2010 年 MCM 特等奖论文

Problem A: The Sweet Spot

Explain the "sweet spot" on a baseball bat. Every hitter knows that there is a spot on the fat part of a baseball bat where maximum power is transferred to the ball when hit. Why isn't this spot at the end of the bat? A simple explanation based on torque might seem to identify the end of the bat as the sweet spot, but this is known to be empirically incorrect. Develop a model that helps explain this empirical finding.

Some players believe that "corking" a bat (hollowing out a cylinder in the head of the bat and filling it with cork or rubber, then replacing a wood cap) enhances the "sweet spot" effect. Augment your model to confirm or deny this effect. Does this explain why Major League Baseball prohibits "corking"?

Does the material out of which the bat is constructed matter? That is, does this model predict different behavior for wood (usually ash) or metal (usually aluminum) bats? Is this why Major League Baseball prohibits metal bats?

The Sweet Spot: A Wave Model of Baseball Bats

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Abstract

We determine the sweet spot on a baseball bat. We capture the essential physicals of the ball-bat impact by taking the ball to be a lossy spring and the bat to be an Euler-Bernoulli beam. To impart some intuition about the model, we begin by presenting a rigid-body model. Next, we use our full model to reconcile various correct and incorrect claims about

the sweet spot found in the literature. Finally, we discuss the sweet spot and the performances of corked and aluminum bats, with a particular emphasis on hoop modes.

我们确定了棒球棒的最佳击球点。我们把棒球视作弹性体，把棒视为欧拉—伯努利梁，从而抓住了棒—球作用过程的物理本质。为了把对该问题的直观判断引入模型当中，我们用一个刚体模型作为初始模型。然后，我们提出了我们的波传播模型，并对一些文献中正确的和不正确的判断做了整合。最后，我们并对塞有软木的棒球棒和铝质棒球棒的最佳击球点进行了讨论。

Introduction

Although a hitter might expect a model of the bat-baseball collision to yield insight into how the bat breaks, how the bat imparts spin on the ball, how best to swing the bat, and so on, we model only the sweet spot.

尽管击球手对球与棒的碰撞过程会做出估计，诸如棒会怎么断裂，棒如何对球施加旋转，怎样最好地挥动棒等等，我们仅仅对最佳击球点进行建模。

There are at least two notions of where the sweet spot should be—an impact location on the bat that either（对最佳击球点至少有两种定义）

- minimizes the discomfort to the hands, or （对手部的痛感最小）
- maximizes the outgoing velocity of the ball （球速最大）

We focus exclusively on the second definition. （我们只专注于第二个定义）

The velocity of the ball leaving the bat is determined by （决定球速的因素）

- the initial velocity and rotation of the ball （球的初速度和角速度）
- the initial velocity and rotation of the bat （棒的初速度和角速度）
- the relative position and orientation of the bat and ball, and（球和棒的相对位置和方向）
- the force over time that the hitter's hands apply on the handle.（手作用于握柄的时间）

We assume that the ball is not rotating and that its velocity at impact is perpendicular to the length of the bat. We assume that everything occurs in a single plane and we argue that the hands' interaction is negligible. In the frame of reference of the center of mass of the bat, the initial conditions are completely specified by

我们假设不考虑球的旋转，球的初速度方向垂直于棒的长度方向，并假设碰撞发生在一个平面内，碰撞过程中手的作用可以忽略。在棒的质心坐标系下，初始条件由

- the angular velocity of the bat（棒的角速度）
- the velocity of the ball, and（球的速度）
- the position of the impact along the bat（作用点的位置）

决定。

The location of the sweet spot depends not on just the bat alone but also on the pitch and on the swing.

最佳击球点的位置不仅取决于棒，还取决于挥动轴和挥动的力度。

The simplest model for the physics involved has the sweet spot at the *center of percussion* [Brody 1986], the impact location that minimizes discomfort to the hand. The model assumes the **bat**[1] to be a rigid body for which there are *conjugate points*: An impact at one will exactly balance the angular recoil and linear recoil at the other. By gripping at one and impacting at the other (the center of percussion), the hands experience minimal shock and the ball exit with high velocity. The center of percussion depends heavily on the moment of inertia and the location of the[1] hands. We cannot accept this model because it both erroneously equates the two definitions of sweet spot and furthermore assumes incorrectly that the bat is a rigid body.

一种最简单的模型将撞击中心视为最佳击球点[Brody 1986]，球击打该处时对手产生的痛感最小。这个模型把棒球棒假设为刚体，棒上存在一对点：在撞击中心处的击打力会平衡握柄端的后坐力。握住一端击打撞击中心会使得手的痛感最小而棒球以高速飞离棒球棒。撞击中心的位置主要取决于转动惯量和手的位置。由于该模型不仅错误地等同两种最佳击球点的定义，还错误地把球假设为刚体，所以我们不能接受这个模型。

Another model predicts the sweet spot to be between nodes of the two lowest natural frequencies of the bat [Nathan 2000]. Given a free bat allowed to oscillate, its oscillations can be decomposed into fundamental modes of various frequencies. Different geometries and materials have different natural frequencies of oscillation. The resulting wave shapes suggest how to excite those modes (e.g. plucking a string at the node of a vibration mode will not excite that mode). It is ambiguous which definition of sweet spot this model uses. Using the first definition, it would focus on the uncomfortable excitations of vibrational modes: Choosing the impact location to be near nodes of important frequencies, a minimum of uncomfortable vibrations will result. Using the second definition, the worry is that energy sent into vibrations of the bat will be lost. This model assumes that the most important energies to model are those lost to vibration.

另一种模型认为最佳击球点位于棒的前两阶固有频率的节点之间[Nathan 2000]。给定一个自由振动的球棒，它的振动可以分解为许多固有频率不同的基准模态。不同的几何尺寸和材料的球棒有着不同的固有频率。模态形状决定了如何激发相应的模态（比如：

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1 原文这里为“ball”，根据上下文应该为“bat”

作用于一根细线的节点处的力不会激发相应的模态）。很难判断这个模型到底用了那种定义。如果是第一种定义，重点就在于选择主要频率的节点附近作为击球点使得对手部的振动最小，如果是第二种定义，那么问题在于球的动能会因为需要传递部分能量给棒的振动而产生损失。这个模型假设损失的这部分能量在建模过程中起着重要作用。

This model raises many questions. Which frequencies get excited and why? The Fourier transform of an impulse in general contains infinitely many modes. Furthermore, wood is a viscoelastic material that quickly dissipates its energy. Is the notion of an oscillating bat even relevant to modeling a bat? How valid is the condition that the bat is free? Ought the system to be coupled with hands on the handle, or the arm's bone structure, or possibly even the ball? What types of oscillations are relevant? A cylindrical structure can support numerous different types of modes beyond the transverse modes usually assumed by this model [Graff 1975].

这个模型引起了很多疑问。哪些频率会受到激发，为什么？毕竟脉冲函数的傅立叶变换包含有无数的模态。而且，木头是一种粘弹性材质，其传递的能量会很快耗散掉。棒是自由的这个条件有多强？应不应该考虑握柄端的手对棒的作用，手的骨骼结构，棒球等的影响？什么样的振动是相关的？除了该模型中的横向振动模态以外，柱形棒还有很多种振动模态[Graff 1975]。

Following the center-of-percussion line of reasoning, how do we model the recoil of the bat? Following the vibration-nodes line of reasoning, how do we model the vibrations of the bat? In the general theory of impact mechanics [Goldsmith 1960], these two effects are the main ones (assuming that the bat does not break or deform permanently). Brody [1986] ignores vibrations, Cross [1999] ignores bat rotation but studies the propagation of the impulse coupled with ball, and Nathan [2000] emphasizes vibration modes. Our approach reconciles the tension among these approaches while emphasizing the crucial role played by the *time-scale* of the collision.

从前面对撞击中心的分析，我们该如何对棒的反应进行建模？从对振动模态的分析来看，我们又该如何对棒的振动进行建模？在撞击动力学的一般理论中[Goldsmith 1960]，这两种效应起主导作用（假设棒不会断裂也不会发生永久变形）。**Brody** [1986]忽略了振动，**Cross** [1999]忽略了棒的转动，转而研究在与球的耦合作用下冲击在棒中的传播，**Nathan** [2000]则专注于振动模态的研究。我们的方法在综合这三种方法的同时，还主要考虑了撞击过程的时间尺度的重要影响。

Our main goal is to understand the sweet spot. A secondary goal is to understand the differences between the sweet spots of different bat types. Although marketers of bats often emphasize the sweet spots, there are other relevant factors: ease of swing, tendency of the bat to break, psychological effects, and so on. We will argue that it doesn't matter to the collision whether the batter's hands are gripping the handle firmly or if the batter follows through on the swing; these circumstances have no bearing on the technique required to swing the bat or how the bat's properties affect it.

我们的首要目标就是理解最佳击球点的含义。次要目标就是理解不同类型的棒球棒的最佳击球点的区别。尽管棒球棒厂商专注于最佳击球点，但是其他因素比如挥舞的轻松程度，棒是否容易断裂，棒球手的心理因素等等也有关系。我们会证明棒球手是否握紧棒球棒，棒球手是否随着挥舞运动等并不重要。这些情况对最佳击球点没有影响。

Our paper is organized as follows. First, we present the Brody rigid-body model, illuminating the recoil effects of impact. Next we present a full computational model based on wave propagation in an Euler-Bernoulli beam coupled with the ball modeled as a lossy spring. We compare this model with others and explore the local nature of impact, the interaction of recoil and vibrations, and robustness to parameter changes. We adjust the parameters of the model to comment on the sweet spots of corked bats and aluminum bats. Finally, we investigate the effect of hoop frequencies on aluminum bat.

