A Portfolio's Common Causal Conditional Risk-neutral PDE

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Abstract. Portfolio's optimal drivers for diversification are common causes of the constituents' correlations. A closed-form formula for the conditional probability of the portfolio given its optimal common drivers is presented, with each pair constituent-common driver joint distribution modelled by Gaussian copulas. A conditional risk-neutral PDE is obtained for this conditional probability as a system of copulas' PDEs, allowing for dynamical risk management of a portfolio as shown in the experiments. Implied conditional portfolio volatilities and implied weights are new risk metrics that can be dynamically monitored from the PDEs or obtained from their solution.

Keywords: causality, Gaussian copula, partial differential equations, portfolio management, risk-neutral measure

1 Common causal conditional risk-neutral PDE

It has been proved that optimal portfolio drivers for diversification must be the common causes of portfolio constituents' correlations [2]. Reichenbach's Common Cause Principle (RCCP) provides a set of independent conditions that variables need to satisfy to be a common cause of a probabilistic correlation [1]. Assume that $(\Omega, \mathcal{F}, P)$ is a standard probability space representing a financial market. A portfolio consists on n financial assets $a_t = [a_{1t}, \ldots, a_{nt}]$ with respective weights $\boldsymbol{w}$. The optimal common causal drivers for the portfolio are selected based on the Commonality Principle [2]. Specifically, the subset of m optimal common causal drivers for a portfolio are selected such that the Reichembach's Common Cause Principle independent conditions have the highest probability [1], from a set of drivers' candidates M belonging to Ω with $M >> m$. The subset of optimal portfolio common drivers $\boldsymbol{D} = \boldsymbol{D_{t-1}} = [D_{t-1,1}, \ldots, D_{t-1,m}]$ satisfying the RCCP conditions make the portfolio constituents, conditional on $\boldsymbol{D}$, independent:

$$P(p_t|\boldsymbol{D_{t-1}}) = P\left(\sum_{i=1}^{n} w_i a_{it} \middle| \boldsymbol{D_{t-1}}\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} w_i D_{t-1,j} P\left(a_{it}|D_{t-1,j}\right) \quad (1)$$

Jeffrey conditionalization, $P\left(a_{it}|D_{t-1,1} \equiv D_{t-1,1}, \ldots, D_{t-1,m} \equiv D_{t-1,m}\right) =$ $= \sum_{j-1}^{m} D_{t-1,i} P(a_{it}|D_{t-1,j})$ is applied in the final step of (1). The joint probabilities of each constituent with respect to each common driver follow bivariate distributions that can be modelled with copulas. In the case of a Gaussian copula, the density is given by:

$$C\left(u_1, u_2\right) = \\ = \frac{1}{\sqrt{1-\rho_{12}^2}} exp\left(-\frac{\rho_{12}^2\left(\Phi^{-1}\left(u_1\right)^2 + \Phi^{-1}\left(u_2\right)^2\right) - 2\rho_{12}\Phi^{-1}\left(u_1\right)\Phi^{-1}\left(u_2\right)}{2\left(1-\rho_{12}^2\right)}\right) \tag{2}$$

The conditional probability is given by:

$$P(X \leq x \mid Y = y) = \frac{\partial}{\partial v} C(u,v)\bigg|_{(F_X(x), F_Y(y))} \tag{3}$$

Applying it to (1):

$$P(p_t|\boldsymbol{D_{t-1}}) = \sum_{i=1}^{n}\sum_{j=1}^{m} w_i D_{t-1,j} \frac{\partial}{\partial D_{t-1,j}} C(F_{a_{it}}(a_{it}), F_{D_{t-1,j}}(D_{t-1,j})) \tag{4}$$

