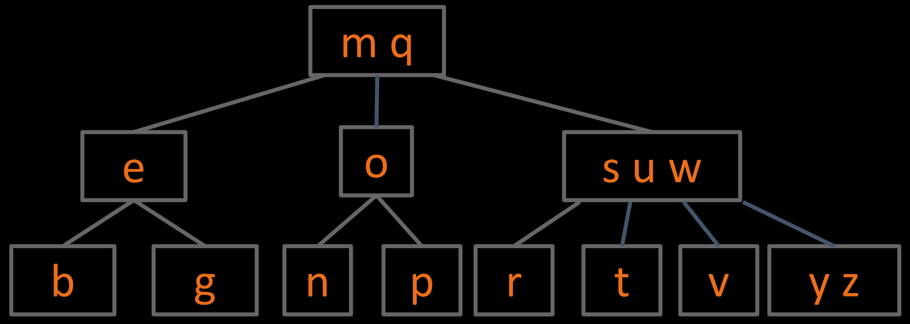
Property #7

Children

- n=3, l=2
- Root is a leaf or has [2, n] children
- Non-leaf nodes have [ceil(n/2), n] children



Properties Summary

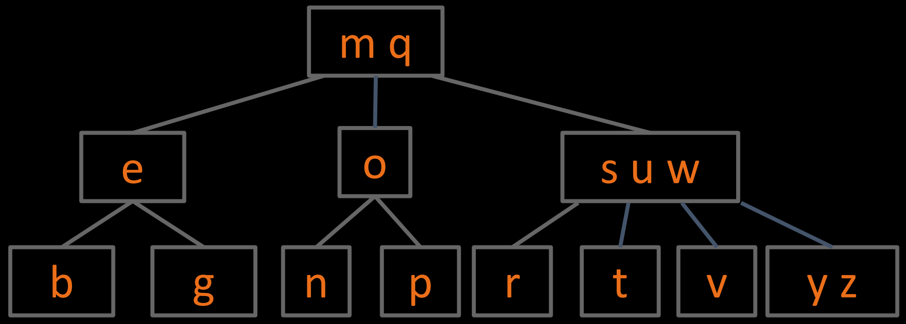


n=4, l=3

- Each Node is a Block Containing Multiple “Keys”
- B Trees are n-ary Trees or have an order “n”
- Children
  - Root is a leaf or has [2, n] children
  - Non-leaf nodes have [ceil(n/2), n] children
  - Maximum children is at most n for all nodes
- Keys
  - All keys are in Sorted Order
  - Non-leaf nodes store up to n-1 keys
  - Leaf nodes have [ceil(l/2), l] keys except when tree has less than l/2 elements (Not strictly enforced)
- All leaves are at same depth, so the tree is always balanced
- Data items are stored in leaves and non-leaf nodes in a B Tree. In a B+ Tree, data is stored in only leaves
- When a node is full Splitting occurs

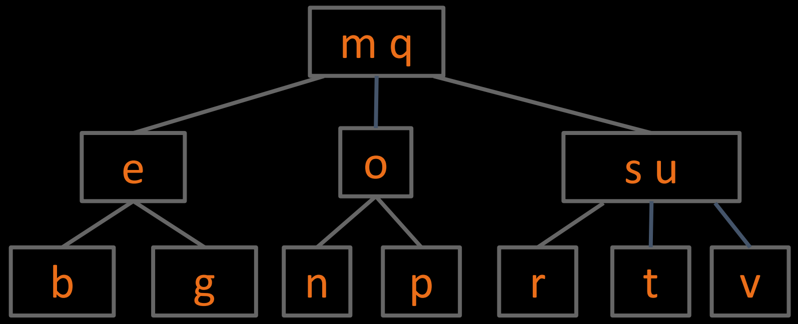
Examples

- B-trees of order **n=4, l=3** are also called a 2-3-4 tree or a 2-4 tree.
  - "2-3-4" refers to the number of children that a node can have, e.g. a 2-3-4 tree node may have 2, 3, or 4 children.
- B-trees of order **n=3, l=2** are also called a 2-3 tree.
- l can be very large in case of file systems



2-3-4 a.k.a. 2-4 Tree:
Max 3 items per node.
Max 4 non-null children per node.

n=4, l=3



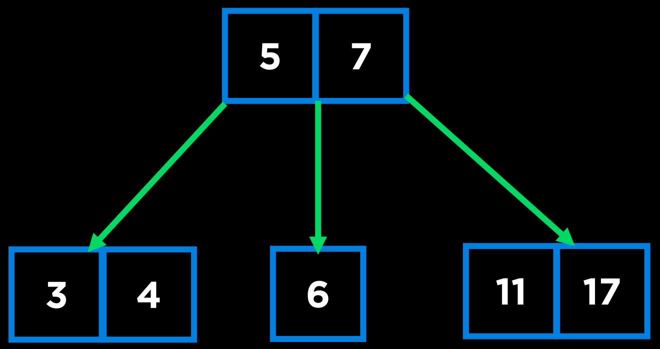
2-3 Tree (L=2):
Max 2 items per node.
Max 3 non-null children per node.

n=3, l=2

https://sp19.datastructur.es/

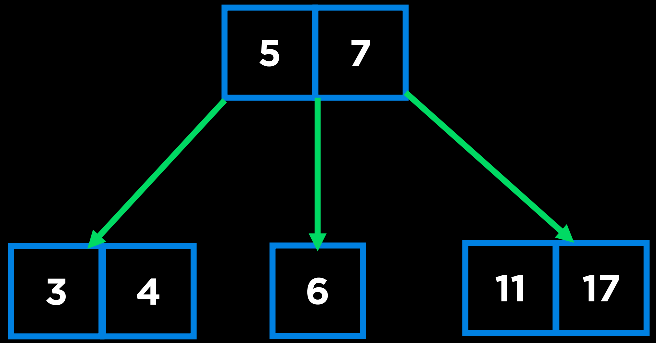
B Tree Insertion

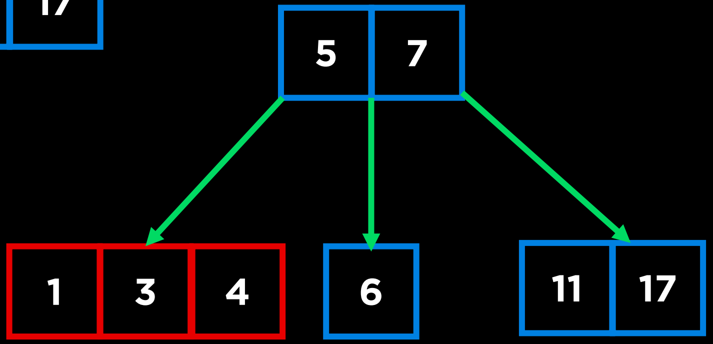
```
2-3 Tree (L=2):
Max 2 items per node.
Max 3 non-null children per node.
Insert 1
```



B Tree

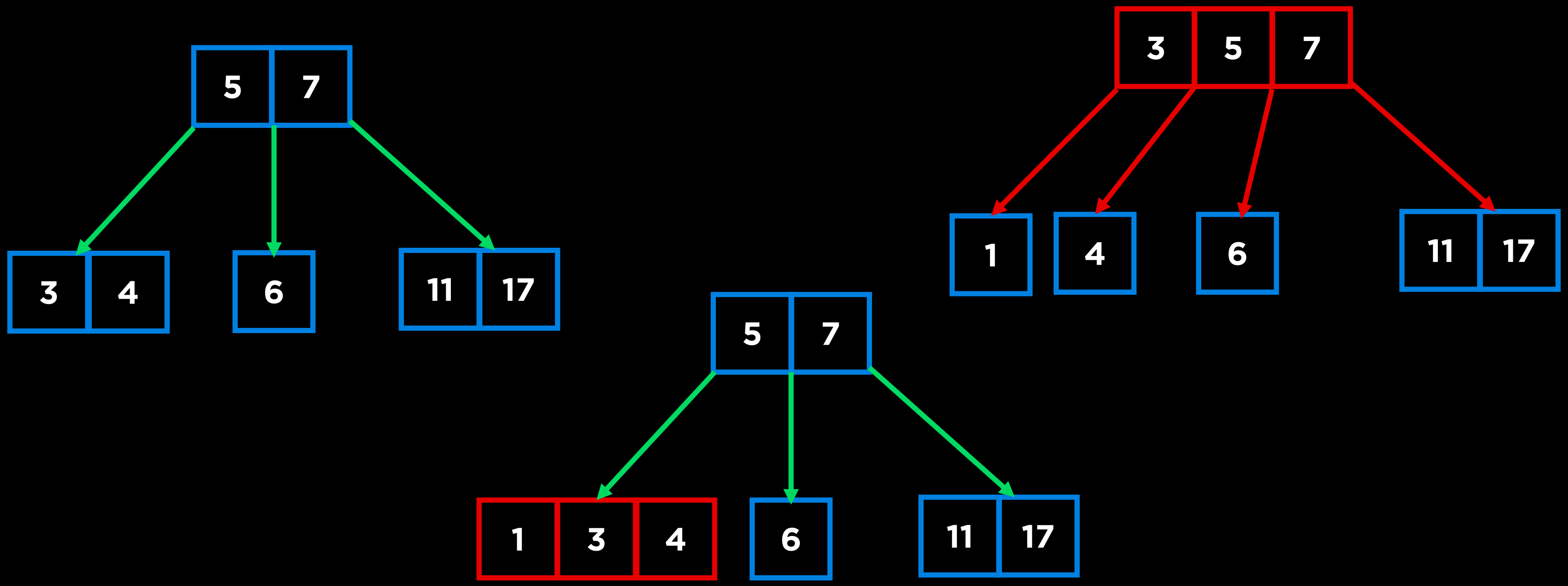
```
2-3 Tree (L=2):
Max 2 items per node.
Max 3 non-null children per node.
Insert 1
```





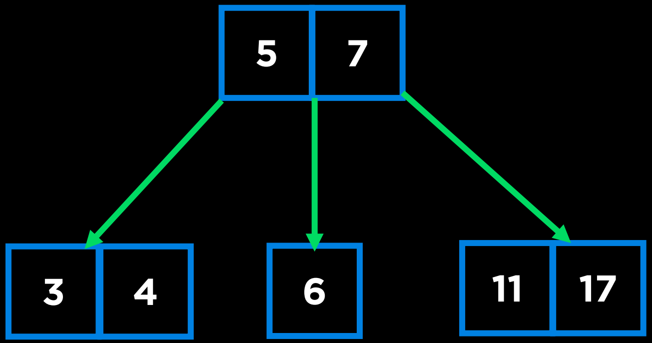
B Tree

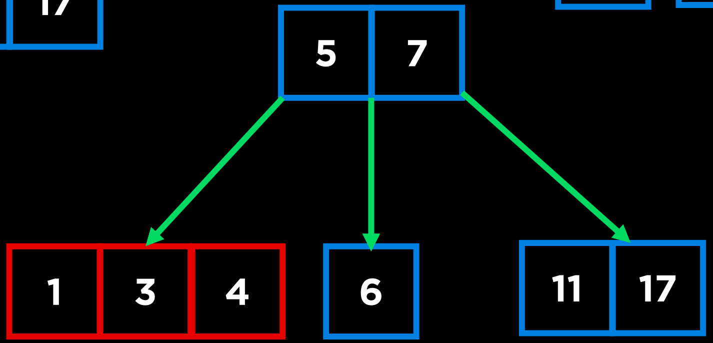
2-3 Tree (L=2):
Max 2 items per node.
Max 3 non-null children per node.
Insert 1

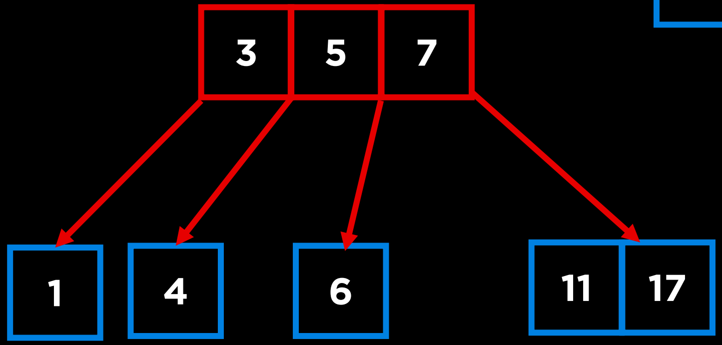


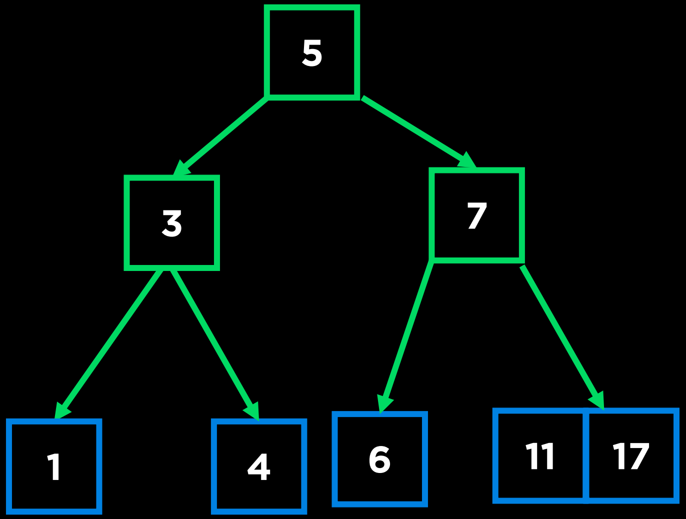
B Tree

2-3 Tree (L=2):
Max 2 items per node.
Max 3 non-null children per node.



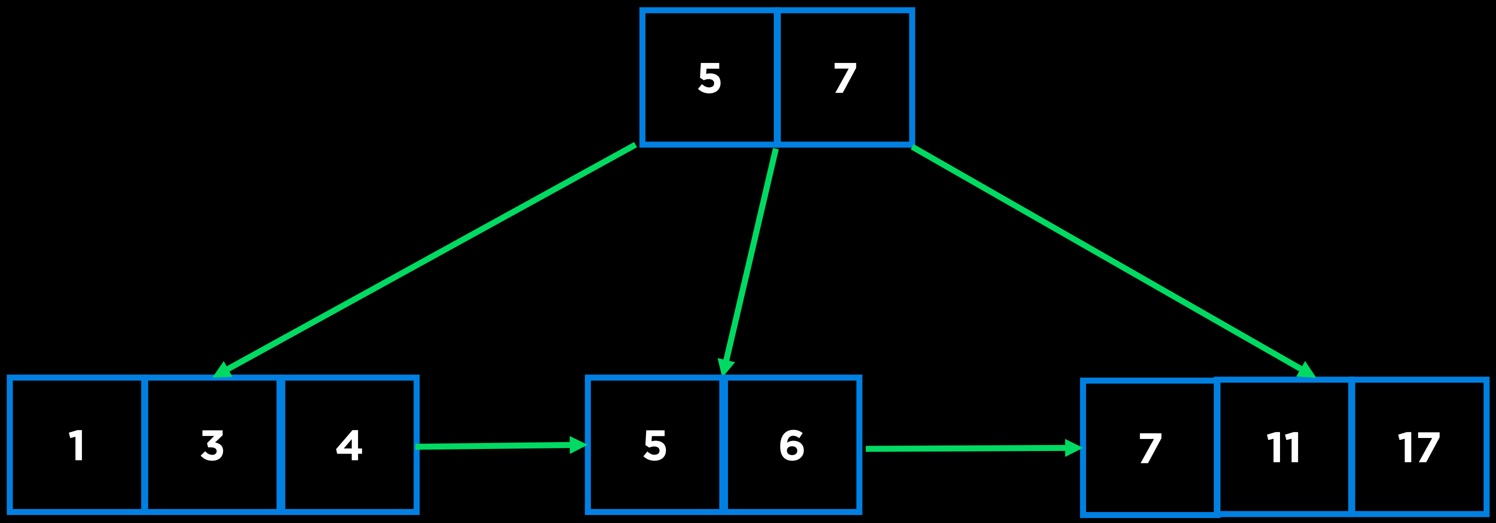






Height is still perfectly balanced!

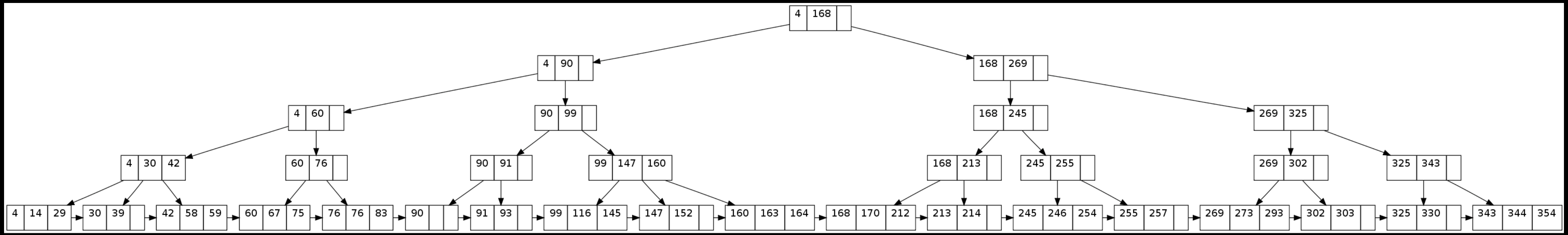
B+ Tree



B Tree + All data is stored in the leaves +
The leaves have pointers to the other leaves forming a linked list for faster traversal

https://en.wikipedia.org/wiki/B%2B_tree

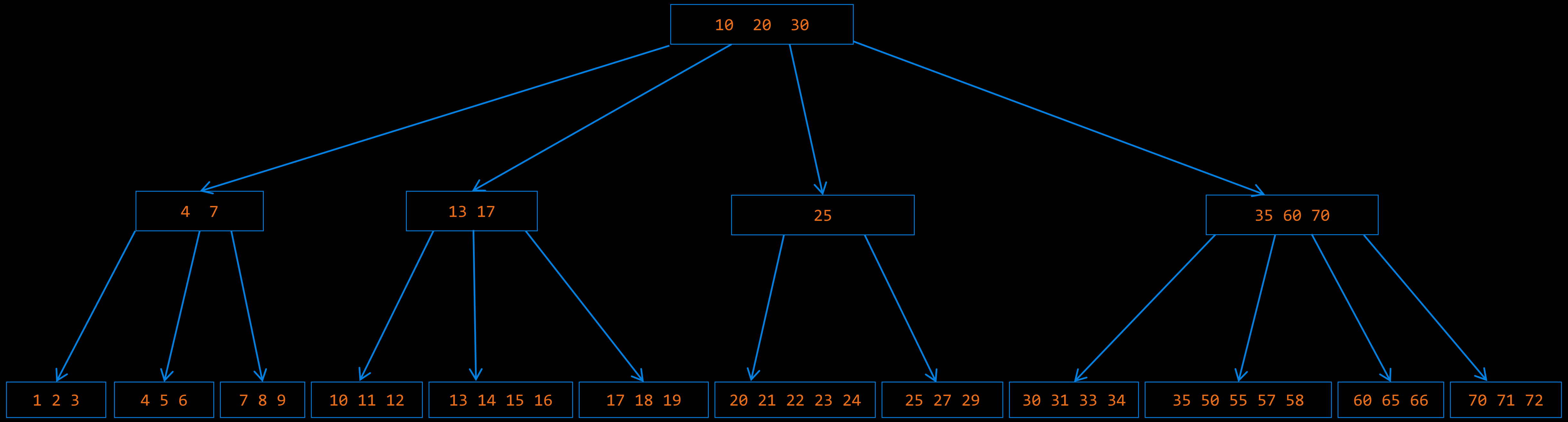
B+ Tree



4 168
4 90
168 269
4 60
90 99
168 245
269 325
4 30 42
60 76
90 91
99 147 160
168 213
245 255
269 302
325 343
4 14 29
30 39
42 58 59
60 67 75
76 76 83
90
91 93
99 116 145
147 152
160 163 164
168 170 212
213 214
245 246 254
255 257
269 273 293
302 303
325 330
343 344 354

B+ Tree Insertion

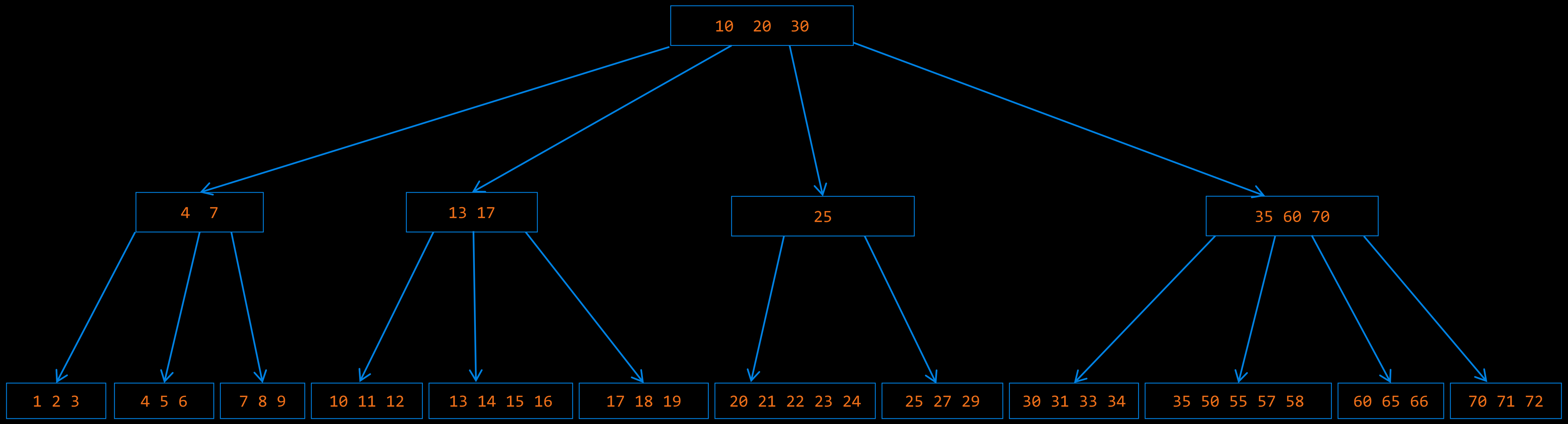
N = 4 (at most 4 children and 3 keys in a non-leaf node),
L = 5 (at most 5 keys in a leaf node)