我们的论文结构如下。首先，我们阐述了 **Brody** 的刚体模型，说明了冲击的反冲效应。然后我们建立了一个纯计算模型，该模型以欧拉—伯努利梁理论为基础，把棒当作弹性体并考虑了与球的耦合作用。我们将此模型与其他模型做了比较，并研究了撞击的本质，反冲与振动之间的相互作用，还验证了参数的鲁棒性。我们通过改变参数对塞有软木的棒球棒和铝制棒球棒进行了分析。最后，我们探讨了铝质棒球棒的 hoop mode（这个“模态”不好翻译）。

A simple Example

We begin by considering only the rigid recoil effects of the bat—ball collision, much as in Brody [1986]. For simplicity, we assume that the bat is perfectly rigid. Because the collision happens on such a short time-scale (around 1ms), we treat the bat as a free body. That is to say, we're not concerned with the batter's hands exerting force on the bat that may be transferred to the ball.

我们开始只考虑一个刚体模型，就像 **Brody** 的模型一样。为了简化，我们假设棒球棒是刚体。由于撞击过程的持续时间很短（大约 1 毫秒），我们进一步把棒视为自由刚体。也就是说，我们不关心棒球手会对球产生力的作用。

The bat has mass M and moment of inertia I about its center of mass. From the reference frame of the center of mass of the bat just before the collision, the ball has initial velocity V_i in the positive x-direction while the bat has initial angular velocity ω_i. In our setup, V_i and ω_i have opposite signs when the batter is swinging at the ball as in **Figure 1**, in which arrows point in the positive directions for corresponding parameters.

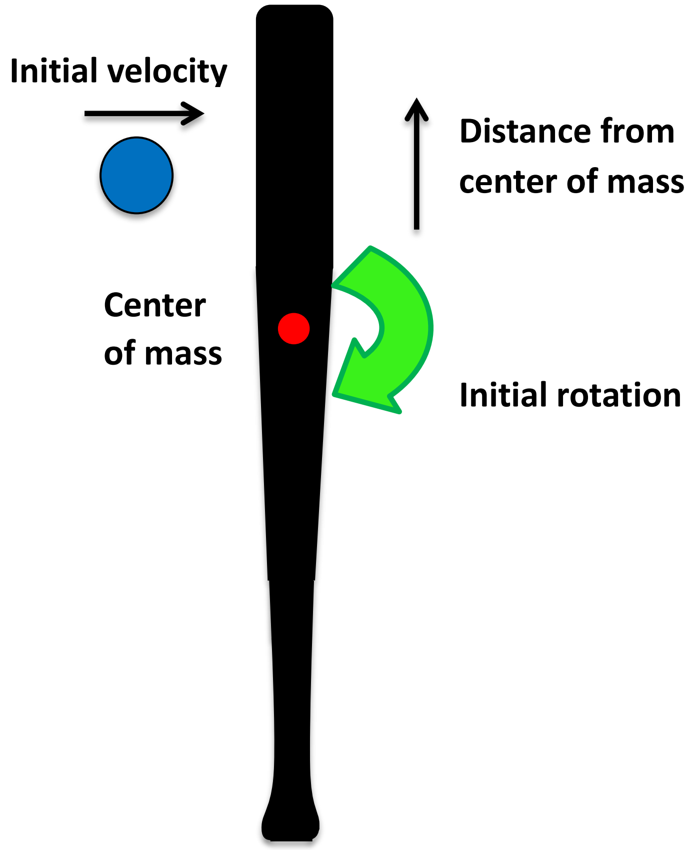

Figure 1 The collision

棒球棒的质量为 M，绕质心的转动惯量为 I。以撞击之前棒的质心坐标系为参考坐标系，球的初速度为v_i，棒的初始角速度为ω_i。假设正方向为 **Figure 1** 中所示。撞击点位于距离棒的质心l处。我们假设碰撞为正碰撞。碰撞之后，球速为v_f，棒球棒质心处的速度为V_f，角速度为ω_f。（这里的物理量均表示大小，不带符号）碰撞过程中，球压缩后反弹，这一过程中将动能转化为势能，并伴随着能量损失，这是由于碰撞是非弹性碰撞。球与球棒在碰撞前后的相对速度之比叫做恢复系数，通常用e表 示，$e=0$表示完全非弹性碰撞，$e=1$表示完全弹性碰撞。在本模型中，我们做出如下两点假设：

- e 对于棒上任何一点都相等（e is constant along the length of the bat)
- e 对于任何v_i都相等（e is constant for all v_i)

给出模型的基本条件之后，我们有如下方程：

动量守恒(Conservation of linear momentum)： $MV_f = m(v_i - v_f)$

角动量守恒(Conservation of angular momentum)： $I(\omega_f - \omega_i) = ml(v_i - v_f$

恢复系数(Definition of coefficient of restitution)： $e(v_i - \omega_i l) = -v_f + V_f + \omega_f l$

由上述式子解出：

$$v_f = \frac{-v_i(e - \frac{m}{M^*}) + \omega_i l(1+e)}{1 + \frac{m}{M^*}}$$

这里，

$$M^* = \frac{M}{1 + \frac{Ml^2}{I}}$$

是棒的有效质量。(The effective mass of the bat)

For calibration purposes, we use the following data, which are typical of a regulation bat connecting with a fastball in Major League Baseball. The results are plotted in **Figure 2**.

为了最佳击球点的位置有一个初步的估计，我们使用以下数据，这些数据都是全美棒球联盟的典型棒球棒的数据。结果如 **Figure 2** 所示。

m	0.145	kg
M	0.83	kg
L	0.84	m
I	0.039	$kg \cdot m^2$
v_i	67	m/s
w_i	-60	rad/s
e	0.55	

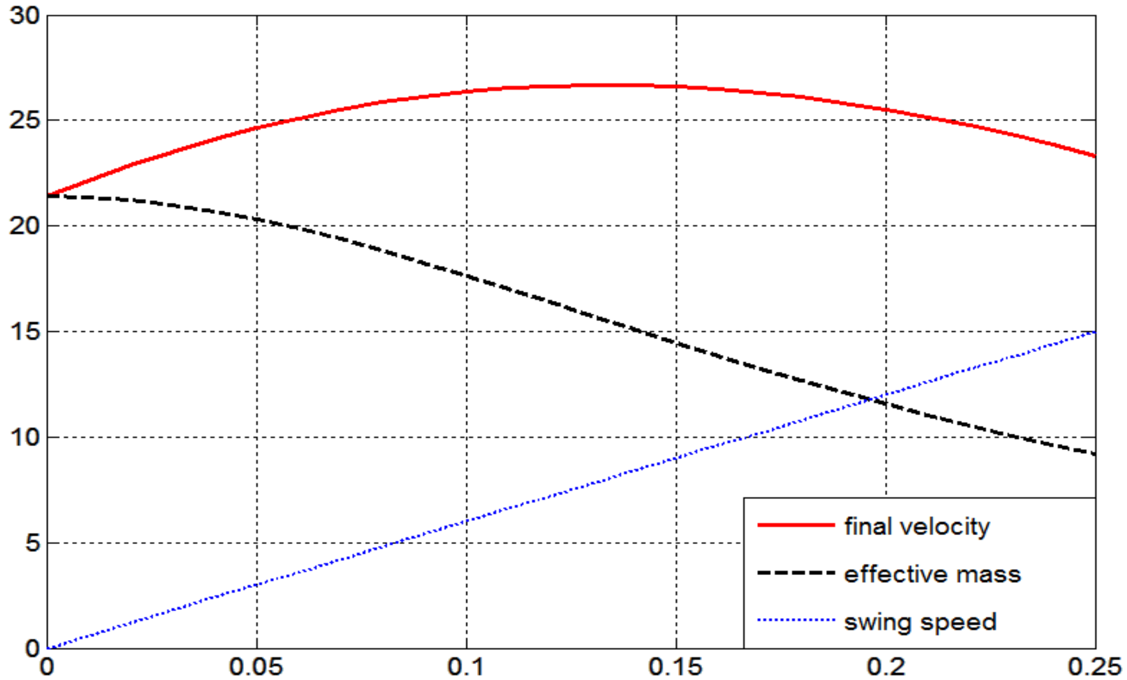

Figure 2 Final velocity v_f (solid arc at top), swing speed $\omega_i l$ (dotted rising line), and effective mass (dashed falling curve) as a function of distance l (in meters) from center of mass.

The maximum exit velocity is 27m/s, and the sweet spot is 13cm from the center of mass. Missing the sweet spot by up to 5cm results in at most 1m/s difference from the maximum velocity, implying a relatively wide sweet spot.

击球后球速最大为 27m/s，最佳击球点位于距离棒质心 13cm 处。远离这一最佳击球点 5cm 时最大速度减小 1m/s，可见最佳击球点的范围较大。

From this example, we see that the sweet spot is determined by a multitude of factors, including the length, mass, and shape of the baseball bat; the mass of the baseball; and the coefficient of restitution between bat and ball. Furthermore, the sweet spot is not uniquely determined by the bat and ball: It also depends on the incoming baseball speed and the batter's swing speed.

从这个例子可以看出，最佳击球点有多个因素决定，包括棒长，棒的质量，棒的形状和棒与球之间的恢复系数。而且最佳击球点并不只是由棒和球决定，还取决于球速和棒球手的挥棒的速度。

Figure 2 also shows intuitively why the sweet spot is located somewhere between the center of mass and the end of the barrel. As the point of collision moves out[2]ward along the bat, the effective mass of the bat goes **down**[2], so that a greater fraction of the initial kinetic energy is put into the bat's rotation. At the same time, the rotation in the bat means that the bat is moving faster than the center of mass (or handle). These two effects work in opposite directions to give a unique sweet spot that's not at either end point.