Using (4), and by computing the partial derivatives of C as in (2):

$$P(p|\boldsymbol{D}) = \\ = \sum_{i=1}^{n}\sum_{j=1}^{m} \frac{-w_i D_j \exp\left(\frac{-\ \rho_{a_i D_j}{}^2\left(x_1{}^2 + x_2{}^2\right) - 2\rho_{a_i D_j} x_1 x_2}{2\left(1-\rho_{a_i D_j}{}^2\right)}\right) 2\rho_{a_i D_j}{}^2 \frac{x_2}{\Phi'(x_2)} - 2\rho_{a_i D_j}\frac{x_1}{\Phi'(x_2)}}{2\left(1-\rho_{a_i D_j}{}^2\right)^{3/2}} \tag{5}$$

with $p = p_t$, $\boldsymbol{D} = \boldsymbol{D_{t-1}}$, $D_j = D_{t-1,j}$, $x_1 = \Phi^{-1}(a_i)$ and $x_2 = \Phi^{-1}(D_j)$. In matrix form (5) is given by:

$$P(p|\boldsymbol{D}) = -\boldsymbol{w^T \Pi D} \tag{6}$$

with $\boldsymbol{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$, $\boldsymbol{\Pi}_{(nxm)} = \begin{bmatrix} \frac{\partial C(a_1,D_1)}{\partial D_1} & \ldots & \frac{\partial C(a_1,D_m)}{\partial D_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial C(a_n,D_1)}{\partial D_1} & \ldots & \frac{\partial C(a_n,D_m)}{\partial D_m} \end{bmatrix}$ and $\boldsymbol{D}_{(mxn)} = \begin{bmatrix} D_1 & \ldots & D_1 \\ \vdots & \ddots & \vdots \\ D_m & \ldots & D_m \end{bmatrix}$.
Deriving (6) with respect to t, the following PDE is obtained:

$$\frac{\partial\ P(p|\boldsymbol{D})}{\partial t} dt = -\boldsymbol{w^T}\left[\left(\frac{\partial \boldsymbol{\Pi}}{\partial \boldsymbol{a}}\frac{\partial \boldsymbol{a}}{\partial t} + \frac{\partial \boldsymbol{\Pi}}{\partial \boldsymbol{D}}\frac{\partial \boldsymbol{D}}{\partial t} + \frac{\partial \boldsymbol{\Pi}}{\partial \boldsymbol{\rho}}\frac{\partial \boldsymbol{\rho}}{\partial t}\right)\boldsymbol{D} + \boldsymbol{\Pi}\frac{\partial \boldsymbol{D}}{\partial t}\right] dt \tag{7}$$

which can be expressed in terms of $P(p|\boldsymbol{D})$ by using (6) as:

$$\left[\frac{\partial\ P(p|\boldsymbol{D})}{\partial t} + \frac{\partial\ P(p|\boldsymbol{D})}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial\ P(p|\boldsymbol{D})}{\partial \boldsymbol{D}}\frac{\partial \boldsymbol{D}}{\partial t} + \left(\frac{\partial\ P(p|\boldsymbol{D})}{\partial \boldsymbol{\rho}} - \boldsymbol{w^T \Pi}\frac{\partial \boldsymbol{D}}{\partial \boldsymbol{\rho}}\right)\frac{\partial \boldsymbol{\rho}}{\partial t}\right] dt = 0 \tag{8}$$

and the partial derivatives can also be obtained from (6) as $\frac{\partial\ P(p|\boldsymbol{D})}{\partial p} = \boldsymbol{w^T}\left(\frac{\partial \boldsymbol{\Pi}}{\partial \boldsymbol{a}}\boldsymbol{D}\right)$, $\frac{\partial\ P(p|\boldsymbol{D})}{\partial \boldsymbol{D}} = \boldsymbol{w^T}\left(\frac{\partial \boldsymbol{\Pi}}{\partial \boldsymbol{D}}\boldsymbol{D} + \boldsymbol{\Pi}\right)$ and $\frac{\partial\ P(p|\boldsymbol{D})}{\partial \boldsymbol{\rho}} = \boldsymbol{w^T}\left(\frac{\partial \boldsymbol{\Pi}}{\partial \boldsymbol{\rho}}\boldsymbol{D} + \boldsymbol{\Pi}\frac{\partial \boldsymbol{D}}{\partial \boldsymbol{\rho}}\right)$.

