

# MrMARTIAN: A Multi-resolution Mass Reconstruction Algorithm Combining Free-form and Analytic Components

SANGJUN CHA  <sup>1</sup> AND M. JAMES JEE  <sup>1,2</sup>

<sup>1</sup>Department of Astronomy, Yonsei University, 50 Yonsei-ro, Seoul 03722, Korea

<sup>2</sup>Department of Physics and Astronomy, University of California, Davis, One Shields Avenue, Davis, CA 95616, USA

(Received; Revised; Accepted)

Submitted to

## ABSTRACT

We present MrMARTIAN (Multi-resolution MAximum-entropy Reconstruction Technique Integrating Analytic Node), a new hybrid strong lensing (SL) modeling algorithm. By incorporating physically motivated analytic nodes into the free-form method MARS, MrMARTIAN enables stable and flexible mass reconstructions while mitigating oversmoothing in the inner mass profile. Its multi-resolution framework increases the degrees of freedom in regions with denser strong lensing constraints, thereby enhancing computational efficiency for a fixed number of free parameters. We evaluate the performance of MrMARTIAN using publicly available simulated SL data and find that it consistently outperforms MARS in recovering both mass and magnification. In particular, it delivers significantly more stable reconstructions when multiple images are sparsely distributed. Finally, we apply MrMARTIAN to the galaxy cluster MACS J0416.1–2403, incorporating two analytic nodes centered on the northeastern and southwestern BCGs. Our mass model, constrained by 412 multiple images, achieves an image-plane rms scatter of  $\sim 0''.11$ , the smallest to date for this dataset.

## 1. INTRODUCTION

As the most massive gravitationally bound structures in the universe, galaxy clusters have been the subject of extensive studies. Galaxy clusters enable the study of a wide range of topics, including the evolution of large-scale structures (e.g., Eckert et al. 2015; Kuchner et al. 2020; HyeongHan et al. 2024), cosmology (e.g., Acebron et al. 2017; Magaña et al. 2018; Caminha et al. 2022), and the properties of dark matter (DM) (e.g., Markevitch et al. 2004; Robertson et al. 2019; Ragagnin et al. 2024). Among various techniques, strong gravitational lensing (SL) is one of the most powerful methods for studying galaxy clusters.

Thanks to its unparalleled sensitivity to fine details in mass distributions, SL enables high-resolution and precise mass reconstructions around cluster cores and their inner profiles (e.g., Zitrin et al. 2013; Kawamata et al. 2016; Sebesta et al. 2016; Bergamini et al. 2021; Cha & Jee 2023). Also, massive SL galaxy clusters serve as cosmic telescopes, magnifying faint background objects

(e.g., Welch et al. 2022; Hsiao et al. 2023; Furtak et al. 2024). In addition, SL can be used to infer the Hubble constant  $H_0$  by measuring time delays between multiple images of time-varying sources such as quasars or supernovae (e.g., Kelly et al. 2015; Wong et al. 2020; Pascale et al. 2025).

A number of SL modeling algorithms exist. Broadly, there are two types of SL modeling methods: parametric and free-form. Parametric methods utilize analytic profiles such as NFW (Navarro et al. 1996), dPIE (Elásdóttir et al. 2007) or PIEMD (Kassiola & Kovner 1993). Parametric methods reconstruct mass distributions by superposing these analytic profiles, with their positions constrained by the locations of cluster member galaxies. In contrast, free-form methods do not assume specific profiles for mass modeling. The mass is reconstructed on a grid without incorporating light distributions as priors.

Both approaches have advantages and disadvantages. Parametric methods typically require fewer free parameters for modeling and can yield numerically stable results even with a small number of multiple images. However, the limited number of degrees of freedom restricts their ability to model complex and unexpected mass dis-

Corresponding author: M. James Jee  
 sang6199@yonsei.ac.kr, mkjee@yonsei.ac.kr

tributions. This limitation would be particularly concerning if the true mass distribution deviates from the light distribution. On the other hand, free-form methods offer excellent flexibility, but their high dimensionality often leads to degeneracies and unphysical features in the resulting mass models.