Figure 2 还直观地表明最佳击球点位于质心和棒的末端之间的原因。随着碰撞点的外移，棒的有效质量**减小**[2]，因此更多的初始动能转化为棒的转动动能。另外由于棒的末端的速度大于质心的速度，这两者产生的效应相反，因此最佳击球点既不在棒的末端也不在棒的质心。

However, this model tells only part of the story. Indeed, some of our starting assumptions contradict each other:

然而，这个模型还只是揭示了问题的一部分。实际上最初的假设是相互矛盾的：

- We treated the bat as a free body because the collision time was so short. In essence, during the 1ms of the collision, the ball "sees" only the local geometry of the bat, not the batter's hands on the handle. On the other hand, we assumed that the bat was perfectly rigid—but that means that the ball "sees" the entire bat.

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2 原文这里描述有效质量为 "goes up"，应该为 "goes down"

由于碰撞时间极短，我们把棒视为自由刚体。实际上，在 1 毫秒的接触时间内，球只对棒的局部起作用，不会影响到握柄端。而另一方面，我们假设棒球棒是刚体，这就意味着球会影响到整个棒球棒。

- We also assumed that is constant along the length of the bat and for different collision velocities. Experimental evidence [Adair 1994] suggests that neither issue can be ignored for an accurate prediction of the location of the sweet spot.
  我们还假设恢复系数对于棒长和任意初速度是常数，实验表明对于精确分析而言，它不是一个常数。

We need a more sophisticated model to address these shortcomings.
我们需要建立一个更完善的模型来弥补这些缺点。

Our Model

We draw from Brody's rigid-body model but more so from Cross [1999]. One could describe our work as an adaption of Cross's work to actual baseball bats. Nathan [2000] attempted such an adaption but was misled by incorrect intuition about the role of vibrations. We describe his approach and error as a way to explain Cross's work and to motivate our work.
我们从 **Brody** 的刚体模型出发，但是更多地采用了 **Cross** 的模型。我们的工作可以看成是 **Cross** 的成果应用于实际棒球棒的变型。**Nathan** 也这样做过，但是他被错误的直觉所误导，对振动所起的作用估计不足。我们描述了他的方法和错误，以此来阐释我们的模型。

Previous Models

Brody's rigid-body model correctly predicts the existence of a sweet spot not at the end of the bat. That model suffers from the fact that the bat is not a rigid body and experience vibrations. One way to account for vibrations is to model the bat as a flexible object. Beam theories (of varying degrees of accuracy and complication) can model a flexible bat. Van Zandt [1992] was the first to carry out such an analysis, modeling the beam as a *Timoshenko beam*, a fourth-order theory that takes into account both shear forces and tensile stresses. The equations are complicated and we will not need them. Van Zandt's model assumes the ball to be uncoupled from the beam and simply takes the impulse of the ball as a given. The resulting vibrations of the bat are used to predict the velocity of the beam at the impact point (by summing the Brody velocity with the velocity of the displacement at the impact point due to vibrations) and hence the exit velocity of the ball from the equations of the coefficient of restitution [Van Zandt 1992].

Brody 的刚体模型正确的表明了最佳击球点的位置并不在棒的末端。但是该模型的缺点在于棒并不是刚体，而且会发生振动。为了考虑这一振动，应该把棒当成弹性体。梁理论（各种梁理论的精度和复杂度不同）适用于这一弹性体。**Van Zandt** 第一个将梁理论用于这个问题的分析，他把棒视为一个铁摩辛柯梁，这一理论是四阶梁理论，考虑了剪应力和轴向应力。该理论的方程形式很复杂，我们在这里不列出来。**Van Zandt** 的模型假设球和棒并不耦合，还简单地假设球的冲击是瞬态的。考虑振动后，撞击点的速度除了包括 **Brody** 模型中的速度外，还要加上振动引起的速度，再由恢复系数的表达式即可求出球的最终速度[Van Zandt 1992]。

Cross [1999] modeled the interaction of the impact of a ball with an aluminum beam, using the less-elaborate Euler-Bernoulli equations to model the propagation of waves. In addition, he provided equations to model the dynamical coupling of the ball to the beam during the impact. After discretizing the beam spatially, he assumed that the ball acts as a lossy spring coupled to the single component of the region of the impact.

Cross 使用了较不精确的欧拉-伯努利梁理论来分析铝质棒球棒受到棒球冲击时的振动波的传播。除此之外，他还提供了球与棒的耦合作用的方程。在将棒球棒离散之后，他假设球也是弹性体，而且球对棒的作用仅仅位于冲击所在区域的单个单元。

Cross’s work was motivated by both tennis rackets and baseball bats, which differ importantly in the time-scale of impact. The baseball bat’s collision lasts only about 1ms, during which the propagation speed of the wave is very important. In this local view of the impact, the importance of the baseball’s coupling with the bat is increased.

Cross 的模型适用于网球拍和棒球棒，但是这二者的撞击时间很不相同。棒球棒与棒球的作用时间仅为 1 毫秒左右，这段时间内振动波的传播速度是很重要的。在撞击的局部区域，球与棒的耦合作用显得尤为重要。

Cross argues that the actual vibrational modes and node points are largely irrelevant because the interaction is localized on the bat. The boundary conditions matter only if vibrations reflect off the boundaries; an impact not close enough to the barrel end of the bat will not be affected by the boundary there. In particular, a pulse reflected from a free boundary returns with the same sign (deflected away from the ball, decreasing the force on the ball, decreasing the exit velocity), but a pulse reflected from a fixed boundary returns with the opposite sign (deflected towards the ball, pushing it back, increasing the exit velocity). Away from the boundary, we expect the exit velocity to be uniform along a non-rotating bat. Cross's model predicts all of these effects, and he experimentally verified them. In our model, we expect similar phenomena, plus the narrowing of the barrel near the handle to act somewhat like a boundary.

Cross 提出，由于球的作用，棒的振动模态和节点位置很大程度上是不相关的。边界条件只在振动波传播到边界的时候才起作用：冲击点离棒的末端足够近时，边界条件才起作用。特别地，振动波经自由边界反射后相位不变（冲击力减小，球速减小），经固定边界反射之后反向（冲击力增大，球速增大）。在远离边界的地方，我们认为对于非转动的棒球棒来说，球速是一致的。**Cross** 的模型考虑了上述所有因素，他还用实验证明了他的考虑。在我们的模型中，我们也认为这样的现象会发生，同时还考虑了握柄端的边界条件。

Nathan's model also attempted to combine the best features of Van Zandt and Cross [Nathan 2000]. His theory used the full Timoshenko theory for the beam and the Cross model for the ball. He even acknowledged the local nature of impact. So where do we diverge from him? His error stems from an over emphasis on trying to separate out the ball's interaction with each separate vibrational mode.

Nathan 的模型还试图综合 **Van Zandt** 和 **Cross** 二者的优点。他采用了完整的铁摩辛柯梁理论和 **Cross** 的棒球模型。他还承认了冲击的局部效应。那么我们和他的区别在哪里呢？他的错误在于过于试图把球的作用和每一个振动模态分离开来。

The first sign of inconsistency comes when he uses the "orthogonality of eigenstates" to determine how much a given impulse excites each mode. The eigenstates are not orthogonal. Many theories yield symmetric matrices that need to be diagonalized, yielding the eigenstates; but Timoshenko's theory does not, due to the presence of odd-order derivatives in its equations. Nathan's story plays out beautifully if only the eigenstates were actually orthogonal; but we have numerically calculated the eigenstates, and they are not even approximately orthogonal. He uses the orthogonality to draw important conclusions:

第一个错误在于他使用模态正交性来确定激发每个模态所需的冲击力。模态不是正交的。很多理论都得到了对称矩阵，对称矩阵与对角矩阵相似，可以求出其特征向量；但是铁摩辛柯梁理论的方程中含有奇次导数项，因此得不到对称矩阵。**Nathan** 的理论在模态具备正交性时很完美，但是我们计算了这些模态之后，发现它们根本不正交。他用模态正交性得出了以下结论：

- The location of the nodes of the vibrational modes are important
  振动模态的节点位置很重要
- High-frequency effects can be completely ignored
  高频率效应可以完全忽略

We disagree with both of these. （这两点我们都不同意）

（这里突然拿出运动方程显得有些突兀，下面给出推导过程）

连续系统的振动方程如下：

$$M\cdot y''+K\cdot y=F(t)$$

离散为 n 自由度系统之后，这里 M 是 $n\times n$ 质量矩阵（对称矩阵），K 是 $n\times n$ 刚度矩阵（正定矩阵），$y(t)$ 是挠度向量，$F(t)$ 是外力向量。本方程的解可由假设法或傅里叶变换得到，这里为了得到自由振动条件下的固有模态及其固有频率，采用假设法，假设方程的解为：

$$y=\phi\cdot\sin(\omega t-\theta)$$

代入无外力项的振动方程后得：

$$(K-\omega^2\cdot M)\cdot\phi\cdot\sin(\omega t-\theta)=0$$

该式对于任何 t 均成立，则有：

$$(K-\omega^2\cdot M)\cdot\phi=0$$

变换后：

$$M^{-1}K\cdot\phi=\omega^2\cdot\phi$$

由此可解出 n 个模态向量 ϕ 和 n 个相应的固有频率 ω。

于是原方程可写成：

$$y''=-M^{-1}K\cdot y+M^{-1}\cdot F$$

令

$$H=-M^{-1}K$$

$$F=M^{-1}\cdot F$$

$$\Phi=[\phi_1\ \phi_2\ \cdots\ \phi_n]_{n\times n}$$

即可得到文中这部分的相关方程，可见 H 为对称矩阵，而非文中所说的反对称矩阵。

还需指出的是，文中的下标均使用了爱因斯坦求和约定（Einstein Summation Convention）。

The correct derivation starts with the following equation of motion, where k is the position of impact, y_i is the displacement and F_i is the external force on the i th segment of the bat, and H_{ij} is a **symmetric**[3] matrix:

$$y_k''(t)=H_{kj}y_j(t)+F_k(t)$$

We write the solutions as $y_k(t)=\Phi_{kn}a_n(t)$, where the **columns**[4] of Φ_{kn} are eigenmodes with eigenvalues $-\omega_n{}^2$. Explicitly, $H_{jk}\Phi_{kn}=-\omega_n{}^2\Phi_{jn}$ and Φ_{kn} indicates the k th component of