On the other hand, by applying Ito's lemma to $P(p|\boldsymbol{D})$:

$$dP(p|\boldsymbol{D}) = \left[\frac{\partial P(p|\boldsymbol{D})}{\partial t} + \frac{1}{2}\left(\frac{\partial^2 P(p|\boldsymbol{D})}{\partial p^2}\boldsymbol{\sigma_p}^2 + \frac{\partial^2 P(p|\boldsymbol{D})}{\partial \boldsymbol{D}^2}\boldsymbol{\sigma_D}^2\right) + \right. \\ \left. + \frac{\partial^2 P(p|\boldsymbol{D})}{\partial p \partial \boldsymbol{D}}\boldsymbol{\sigma_p}\boldsymbol{\sigma_D}\right] dt + \frac{\partial P(p|\boldsymbol{D})}{\partial p} dp_t + \frac{\partial P(p|\boldsymbol{D})}{\partial \boldsymbol{D}} d\mathbf{D}_{\boldsymbol{t}} \tag{9}$$

By substituting the PDE given in (8) into the Ito's lemma derivation of $P(p|\boldsymbol{D})$ in (9):

$$dP(p|\boldsymbol{D}) = \left[\frac{1}{2}\left(\frac{\partial^2 P(p|\boldsymbol{D})}{\partial p^2}\boldsymbol{\sigma_p}^2 + \frac{\partial^2 P(p|\boldsymbol{D})}{\partial \boldsymbol{D}^2}\boldsymbol{\sigma_D}^2\right) + \frac{\partial^2 P(p|\boldsymbol{D})}{\partial p \partial \boldsymbol{D}}\boldsymbol{\sigma_p}\boldsymbol{\sigma_D}\right] dt - \\ - \left(\frac{P(p|\boldsymbol{D})}{\boldsymbol{D}}\frac{\partial\ \boldsymbol{D}}{\partial \boldsymbol{\rho}} + \frac{\partial\ P(p|\boldsymbol{D})}{\partial \boldsymbol{\rho}}\right) d\boldsymbol{\rho_t} \tag{10}$$

with $\boldsymbol{\rho_t} = \frac{\boldsymbol{\sigma_{p,D}}}{\boldsymbol{\sigma_p}\boldsymbol{\sigma_D}}$, the vector of covariances between constituents and common drivers as $\boldsymbol{\sigma_{p,D}}$, and the variance-covariance matrices for the common drivers and portfolio respectively, $\boldsymbol{\sigma_D}$ and $\boldsymbol{\sigma_p}$. Assuming $\boldsymbol{\rho_t}$ can be modelled as an Geometric Brownian Motion $d\boldsymbol{\rho_t} = \boldsymbol{\mu_\rho}\boldsymbol{\rho_t}dt + \boldsymbol{\sigma_\rho}\boldsymbol{\rho_t}d\boldsymbol{W_\rho^t}$ and substituting in (10):

$$dP(p|\boldsymbol{D}) = \left[\frac{1}{2}\left(\frac{\partial^2 P(p|\boldsymbol{D})}{\partial p^2}\boldsymbol{\sigma_p}^2 + \frac{\partial^2 P(p|\boldsymbol{D})}{\partial \mathbf{D}^2}\boldsymbol{\sigma_D}^2\right) + \frac{\partial^2 P(p|\boldsymbol{D})}{\partial p \partial \boldsymbol{D}}\boldsymbol{\sigma_p}\boldsymbol{\sigma_D} - \right. \\ \left. - \left(\frac{P(p|\boldsymbol{D})}{\boldsymbol{D}}\frac{\partial\ \boldsymbol{D}}{\partial \boldsymbol{\rho}} + \frac{\partial\ P(p|\boldsymbol{D})}{\partial \boldsymbol{\rho}}\right)\boldsymbol{\mu_\rho}\boldsymbol{\rho_t}\right] dt - \left(\frac{P(p|\boldsymbol{D})}{\boldsymbol{D}}\frac{\partial\ \boldsymbol{D}}{\partial \boldsymbol{\rho}} + \frac{\partial\ P(p|\boldsymbol{D})}{\partial \boldsymbol{\rho}}\right)\boldsymbol{\sigma_\rho}\boldsymbol{\rho_t}d\boldsymbol{W_\rho^t} \tag{11}$$