Recently, Cha & Jee (2022) introduced a new grid-based free-form MAximum-entropy ReconStruc-tion (MARS) algorithm, which employs maximum cross-entropy regularization. Thanks to this additional constraint, MARS can provide a quasi-unique solution and suppress spurious fluctuations (Cha & Jee 2023; Cha et al. 2024, 2025). However, despite these successes, MARS still has limitations. One such limitation is the smoothing of density peaks, which arises from both regularization and the finite grid size, especially in regions where SL constraints are sparse. Another limitation is the model’s dependence on priors in regions lacking SL constraints. Since free-form models are solely constrained by the multiple images, an asymmetric or sparse distribution of SL constraints can introduce biases in the reconstructed mass distribution. (Natarajan et al. 2024).

In this study, we propose a new hybrid Multi-resolution MAximum-entropy Reconstruction Technique Integrating Analytic Node (**MrMARTIAN**) to mitigate the aforementioned drawbacks of MARS. Although we choose a truncated NFW profile (Baltz et al. 2009) as an analytic node in this study, **MrMARTIAN** can use a wide range of analytic profiles. By combining both free-form and parametric approaches, **MrMARTIAN** can leverage the advantages of both methods. In addition, we implement a multi-resolution framework in lens modeling. By assigning a higher resolution grid (i.e., more degrees of freedom) to regions with denser SL constraints, the total number of free parameters can be reduced without a significant loss of reconstruction quality. This enhancement enables **MrMARTIAN** to achieve improved computational efficiency.

To evaluate the performance of the **MrMARTIAN** algorithm, we reconstruct lens models of simulated galaxy clusters. We use the publicly available synthetic clusters Ares and Hera from Meneghetti et al. (2017), which are designed to mimic galaxy clusters observed by the Hubble Frontier Fields (HFF) program (Coe et al. 2015; Lotz et al. 2017). We also apply **MrMARTIAN** to the real galaxy cluster MACS J0416.1-2403 ( $z = 0.396$ , hereafter MACSJ0416). As one of the targets of the HFF program, MACSJ0416 is well-known for its large number of multiple images (Richard et al. 2021; Cha & Jee 2023; Bergamini et al. 2023). Moreover, MACSJ0416 has also been observed by JWST as part of the Prime Ex-

tragalactic Areas for Reionization and Lensing Science (PEARLS; Windhorst et al. 2023) and the CANadian NIRISS Unbiased Cluster Survey (CANUCS; Willott et al. 2022) programs. Based on the HFF and JWST data, lensing studies of MACSJ0416 have been published by both collaborations (Diego et al. 2023, 2024; Rihtarič et al. 2025). Rihtarič et al. (2025) report 415 multiple images across the MACSJ0416 field of view (FOV), which is the largest SL dataset to date. This extensive dataset makes MACSJ0416 an ideal testbed for evaluating whether **MrMARTIAN** is applicable in the current JWST era when hundreds of multiple images become available.

This paper is organized as follows. In §2, we introduce the data used in the lensing analysis. We describe our new hybrid algorithm for mass reconstruction in §3. We present and discuss our results of MACSJ0416 in §4 and §5, respectively. We conclude in §6. Unless stated otherwise, this paper assumes a flat  $\Lambda$ CDM cosmology with the dimensionless Hubble parameter  $h = 0.7$  and the matter density parameter  $\Omega_M = 1 - \Omega_\Lambda = 0.3$ . The plate scale at the cluster redshift ( $z = 0.396$ ) is  $5.34 \text{ kpc arcsec}^{-1}$ . We make our lens model publicly available<sup>1</sup>.