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3 原文这里为“asymmetric”，这里的 H 矩阵应该为对称矩阵而非反对称矩阵

4 原文这里为“rows”，应该为“columns”，矩阵 Φ 中的每一列对应一个模态向量

the n th eigenmode. Then we write the equation of motion:

$$\Phi_{kn} a_n''(t) + \Phi_{kn} \omega_n^2 a_n(t) = F_k = \Phi_{kn} \Phi_{nj}^{-1} F_j$$

$$a_n''(t) + \omega_n^2 a_n(t) = \Phi_{nk}^{-1} F_k$$

In the last step, we used the fact that the eigenmodes form a complete basis.（这里运用了模态矩阵Φ的正交性）

Nathan's paper uses on the right-hand side simply $\Phi_{kn} F_k$ scaled by a normalization constant. At first glance, this seems like a minor technical detail, but the physics here is important. We calculate that the $\Phi_{nk}^{-1} F_k$ terms stay fairly large for even high value of n, corresponding to the high frequency mode (k is just the position of impact). This means that there are significant high frequency components, at least at first. In fact, the high frequency modes are necessary for the impulse to propagate slowly as a wave packet. In Nathan's model, only the lowest standing modes are excited; so the entire bat starts vibrating as soon as the ball hits. This contradicts his earlier belief in localized collision (which we agree with) that the collision is over so quickly that the ball "sees" only part of the bat. Nathan also claims that the sweet spot is related to the nodes of the lowest mode, which contradicts locality: The location of the lowest order nodes depends on the geometry of the entire bat, including the boundary conditions at the handle.

Nathan 在他的论文里对右端项使用了正交化常数。乍一看这只是一个小的数学处理，但是这里的物理意义是很重要的。我们计算后发现，即使在n很大的时候，右端项$\Phi_{nk}^{-1} F_k$对应的数值依然很大，这和高频模态有关（k是冲击点位置）。实际上，高频模态对冲击的慢速传播是必要的。在 **Nathan** 的模型里，只有最低的几阶模态受到激发，因此整个棒球棒在击打球的瞬间就开始振动了。这和他之前认为的撞击是局部的（这一点正是我们同意的）相矛盾，撞击很快就结束了以至于球与棒的作用仅限于局部区域。**Nathan** 还声称最佳击球点的位置和最低阶模态的节点位置有关，这也和局部撞击相矛盾：最低阶模态的节点取决于棒的几何形状，包括端部的边界条件在内。

While the inconsistency in the Nathan model may cancel out, we build our model on a more rigorous footing. For simplicity, we use the Euler-Bernoulli equations rather the full Timoshenko equations. The difference is that the former ignore shear forces. This should be acceptable; Nathan points out that his model is largely insensitive to the shear modulus. We solve the differential equations directly after discretizing in space rather than decomposing into modes. In these ways, we are following the work of Cross [1999].

Nathan 的模型的缺点可能抵消它的长处，因此我们在更加严谨的基础上建立了我们的模型。为简化起见，我们采用欧拉—伯努利方程而不是完整的铁摩辛柯方程。前者忽略了剪切力。这一点是可以接受的；**Nathan** 指出他的模型对剪切模量很不敏感。我们直接对该偏微分方程离散求解，而没有采用模态分解。从这个意义上来说，我们沿用了 **Cross** 的方法。

On the other hand, our model extends Cross’s work in several key ways:

另一方面，我们的模型在以下几个方面扩展了 **Cross** 的工作：

- We examine parameters much closer to those relevant to baseball. Cross’ model focused on tennis. Featuring an aluminum beam of width 0.6cm being hit with a ball of 42g at around 1m/s. For baseball, we have an aluminum or wood bat of radius width 6cm being hit with a ball of 145g travelling at 40m/s (which involves 5000 times as much impact energy).
  我们采用的是棒球的参数。Cross 的模型则专注于网球，以一根直径为 0.6cm 的铝制网球拍击打一个重 42g、速度 1m/s 的网球。我们用的是直径 6cm 的铝制或木质棒球棒，击打一个重 145g、速度 40m/s 的棒球（动能为前者的 5000 倍）。
- We allow for a varying cross section, an important feature of a real bat.
  我们考虑了截面沿棒长的变化，这是实际棒球棒的一个重要特点。
- We allow the bat to have some initial angular velocity. This will let us scrutinize the rigid-body model prediction that higher angular velocities lead to the maximum power point moving farther up the barrel.
  我们考虑棒球棒有初始角速度。这样可以检验刚体模型中认为角速度增大会使最佳击球点外移这个结论是否正确。

To reiterate, the main features of our model are

再次重申，我们模型的主要特点是：

- an emphasis on the ball coupling with the bat
  重点考虑球与棒的耦合作用
- Finite speed of wave propagation in a short time-scale, and
  振动波的传播速度有限
- Adaption to realistic bats
  更接近真实的棒球棒

These are natural outgrowths of the approaches in the literature.

这些是文献中所述方法的自然延伸。

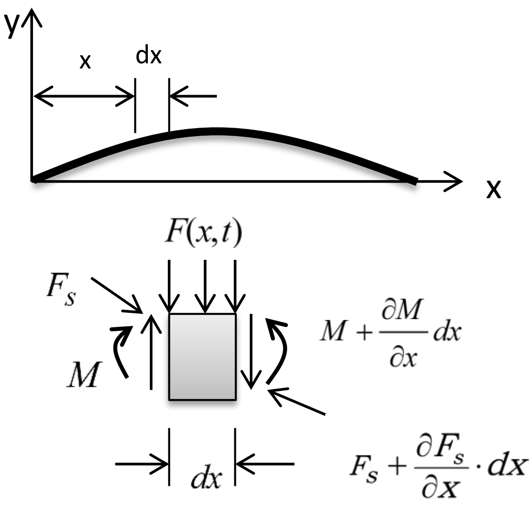
Mathematics of Our Model

（下面给出欧拉—伯努利方程的推导过程）

取梁的微元段 dx 列力平衡方程和力矩平衡方程

y 方向力平衡：

$$\rho_l dx\frac{\partial^2 y}{\partial t^2}=F_s-(F_s+\frac{\partial F_s}{\partial x}dx)-F(x,t)dx$$



化简得：

$$\rho_l\frac{\partial^2 y}{\partial t^2}=-\frac{\partial F_s}{\partial x}-F(x,t)$$

力矩平衡：

$$M+F_s dx-(M+\frac{\partial M}{\partial x}dx)-F(x,t)dx\frac{dx}{2}=0$$

化简得：

$$F_s=\frac{\partial M}{\partial x}$$

由材料力学：

$$M=EI\frac{\partial^2 y}{\partial x^2}$$

综上得：

$$\rho_l\frac{\partial^2 y}{\partial t^2}+\frac{\partial^2}{\partial x^2}(EI\frac{\partial^2 y}{\partial x^2})+F(x,t)=0$$

Our equations are a discretized version of the Euler-Bernoulli equations:

$$\rho\frac{\partial^2 y(z,t)}{\partial t^2}=F(z,t)+\frac{\partial^2}{\partial z^2}(YI\frac{\partial^2 y(z,t}{\partial z^2}$$

此方程应该为：

$$\rho_l\frac{\partial^2 y}{\partial t^2}+\frac{\partial^2}{\partial z^2}(YI\frac{\partial^2 y}{\partial z^2})+F(z,t)=0 \qquad (1)$$

Where

ρ is the mass density,（质量密度）（应该为线密度 ρ_l）

$y(z,t)$ is the displacement, （挠度）

$F(z,t)$ is the external force (in our case, applied by the ball), （外力）

Y is the Young's modulus of the material (a constant), and （杨氏模量）

I is the second moment of area ($\pi R^4/4$ for a solid disc). （截面惯性矩）

（A是棒的截面积，z是棒的自然坐标，对z离散，步长为Δ，采用中心差分）

We discretize z in steps of Δ. The only force is from the ball, in the negative direction to the k th segment. Our discretized equation is:

$$\rho_l \Delta \frac{d^2 y_i}{dt^2} = -\delta_{ik} F(t) - \frac{Y}{\Delta^3}[I_{i-1}(y_{i-2} - 2y_{i-1} + y_i)$$

$$-2I_i(y_{i-1} - 2y_i + y_{i+1}) + I_{i+1}(y_i - 2y_{i+1} + y_{i+2})]$$

● 这里把第 k 个单元（即棒球作用的单元）上的分布力等效为集中力（乘以单元步长）

Our dynamic variables are y_1 through y_N. For a fixed left end, we pretend that $y_{-1} = y_0 = 0$. For a free left end, we pretend that

$$y_1 - y_0 = y_0 - y_{-1} = y_{-1} - y_{-2}$$

The conditions on the right end are analogous. These are the same conditions that Cross uses.

Finally, we have an additional variable for the ball's position (relative to some zero point) $w(t)$. Initially, $w(t)$ is positive and $w'(t)$ is negative, so the ball is moving from the positive direction towards the negative. Let $u(t) = w(t) - y_k(t)$. This variable represents the compression of the ball, and we replace $F(t)$ with $F(u(t), u'(t))$. Initially, $u(t) = 0$ and $u'(t) = -v_{ball}$. The force between the ball and the bat takes the form of hysteresis curves such as the ones shown in **Figure 3**.