B+ Tree Insertion

N = 4 (at most 4 children and 3 keys in a non-leaf node),
L = 5 (at most 5 keys in a leaf node)

Add 40



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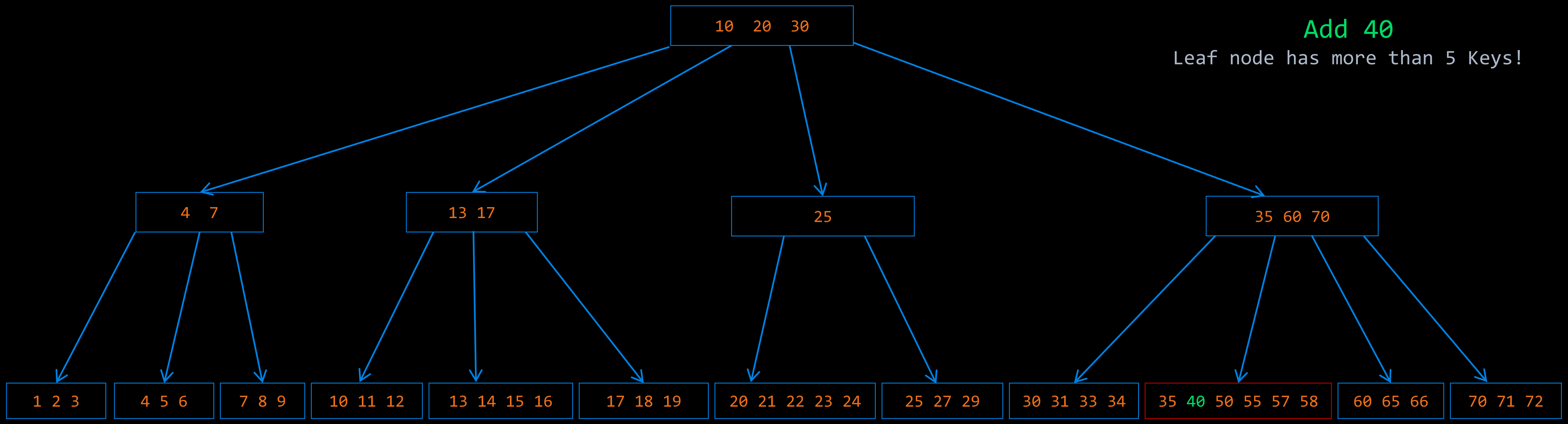
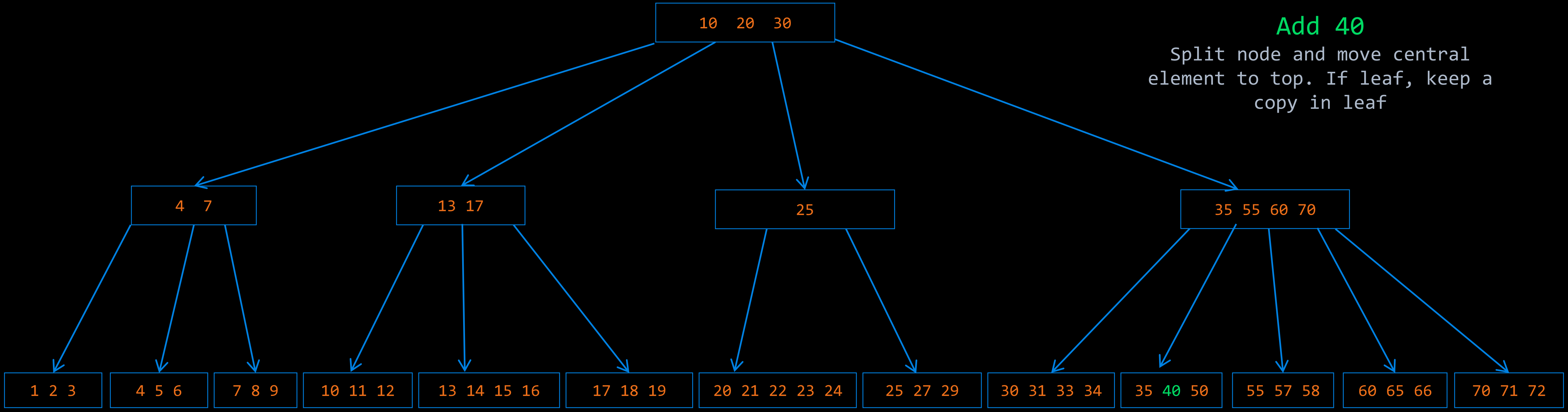


Add 40

Leaf node has more than 5 Keys!

B+ Tree Insertion

N = 4 (at most 4 children and 3 keys in a non-leaf node),
L = 5 (at most 5 keys in a leaf node)



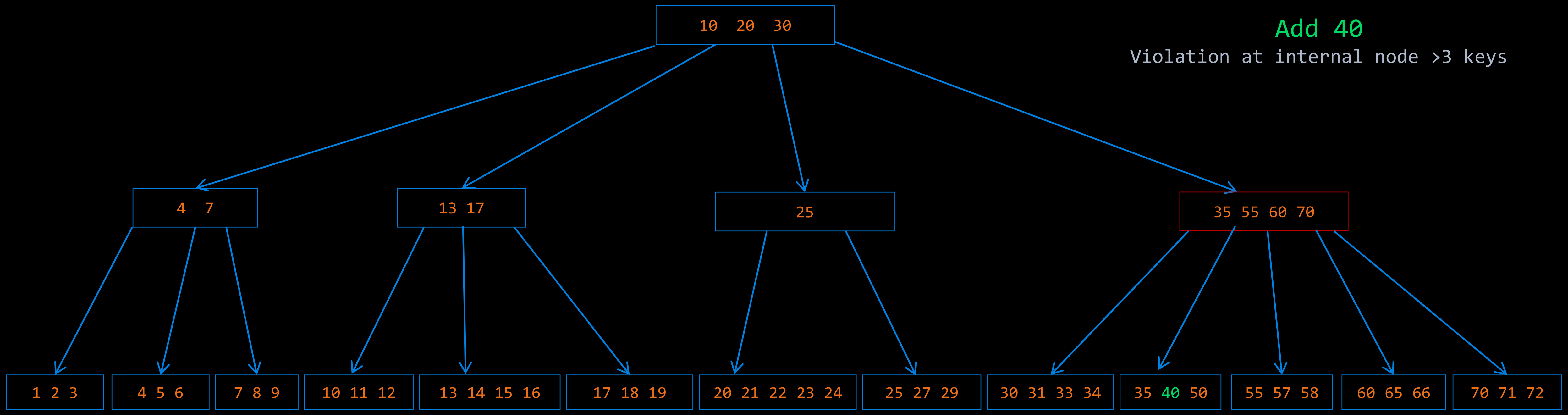
Add 40

Split node and move central element to top. If leaf, keep a copy in leaf

B+ Tree Insertion

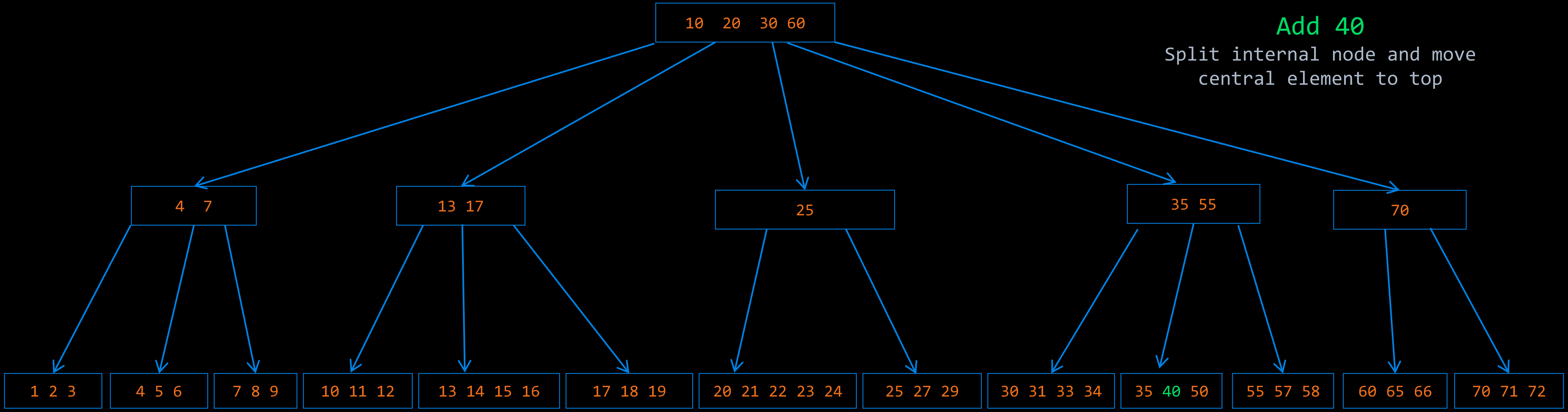
N = 4 (at most 4 children and 3 keys in a non-leaf node),
L = 5 (at most 5 keys in a leaf node)

Add 40
Violation at internal node >3 keys



B+ Tree Insertion

N = 4 (at most 4 children and 3 keys in a non-leaf node),
L = 5 (at most 5 keys in a leaf node)



Add 40

Split internal node and move central element to top

B+ Tree Insertion

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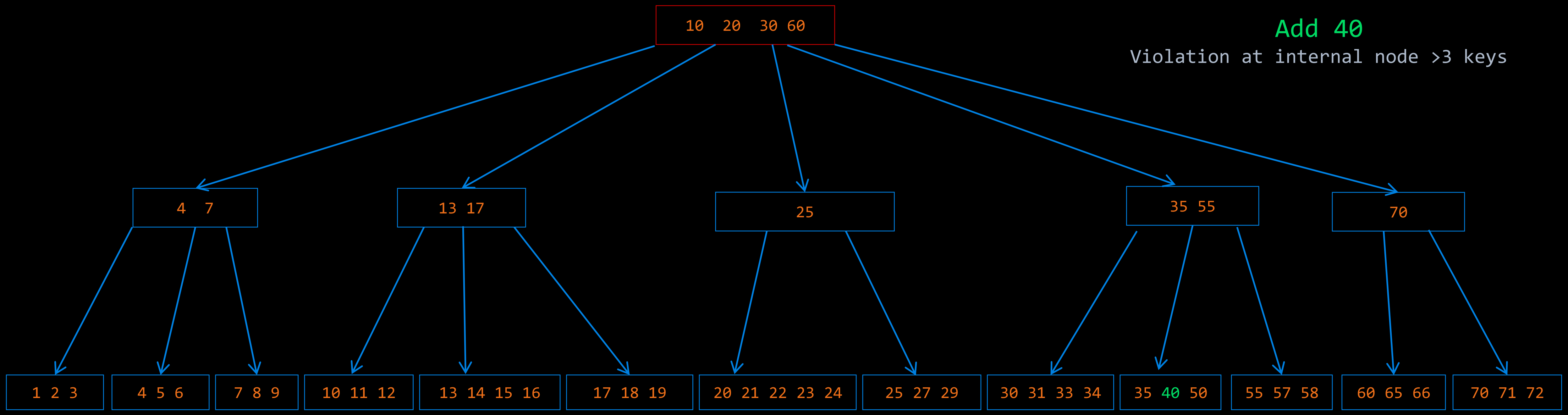
L = 5 (at most 5 keys in a leaf node)

Add 40
Violation at internal node >3 keys



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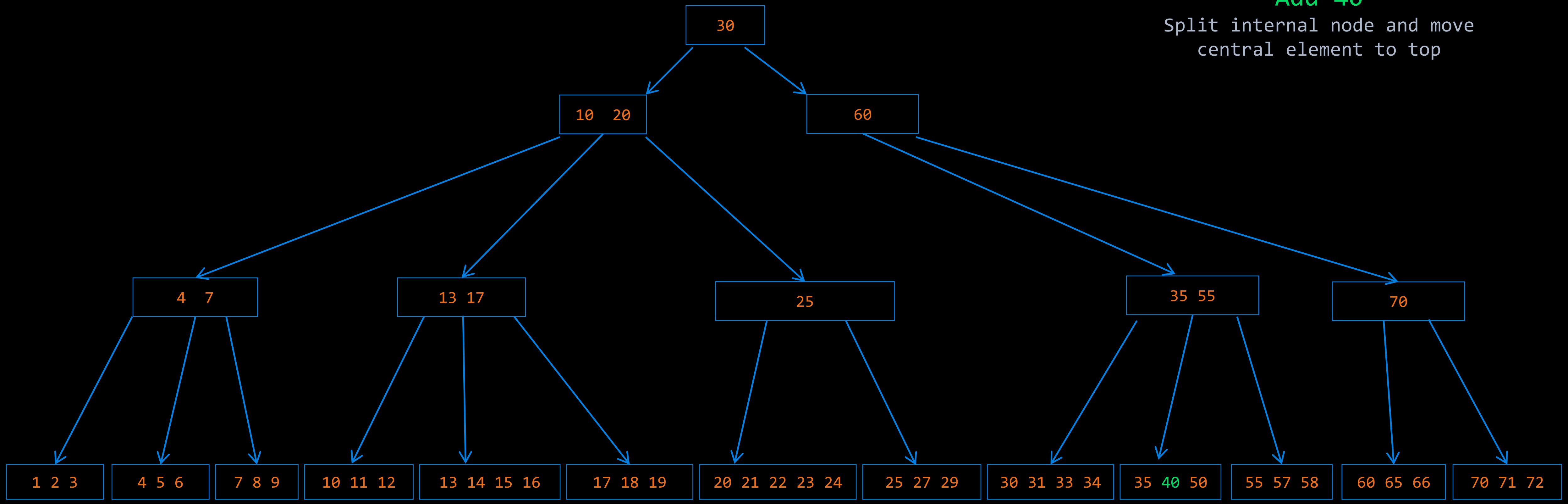


B+ Tree

`A completely full B+ Tree with N=3 and L=3 and height = 2 (has level 0, 1, 2)`
`has how many unique values?`

B+ Tree

A completely full B+ Tree with N=3 and L=3 and height = 2 (has level 0, 1, 2) has how many unique values?

Level 0 - 2 values
Level 1 – 3 nodes, each with 2 values = 6 values
Level 2 – 3*3 = 9 nodes, each with 3 values = 27 values - 6 – 2 = 19

2+6+19=27

Or, root has 3 children, each child has 3 children. So 9 leaves. Each leaf has 3 values. So 27 values.

Use Case

- Hard Drives, Databases, Filesystems
- Indexing Example
  - Tracks, Sectors and Blocks

Performance

- Height: Between ~$\log_{l+1}(n)$ and ~$\log_2(n)$
- Largest possible height is all non-leaf nodes have l/2 items
- Smallest possible height is all nodes have l items
- Overall height is therefore O(log n)
- Search Time: O(hl) ~ O(l log (n)), where
  - h is height of tree
  - n is maximum number of children
  - l is maximum number of keys
- Search Time: O(log (n)), as l is a constant