## 2. DATA

### 2.1. Mock Cluster Data

We use publicly available SL data for two simulated galaxy clusters, Ares and Hera, from Meneghetti et al. (2017)<sup>2</sup>. Ares is generated using the semianalytic code MOKA (Giocoli et al. 2012) and is located at  $z = 0.5$ . Hera, at  $z = 0.507$ , is created from a zoom-in, high-resolution resimulation of a cosmological simulation presented in Planelles et al. (2014). Ares has two major halos: one to the northwest with mass  $M \sim 8.8 \times 10^{14} h^{-1} M_\odot$ , and the other to the southeast with mass  $M \sim 1.32 \times 10^{15} h^{-1} M_\odot$ , separated by a projected distance of  $\sim 400 h^{-1} \text{ kpc}$ . Hera also shows a bimodal structure, with a total mass of  $\sim 9.4 \times 10^{14} M_\odot$  and a projected separation of  $\sim 130 h^{-1} \text{ kpc}$ .

We use 242 multiple images from 85 sources to reconstruct Ares, and 65 images from 19 sources for Hera. In addition, we include analytic nodes in the lens models. Degeneracies can arise between the grid and the analytic node components. To reduce degeneracy, the number of node components should be kept minimal. Since our free-form reconstructions of both Ares and Hera reveal that each is composed of two cluster-scale mass struc-

<sup>1</sup> <https://github.com/sang6199/MACS-J0416.1-2403-result-maps>

<sup>2</sup> <http://pico.oabo.inaf.it/~massimo/Public/FF/index.html>

tures (Cha & Jee 2022), we employ two analytic nodes for each cluster.

### 2.2. MACSJ0416 Data

#### 2.2.1. JWST NIRCam Images

We use publicly available JWST NIRCam imaging data observed under the GTO programs 1176 (PI: Rogier A. Windhorst) and 1208 (PI: Chris J. Willott). We briefly outline our reduction process here. Data reduction followed the JWST pipeline (Bushouse et al. 2024), starting with the calibration reference file `jwst_1241.pmap`<sup>3</sup>. To correct snowball artifacts caused by cosmic ray hits, we applied the method described in Bagley et al. (2023). Additionally, we applied the techniques outlined in Bagley et al. (2023) to remove 1/f noise and wisp artifacts. A custom flat-field image, generated from approximately 200 science images, was used to account for detector response variations. The processed images were then combined using the Drizzle algorithm with a pixel scale of 0''.02/pixel, yielding high-resolution outputs for further analysis.

#### 2.2.2. SL Data

We adopt the multiple-image catalog published by the CANUCS team<sup>4</sup> (Rihtaršič et al. 2025), which compiles 415 multiple images in total, combining deep JWST/NIRCam and NIRISS observations with previous MUSE- and HST-based identifications. The catalog includes 38 newly identified images in 15 systems. While Rihtaršič et al. (2025) constructed their lensing model using only the gold sample, which consists of 303 multiple images with spectroscopic redshifts, we use both the gold and silver samples, the latter of which do not have spectroscopic redshifts. Among the silver images, we exclude system K81, as its inclusion introduces unphysical fluctuations when the model forces its multiple images to converge to a level of scatter comparable to that of the others. Figure 1 presents the distribution of the 412 multiple images used in this study, consisting of 347 gold and 65 silver images. The source redshifts of silver images are treated as free parameters and determined during the lens model optimization.

As with the simulated clusters, we prefer to keep the number of analytic nodes minimal to mitigate degeneracies between the grid and analytic components. Motivated by our free-form reconstruction in Cha & Jee (2023), we place two analytic nodes on the northern and southern BCGs of MACS J0416, as indicated by the yellow crosses in Figure 1.

<sup>3</sup> <https://jwst-crds.stsci.edu/>

<sup>4</sup> <https://niriss.github.io/lensing.html>

## 3. METHOD

### 3.1. Lensing Theory

Here, we briefly summarize a lensing theory. For more details, we refer readers to review papers (e.g., Bartelmann & Schneider 2001; Kochanek 2006; Kneib & Natarajan 2011; Hoekstra et al. 2013; Natarajan et al. 2024).