最后我们再对棒球相对于棒的位置引入一个变量$w(t)$。初始时，$w(t)$为正，$w'(t)$为负，因此棒球向负方向运动。令$u(t) = w(t) - y_k(t)$，$u(t)$表示球的压缩量，作用力是时间的函数，因此也是$u(t)$和$u'(t)$的函数。初始时刻，$u(t) = 0$，$u'(t) = -v_{ball}$。球与棒的作用力与球的压缩量的关系如图 3 所示。

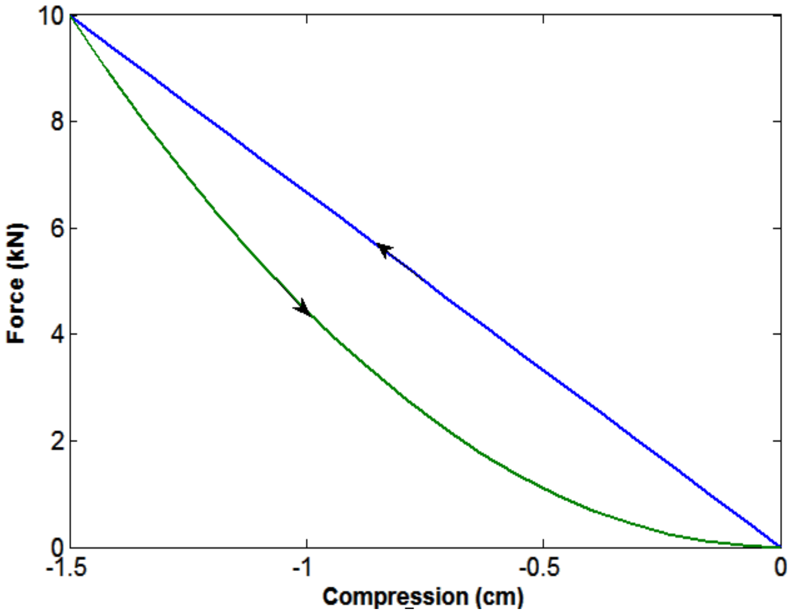


Figure 3 A hysteresis curve used in our modeling, with maximum compression 1.5 cm

The higher curve is taken when $u'(t) < 0$ (compression) and the lower curve when $u'(t) > 0$ (expansion). When $u(t) > 0$, the force is zero. The equation of motion for the ball is then

$$w''(t) = u''(t) + y_k''(t) = F(u(t), u'(t)) \qquad \text{(2)}$$

We have eliminated the variable w.

图中上方的曲线为压缩过程，$u'(t) < 0$，下方的曲线为扩张过程，$u'(t) > 0$。当$u(t) > 0$ 时，作用力为零。 球的运动方程如下：

$$w''(t) = u''(t) + y_k''(t) = F(u(t), u'(t))$$

这样我们就消去了 w。

We have yet to specify the function $F(u(t), u'(t))$. As can be seen in videos [Baseball Research Center n.d.], the ball compress significantly (often more than 1cm) in a collision. The compression and decompression is lossy. We could model this loss by subtracting a fraction of the ball's energy after the collision; that approach is good enough for many purposes, but we instead follow Nathan and use a nonlinear spring with hysteresis.

但是我们还没有确定函数 $F(u(t), u'(t))$。正如在视频里看到的，棒球在撞击过程中显著压缩变形。这一过程伴随能量损失。计算碰撞前后的能量之差就可以得到这一损失。这个方法有很多好处，但是我们仍然沿用 **Nathan** 的方法，采用带有滞后效应的非线性弹性体。

Since $W = \int F dx$, the total energy lost is the area between the two curves in **Figure 3**. A

problem with creating hysteresis curves is that one does not know the maximum compression (i.e. where to start drawing the bottom curve) until after solving the equations of motion. In practice, we solve the equation in two steps.
损失的总能量可以用图 3 中的曲线围成的面积所表示。但是构造该曲线的问题在于，在解出运动方程之前，无法确定最大压缩量，也就无法确定该曲线的起点位置。实际中，我们用两步来解方程。

The main assumptions of our model derive from the main assumptions of each equation:
我们模型的主要假设就是对这两个方程（方程（1）和（2））的假设：

- The first is the exact form of the hysteresis curve of the ball. Cross [1999] argues that the exact form of the curve is not very important as long as the duration of impact, magnitude of impulse, maximum compression of the ball, and energy loss are roughly correct.
  首先就是棒球的滞后曲线的形式。**Cross** 认为曲线的形式并不重要，只要冲击的持续时间，冲击的大小，棒球的最大压缩量和能量损失大概正确。
- Both the Timoshenko and Euler—Bernoulli theories ignore azimuthal and longitudinal waves. This is a fundamental assumption built into all of the approaches in the literature. Assuming that the impact of the ball is transverse and the ball does not rotate, the assumption is justified.
  铁摩辛柯和欧拉伯—努利梁理论都忽略了方位波和纵波。这是文献中所有方法的基本假设。假设棒球的冲击是横向的，棒球不发生旋转，这样假设是合理的。

The assumptions of our model are the same as those in the literature, so they are confirmed by the literature's experiments.

由于方程（2）对应的压缩量与时间的关系不可知，故不能对时间进行离散，可采用“线上法”求解，这里给出全部求解过程

对方程（1），离散后的棒的振动方程为：

$$\frac{d^2 y_i}{dt^2} = -\frac{\delta_{ik}}{\rho_l \cdot \Delta} F(t) - \frac{Y}{\rho_l \cdot \Delta^4}[I_{i-1}(y_{i-2} - 2y_{i-1} + y_i) - 2I_i(y_{i-1} - 2y_i + y_{i+1}) + I_{i+1}(y_i - 2y_{i+1} + y_{i+2})]$$

$$i = 1, 2 \cdots, N \qquad (3)$$

对方程（2），离散后的球的运动方程为：

$$\frac{d^2u}{dt^2}=(\frac{1}{m_{ball}}+\frac{1}{\rho_l\Delta})F(u,\frac{du}{dt})+\frac{Y}{\rho_l\Delta^4}[y_{k-2}\cdot I_{k-1}+y_{k-1}\cdot(-2I_{k-1}-2I_k)$$

$$+y_{k+1}(-2I_k-2I_{k+1})+y_{k+2}\cdot I_{k+1} \qquad (4)$$

联立方程（3）和（4）求解。由图3， 假设作用力与压缩量的关系曲线的形式，假设压缩段为直线，松弛段为：

$$Force=10\times(Compression/1.5)^2$$

将方程（3）和（4）联写成矩阵形式

从而转化为常微分方程组的初值问题，可利用 **Matlab** 的相关函数求解

$$\begin{bmatrix} \frac{d^2y_{-2}}{dt^2} \\ \frac{d^2y_{-1}}{dt^2} \\ \frac{d^2y_0}{dt^2} \\ \frac{d^2y_1}{dt^2} \\ \vdots \\ \frac{d^2y_k}{dt^2} \\ \vdots \\ \frac{d^2y_{N+2}}{dt^2} \\ \frac{d^2u}{dt^2} \end{bmatrix}=\frac{Y}{\rho_l\Delta^4}\cdot T_{(N+6)\times(N+6)}\cdot\begin{bmatrix} y_{-2} \\ y_{-1} \\ y_0 \\ y_1 \\ \vdots \\ y_{k-2} \\ y_{k-1} \\ y_k \\ y_{k+1} \\ y_{k+2} \\ \vdots \\ y_{N+2} \\ F(t) \end{bmatrix}$$

需注意的是，上式左端未知量为加速度项，右端未知量为位移项，在球棒作用期间，位移的变化远远快于加速度的变化，因此这是一个刚性问题（ 上式中还应将 $F(t)$ 表示成 $u(t)$ 的函数 ）

其中，

$$T=\begin{pmatrix} 0 & \cdots & & & & & & & 0 \\ 0 & \cdots & & & & & & & 0 \\ 0 & \cdots & & & & & & & 0 \\ 0 & \cdots & I_{k-1} & (-2I_{k-1}-2I_k) & (I_{k-1}+4I_k+I_{k+1}) & (-2I_k-2I_{k+1}) & I_{k+1} & \cdots & \dfrac{\Delta^3}{Y} \\ 0 & \cdots & I_{k-1} & (-2I_{k-1}-2I_k) & (I_{k-1}+4I_k+I_{k+1}) & (-2I_k-2I_{k+1}) & I_{k+1} & \cdots & \dfrac{1}{m_{ball}}+\dfrac{1}{\rho_l\cdot\Delta} \end{pmatrix}.$$

写成便于用 **ode** 求解的形式是：

$$P=T'\cdot Q$$

其中，

$$P=[y'_{-2},\ y'_{-1},\ \cdots y'_{N+2},\ u',\ y''_{-2},\ y''_{-1},\ \cdots y''_{N+2},\ u'']^T{}_{2\cdot(N+6)}$$

$$T'=\begin{pmatrix} 0 & I \\ T & 0 \end{pmatrix}_{2(N+6)\times 2(N+6)}$$

$$Q=[y_{-2},\ y_{-1},\ \cdots y_{N+2},\ u,\ y'_{-2},\ y'_{-1},\ \cdots y'_{N+2},\ u']^T{}_{2\cdot(N+6)}$$

设球压缩到最大对应的时间为t_1，从时刻t_1到球离开棒的时间为t_2

则

$$F(u)=\begin{cases} -10u/1.5, & 0<t<t_1 \\ 10\times(u/1.5)^2, & t_1<t<t_2 \end{cases}$$

初始条件为:

$$y_{-2}=y_{-1}=y_0=y_1=\cdots y_{N+2}=u=0$$

$$y_i'=(i-1)\Delta\cdot w,\qquad i=1,2\cdots N$$

其中w为棒的角速度

这里由于t_1和t_2的值未知，而$t_1+t_2\approx 1.4ms$，只好对t_1和t_2的值进行假设。

Simulation and Analysis

Simulation Results

Our model's two main features are wave propagation in the bat and nonlinear compression/decompression of the ball. The latter is illustrated by the asymmetry of the plot in **Figure 4a**. This plot also reveals the time-scale of the collision: The ball leaves the bat 1.4ms after impact. During and after collision, shock waves propagate through the bat.
我们模型的两个主要特点就是同时考虑振动波在棒中的传播和球的压缩/扩张过程。后者在图 4a 中得到了说明。该图还表明了碰撞过程的时间尺度：棒球在碰撞发生后 1.4 毫秒离开棒球棒。碰撞期间和碰撞结束后，振动波在棒球棒中传播。