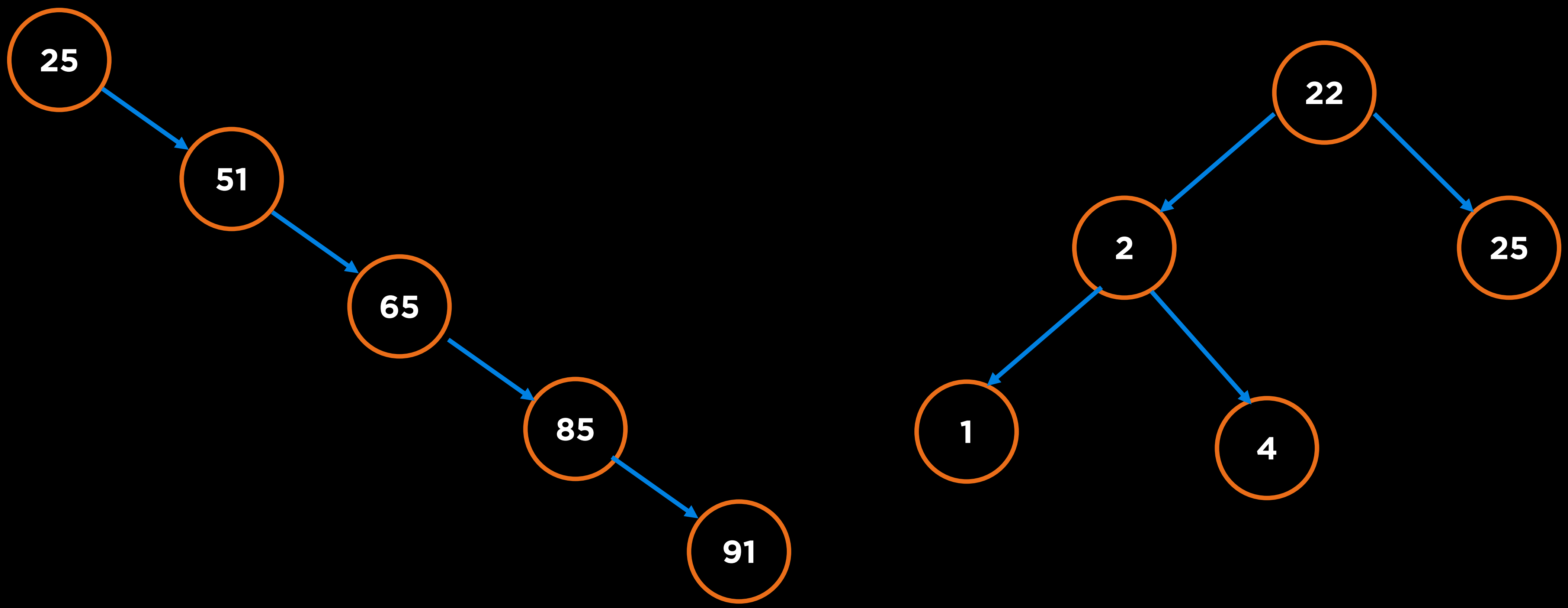
Mentimeter

menti.com

4152 2550

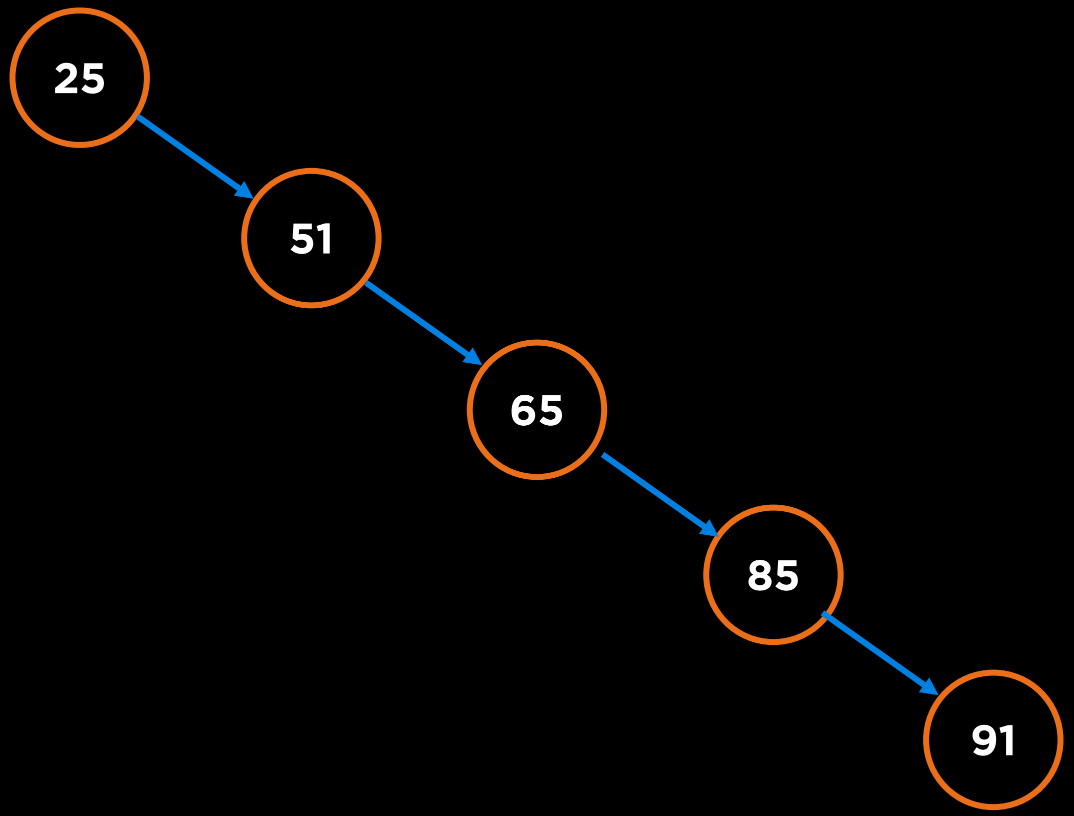
Splay Tree

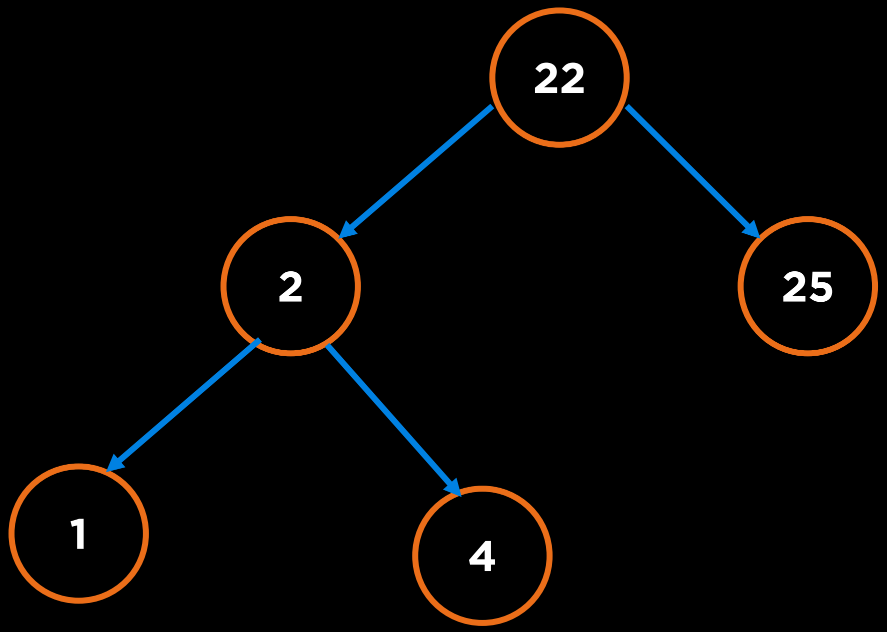
Searching for Random Inputs



O(n) or O(log n) in BST or AVL Tree

Searching for Non-Random Inputs

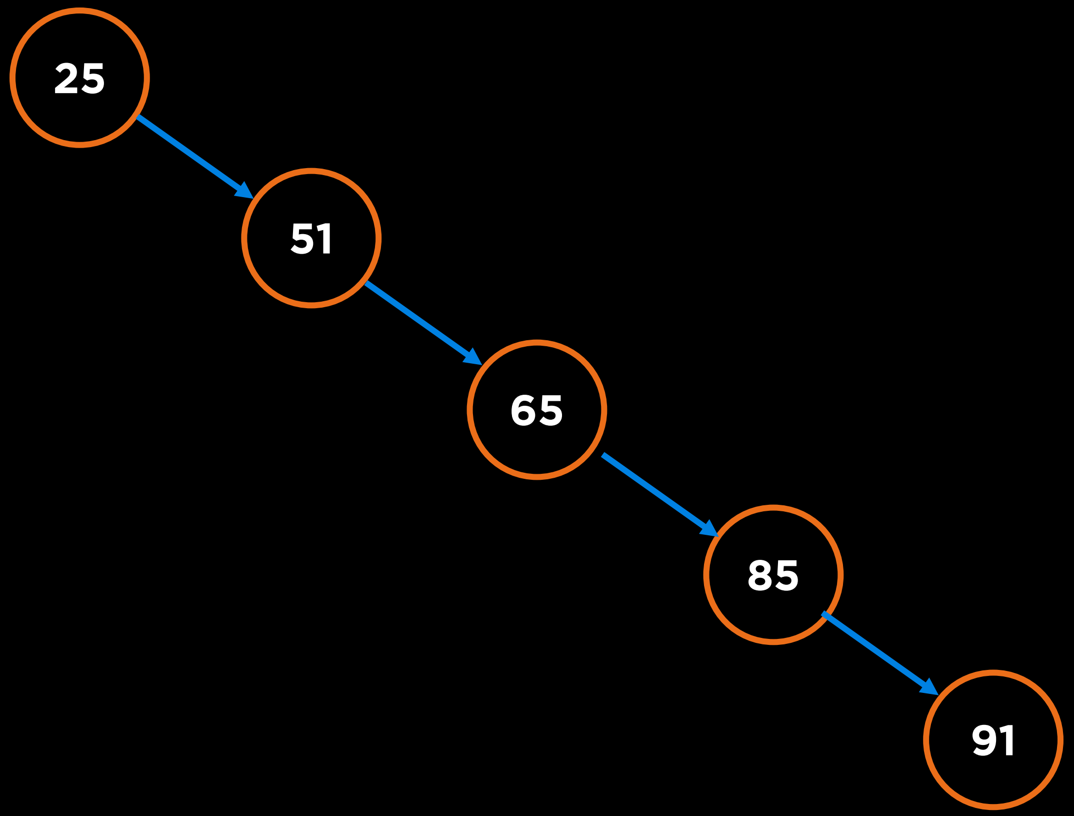




O(n) or O(log n) in BST or AVL Tree

What if the Input is non-Random?

Enter Splay Tree

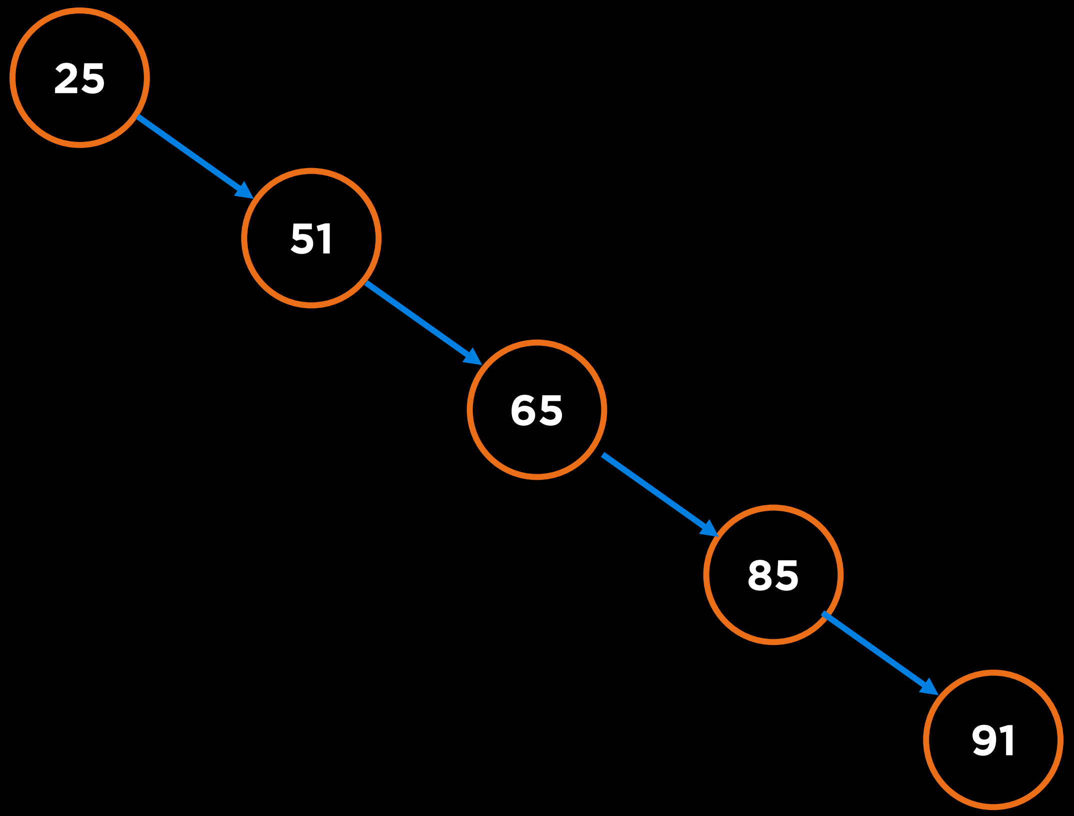


If we are searching 91 again and again, bring it closer to root!

Simple Rotation won't work!

Special rotations involving grandparent, parent and child.

Enter Splay Tree



A splay tree is a self-balancing binary search tree with the additional property that recently accessed elements are quick to access again.

Splay Tree

- A binary search tree with the additional property that recently accessed elements are quick to access again

- For many sequences of non-random operations, splay trees perform better than other search trees

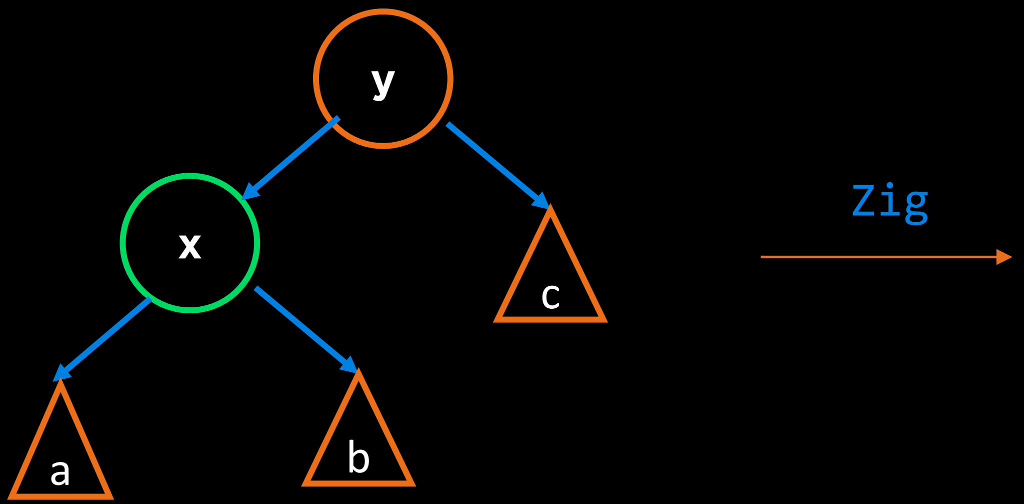
- The splay tree was invented by Daniel Sleator and Robert Tarjan

- All normal operations on a binary search tree are combined with one basic operation, called splaying. Splaying the tree for a certain element rearranges the tree so that the element is placed at the root of the tree.

https://en.wikipedia.org/wiki/Splay_tree

Splay Tree: Zig Rotation

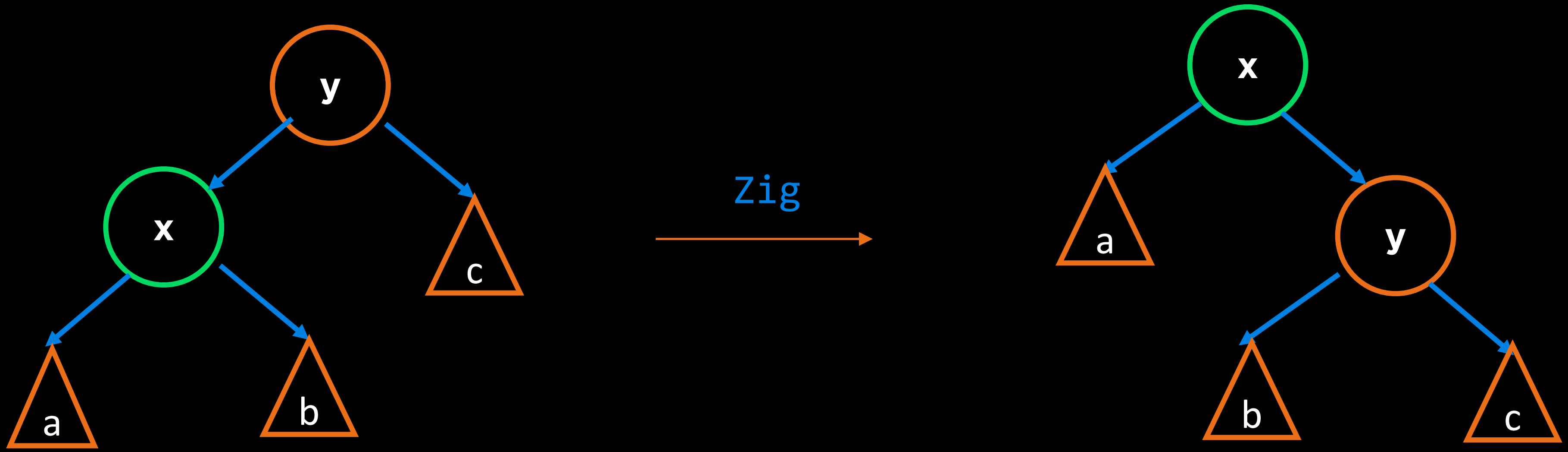
Search x
Splay(x)



Zig = Right Rotation (Splay(x), x is left of parent and has no grandparent)

Splay Tree: Zig Rotation

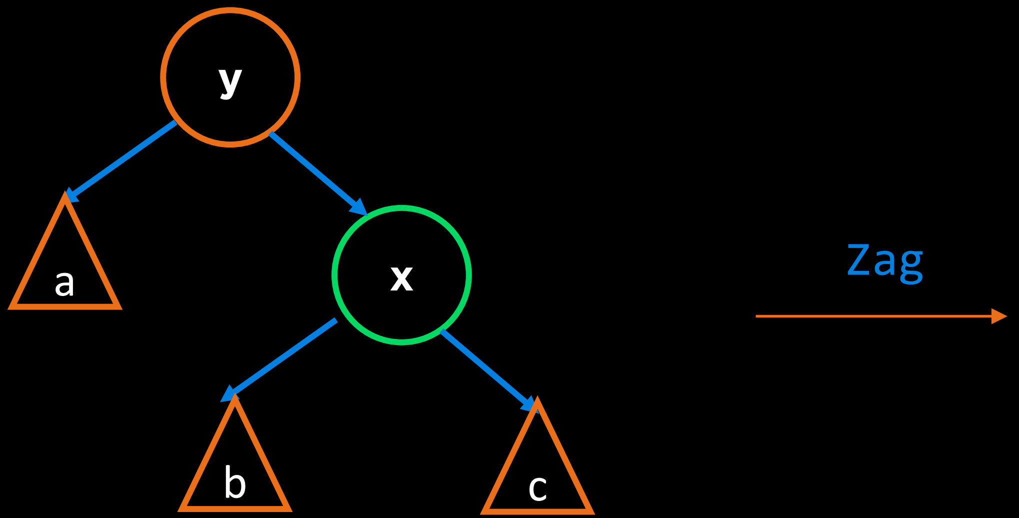
Search x
Splay(x)



Zig = Right Rotation (Splay(x), x is left of parent and has no grandparent)

Splay Tree: Zag Rotation (Zig)

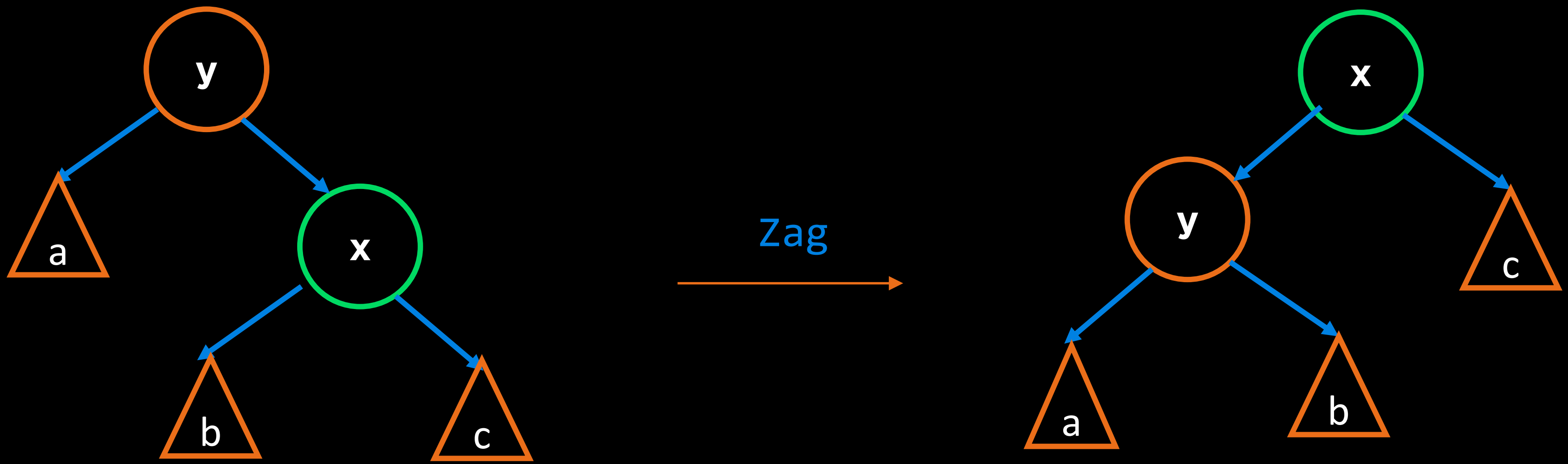
Search x
Splay(x)



Zag = Left Rotation (Splay(x), x is right of parent and has no grandparent)

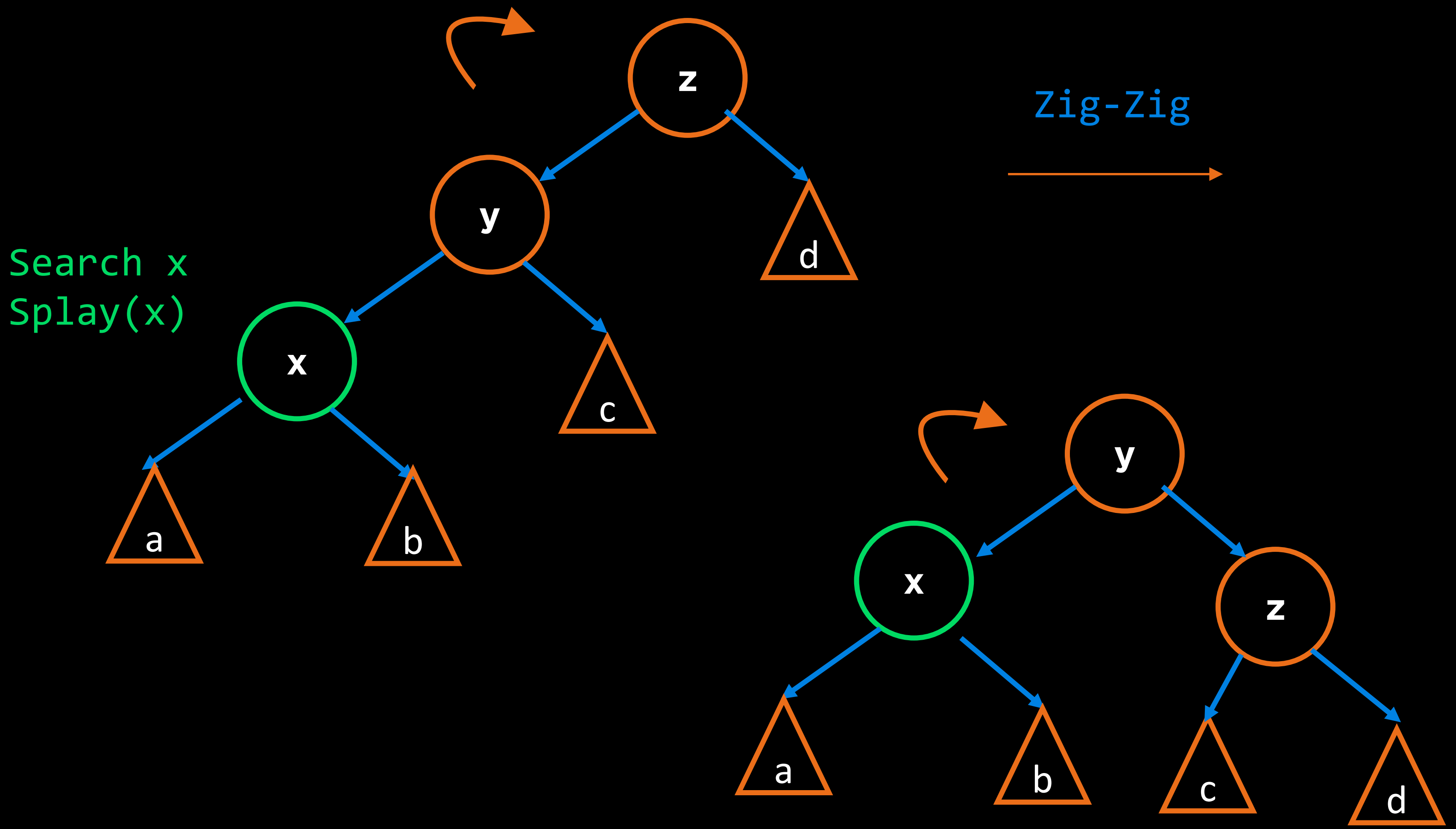
Splay Tree: Zag Rotation (Zig)

Search x
Splay(x)



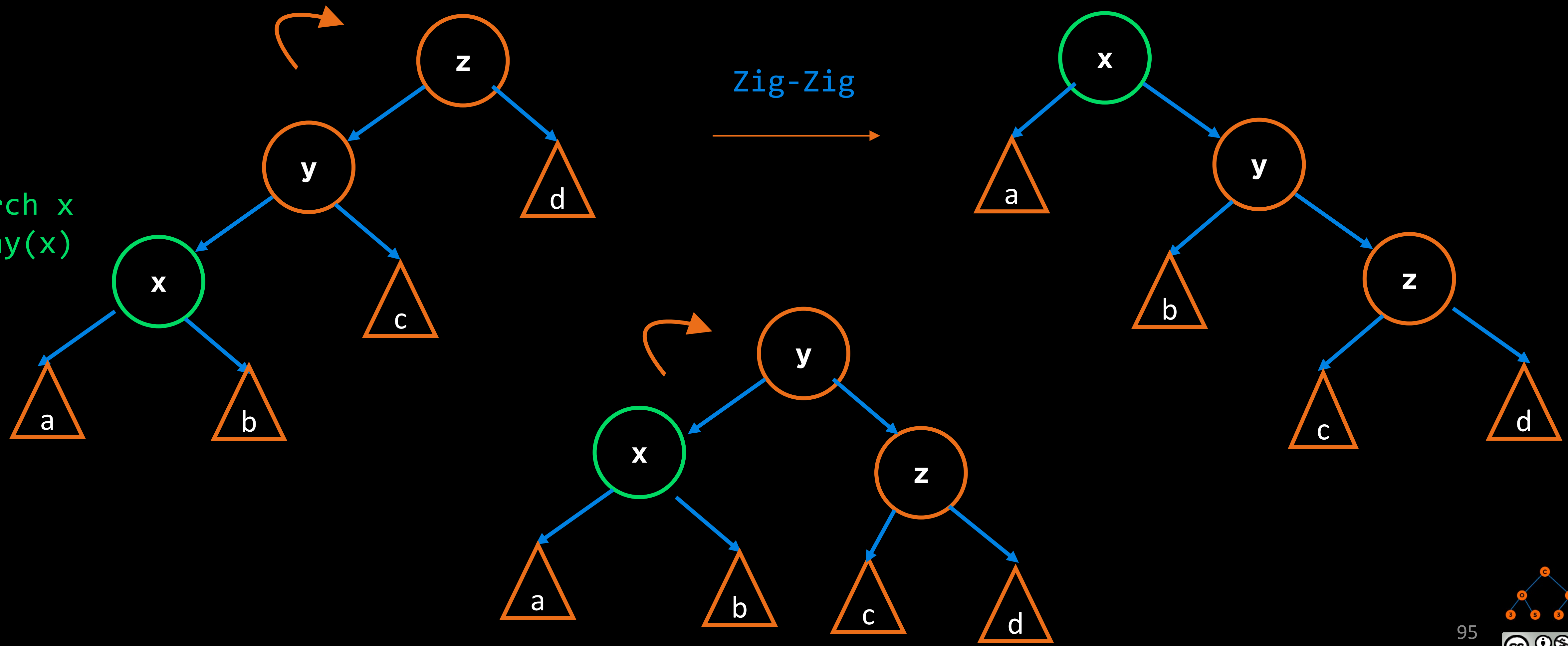
Zag = Left Rotation (Splay(x), x is right of parent and has no grandparent)