The lens equation, which describes the relation between the source position  $\beta$  and the observed image position  $\theta$  is computed by the following:

$$\beta = \theta - \alpha(\theta), \quad (1)$$

where  $\alpha$  indicates the deflection angle. The deflection angle can be obtained through two different approaches. One of them is the derivative of the lensing potential  $\varphi$ :

$$\alpha = \nabla \varphi. \quad (2)$$

Alternatively, the deflection angle can also be computed by convolving the convergence  $\kappa$ ,

$$\alpha(\theta) = \frac{1}{\pi} \int \kappa(\theta') \frac{\theta - \theta'}{|\theta - \theta'|^2} d\theta', \quad (3)$$

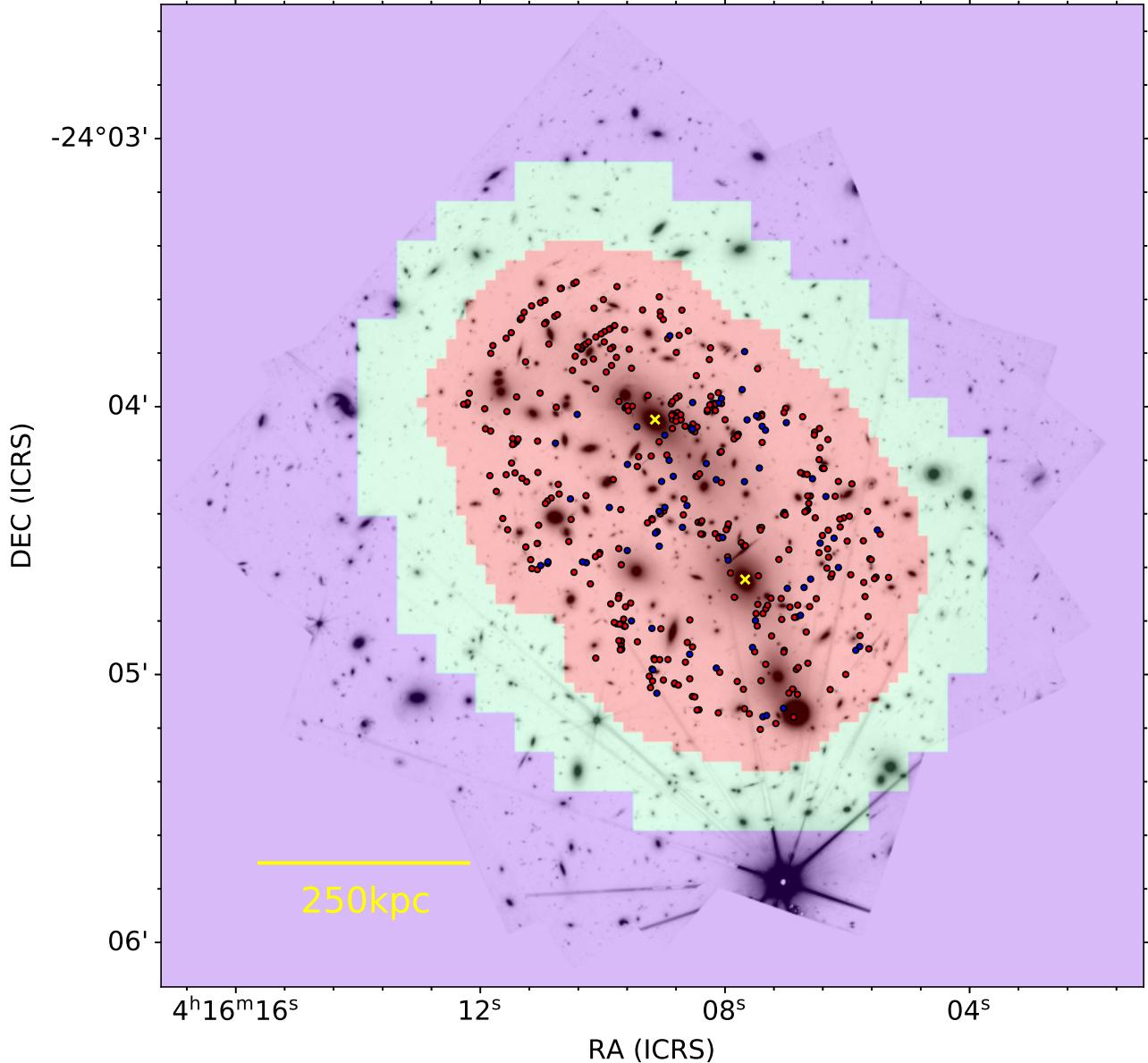
where convergence  $\kappa$  is the unitless surface mass density defined as:

$$\kappa = \frac{\Sigma}{\Sigma_c}, \quad \Sigma_c = \frac{c^2 D_s}{4\pi G D_d D_{ds}}. \quad (4)$$

$\Sigma(\Sigma_c)$  is the (critical) surface mass density.  $c$  and  $G$  are the speed of light and gravitational constant, respectively.  $D_{d(s)}$  indicates the angular diameter distance between the observer and lens (source), while  $D_{ds}$  denotes the angular diameter distance between the source and the lens.

### 3.2. Truncated Pseudo-Elliptical NFW Profile

MrMARTIAN employs an analytic node in the lens modeling. In this study, we use truncated NFW profiles. Although the standard NFW profile (Navarro et al. 1996) has been widely used in both WL and SL analysis, the divergent total mass of the NFW profile is physically unrealistic and may introduce biases in the modeling process (Oguri & Hamana 2011). Truncation of the analytic profile is usually used to describe tidal effects on galaxy-scale DM halos in dense environments such as galaxy clusters (e.g., Limousin et al. 2007, 2009; Baxter et al. 2017). Moreover, the truncated NFW profile ensures a finite total mass, addressing the divergent total mass of the standard NFW profile. In this work, we follow the formulae described in Baltz et al. (2009) to implement the smoothly truncated NFW profile.



**Figure 1.** Multiple-image distributions and resolution levels for the reconstruction of MACSJ0416. The red (blue) circles indicate the positions of gold (silver) images, while the yellow crosses represent the initial locations of analytic nodes. The red, green, and purple shaded regions illustrate the resolutions of  $1.^{\prime\prime}1/\text{pix}$ ,  $2.^{\prime\prime}2/\text{pix}$ , and  $8.^{\prime\prime}8/\text{pix}$ , respectively. The mosaic image is created using the F200W filter.

In addition to truncation, we also consider the ellipticity of the analytic nodes. However, since there is no general analytic expression for an elliptical NFW profile, we adopt a pseudo-elliptical approach. The pseudo-elliptical NFW profile has an elliptical lensing potential, not an elliptical convergence  $\kappa$ . In this paper, we adopt the description in Golse & Kneib (2002) and Dímet-Montoya et al. (2012). A brief description of the truncated NFW profile and the pseudo-elliptical approach is summarized in Appendix A. For more details, we refer

readers to the following papers: Golse & Kneib (2002); Baltz et al. (2009); Dímet-Montoya et al. (2012).