In this example, the bat struck 60cm from the handle. What does the collision look like at 10cm from the handle? **Figure 4b** shows the answer: The other end of the bat does not feel anything until about 0.4ms and does not feel significant forces until about 1.0ms. By the time that portion of the bat swings back (almost 2.0ms), the ball has already left contact with the bat. This illuminates an important point: We are concerned only with forces on the ball that act within the 1.4ms time-frame of the collision. Any waves taking longer to return to the impact location do not affect exit velocity.
在本例中，击球点离握柄处 60cm。在距离握柄 10cm 处的情况怎么样呢？图 4b 给出了答案：该处知道撞击发生后大约 0.4 毫秒才发生振动，直到大约 1.0 毫秒后才达到最大位移。到位移重新达到最小值时（撞击发生 2.0 毫秒之后），球已经飞离了棒。这也说明了一点：我们只关心在碰撞 1.4 毫秒之内的作用。在这一时间之外的任何振动波都不会影响球的最终速度。

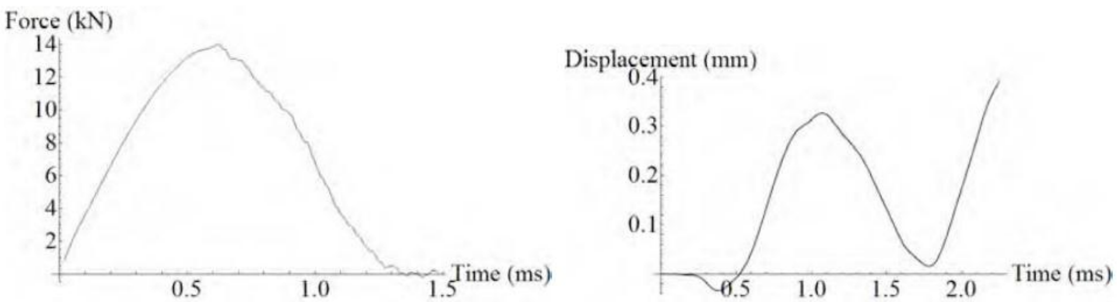


Figure 4.

a. Left: The force between the ball and the bat as a function of time; the impulse lasts 1.4 ms.
b. Right: The waveform of $y_{10}(t)$ when the bat is struck at 60 cm. The impulse reaches this chunk at around 0.4 ms but does not start moving significantly until later.

Having demonstrated the basic features of our model, we now replicate some of Cross's results but with baseball-like parameters. In **Figure 5a**, we show that the effects of fixed- vs.

free-boundary conditions are in agreement with Cross's model.
在说明了我们模型的基本特点之后，我们现在可以利用 **Cross** 的结果，只不过用的是棒球的参数。在图 5a 中，我们表明在固端边界和自由边界条件下的结果都和 **Cross** 的模型吻合得很好。

As we expected, fixed boundaries enhance the exit velocity and free boundaries reduce them. From this result, we see the effect of the shape of the bat. The handle does indeed act like a free boundary. The distance between the boundaries is too small to get a flat zone in the exit velocity vs. position curve. If we extend the barrel by 26cm. a flat zone develops (**Figure 5b**; notice the change in axes). Intuitively, this flat zone exists because the ball "sees" only the local geometry of the bat and the boundaries are too far away to have a substantial effect.
正如我们预料的，固端边界增大了球速，自由边界减小了球速。从这一结果我们可以看出棒的形状的影响。握柄端的作用相当于一个自由边界。两个边界之间的距离较短时，最佳击球区较窄，如果增大棒长 26cm，最佳击球区会变宽（图 5b）。直观的解释是，在两端边界距离较远时，棒球的作用仅限于冲击的局部区域，最佳击球区就更加平缓。

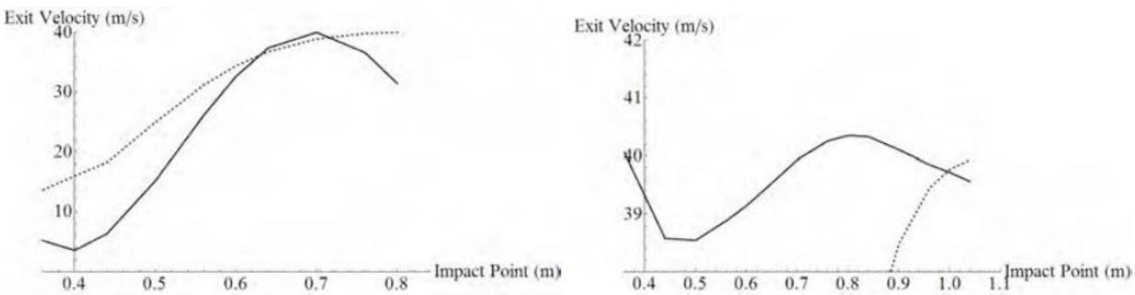


Figure 5.

a. Left: Exit velocity vs. impact position for a free boundary (solid line) and for a fixed boundary (dashed line), with barrel end fixed but handle end free, for an 84-cm bat
b. Right: The same graph for a free 110-cm bat.

From now on, we use an 84-cm bat free on both ends, where position zero denotes the handle end. In this base case, the sweet spot is at 70cm. We investigate the dependence of the exit speed on the initial angular velocity. According to rigid-body models, the sweet spot is exactly at the center of mass if the bat has no angular velocity. In **Figure 6**, we present the results of changing the angular velocity. Our results contrast greatly with the simple example presented earlier. While the angular-rotation effect is still there, the effective mass plays only a negligible role in determining the exit speed. In other words, the bat is not a rigid body because the entire bat does not react instantly. The dominating effect is from the boundaries: the end of the barrel and where the barrel tapers off. These free ends cause a significant drop in exit velocity. Increasing the angular velocity of the bat increases the exit velocity, in part just because the impact velocity is greater (by a factor of w_i times the distance from the center of mass of the bat).

现在我们采用两端自由的 84cm 长的棒，棒的握柄端的坐标记为 0。这时最佳击球点的位置为 70cm。我们研究了球速对初始角速度的依赖性。根据刚体模型，在棒球棒没有初始角速度时，最佳击球点就在棒的质心位置。在图 6 中，我们给出了改变角速度后的结果，该结果与之前的刚体模型相差很远。尽管棒的转动效应仍然存在，但是有效质量对最终球速的作用几乎可以忽略。换句话说，由于整个棒并不是同时反应，所以棒球棒不是刚体。起主要作用的是边界条件：棒的末端和棒的细端。这些自由边界使得球速显著减小。增大棒的角速度能增大球速，只是因为作用点位置的速度增大了（这个速度等于角速度 w_i 乘以作用点距离质心的距离 l ）。

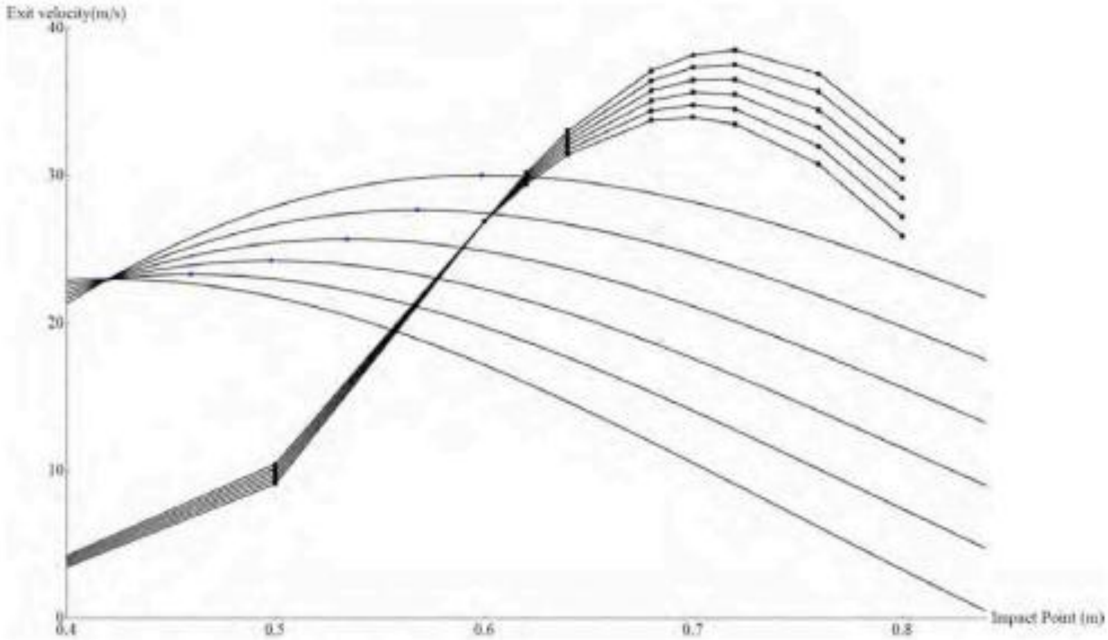


Figure 6. Exit velocity vs. impact position at various initial angular velocities of the bat. Our model predicts the solid curves, while the dashed lines represent the simple model. The dots are at the points where Brody's solution is maximized.

In **Figure 7a**, we show that near the sweet spot (at 0.7m), in increasing angular velocity actually decreases the excess exit velocity (relative to the impact velocity). We should expect this, since at higher impact velocity, more energy is lost to the ball's compression and decompression. To confirm this result, we also recreate the plot in **Figure 7b** but without the hysteresis curve—in which case this effect disappears. This example is one of the few places where the hysteresis curve makes a difference, confirming experimental evidence [Adair 1994; Nathan 2003] that the coefficient of restitution decreases with increasing impact velocity.