Splay Tree: Zig Zig Rotation



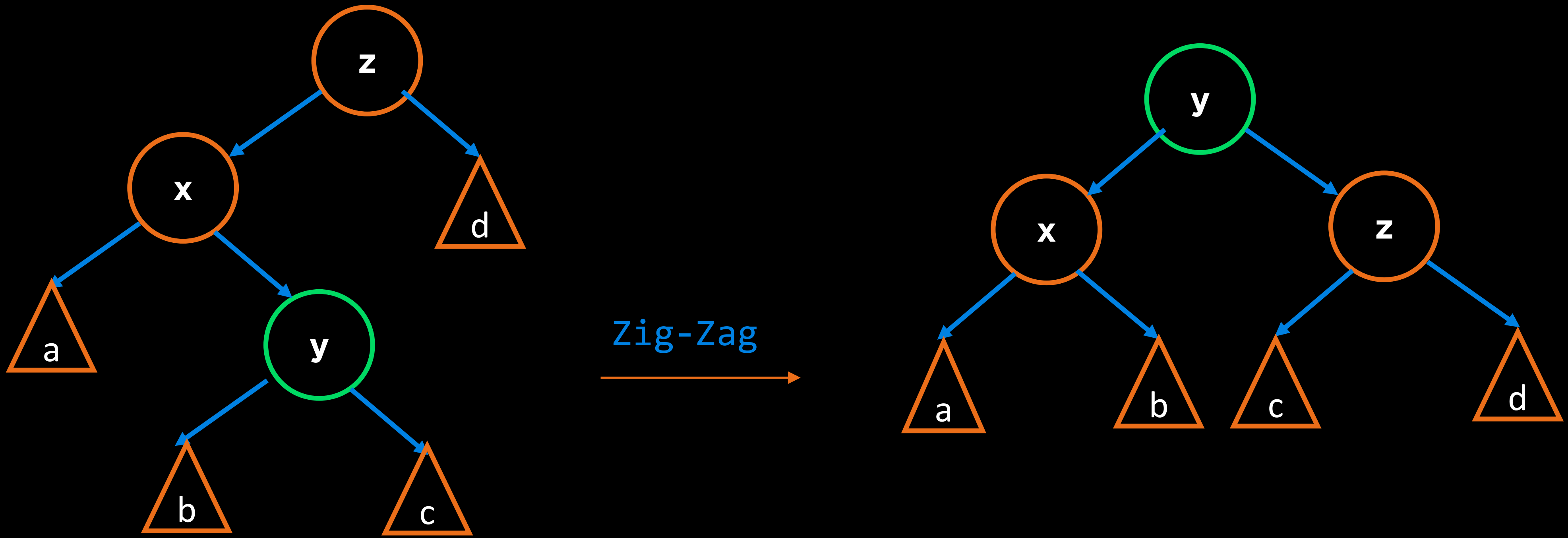
Splay Tree: Zig Zig Rotation

Search x
Splay(x)

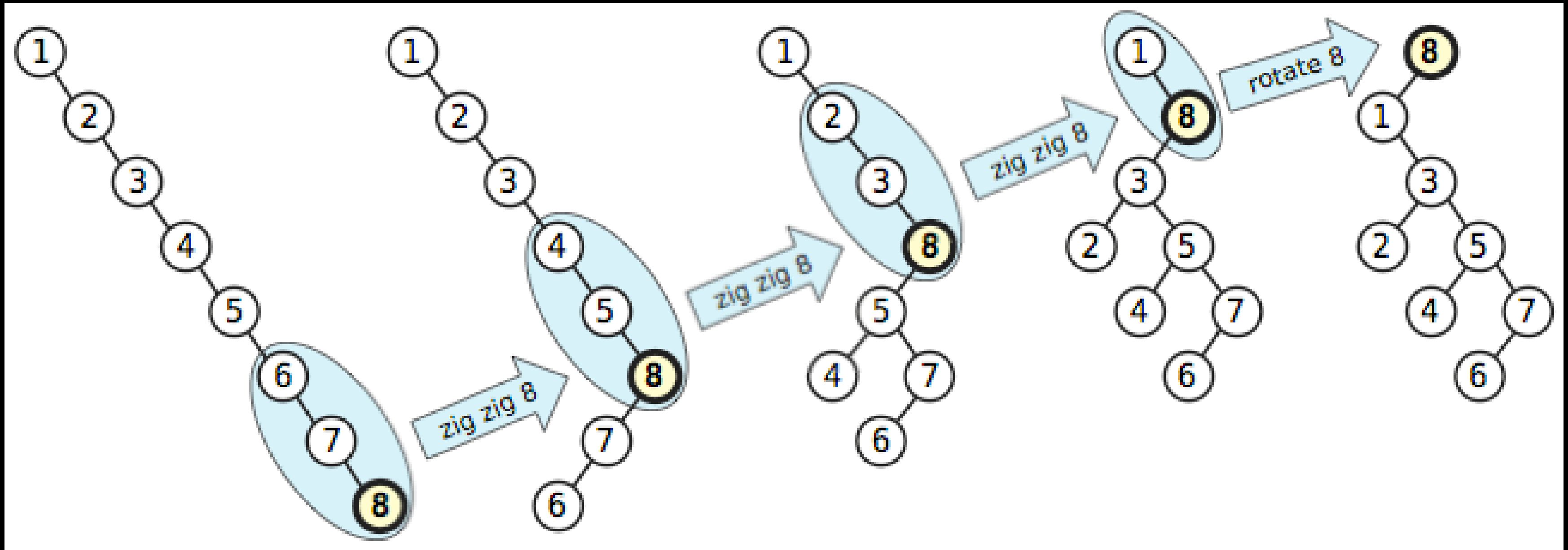


Splay Tree: Zig Zag Rotation

Search y
Splay(y)



Splay Tree: Basic Idea



The basic idea of a splay tree is that after a node is accessed, it is pushed to the root via a series of rotations. And it does manage to shorten the tree.

Start at bottom and move up! Splay(N) till N.Parent == null

Splay Tree: Insert/Search

- Same as BST followed by a Splay Operation on the searched node or newly inserted node

- Splay(N)
  - Determine proper case for rotation and apply
    - Zig Zig
    - Zig Zag
    - Zig
  - If N.Parent != null:
    - Splay(N)

Splay Tree: Performance

- A splay tree is a data structure that guarantees that m tree operations will take O(m log n) time, where n is number of nodes

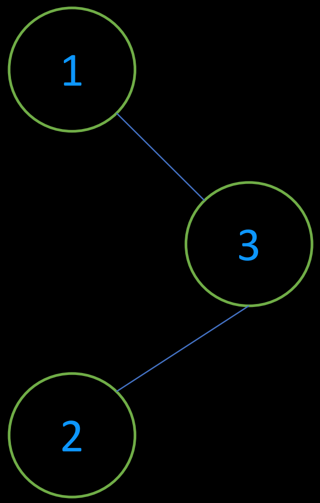
- On average, a tree operation is O(log n)

- In the worst case, an operation is O(n), but subsequent operations are fast

- Implemented in Cache and Garbage Collection

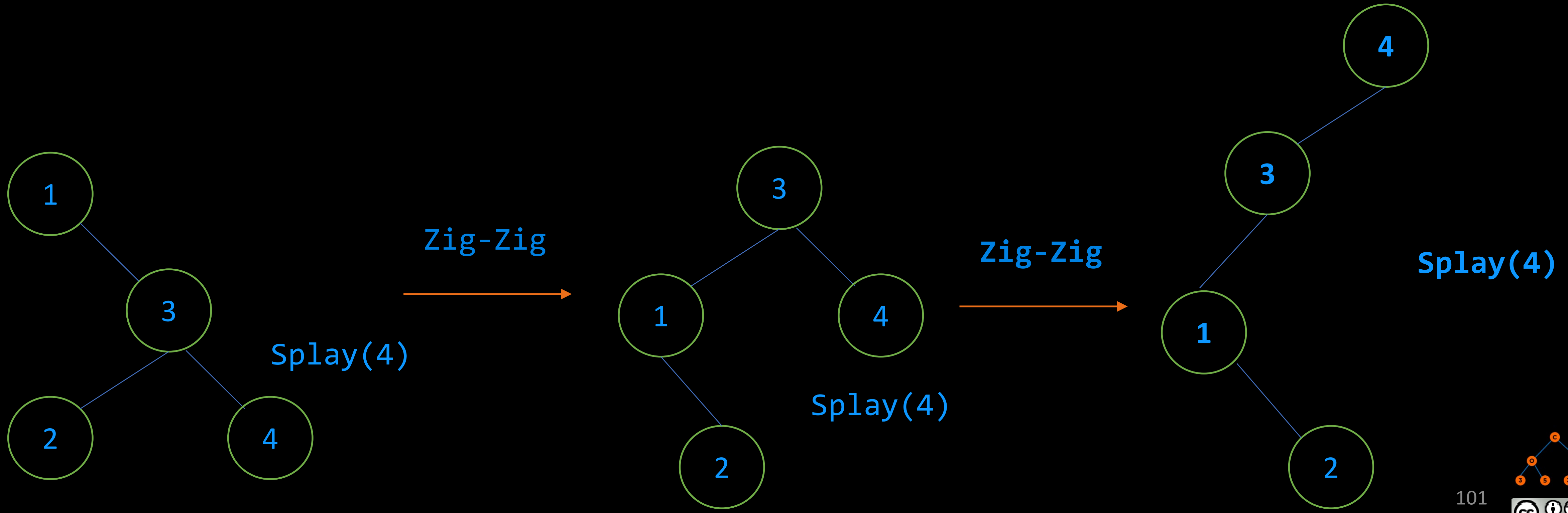
Mentimeter

What will be the value of root node if we insert 4 into this Splay Tree?



Mentimeter

What will be the value of root node if we insert 4 into this Splay Tree?



Resources

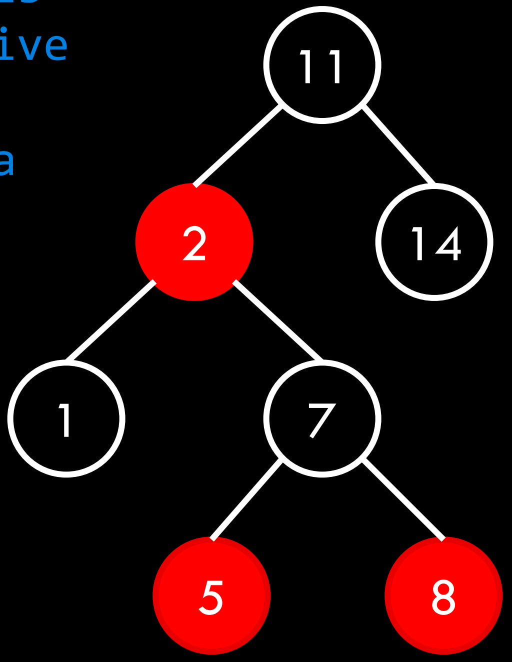
- B+ Tree Visualization: https://www.cs.usfca.edu/~galles/visualization/BPlusTree.html
- Splay Tree Visualization: https://www.cs.usfca.edu/~galles/visualization/SplayTree.html
- Original Paper, Splay Tree: https://www.cs.cmu.edu/~sleator/papers/self-adjusting.pdf
- https://stackoverflow.com/questions/7467079/difference-between-avl-trees-and-splay-trees

Red Black Tree

Red Black Tree Properties

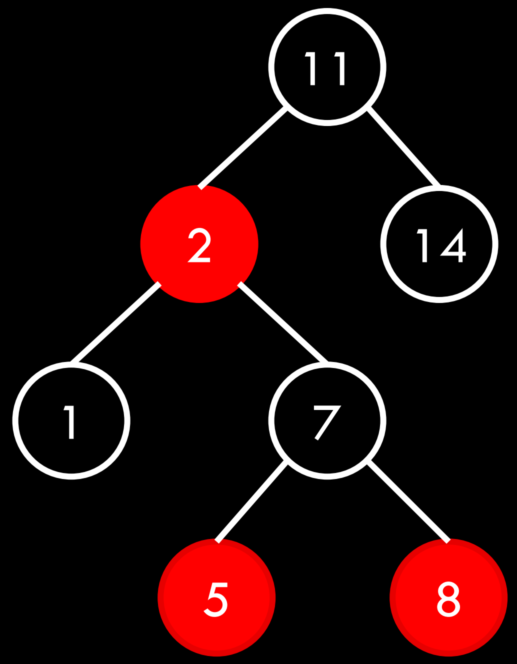
A red-black tree maintains the following invariants:

1. A node is either red or black
2. The root is always black
3. A red node always has black children (a null reference is considered to refer to a black node) Or No two consecutive Red nodes
4. The number of black nodes in any path from the root to a leaf is the same
5. Null nodes are attached to the leaves and are black



Red Black Tree General Idea

- Color is stored as a Boolean in the TreeNode class
- Fixing the tree invariants by rotation or through color flips



Red Black Tree Insertion

- Insert the item into the binary search tree as usual

- Color it red

- If the tree is empty, color it black and make it a root, we are done

Red Black Tree Insertion

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- Color it red
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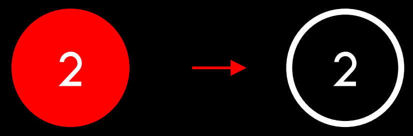
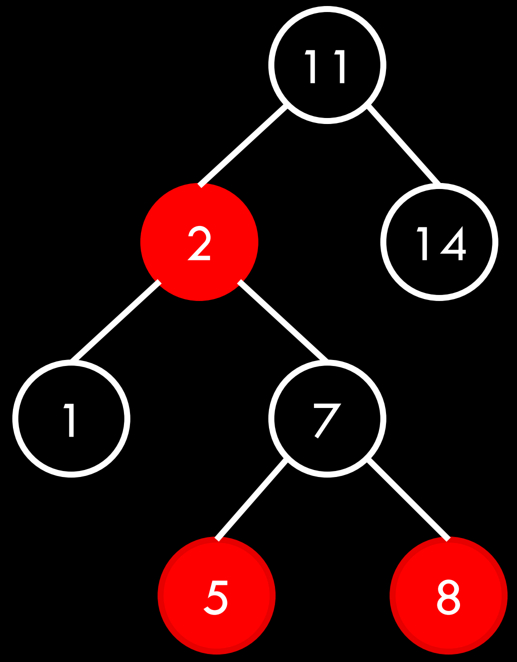


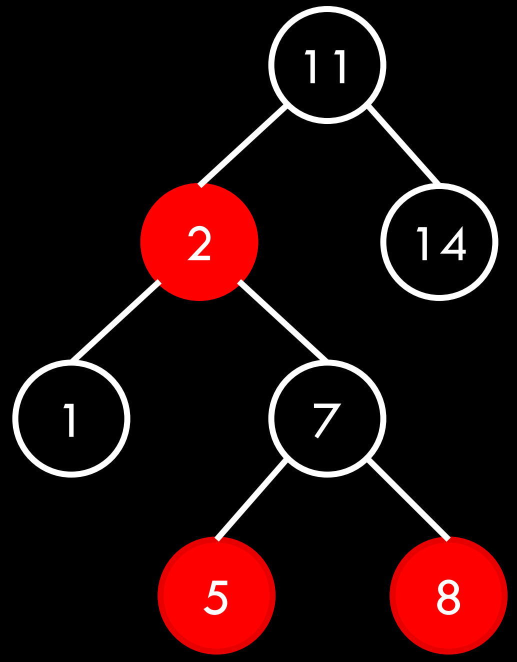
Red Black Tree Insertion

- Insert the item into the binary search tree as usual
- Color it red
- If the parent is black, we are done



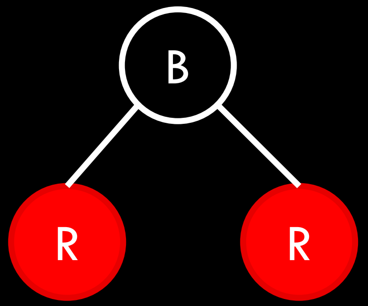
Red Black Tree Insertion

- Insert the item into the binary search tree as usual
- Color it red
- If the parent is red, look at the aunt/uncle or parent's sibling

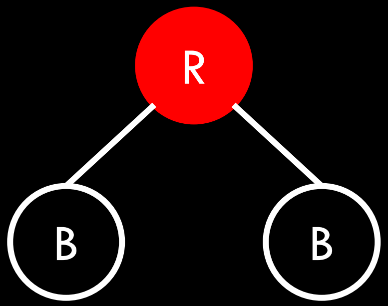


Red Black Tree Insertion

- If the uncle is red, flip colors
- If the uncle is black, rotate
- After Rotation



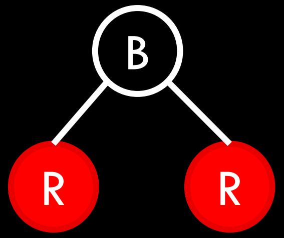
After Color Flip (P, GP)

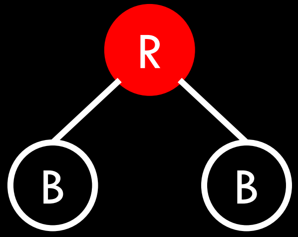


Example

Insert 3

- If the uncle is red, flip colors
- If the uncle is black, rotate
- After Rotation After Color Flip (P, GP)





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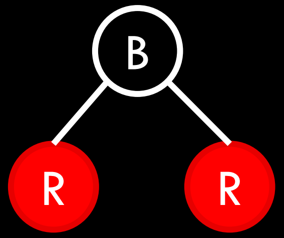
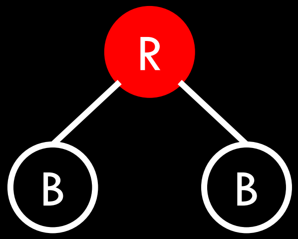
5. Null nodes are attached to the leaves and are black

Example

Insert 3

- If the uncle is red, flip colors
- If the uncle is black, rotate
- After Rotation After Color Flip (P, GP)





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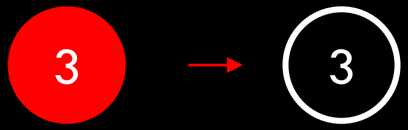
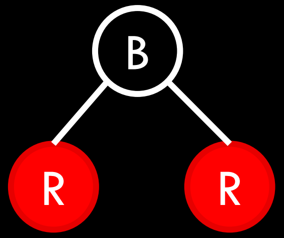
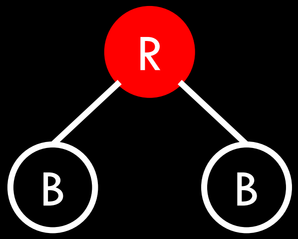
Example

Insert 3



- If the uncle is red, flip colors
- If the uncle is black, rotate
- After Rotation After Color Flip (P, GP)





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Example

Insert 1

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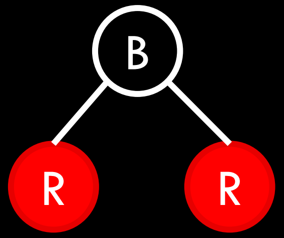
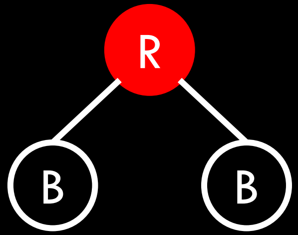
- If the uncle is black, rotate
- After Rotation After Color Flip (P, GP)





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Example

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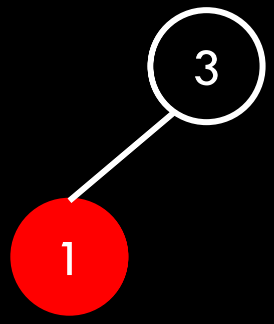

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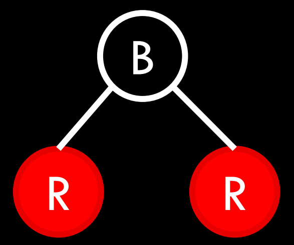

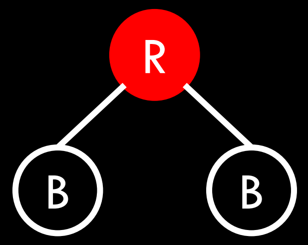

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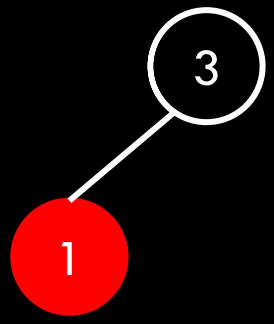
Example

Insert 5



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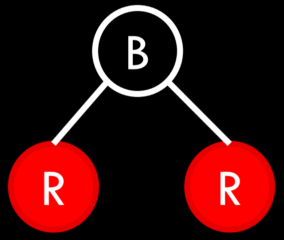