### 3.3. MrMARTIAN: Hybrid Lens Modeling Algorithm

MrMARTIAN reconstructs mass distributions by combining both grid cells and analytic nodes. The convergence  $\kappa$  and deflection angle  $\alpha$  are computed as the sum of

**Table 1.** Prior ranges of the free parameters.

Parameter	Prior	Unit
$\kappa$ (grid cell)	[-10, 10]	-
Halo position ( $x, y$ )	[-15, 15]	arcsecond
Concentration ( $c$ )	[0.01, 30]	-
Scale radius ( $r_s$ )	[1, 1000]	kpc
Truncation ratio ( $\tau$ )	[0.01, 30]	-
Rotation angle ( $i$ )	$[-\infty, \infty]$	degree
Elliptical parameter ( $\epsilon$ )	[0, 0.25]	-
Model redshift ( $z_{\text{model}}$ )	[0.496, 15]	-

NOTE—<sup>1</sup>The coordinates are defined with respect to the position of each BCG.

contributions from grid cells and analytic nodes:

$$\begin{aligned}\kappa(\boldsymbol{\theta}) &= \kappa_{\text{grid}}(\boldsymbol{\theta}) + \sum_{n=1}^N \kappa_{\text{prof},n}(\boldsymbol{\theta}), \\ \boldsymbol{\alpha}(\boldsymbol{\theta}) &= \boldsymbol{\alpha}_{\text{grid}}(\boldsymbol{\theta}) + \sum_{n=1}^N \boldsymbol{\alpha}_{\text{prof},n}(\boldsymbol{\theta}),\end{aligned}\quad (5)$$

where the subscripts “grid” and “prof” indicate values of  $\kappa$  and  $\boldsymbol{\alpha}$  derived from grid cells and analytic profiles, respectively.  $N$  is the total number of analytic profiles and  $n$  represents the  $n^{\text{th}}$  profile. Each analytical node is described by seven free parameters: halo position ( $x, y$ ), concentration  $c$ , scale radius  $r_s$ , truncation ratio  $\tau$ , elliptical parameter  $\epsilon$ , and rotation angle  $i$  measured counterclockwise from the x-axis. We set a flat prior for each parameter, as summarized in Table 1.

Unlike typical parametric methods, MrMARTIAN optimizes halo parameters purely based on strong lensing data, without relying on optical priors such as scaling relations. As noted in §2, degeneracies can arise between the grid and analytic node components because their individual contributions to the total deflection cannot be uniquely separated. Therefore, the risk of degeneracies would increase if we apply a large number of analytic halos. To reduce the risk of degeneracy, we keep the number of analytic nodes minimal.

The MrMARTIAN algorithm minimizes the following target function:

$$f = \chi_{\text{SL}}^2 + rR, \quad (6)$$

where  $\chi_{\text{SL}}^2$  is the  $\chi^2$  for SL and  $R$  is the regularization term. The parameter  $r$  determines the relative importance between  $\chi_{\text{SL}}^2$  and  $R$ .  $\chi_{\text{SL}}^2$  is computed as

$$\chi_{\text{SL}}^2 = \sum_{i=1}^I \sum_{j=1}^J \frac{(\boldsymbol{\theta}_{i,j} - \boldsymbol{\alpha}_{i,j}(z) - \boldsymbol{\beta}_i)^2}{\sigma_i^2}, \quad (7)$$

where

$$\boldsymbol{\beta}_i = \frac{1}{J} \sum_{j=1}^J (\boldsymbol{\theta}_{i,j} - \boldsymbol{\alpha}_{i,j}(z)). \quad (8)$$

$I$  and  $J$  are the total number of SL image systems and the number of lensed images from each system, respectively. Here,  $\sigma$  does not represent the positional uncertainty, but rather the amplitude of the source-plane scatter, evaluated using the multiple image positions and the virtual knots (see Figure 1 in Cha & Jee 2022).