图 7a 表明，在最佳击球点（70cm）附近，增大角速度会降低相对球速（球速减去作用点速度）。这一点是可以预见的，因为更高的作用速度意味着更多的能量损失在棒球的压缩和扩张过程中。为了验证这一结论，我们还在图 7b 中给出了不考虑迟滞现象后的

同一曲线。这个例子只是少数考虑迟滞现象的例子之一，它验证了恢复系数随着角速度的增大而减小的实验结果[Adair 1994; Nathan 2003]。

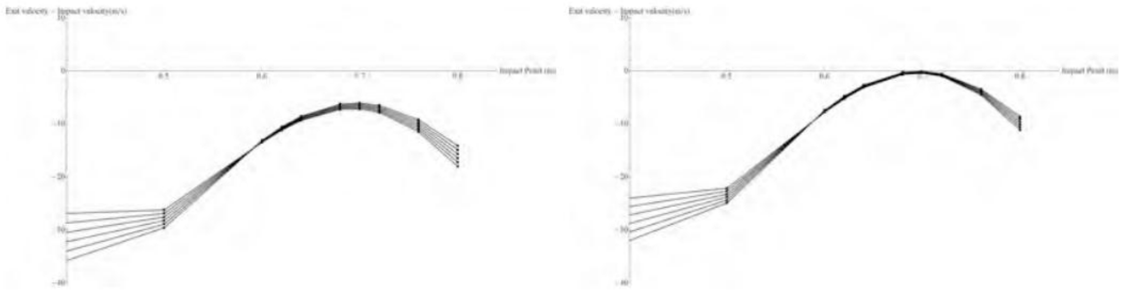


Figure 7.

Exit velocity minus impact velocity vs. impact position, for initial angular velocities of the bat.
a. Left: Near the center of mass, higher angular velocity gives higher excess exit velocity, but towards the sweet spot the lines cross and higher angular velocity gives lower excess exit velocity.
b. Right: The same plot without a hysteresis curve. The effect disappears.

The results for angular velocity contrast with the simple model. As evident from **Figure 8**, the rigid-body model greatly overestimates this effect for large angular velocities.
同样的角速度分析应用于刚体模型时给出了不同的结论。图 8 的结果显示刚体模型过高估计了角速度的这一作用（增大角速度会降低球速）。

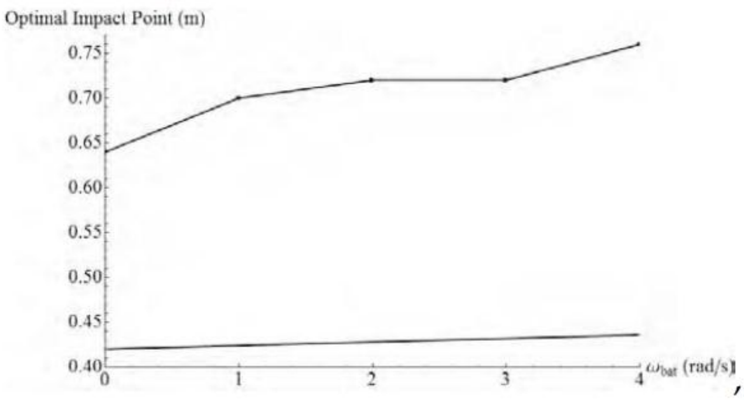


Figure 8. Optimal impact position vs. angular velocity. The straight line is the rigid-body prediction, while the points are our model's prediction.

Parameter Space Study

There are various adjustable parameters in our model. For the bat, we use density $\rho = 649 kg/m^3$ and Young's modulus $Y = 1.814 \times 10^{10} N/m^2$.

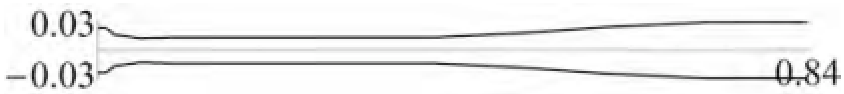


Figure 9. The profile of our bat.

These values, as well as our bat profile (**Figure 9**), were used by Nathan as typical values

for a wooden bat. While these numbers are in good agreement with other sources, we will see that these numbers are fairly special. As a result of our bat profile, the mass is 0.831 kg and the moment of inertia around the center of mass (at 59.3cm from the handle of our 84cm bat) is $0.039 kg \cdot m^2$. We let the 145-g ball's initial velocity be 40m/s, and set up our hysteresis curve so that the compression phase is linear with spring constant $7 \times 10^5 N/m$.
这些参数值和棒球棒形状（图 9）是 **Nathan** 采用的典型木质棒球棒的参数值。我们发现这些值很特殊。棒的质量为 0.831kg，绕质心（距离握柄端 59.3cm）的转动惯量为 $0.039 kg \cdot m^2$。球质量为 145g，速度为 40m/s，弹性常数为$7 \times 10^5 N/m$。

- We vary the density of the bat and see that the density value occupies a narrow region that gives peaked exit velocity curves (see **Figure 10**).
  棒的密度对最佳击球区的影响很小

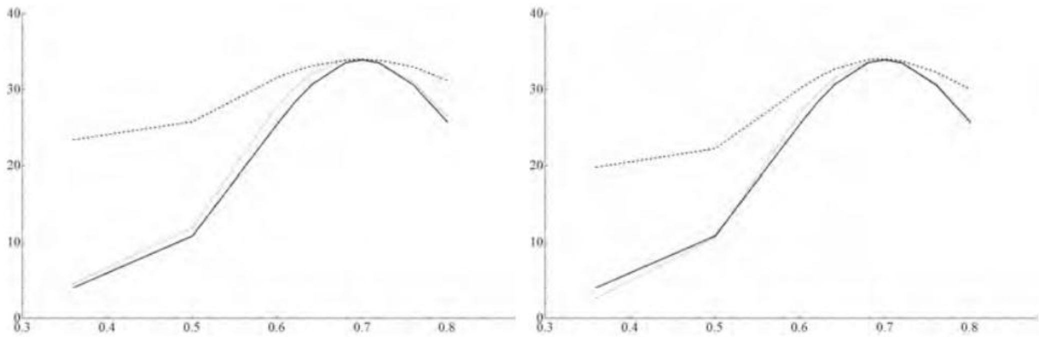


Figure 10.
Exit velocity vs. impact position for various densities. The solid line is the original $\rho = 649 \mathrm{kg/m^3}$.
a. Left: Dotted is $\rho = 700$, dashed is $\rho = 1000$. **b.** Right: Dotted is $\rho = 640$, dashed is $\rho = 500$.

- We also vary the Young's modulus and shape of the bat to similar effect (see **Figure 11**). The bat that varying any of Nathan's parameters makes the resulting exit velocity vs. location plot less peaked means that baseball bats are specially designed to have the shape shown in **Figure 9** (or else the parameters were picked in a special way).
  改变杨氏模量和棒的形状。

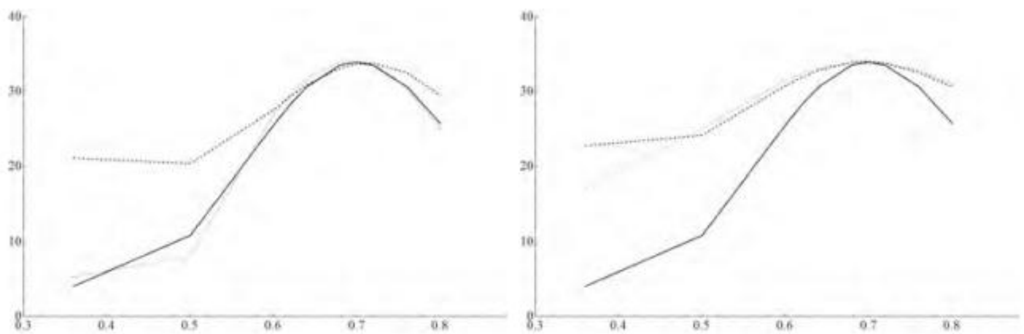


Figure 11.
a. Left: Varying the value of Y. Solid is $Y = 1.1814 \times 10^{10} \mathrm{N/m^2}$; dashed is 1.25 times as much, while dotted is 0.8 times.
b. Right: Varying the shape of the bat. Solid is the original shape; dashed has a thicker handle region, while dotted has a narrower handle region.

- Finally, we vary y, the speed of the ball (see **Figure 12**). The exit velocity simply scales with the input velocity, as expected.

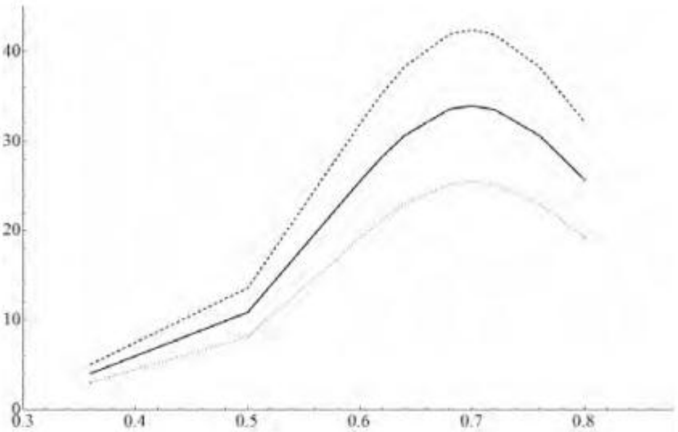


Figure 12. Varying the speed of the ball. Solid is the original 40 m/s, dashed is 50 m/s, while dotted is 30 m/s.