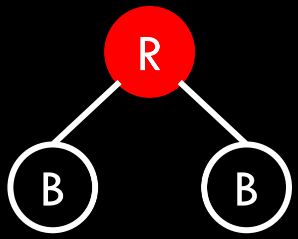

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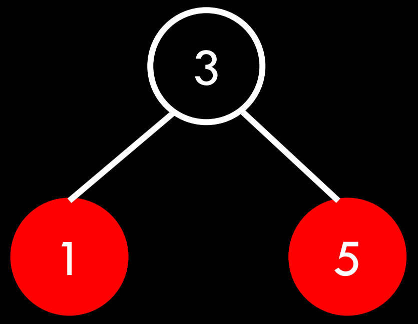

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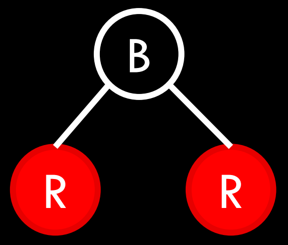

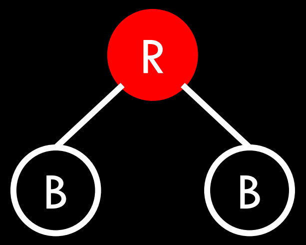

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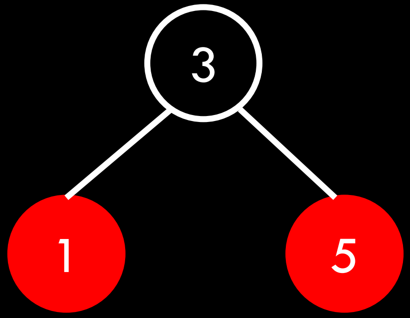
Example

Insert 7



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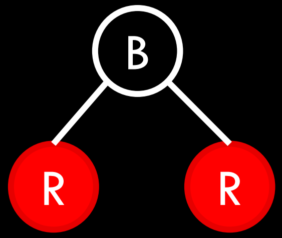

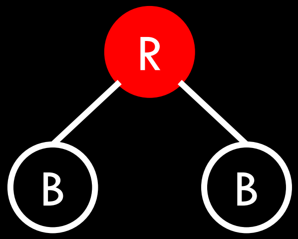

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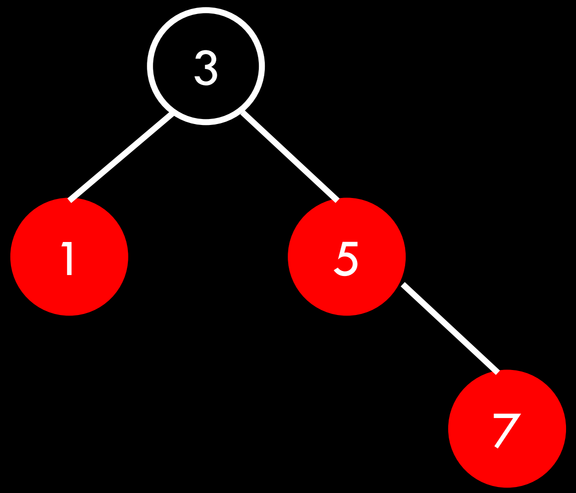

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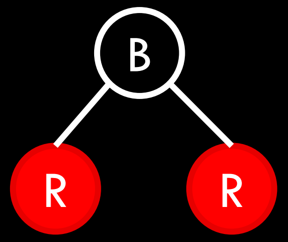

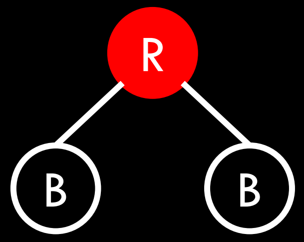

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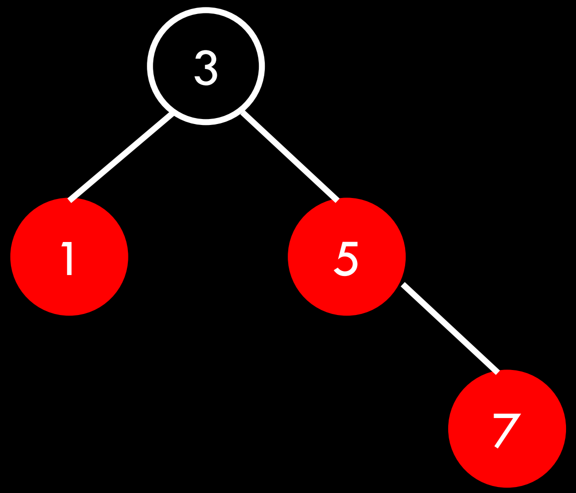

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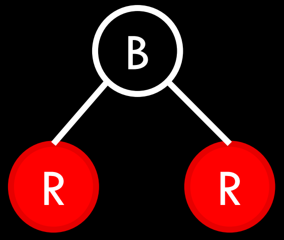

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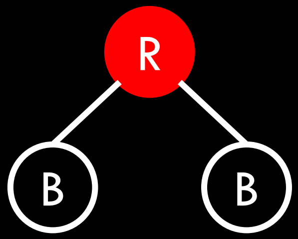

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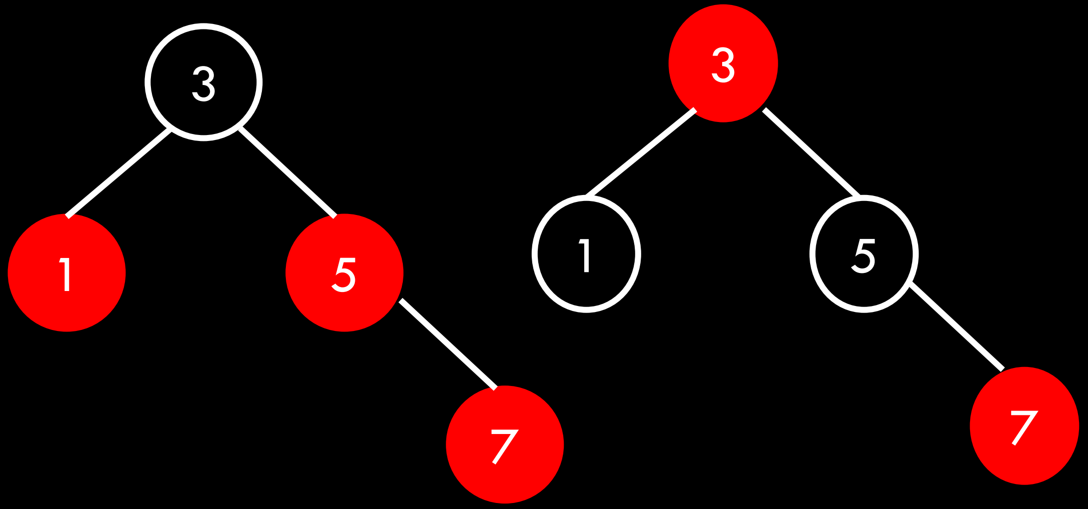

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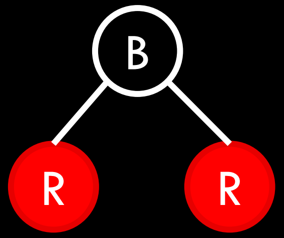

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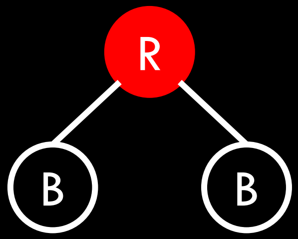

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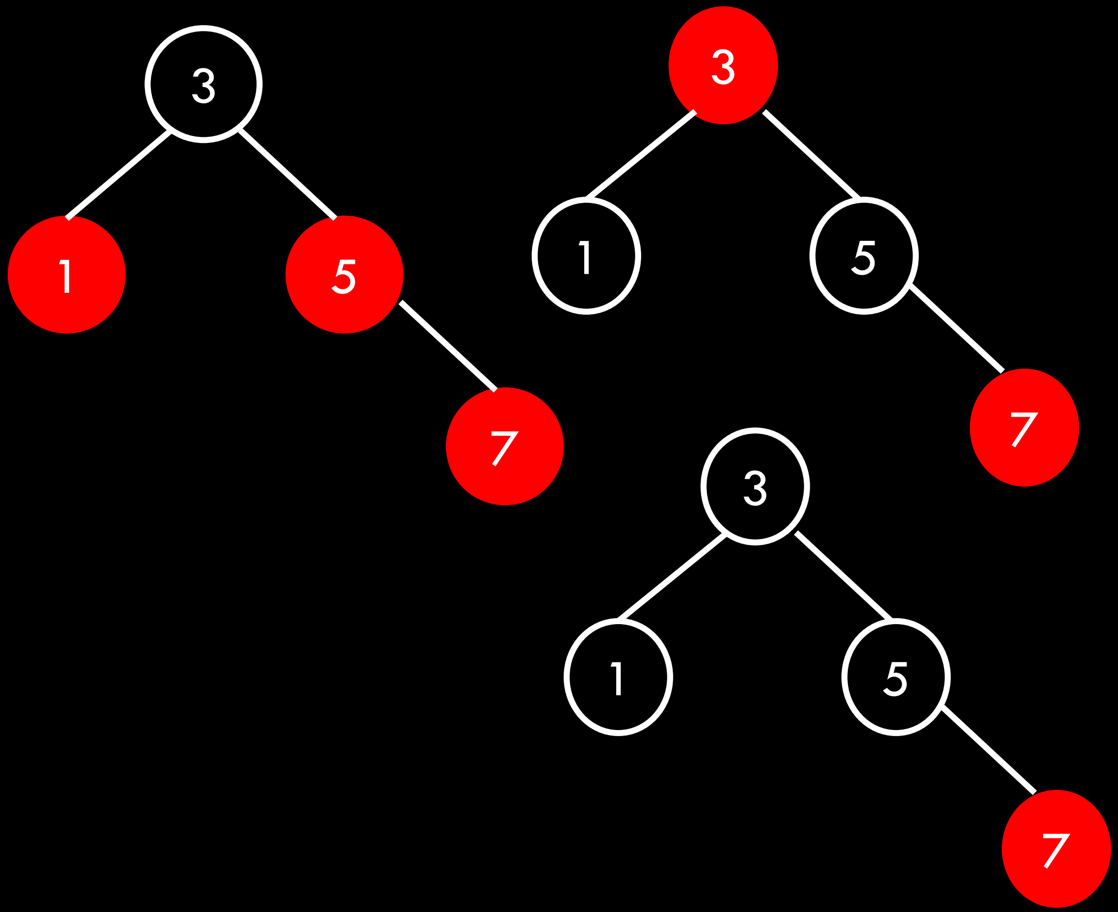

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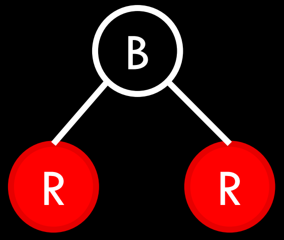

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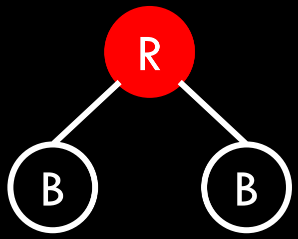

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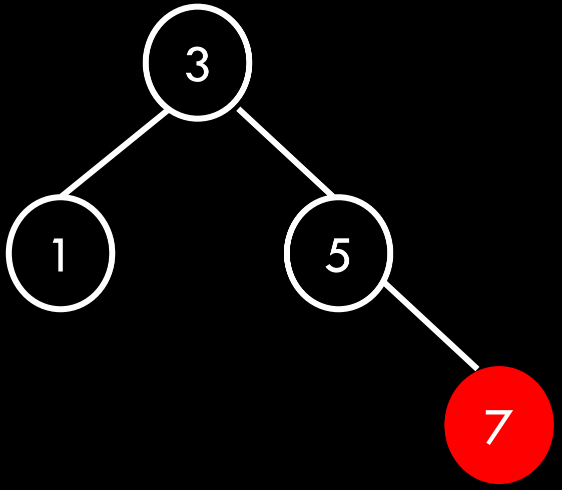
Example

Insert 6



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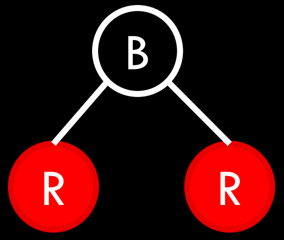
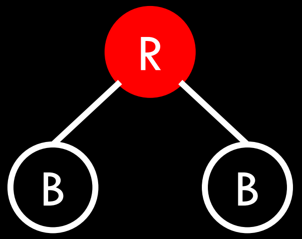


After Color Flip (P, GP)



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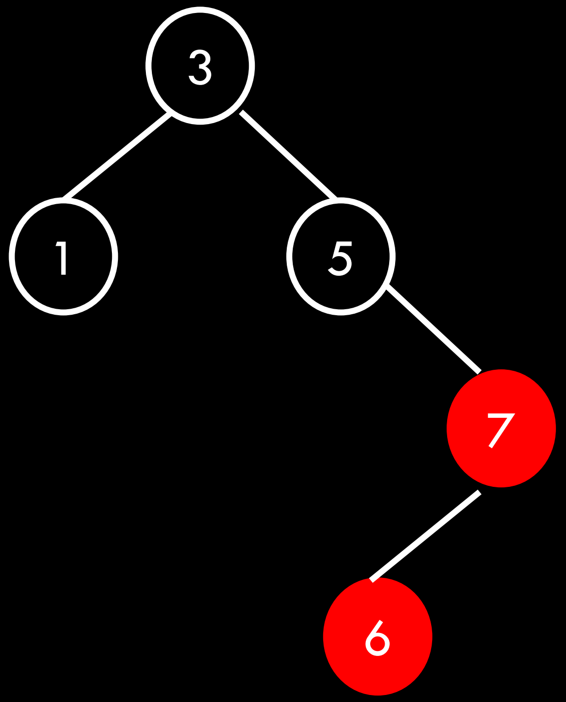

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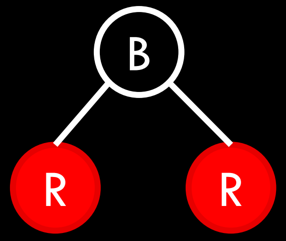

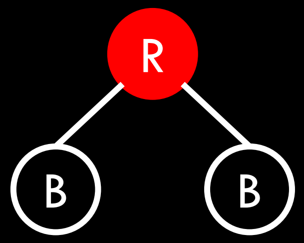

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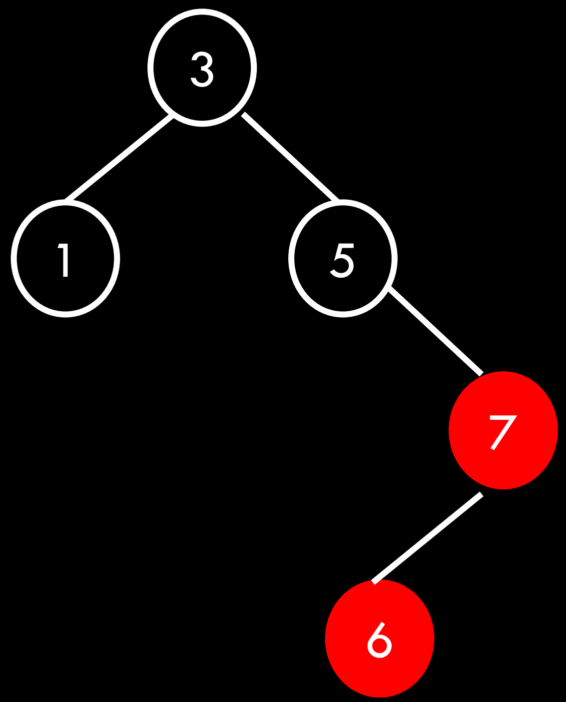

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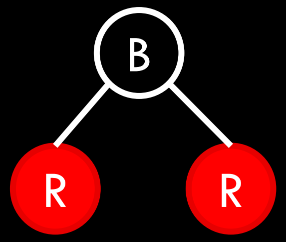

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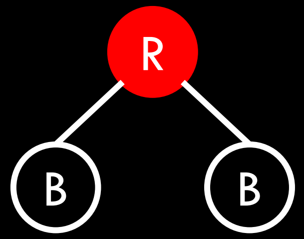

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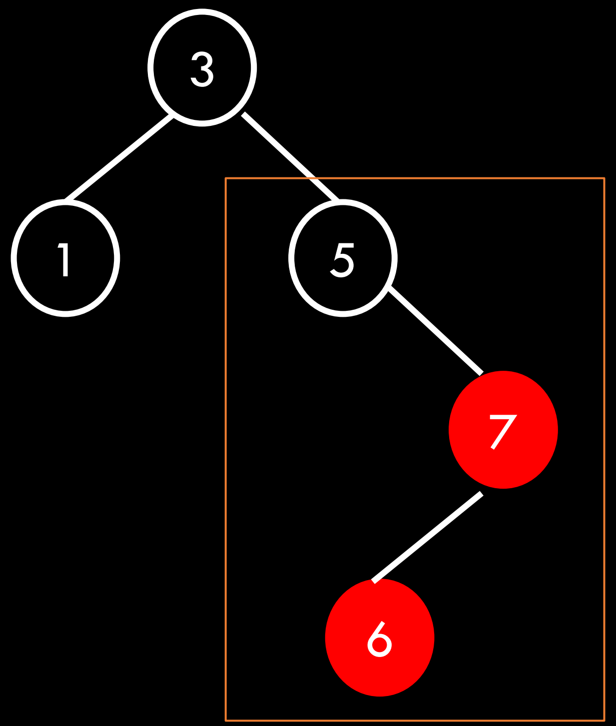

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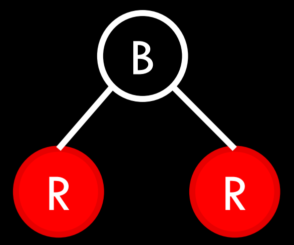

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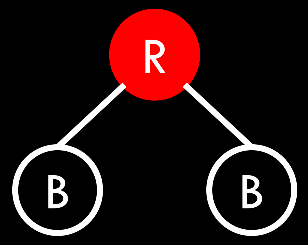

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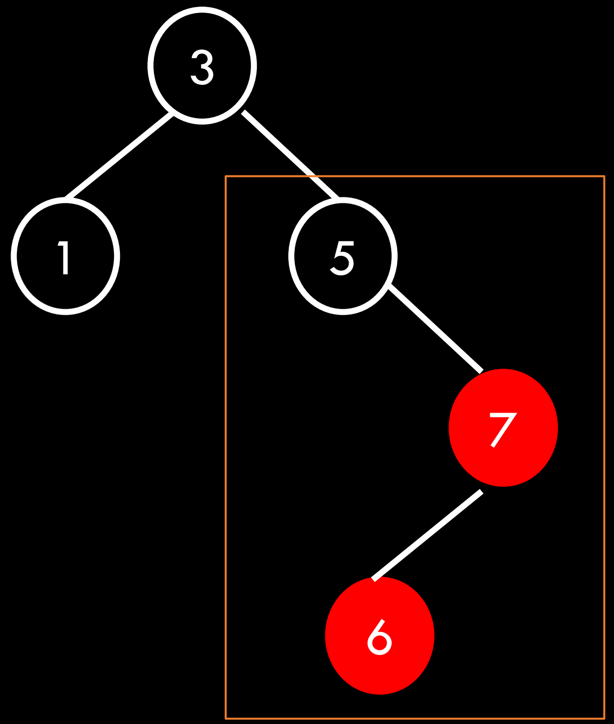

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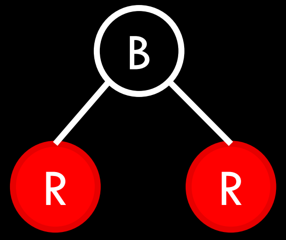

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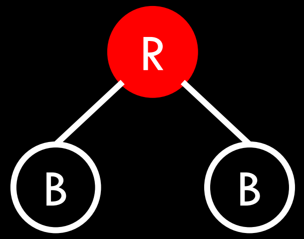

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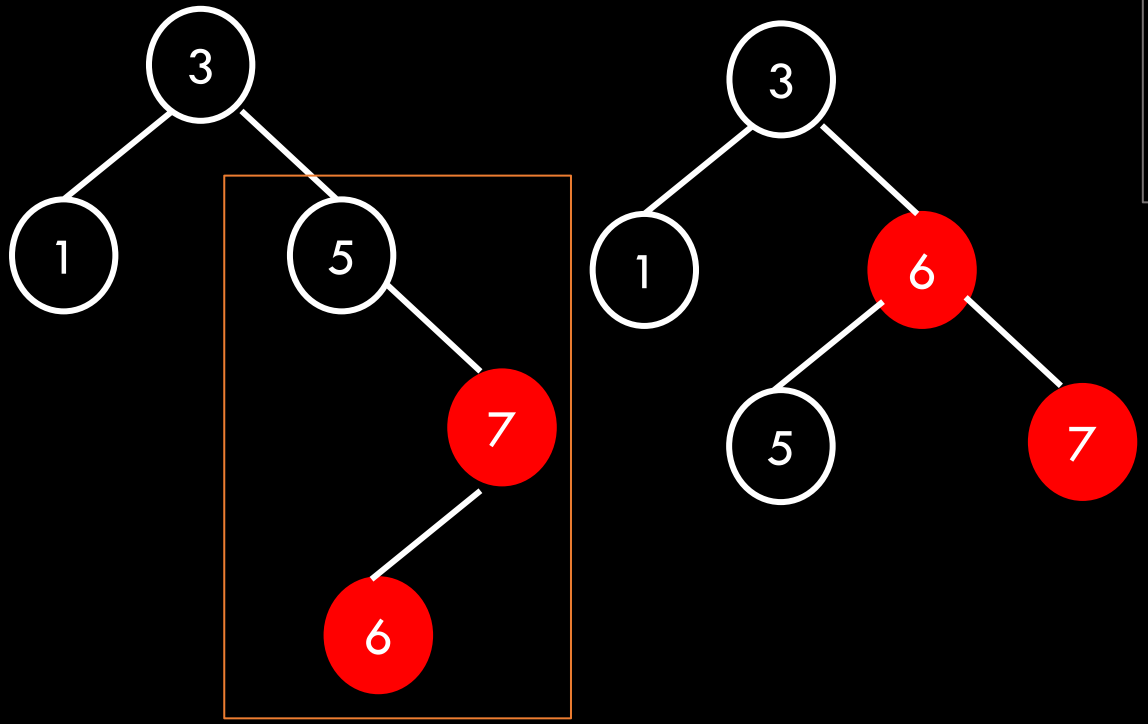