For regularization, MrMARTIAN improves upon the MARS regularization method by allowing negative  $\kappa$  values, following the scheme of Hobson & Lasenby (1998). This additional flexibility enables MrMARTIAN to preserve the overall mass density profile set by the analytic components while accommodating local perturbations that are either lower or higher than those predicted by the profile. Specifically, the  $\kappa$  value in each grid cell is decomposed into positive ( $\kappa_p$ ) and negative ( $-\kappa_n$ ) components, such that  $\kappa_{\text{grid}} = \kappa_p + (-\kappa_n)$ , and the regularization term is computed as follows:

$$R = \sum \left( p_p + p_n - \psi + \kappa_{\text{grid}} \ln \frac{\psi + \kappa_{\text{grid}}}{2p_p} \right), \quad (9)$$

where  $p_p$  ( $p_n$ ) indicates the prior of  $\kappa_p$  ( $\kappa_n$ ). Similar to MARS, MrMARTIAN updates prior iteratively. At each minimization epoch, the prior is computed by smoothing the  $\kappa_{\text{grid}}$  from the previous epoch using a Gaussian kernel.  $\psi$  is defined as:

$$\psi = (\kappa_{\text{grid}}^2 + 4p_p p_n)^{1/2}. \quad (10)$$

For multiple images without spectroscopic redshifts, we consider their redshifts as free parameters in the modeling process. As in Cha & Jee (2023) and Cha et al. (2024), we set a flat prior of  $z_{\text{model}} = [z_{\text{cluster}} + 0.1, 15]$ , where  $z_{\text{cluster}} = 0.396$  (see Table 1). Unless stated otherwise, the presented  $\kappa$  field is scaled to  $D_{ds}/D_s = 1$ , whose corresponding critical surface mass density (i.e.,  $\kappa = 1$ ) is  $1.51 \times 10^9 M_{\odot} \text{kpc}^{-2}$ .

### 3.4. Multi-resolution Approach

MrMARTIAN employs a multi-resolution approach for lens modeling. By assigning a higher degree of freedom to regions with SL constraints, the number of free parameters can be reduced without significant loss of information. We apply a higher-resolution grid in regions strongly constrained by multiple images, and a lower-resolution grid where the constraints are weaker. At each iteration, the MrMARTIAN algorithm oversamples low-resolution  $\kappa$  regions to the highest resolution when calculating deflection angles via convolution. It then

**Table 2.** Sampling ranges for the initial conditions.

Parameter	Prior	Unit
$\kappa$ (grid cell)	[-0.5, 0.5]	-
Halo position <sup>1</sup> ( $x, y$ )	[-10, 10]	arcsecond
Concentration ( $c$ )	[0.01, 3]	-
Scale radius ( $r_s$ )	[1, 100]	kpc
Truncation ratio ( $\tau$ )	[0.01, 30]	-
Rotation angle ( $i$ )	[0, 180]	degree
Elliptical parameter ( $\epsilon$ )	[0, 0.25]	-
Model redshift ( $z_{model}$ )	[0.496, 15]	-

NOTE—<sup>1</sup>The coordinates are defined with respect to the position of each BCG.

computes the model evaluation using Function 6 and updates parameters accordingly for the next iteration.

We apply three resolution levels to Ares and Hera. To avoid artifacts caused by large differences in pixel size, especially at the resolution boundaries, we include an intermediate level between the highest and lowest resolutions. The same three-level resolution scheme is applied to the modeling of MACS J0416, using pixel scales of 1''.1, 2''.2, and 8''.8. Figure 1 displays the regions corresponding to each resolution level. As a result, the number of free parameters for the  $\kappa$  grid is reduced to 9928, which is  $\sim 13\%$  of the 78400 parameters required without the multi-resolution approach. The total number of free parameters used in the reconstruction is 9967:  $N_{\text{grid}} = 9928$ ,  $N_{\text{prof}} = 7 \times 2 = 14$ , and  $N_{\text{redshift}} = 25$ .