The common causal conditional risk neutral PDE is given by equating the drift part in (11) to the drift of the common causal drivers' processes times the conditional probability of the portfolio given these set of common causal drivers:

$$\left[\frac{1}{2}\left(\frac{\partial^2 P(p|\boldsymbol{D})}{\partial p^2}\sigma_p^2 + \frac{\partial^2 P(p|\boldsymbol{D})}{\partial \mathbf{D}^2}\boldsymbol{\sigma_D}^2\right) + \frac{\partial^2 P(p|\boldsymbol{D})}{\partial p \partial \boldsymbol{D}}\sigma_p\boldsymbol{\sigma_D}\right] dt = \left[\boldsymbol{\mu_D}P(p|\boldsymbol{D}) + \boldsymbol{\Delta}\right] dt \tag{12}$$

and there is one condition, given by the following PDE, such that the PDE in (11) does not have a Brownian Motion component:

$$\frac{P(p|\boldsymbol{D})}{\boldsymbol{D}}\frac{\partial \boldsymbol{D}}{\partial \boldsymbol{\rho}} = -\frac{\partial P(p|\boldsymbol{D})}{\partial \boldsymbol{\rho}} \tag{13}$$

Parameter $\boldsymbol{\Delta}$ in (12) is the deviation from risk neutrality of the assumed conditional risk neutrality. We assume now for simplicity is zero. It can work as a loss function for a learning/calibration, which is left as future work.

If the partial derivatives computed from (6) are substituted in (13):

$$\frac{-\boldsymbol{w^T}\boldsymbol{\Pi}\boldsymbol{D}}{\boldsymbol{D}}\frac{\partial \boldsymbol{D}}{\partial \boldsymbol{\rho}} = -\boldsymbol{w^T}\left(\frac{\partial \boldsymbol{\Pi}}{\partial \boldsymbol{\rho}}\boldsymbol{D} + \boldsymbol{\Pi}\frac{\partial \boldsymbol{D}}{\partial \boldsymbol{\rho}}\right) \tag{14}$$

which simplifies to:

$$\boldsymbol{\Pi}\frac{\partial \boldsymbol{D}}{\partial \boldsymbol{\rho}} = \left(\frac{\partial \boldsymbol{\Pi}}{\partial \boldsymbol{\rho}}\boldsymbol{D} + \boldsymbol{\Pi}\frac{\partial \boldsymbol{D}}{\partial \boldsymbol{\rho}}\right) \tag{15}$$

This gives another condition for the PDE, $\frac{\partial \boldsymbol{\Pi}}{\partial \boldsymbol{\rho}}\boldsymbol{D} = 0$, which can be derived analytically from Π in (6) and consists of a system of n equations.

Now, the partial derivatives of $P(p|\boldsymbol{D})$ in (12) are computed using the expression in (6):

$$\frac{\partial^2 \ P(p|\boldsymbol{D})}{\partial p^2} = \boldsymbol{w^T}\left(\frac{\partial^2 \boldsymbol{\Pi}}{\partial \boldsymbol{a}^2}\boldsymbol{D} + \frac{\partial \boldsymbol{\Pi}}{\partial \boldsymbol{a}}\frac{\partial \boldsymbol{D}}{\partial \boldsymbol{a}} + \frac{\partial \boldsymbol{\Pi}}{\partial \boldsymbol{a}}\frac{\partial \boldsymbol{D}}{\partial \boldsymbol{a}} + \boldsymbol{\Pi}\frac{\partial^2 \boldsymbol{D}}{\partial \boldsymbol{a}^2}\right) = \boldsymbol{w^T}\left(\frac{\partial^2 \boldsymbol{\Pi}}{\partial \boldsymbol{a}^2}\boldsymbol{D}\right) \tag{16}$$

$$\frac{\partial^2 \ P(p|\boldsymbol{D})}{\partial \boldsymbol{D}^2} = \boldsymbol{w^T}\left(\frac{\partial^2 \boldsymbol{\Pi}}{\partial \boldsymbol{D}^2}\boldsymbol{D} + 2\frac{\partial \boldsymbol{\Pi}}{\partial \boldsymbol{D}}\right); \frac{\partial^2 \ P(p|\boldsymbol{D})}{\partial \boldsymbol{\rho}^2} = \boldsymbol{w^T}\left(\frac{\partial \boldsymbol{\Pi}}{\partial \boldsymbol{\rho}}\frac{\partial \boldsymbol{D}}{\partial \boldsymbol{\rho}} + \boldsymbol{\Pi}\frac{\partial^2 \boldsymbol{D}}{\partial \boldsymbol{\rho}^2}\right) \tag{17}$$

$$\frac{\partial^2 P(p|\boldsymbol{D})}{\partial p \partial \boldsymbol{D}} = \boldsymbol{w^T}\left(\frac{\partial \boldsymbol{\Pi}}{\partial \boldsymbol{a}} + \frac{\partial \boldsymbol{\Pi}}{\partial \boldsymbol{a}\partial \boldsymbol{D}}\boldsymbol{D}\right) \tag{18}$$