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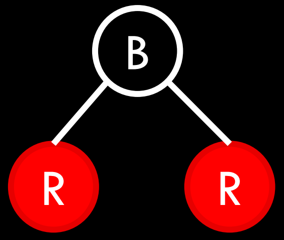

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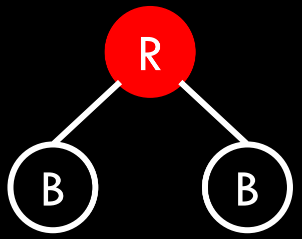

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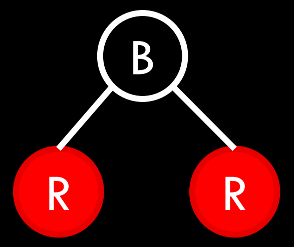

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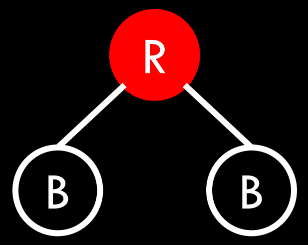

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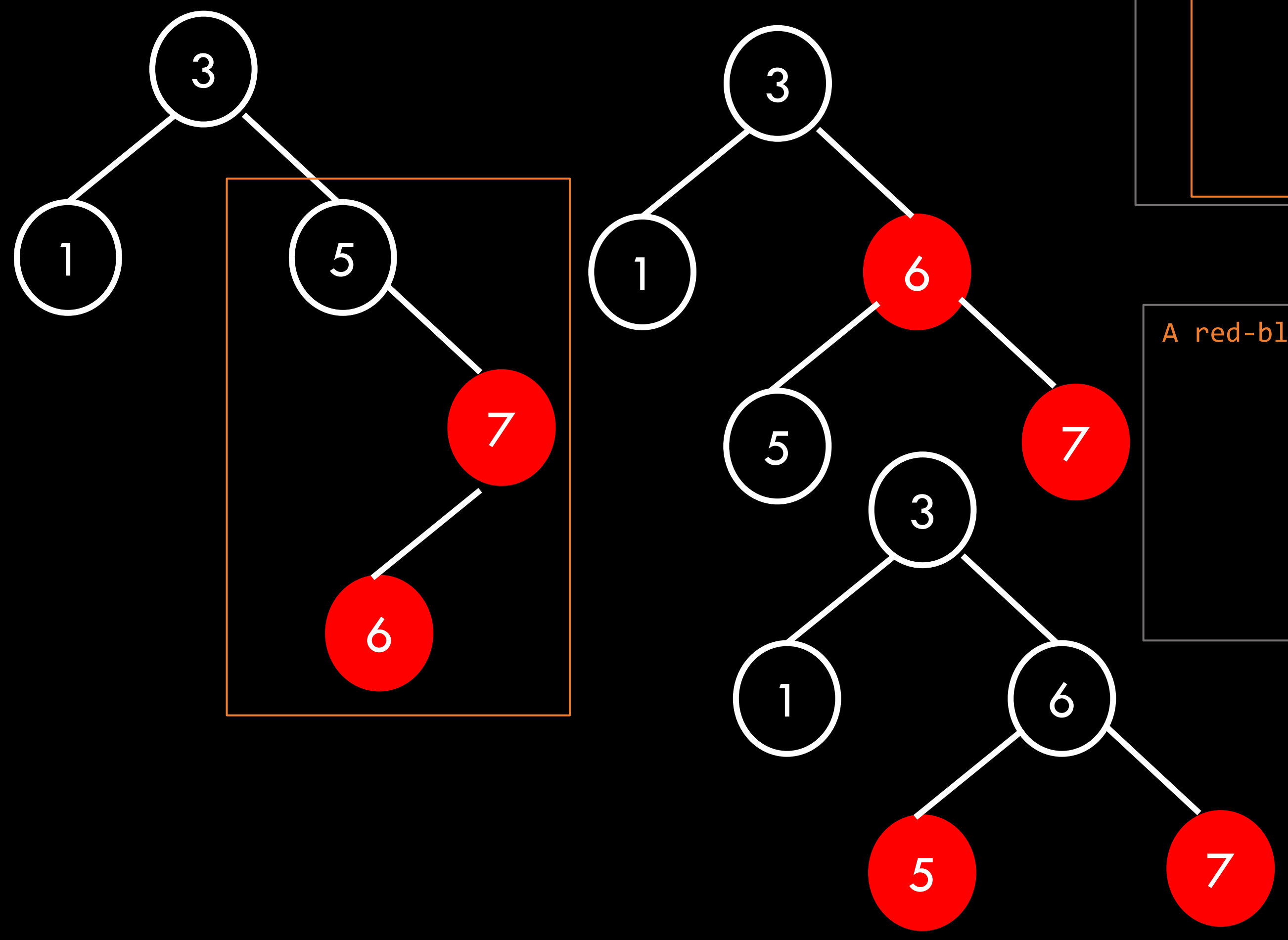

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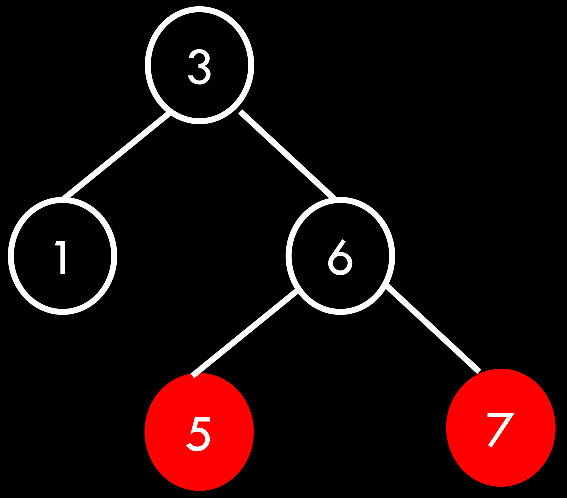
Example

Insert 8



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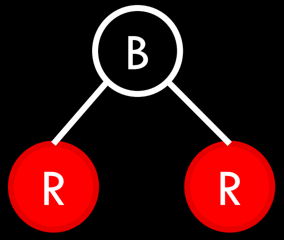

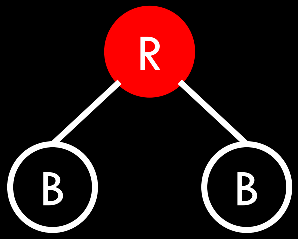

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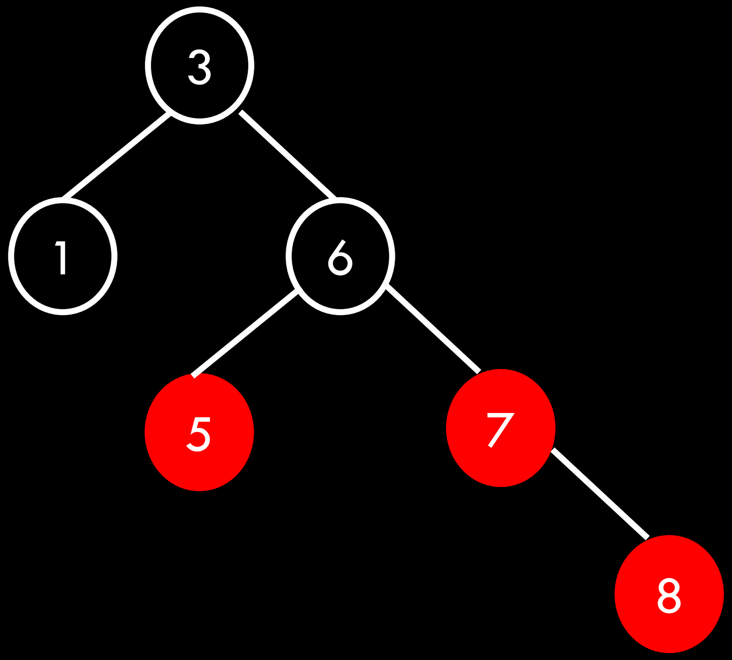

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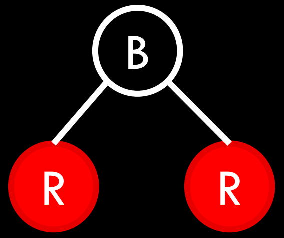

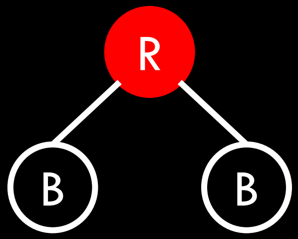

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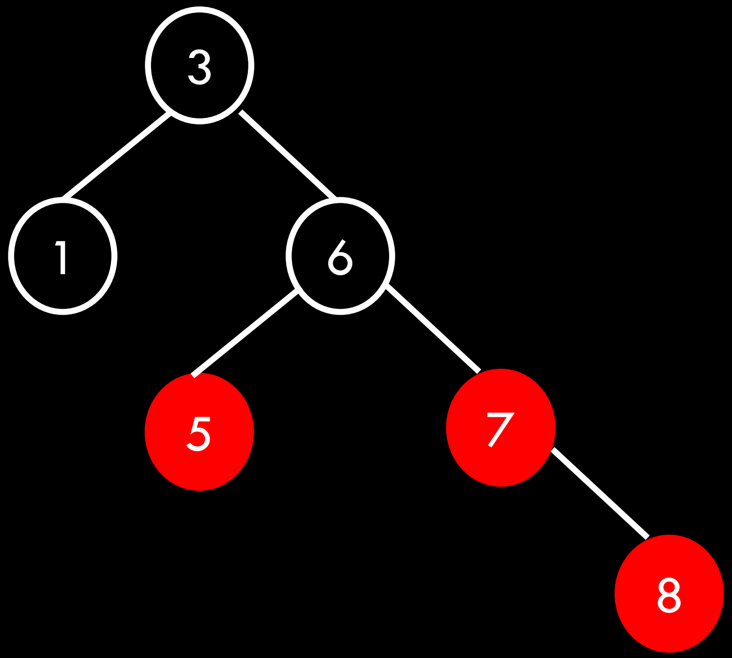

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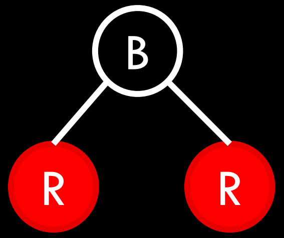

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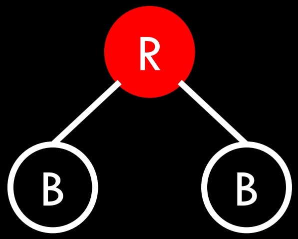

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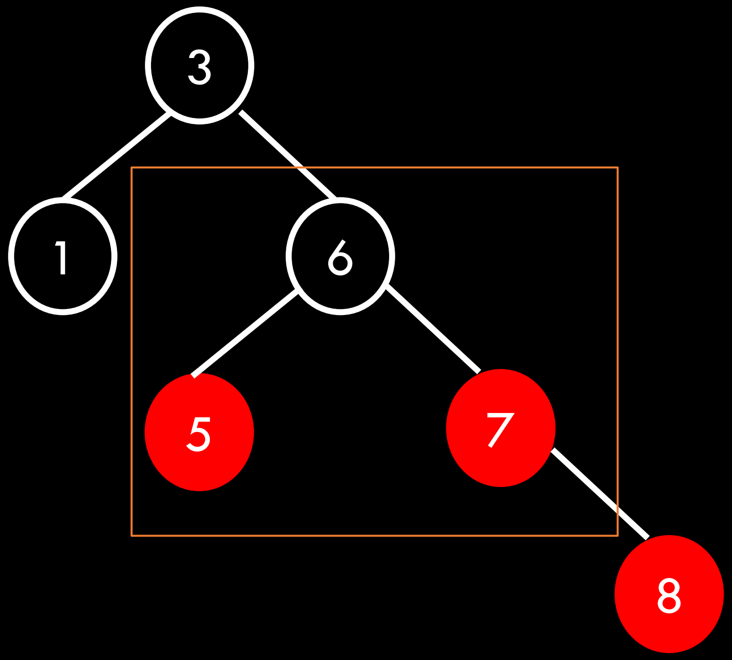

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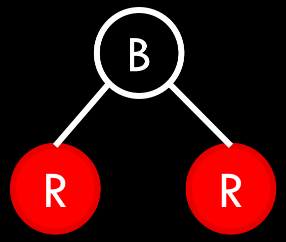

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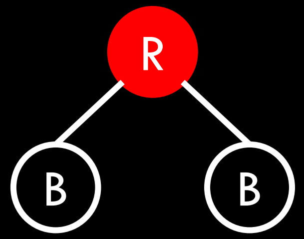

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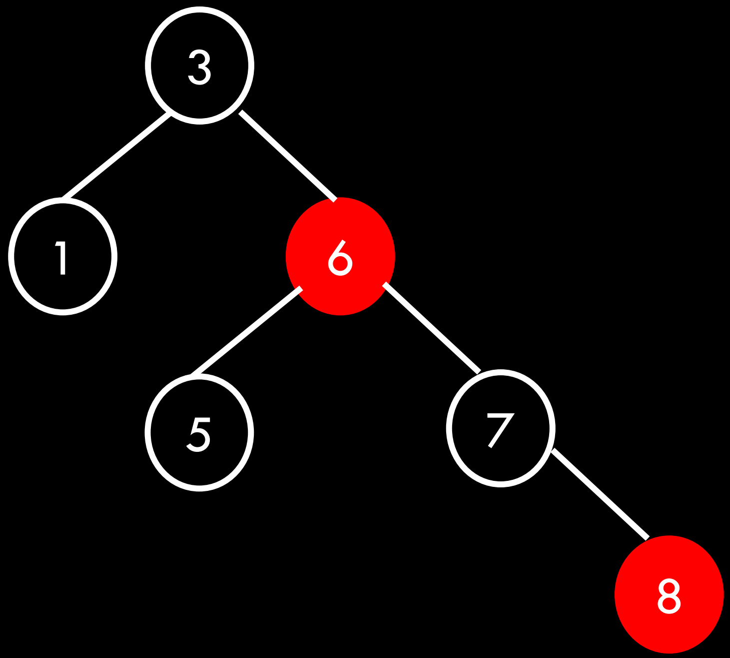

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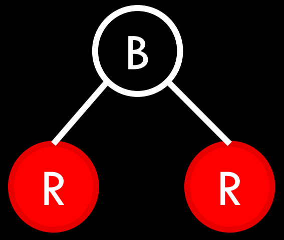

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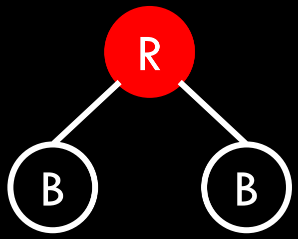

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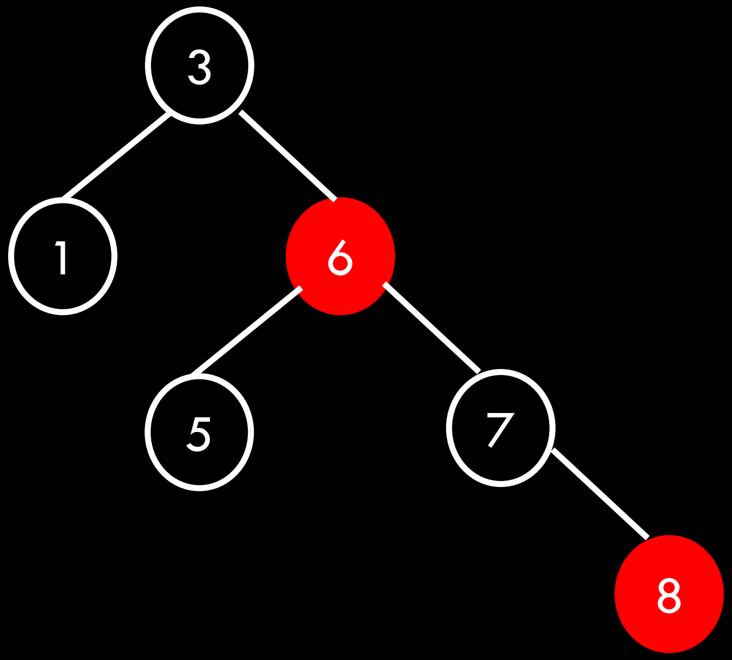

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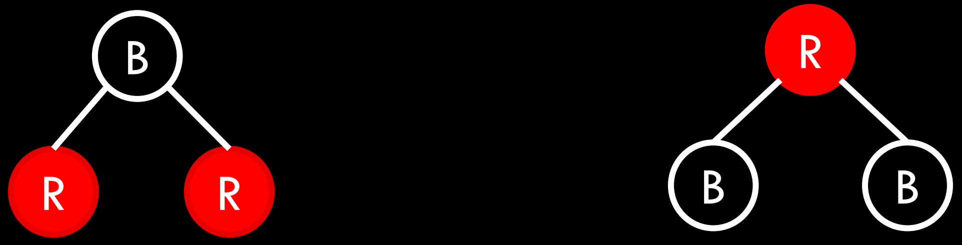

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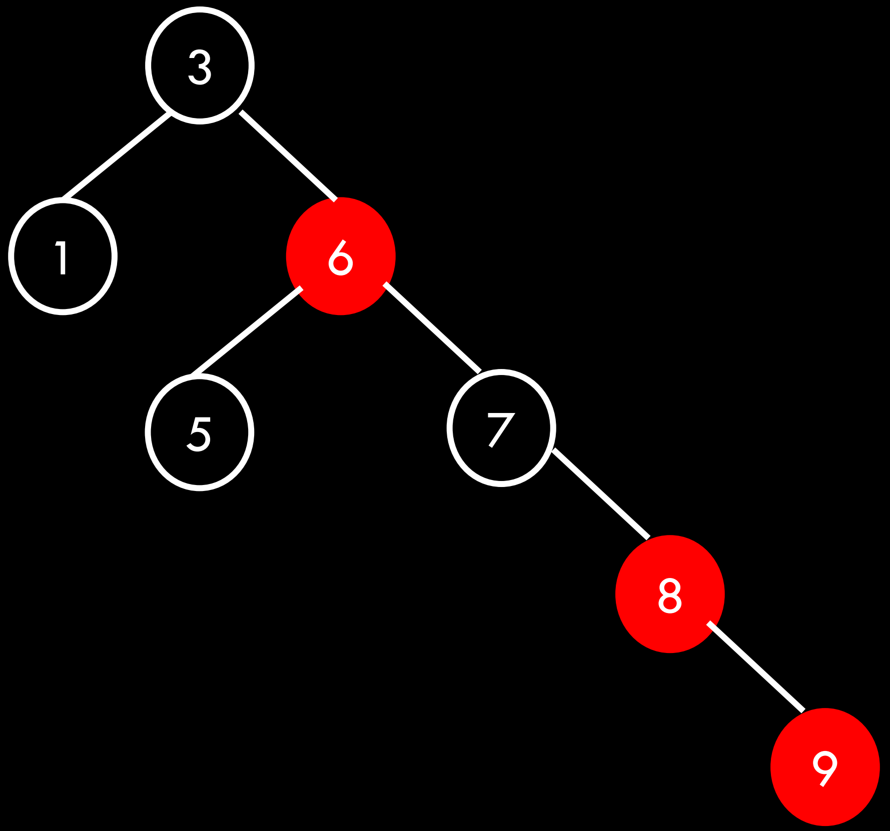

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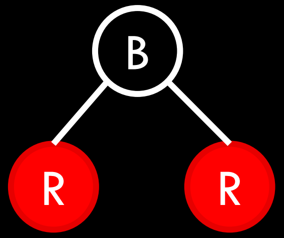

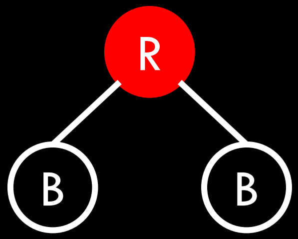

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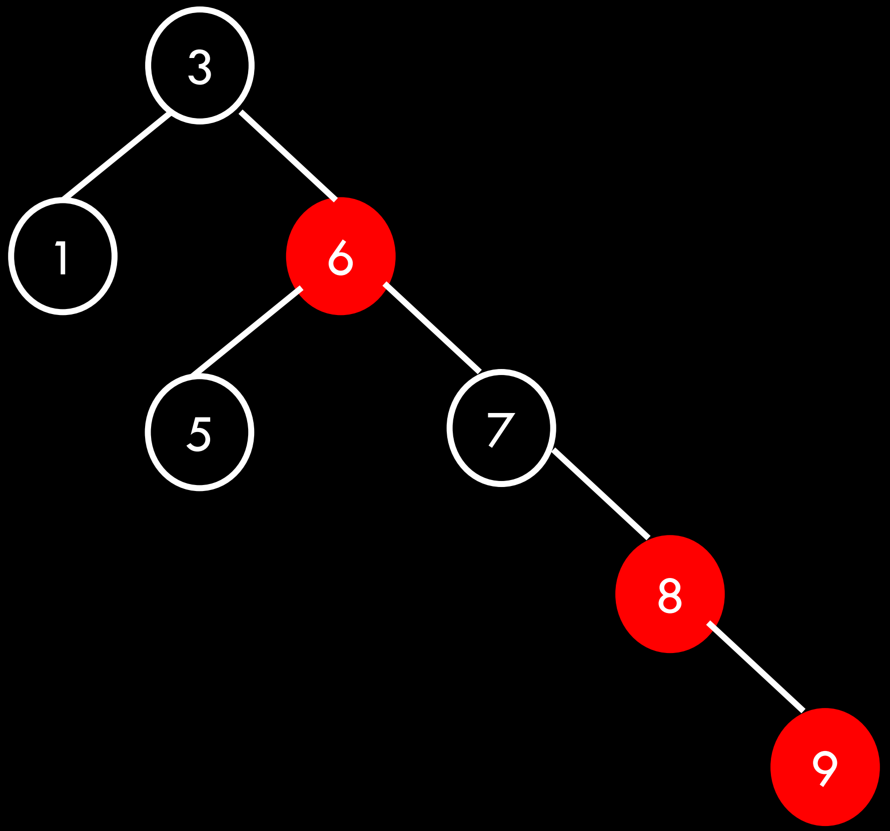