Corked Bat

We model a corked bat as a wood bat with the barrel hollowed out, leaving a shell 1cm or 1.5cm thick. The result is shown in **Figure 13a**. The exit velocities are higher, but this difference is too small to be taken seriously. This result agrees with the literature: The only advantages of a corked bat are the changes in mass and in moment of inertia.
对塞有软木的棒球棒进行建模分析后结果如图 13a 所示。最终球速是增加了，但是增加得非常小，几乎可以忽略不计。这也和文献中的结论相吻合：塞有软木的棒球棒的唯一好处就是改变了棒的质量和转动惯量。

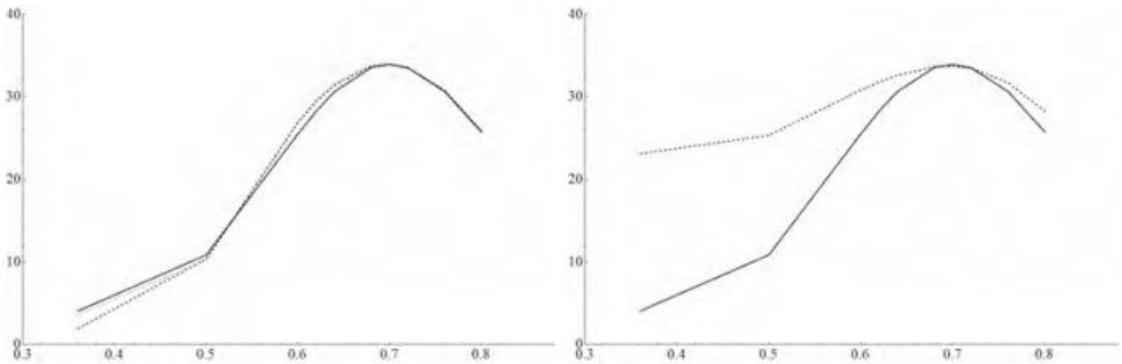


Figure 13. Exit velocity vs. distance of point of collision on the bat from the handle end.
a. Corked bat. **b.** Aluminum bat.

Aluminum Bat

We model an aluminum bat as a 0.3cm-thick shell with a density of $2700 kg/m^3$ and Young's modulus $6.9\times10^{10} N/m^2$. The aluminum bat performs much better than the wood bat (**Figure 13b**). It has the same sweet spot (70cm) and similar sweet-spot performance, but the exit velocity falls off more gradually away from the sweet spot.

铝质棒球棒由密度为$2700 kg/m^3$，杨氏模量为$6.9\times10^{10} N/m^2$，0.3cm 厚的铝质外壳构成。铝质棒球棒比木质棒球棒的表现好的多（图 13b）。它们的最佳击球点位置相同（70cm），但是对铝质棒球棒，最终球速在远离最佳击球点的位置下降的更慢。

To gain more insight, we animated the displacement of the bat vs. time; we present two frames of the animation inn **Figure 14**. The aluminum bat is displaced less (absorbing less energy). More importantly, in the right-hand diagram of **Figure 14**, the curve for the wood bat is still moving down and left, while the aluminum bat travels faster and returns in time to give energy back to the ball. By the time the pulse for the wood bat returns to the impact location, the ball has already left the bat.

为了更深入地分析，我们绘出了位移—时间动画；在图 14 中给出了两帧画面。铝质棒球棒的位移较小（吸收了更少的能量）。更重要的是，在图 14 的右图中，木质棒球棒的曲线还远离横轴时，铝质棒球棒的曲线已经回到原位置，从而把更多的能量传递给棒球。到振动波传回到木质棒球棒的击球位置时，球已经飞离棒球棒了。

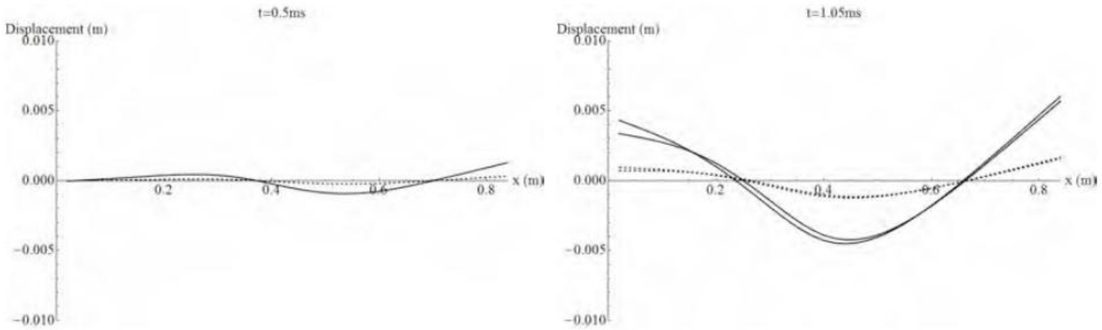


Figure 14. Plots of the displacement of an aluminum bat (dashed) and wood bat (solid) being hit by a ball 60 cm from the handle end. The diagram on the right shows two frames superimposed ($t = 1.05$ ms and $t = 1.10$ ms) so as to show the motion. The rigid translation and rotation has been removed from the diagrams.

In the literature, the performance of aluminum bats is often attributed to a “trampoline effect”, in which the bat compress on impact and then springs back before the end of the collision [Russell 2003]. This effect would improve aluminum-bat performance further. The trampoline effect involves exciting so-called “hoop modes”, modes with an azimuthal dependence, which our model cannot simulate directly. For an aluminum bat, one could conceivably use wave equations for a cylindrical sheet (adjusting for the changing radius) and then solve the resulting partial differential equations in three variables. Analysis of such a complex system of equations is beyond the scope of this paper.

在文献中，铝质棒球棒的表现经常被比喻为蹦床效应，即棒在撞击结束之前就回弹 [Russell 2003]。这一效应进一步加强了棒—球之间的作用。蹦床效应激发了所谓的“环形模态（hoop modes）”，这一模态与方位有关，我们的模型不能对此作出直接模拟。对于铝质棒球棒而言，可以列出其振动方程（考虑沿棒长变化的半径），再与球的运动方程联立解出三个未知量。这样一个复杂的方程组超出了本文的研究范围。

Instead, we artificially insert a hoop mode by hanging a mass from a spring at the spot of the bat where the ball hits. We expect the important modes to be the ones with periods near the collision time (1.4ms, corresponding to 714Hz). We find that this mode does affect the sweet spot, although the exact change does not seem to follow a simple relationship with the frequency. Our results, as shown in **Figure 15**, show that hoop modes around 700Hz do enhance the exit velocity. They not only make the sweet spot wider but also shift it slightly toward the barrel end of the bat.

相反，我们通过在击球点附加一个弹性质量体人为地引入环形模态（hoop modes）。我们预计该模态的周期与撞击时间相近（1.4ms，相应的频率为 714Hz）。我们发现这一模态确实影响了最佳击球点，但是这一影响似乎并不与频率呈简单的关系。我们的结果，如图 15 所示，环形模态（hoop modes）约 700Hz 左右，确实增大了最终球速。该模态不仅使最佳击球区更广，还使最佳击球点的位置向棒的末端偏离。

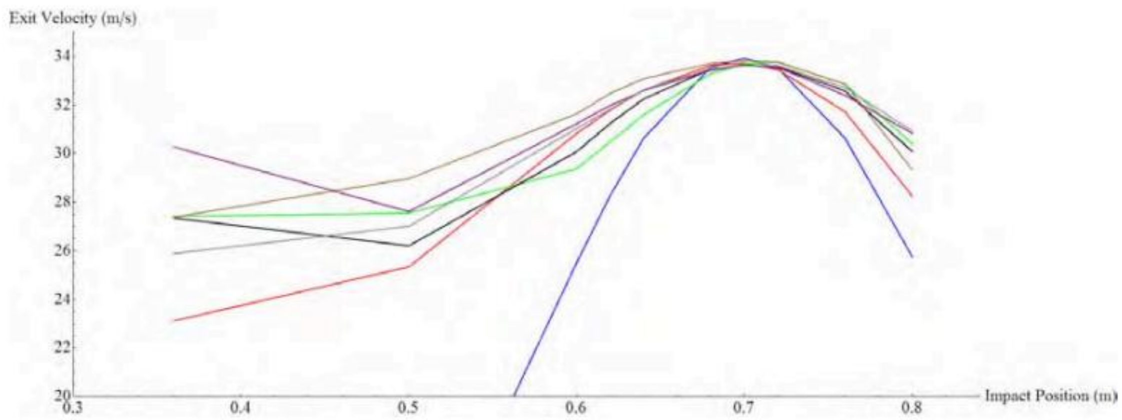


Figure 15. Exit velocity vs. impact position at different hoop frequencies. The lines from bottom to top at the left edge (color) are: (blue, starts off the chart) wood bat, (next higher, red) no hoop mode, (gray) 2000 Hz, (black) 500 Hz, (green) 300 Hz, (brown) 800 Hz, and (purple) 1250 Hz.

Conclusion

We model a ball-bat collision by using Euler-Bernoulli equations for the bat and hysteresis curves for the baseball. By doing so, we reconcile the literature by emphasizing the role of the time-scale of the collision and how the ball “sees” only a local region of the bat because of the finite speed of wave propagation. As a result, the sweet spot is farther out in our model than the rigid-body recoil model predicts.

我们对棒球棒采用了欧拉—伯努利方程，对棒球采用了迟滞曲线来建模。我们着重考虑了冲击的时间尺度和由于波速有限带来的冲击局部效应，这样就整合了文献中的优缺点。我们的结论是最佳击球点的位置比刚体模型认为的更远。

We vary the input parameters and show that the effects are in line with intuition and key results in previous experimental work.

Finally, we show that aluminum bats have wider sweet spot than wooden bats.

We offer several suggestions for improvements and extensions:

我们对模型的改进和扩展提出了几点建议：

- The ball is assumed to be non-rotating with head-on impact; rotating balls and off-center collisions excite torsional modes in the bat that we ignore and make the problem nonplanar.

  这里假设棒球不旋转，而且是正碰；考虑球的转动和偏心碰撞会激发棒扭转振动的相应模态，使问题不再是平面问题。
- We neglect shear forces in the bat.

  我们忽略了棒的剪切力**（即可以采用铁摩辛柯梁理论）**
- Our analysis of hoop modes is rather cursory.

  我们对的分析很粗糙。

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