substituting (16-18) in (12), and assuming $\boldsymbol{\Delta} = 0$:

$$\boldsymbol{w^T}\left[\frac{1}{2}\left(\left(\frac{\partial^2 \boldsymbol{\Pi}}{\partial \boldsymbol{a}^2}\boldsymbol{D}\right)\boldsymbol{\sigma_p}^2 + \left(\frac{\partial^2 \boldsymbol{\Pi}}{\partial \boldsymbol{D}^2}\boldsymbol{D} + 2\frac{\partial \boldsymbol{\Pi}}{\partial \boldsymbol{D}}\right)\boldsymbol{\sigma_D}^2\right) + \left(\frac{\partial \boldsymbol{\Pi}}{\partial \boldsymbol{a}} + \frac{\partial \boldsymbol{\Pi}}{\partial \boldsymbol{a}\partial \boldsymbol{D}}\boldsymbol{D}\right)\boldsymbol{\sigma_p}\boldsymbol{\sigma_D}\right] = \\ = -\boldsymbol{\mu_D}\boldsymbol{w^T}\boldsymbol{\Pi}\boldsymbol{D} \tag{19}$$

with $\boldsymbol{\sigma_p}^2 = \begin{bmatrix} w_1\sigma_1^2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & w_n\sigma_n^2 \end{bmatrix}$. A PDE that does not depend on the weights of the portfolio, except from the term $\boldsymbol{\sigma_p}$ with weights not squared, is obtained:

$$\frac{1}{2}\left(\left(\frac{\partial^2 \boldsymbol{\Pi}}{\partial \boldsymbol{a}^2}\boldsymbol{D}\right)\boldsymbol{\sigma_p}^2 + \left(\frac{\partial^2 \boldsymbol{\Pi}}{\partial \boldsymbol{D}^2}\boldsymbol{D} + 2\frac{\partial \boldsymbol{\Pi}}{\partial \boldsymbol{D}}\right)\boldsymbol{\sigma_D}^2\right) + \left(\frac{\partial \boldsymbol{\Pi}}{\partial \boldsymbol{a}} + \frac{\partial \boldsymbol{\Pi}}{\partial \boldsymbol{a}\partial \boldsymbol{D}}\boldsymbol{D}\right)\boldsymbol{\sigma_p}^2\boldsymbol{\sigma_D} + \\ + \boldsymbol{\mu_D}^T\boldsymbol{D}\boldsymbol{\Pi} = 0 \tag{20}$$

Expressions are given for the partial derivatives in (20), which can be computed with Kronecker products and substituted back in (20):

$$\boldsymbol{D}^T \frac{\partial^2 \boldsymbol{\Pi}}{\partial \boldsymbol{a}^2} {\boldsymbol{\sigma}_{\boldsymbol{p}}}^2 =$$

$$= \boldsymbol{D}^T_{(nxm)} \left(\left(\frac{\partial}{\partial a_1}, \ldots, \frac{\partial}{\partial a_n} \right) \oplus \left(\frac{\partial}{\partial a_1}, \ldots, \frac{\partial}{\partial a_n} \right) \oplus \boldsymbol{\Pi}_{(nxm)} \right)_{(nx(n^2m))} \boldsymbol{\sigma}_{\boldsymbol{p}(nxn)}^2 =$$

$$= \boldsymbol{D}^T_{(nxm)} \begin{bmatrix} \frac{\partial^3 C(a_1,\mathrm{D}_1)}{\partial \mathrm{D}_1 \partial a_1^2} D_1 & \ldots 0 \ldots & \frac{\partial^3 C(a_n,\mathrm{D}_1)}{\partial \mathrm{D}_1 \partial a_n^2} D_1 \\ \vdots & \ddots \vdots \ddots & \vdots \\ \frac{\partial^3 C(a_1,\mathrm{D}_m)}{\partial \mathrm{D}_m \partial a_1^2} D_m & \ldots 0 \ldots & \frac{\partial^3 C(a_n,\mathrm{D}_m)}{\partial \mathrm{D}_m \partial a_n^2} D_m \end{bmatrix}_{(mx(mn^2)} \boldsymbol{\sigma}_{\boldsymbol{p}(n^2x1)}^2 \tag{21}$$