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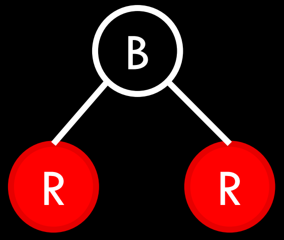

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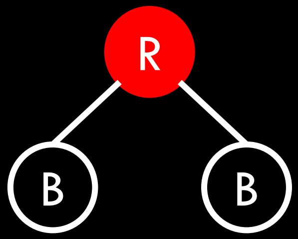

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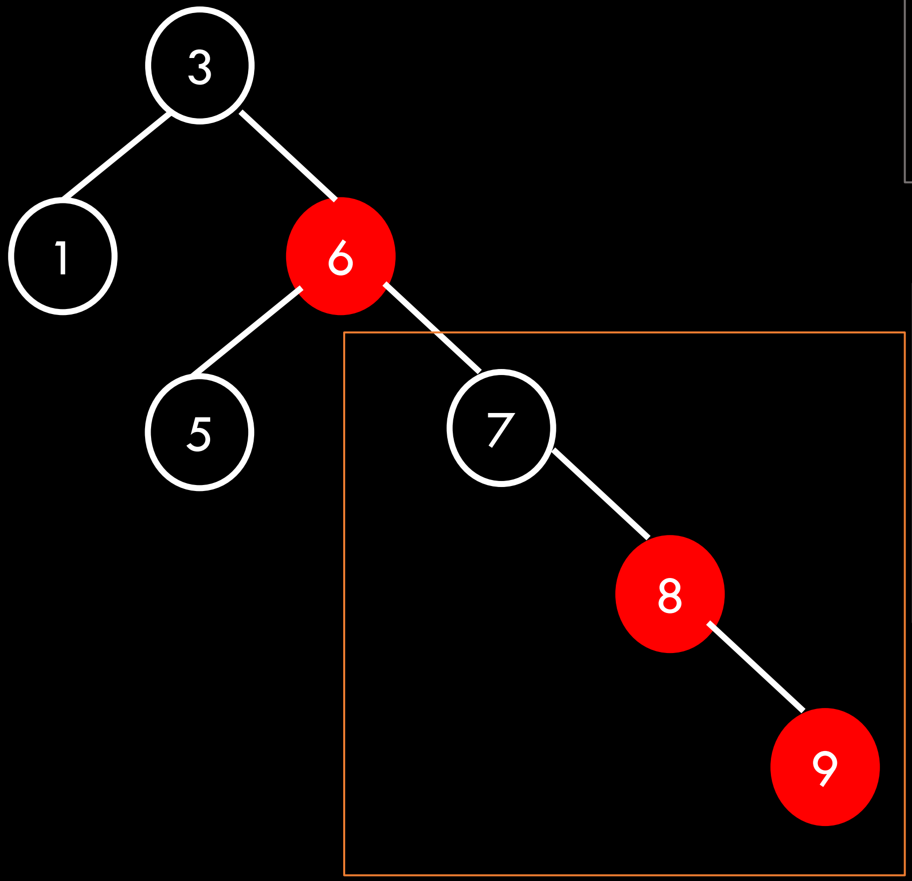

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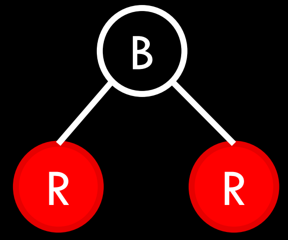

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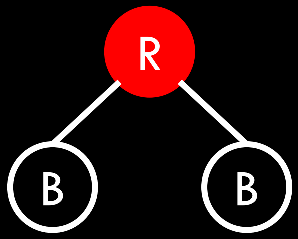

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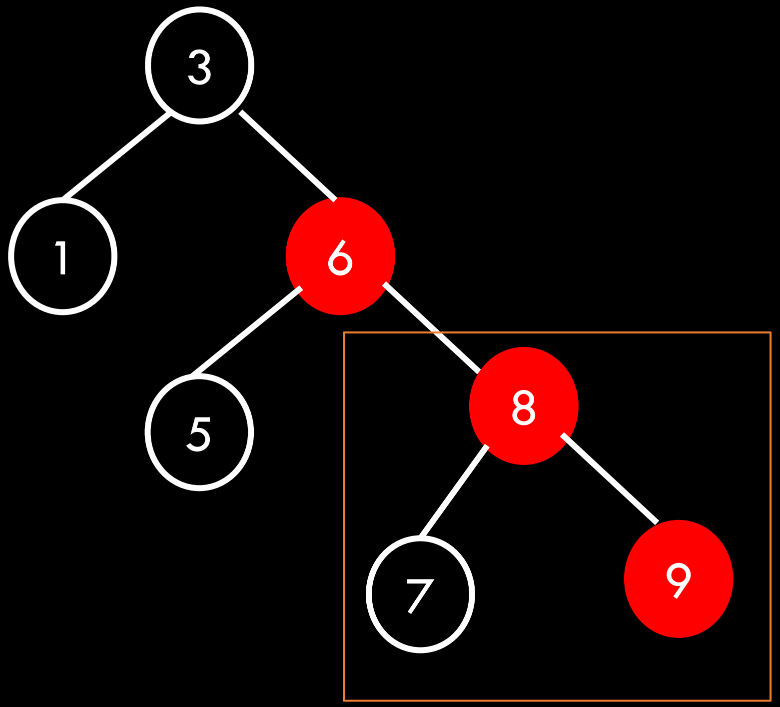

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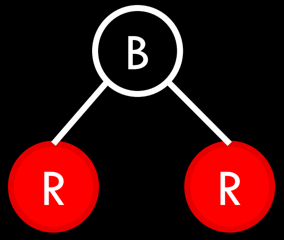

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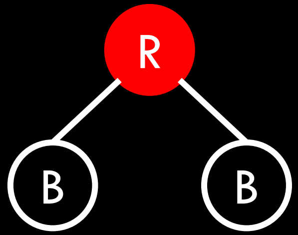

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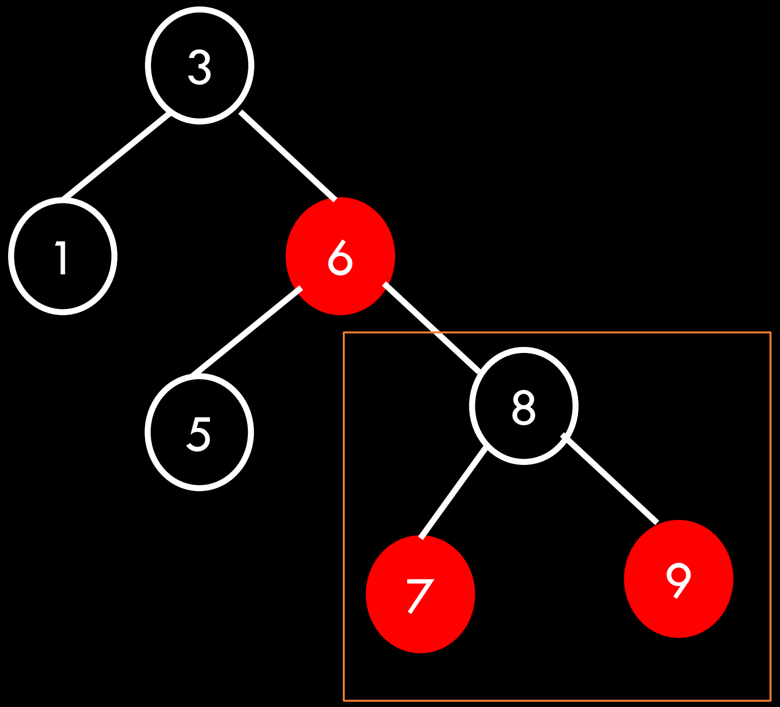

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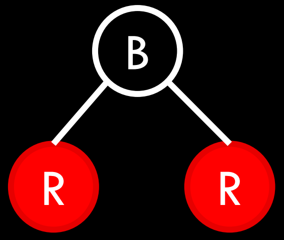

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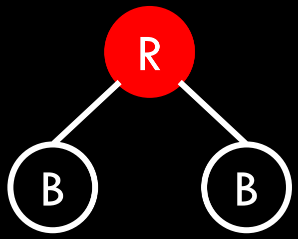

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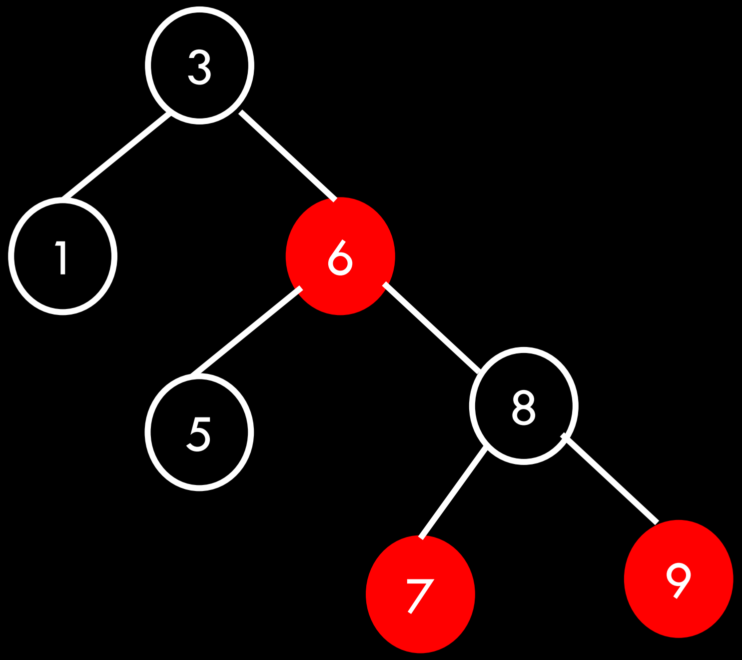

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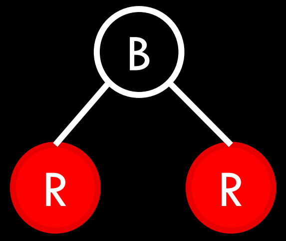

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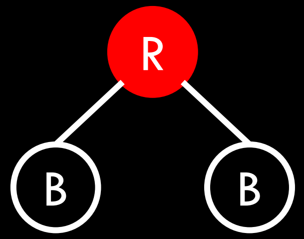

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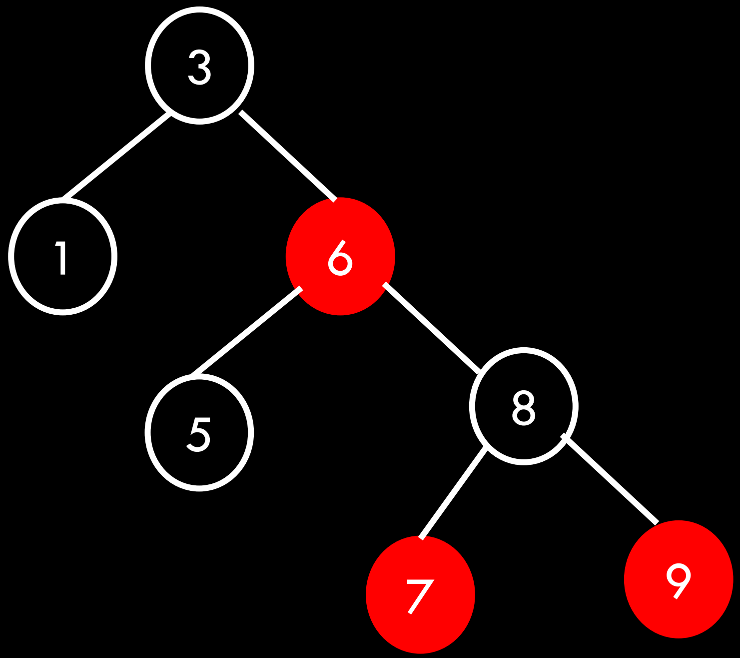

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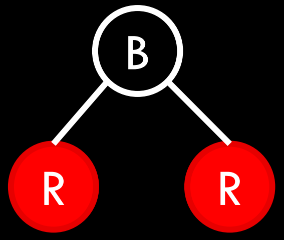

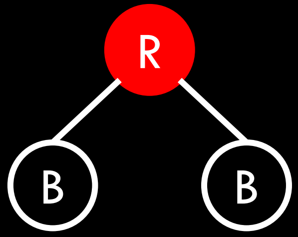

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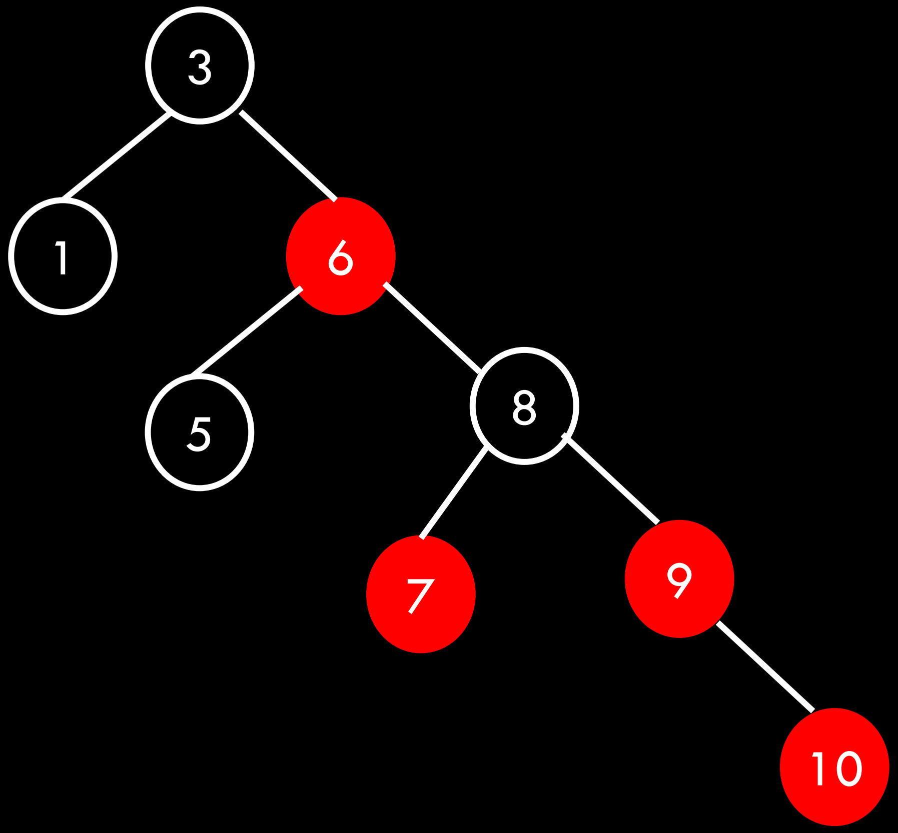

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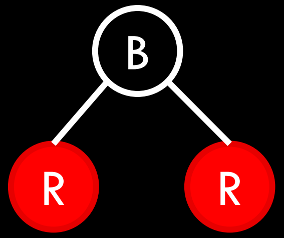

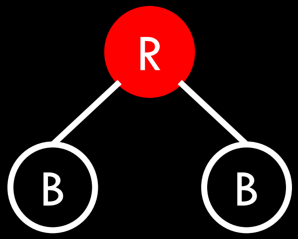

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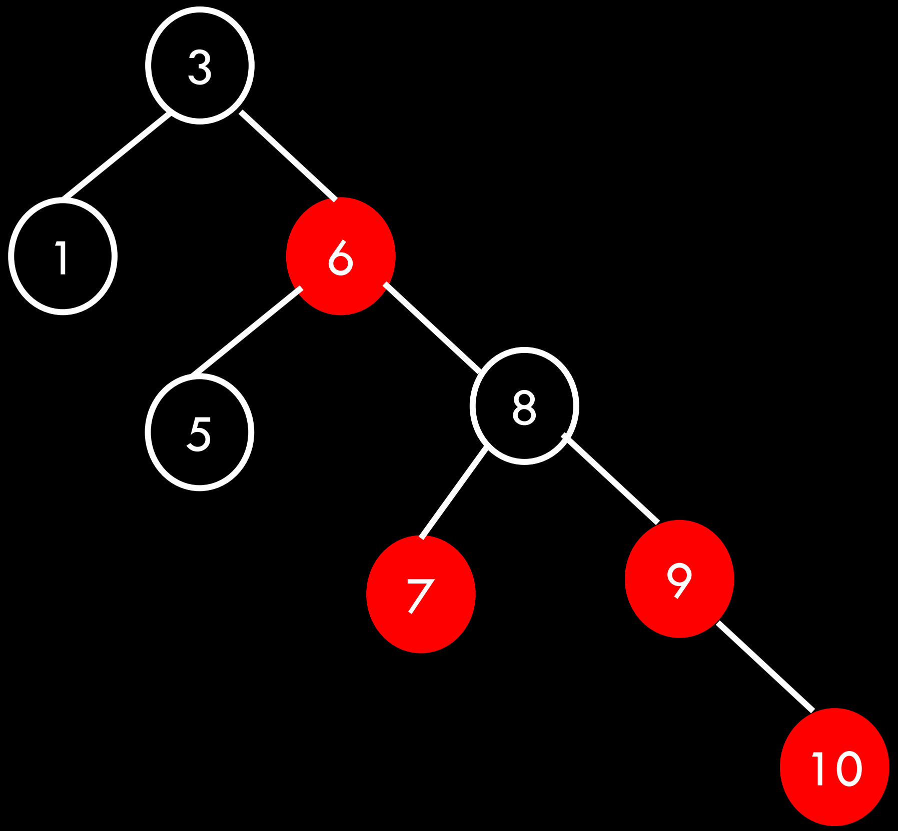

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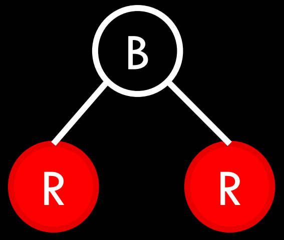

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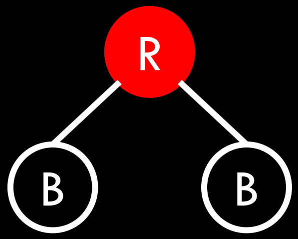

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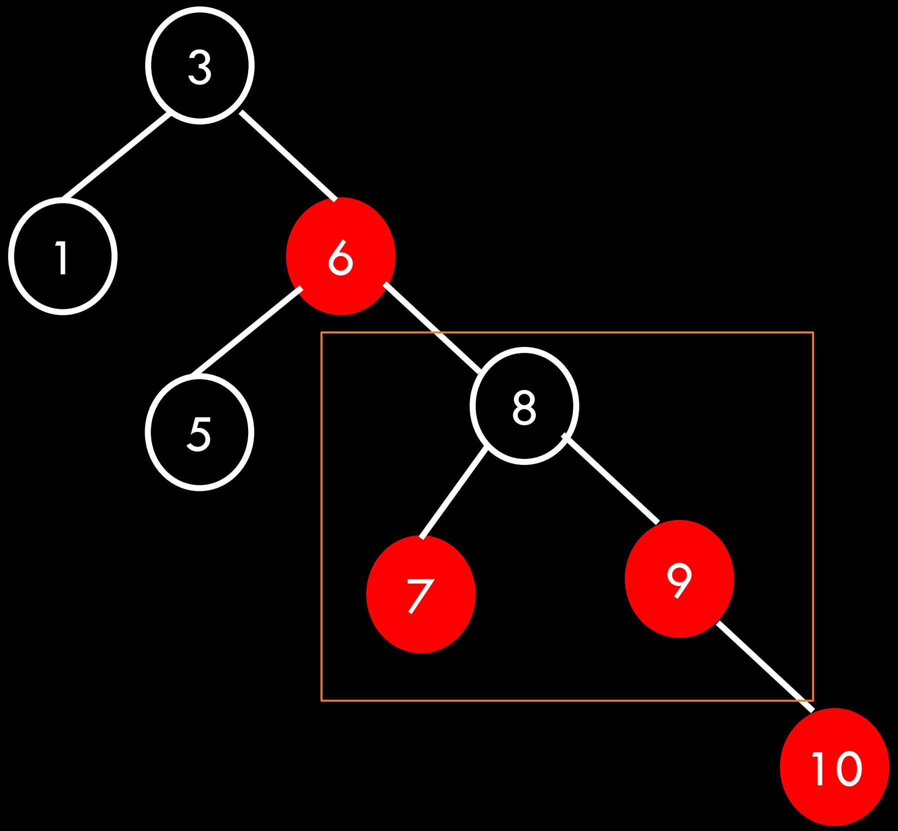

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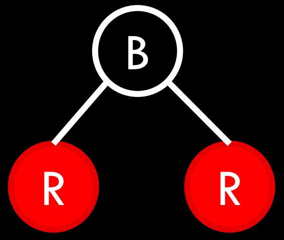

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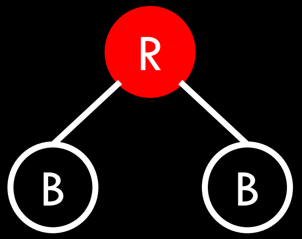