$$\frac{\partial^2 \boldsymbol{\Pi}^T}{\partial \boldsymbol{D}^2} {\boldsymbol{\sigma}_{\boldsymbol{D}}}^2 \boldsymbol{D} =$$

$$= \left(\left(\frac{\partial}{\partial D_1}, \ldots, \frac{\partial}{\partial D_m} \right) \oplus \left(\frac{\partial}{\partial D_1}, \ldots, \frac{\partial}{\partial D_m} \right) \oplus \boldsymbol{\Pi}^T \right)_{(mx(nm))} \left({\boldsymbol{\sigma}_{\boldsymbol{D}}}^2 \boldsymbol{D} \right)_{(mxn)} =$$

$$= \begin{bmatrix} \frac{\partial^3 C(a_1,\mathrm{D}_1)}{\partial \mathrm{D}_1^3} & \cdots & 0 & \cdots & \frac{\partial^3 C(a_n,\mathrm{D}_1)}{\partial \mathrm{D}_1^3} & 0 & \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\partial^3 C(a_1,\mathrm{D}_m)}{\partial \mathrm{D}_m^3} & \cdots & 0 & \cdots & \frac{\partial^3 C(a_n,\mathrm{D}_m)}{\partial \mathrm{D}_m^3} \end{bmatrix}_{(mx(mn))} \left({\boldsymbol{\sigma}_{\boldsymbol{D}}}^2 \boldsymbol{D} \right)_{(mxn)} \tag{22}$$

$${\boldsymbol{\sigma}_{\boldsymbol{D}}}^2 \frac{\partial \boldsymbol{\Pi}^T}{\partial \boldsymbol{D}} = \boldsymbol{\sigma}_{\boldsymbol{D}(mxm)}^2 \begin{bmatrix} \frac{\partial^2 C(a_1,\mathrm{D}_1)}{\partial \mathrm{D}_1^2} & \cdots & 0 & \cdots & \frac{\partial^2 C(a_n,\mathrm{D}_1)}{\partial \mathrm{D}_1^2} & 0 & \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\partial^2 C(a_1,\mathrm{D}_m)}{\partial \mathrm{D}_m^2} & \cdots & 0 & \cdots & \frac{\partial^2 C(a_n,\mathrm{D}_m)}{\partial \mathrm{D}_m^2} \end{bmatrix}_{(mx(mn))} \tag{23}$$

$$\boldsymbol{\sigma}_{\boldsymbol{D}} \frac{\partial \boldsymbol{\Pi}^T}{\partial \boldsymbol{a}} {\boldsymbol{\sigma}_{\boldsymbol{p}}}^2 = \boldsymbol{\sigma}_{\boldsymbol{D}(mxm)} \begin{bmatrix} \frac{\partial^2 C(a_1,\mathrm{D}_1)}{\partial \mathrm{D}_1 \partial a_1} & \ldots 0 \ldots 0 \ldots & \frac{\partial^2 C(a_n,\mathrm{D}_1)}{\partial \mathrm{D}_1 \partial a_n} \\ \vdots & \ddots \vdots \cdots \vdots \ddots & \vdots \\ \frac{\partial^2 C(a_1,\mathrm{D}_m)}{\partial \mathrm{D}_m \partial a_1} & \ldots 0 \cdots 0 \ldots & \frac{\partial^2 C(a_n,\mathrm{D}_m)}{\partial \mathrm{D}_m \partial a_n} \end{bmatrix}_{(mxn^2)} \boldsymbol{\sigma}_{\boldsymbol{p}(n^2x1)}^2 \tag{24}$$

$$\boldsymbol{D}^T \boldsymbol{\sigma}_{\boldsymbol{D}} \frac{\partial \boldsymbol{\Pi}^T}{\partial \boldsymbol{a} \partial \boldsymbol{D}} {\boldsymbol{\sigma}_{\boldsymbol{p}}}^2 =$$

$$= \boldsymbol{D}^T_{(nxm)} \boldsymbol{\sigma}_{\boldsymbol{D}(mxm)} \left[\left(\frac{\partial}{\partial a_1}, \ldots, \frac{\partial}{\partial a_n} \right) \oplus \left(\frac{\partial}{\partial D_1}, \ldots, \frac{\partial}{\partial D_m} \right) \oplus \boldsymbol{\Pi} \right] \boldsymbol{\sigma}_{\boldsymbol{p}(nxn)}^2 =$$