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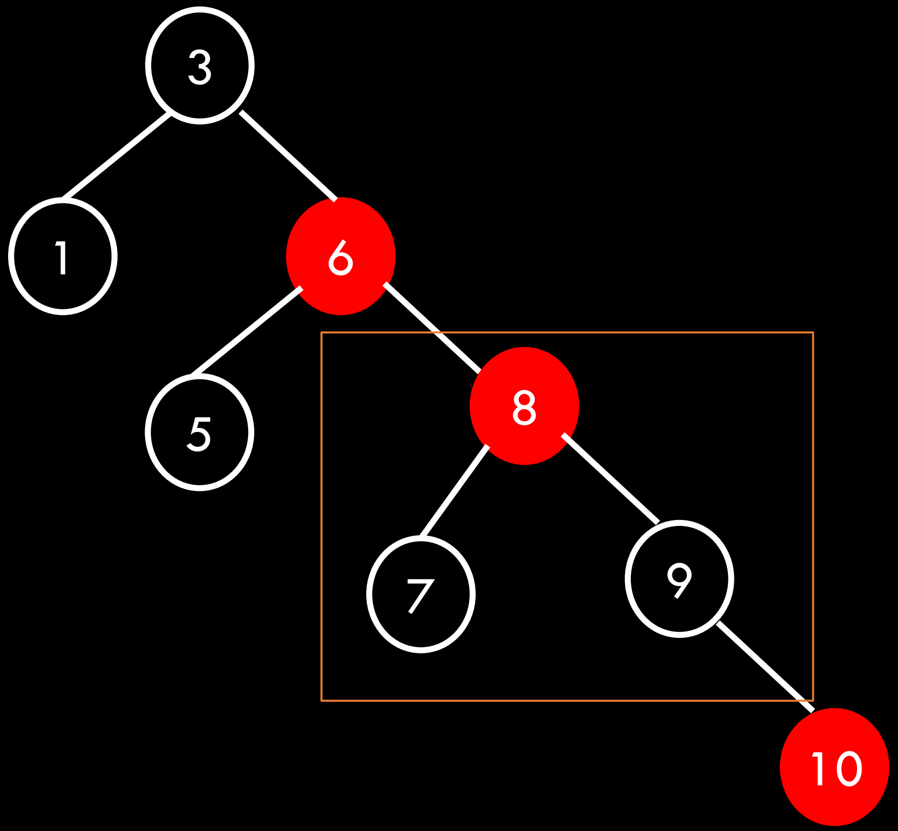

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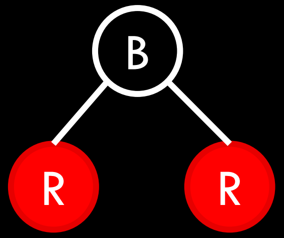

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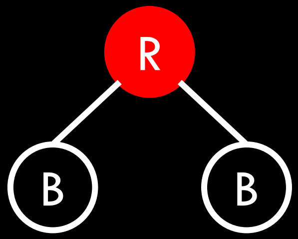

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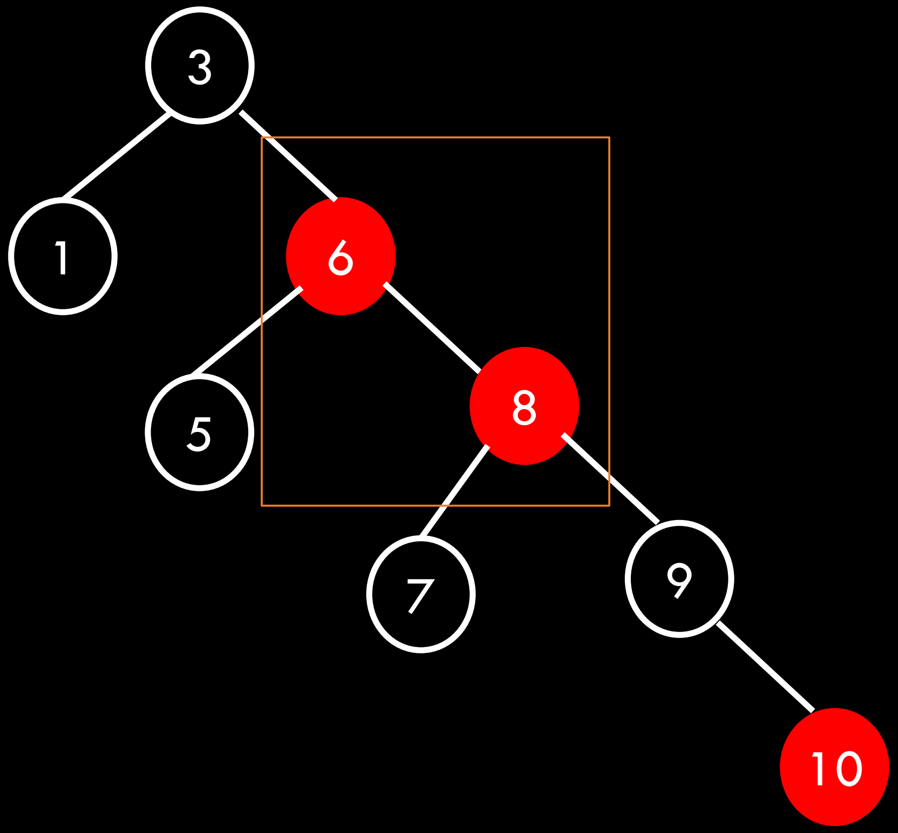

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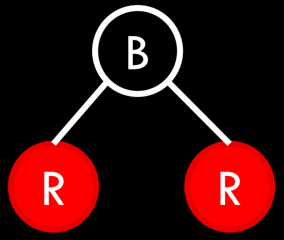

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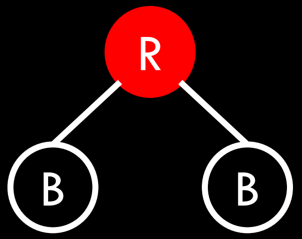

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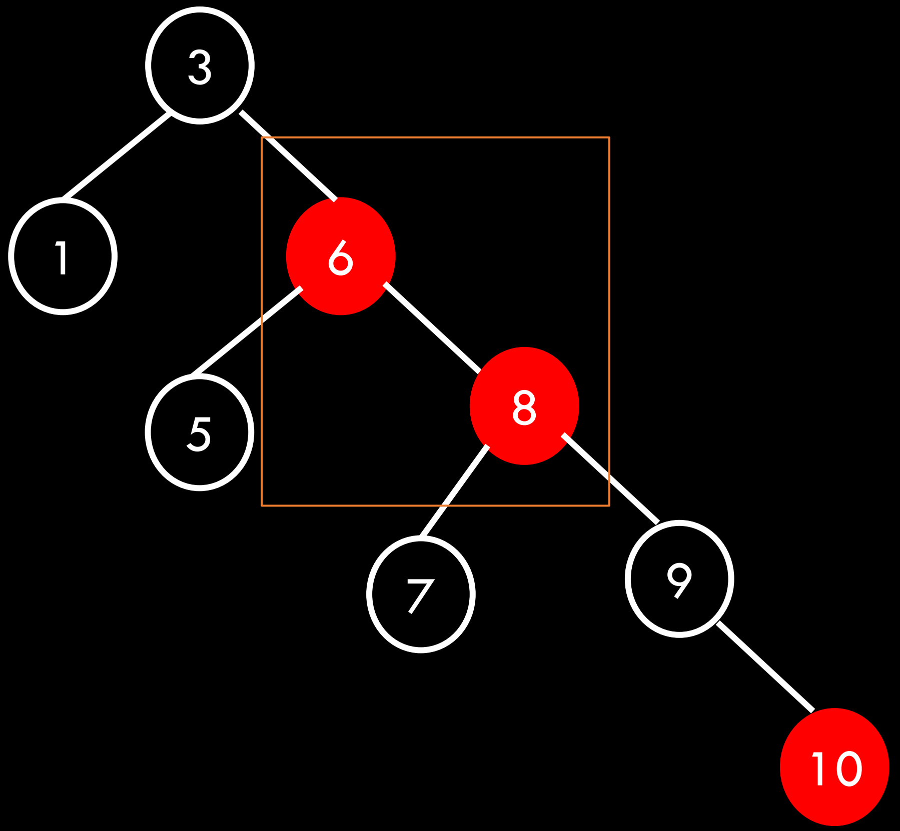

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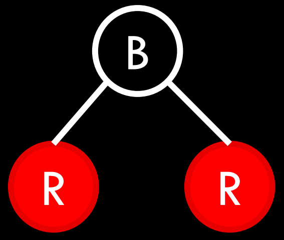

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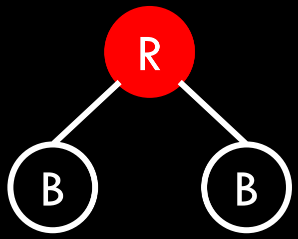

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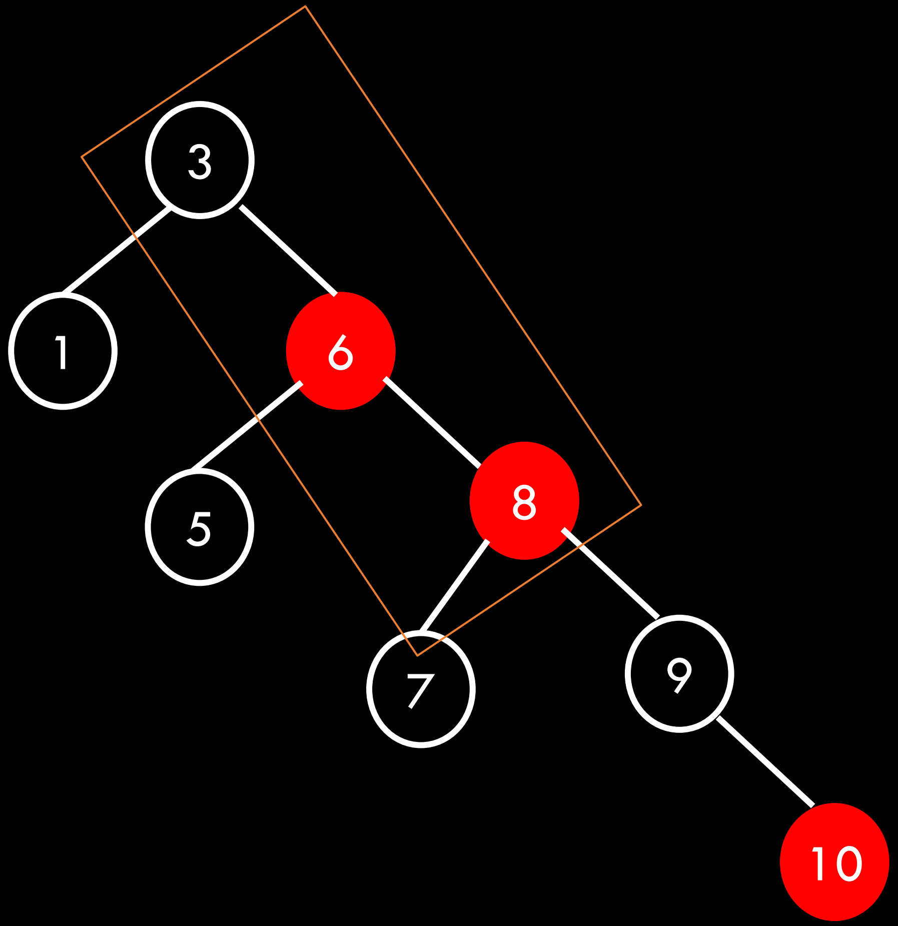
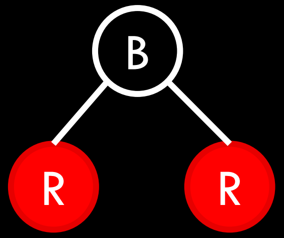
3. A red node always has black children (a null reference is considered to refer to a black node) Or No two consecutive Red nodes
4. The number of black nodes in any path from the root to a leaf is the same
5. Null nodes are attached to the leaves and are black

Example

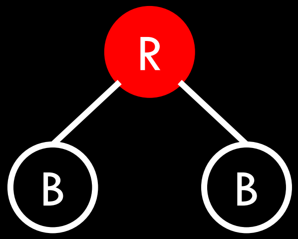
Insert 10



- If the uncle is red, flip colors
- If the uncle is black, rotate
- After Rotation



After Color Flip (P, GP)



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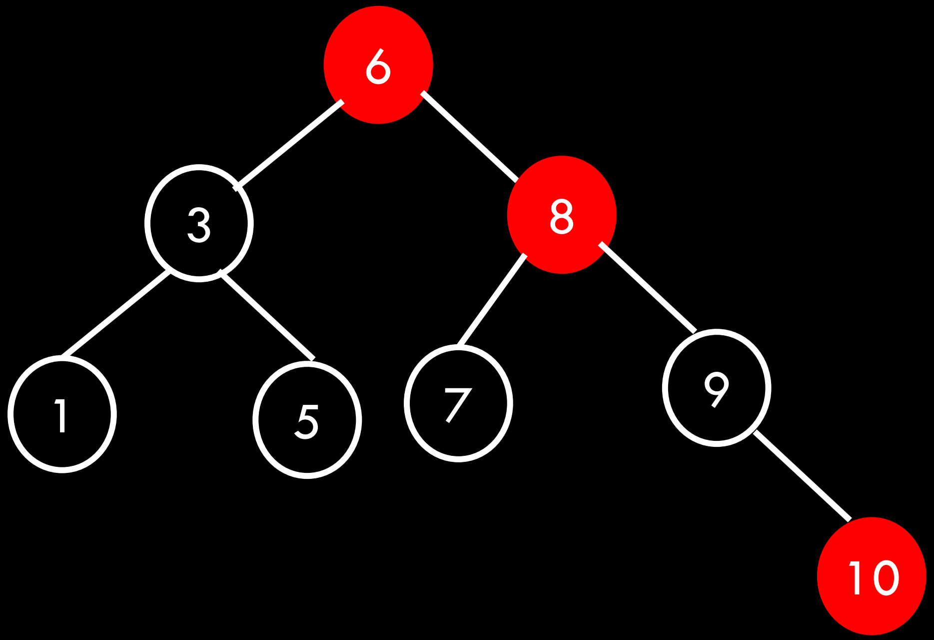

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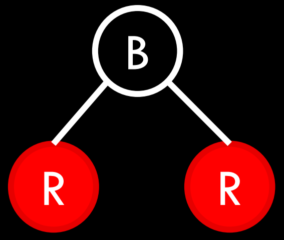

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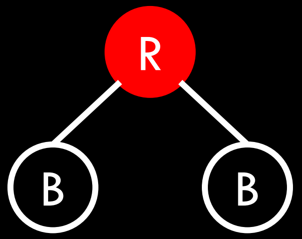

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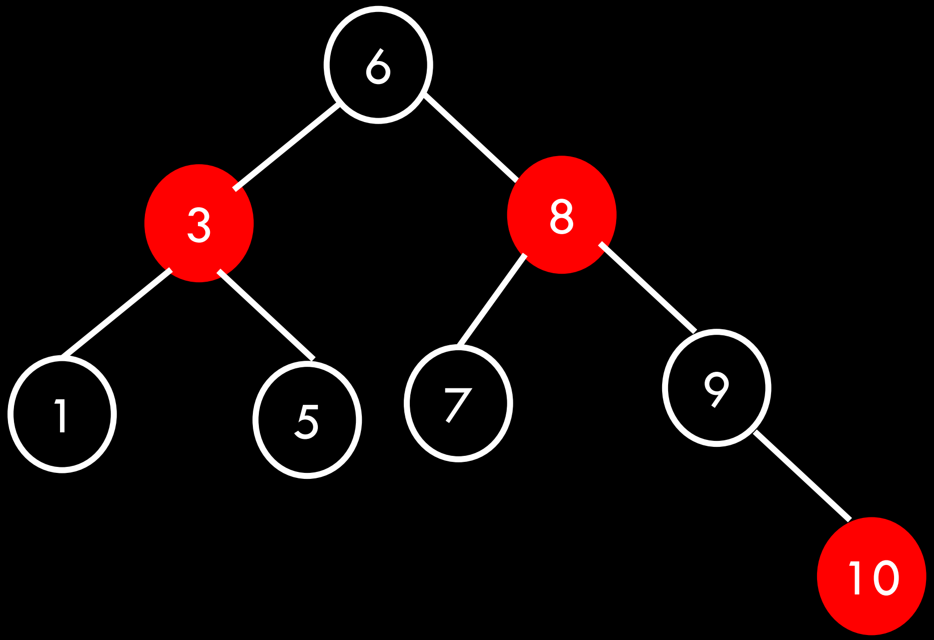

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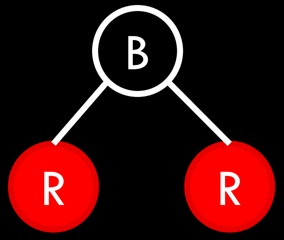
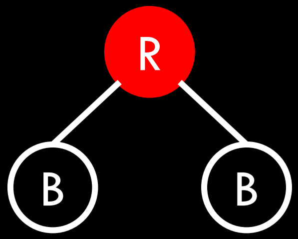


After Color Flip (P, GP)



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Red Black Tree Insertion

```
RBTreeBalance(tree, node)
{   if (node->parent == null)
    {      node->color = black
           return     }
    if (node->parent->color == black)
           return
    parent = node->parent
    grandparent = RBTreeGetGrandparent(node)
    uncle = RBTreeGetUncle(node)
    if (uncle != null && uncle->color == red)
    {      parent->color = uncle->color = black
           grandparent->color = red
           RBTreeBalance(tree, grandparent)
           return     }
    if (node == parent->right && parent == grandparent->left)
    {      RBTreeRotateLeft(tree, parent)
           node = parent
           parent = node->parent      }
     else if (node == parent->left && parent == grandparent->right)
     {      RBTreeRotateRight(tree, parent)
            node = parent
            parent = node->parent    }
     parent->color = black
     grandparent->color = red
     if (node == parent->left)
        RBTreeRotateRight(tree, grandparent)
     else
        RBTreeRotateLeft(tree, grandparent) }
```

Red Black Tree Insertion

```
1. Search (top-down) and insert the new item u as in a Binary Search Tree.
2. Return (bottom-up) and
2.1 If u is root, make it black and the algorithm ends or
2.2 if its parent t is black, the algorithm ends
2.3 If both u and its parent t are red, do one of the following:
2.3.1. [change colors] If t and its sibling v are red:
         Color t and v black and their parent p red.
         Continue the algorithm with p if necessary.
2.3.2. [rotations] If t is red and v black, perform a rotation.
         After the rotation, p and its new parent exchange their colors.
         There are no longer two consequtive red nodes in the tree.
ROTATION:
1 While the recursion returns, keep track of
    node p,
    p's child t and
    p's grandchild u within the path from inserted node to p.
2 If rotation is needed in p, do one of the following rotations:
    if (p.left == t) and (p.left.left == u), single rotation right in p;
    if (p.right == t) and (p.right.right == u), single rotation left in p;
    if (p.left == t) and (p.left.right == u), LR-double rotation in p; or
    if (p.right == t) and (p.right.left == u), RL-double rotation in p.
```

https://opendsa-server.cs.vt.edu/ODSA/Books/Everything/html/RedBlack.html

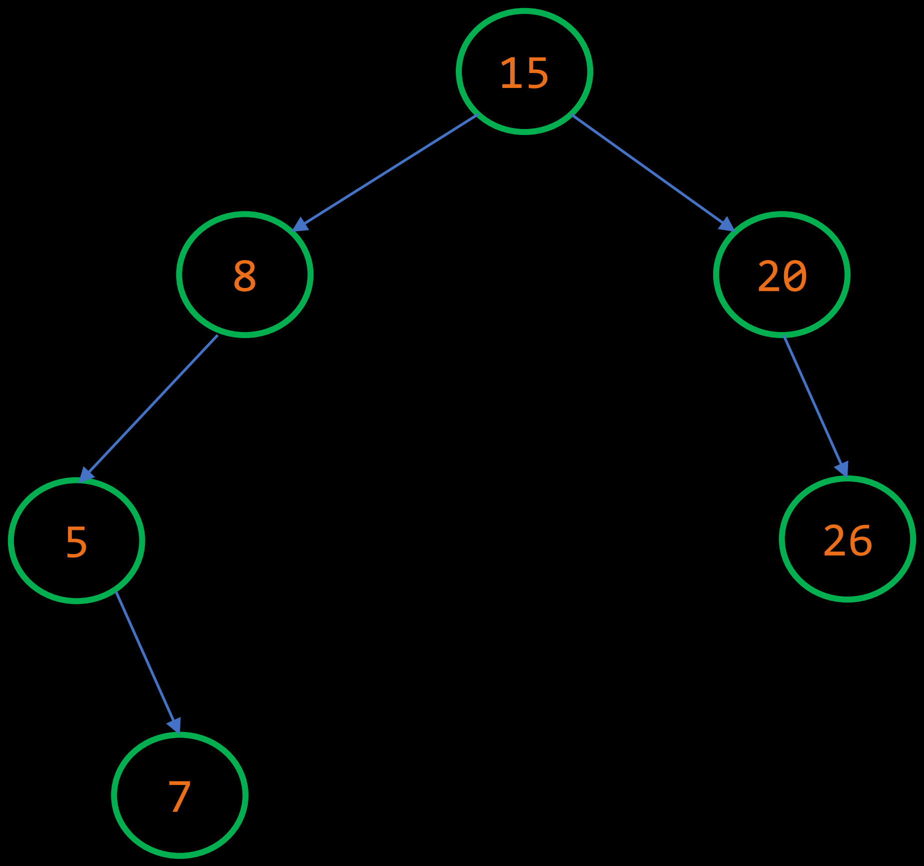
Use Case

- Tree Set, Tree Map, Hash Maps are backed up by a Red Black Tree
- C++ STL

Performance

	Average	Worst case
▪ Space	O(n)	O(n)
▪ Search	O(log n)	O(log n)
▪ Insert	O(log n)	O(log n)
▪ Delete	O(log n)	O(log n)

Balanced Trees



1. Is this an AVL Tree?

2. If it is not AVL Tree, how we should rotate the tree to make it balance?

Questions

Questions