$$= \boldsymbol{D}^T_{(nxm)} \boldsymbol{\sigma}_{\boldsymbol{D}(mxm)} \begin{bmatrix} \frac{\partial^3 C(a_1,\mathrm{D}_1)}{\partial \mathrm{D}_1^2 \partial a_1} & \ldots 0 \ldots & \frac{\partial^3 C(a_n,\mathrm{D}_1)}{\partial \mathrm{D}_1^2 \partial a_n} \\ \vdots & \ddots \vdots \ddots & \vdots \\ \frac{\partial^3 C(a_1,\mathrm{D}_m)}{\partial \mathrm{D}_m^2 \partial a_1} & \ldots 0 \ldots & \frac{\partial^3 C(a_n,\mathrm{D}_m)}{\partial \mathrm{D}_m^2 \partial a_n} \end{bmatrix}_{(mx(mn^2)} \boldsymbol{\sigma}_{\boldsymbol{p}(n^2x1)}^2 \tag{25}$$

Partial derivatives computed from (2), $u_1 = a_i$ and $u_2 = D_j$ $\forall i = 1, \ldots, n; j = 1, \ldots, m$, $\boldsymbol{\sigma_D} = Cov(D_t^k, D_t^q) = e^{(\mu_{Dk}+\mu_{Dq})t}(e^{\rho_{Dk,Dq}\sigma_{Dk}\sigma_{Dq}t} - 1)$, and $\boldsymbol{\sigma_p} = Cov(a_t^k, a_t^q) = e^{(\mu_{ak}+\mu_{aq})t}(e^{\sigma_{ak}\sigma_{aq}t} - 1)$ $\forall k, q = 1, \ldots, m$, and $1, \ldots, n$ respectively.

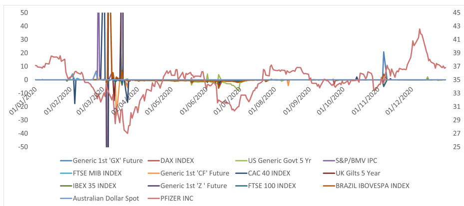

Fig. 1. Copulas' PDEs values for Phyzer - 2020

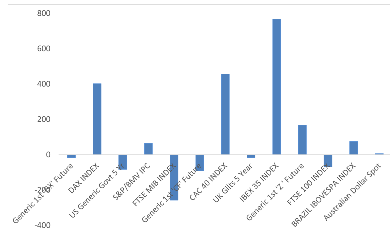

Fig. 2. Sum of values for 2020 (Phyzer)

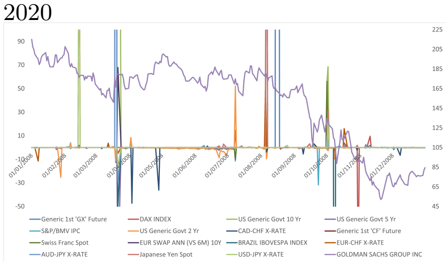

Fig. 3. Copulas' PDEs values for Goldman Sachs (GS) - 2008

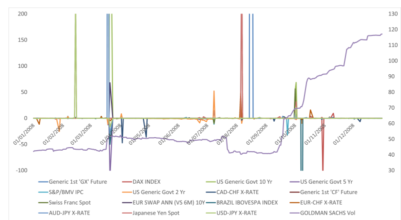

Fig. 4. GS 2008 Volatility

Notes and Comments. A system of copulas' PDEs for each pair of constituent/ common driver is obtained. Each PDE can be used for online risk management at constituent level, as seen in Figures 1-4. Copulas' PDEs are computed from data on a rolling window basis. Deviations from theoretical values are shown in these figures for two constituents. Large deviations are related to increasing volatility (Figure 4). Deviations can be integrated over a period and used for portfolio optimization (Figure 2), or can be used for online risk exposure monitoring and crisis detection. Inspections of the derivatives' orders in the PDEs deviations are needed for accurate portfolio hedging. Experiments are shown for the equal-weights case. Solving for the weights with the online PDE system can give implied market hedges. With the PDEs solution, implied weights based on the conditional risk-neutral common causal measure could be obtained for longer horizons.

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