### 3.5. Uncertainty Estimation Method

The MARS algorithm estimates uncertainties by calculating the Hessian matrix of the target function  $f$  in Equation 6 (see Section 2.2 in Cha & Jee 2022 for more details). While the Hessian-based approach is easy to implement and widely used (e.g., Bridle et al. 1998; Jee et al. 2007; Cha & Jee 2022), it has some limitations. One limitation is the assumption that the posterior distributions of the free parameters follow a normal distribution. This assumption is generally invalid, especially for halo parameters (e.g., Jullo et al. 2007; Jullo & Kneib 2009; Finner et al. 2017; Bergamini et al. 2019; Kim et al. 2021).

In this study, we estimate uncertainties by generating an ensemble of possible solutions, a technique commonly used in free-form methods (e.g., Diego et al. 2007; Lam et al. 2014; Williams & Liesenborgs 2019; Ghosh et al. 2021; Perera et al. 2025). Although ensemble-based approaches are more flexible and do not rely on

Gaussian assumptions, they are computationally expensive. This made them an inefficient option for MARS in its earlier versions. However, with the current improvement—most notably the significant reduction in the number of free parameters—such an approach has now become feasible within the MrMARTIAN framework. We reconstruct 200 realizations from randomly generated initial conditions, sampled within the ranges listed in Table 2.

## 4. RESULT

### 4.1. Mock Cluster Result

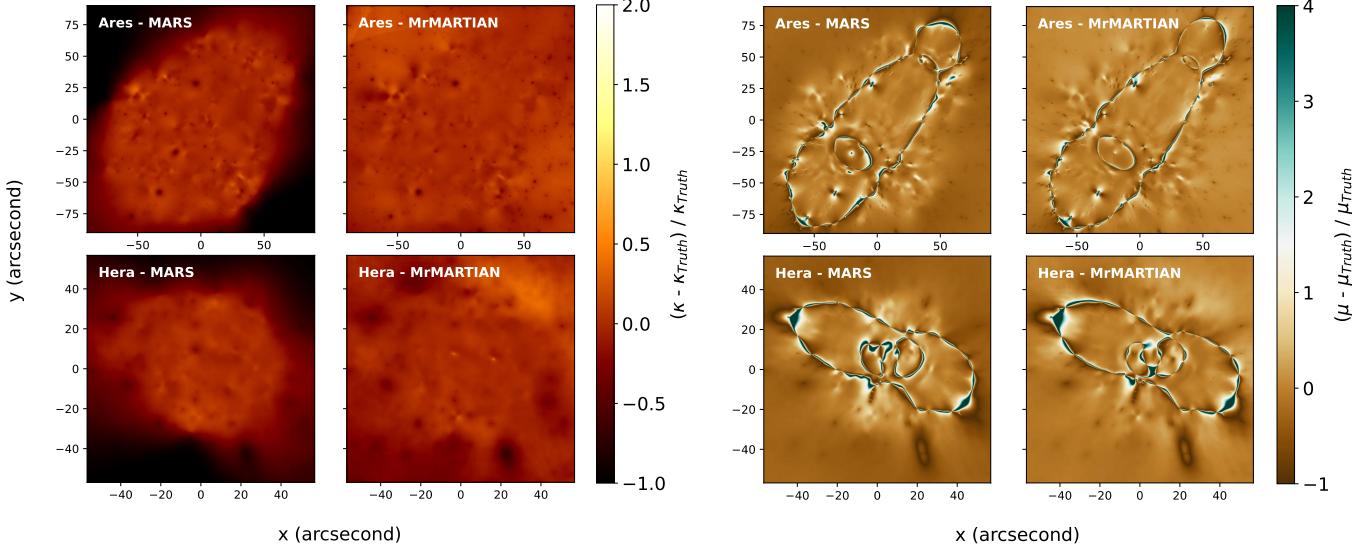
To investigate the effect of employing analytic nodes for the reconstruction, we compare the lens models of Ares and Hera using MrMARTIAN with the ground truth maps, as well as with results by MARS. In this paper, we adopt a  $\kappa = 0$  grid as the initial condition<sup>5</sup>. We display the central regions of  $180'' \times 180''$  for Ares and  $110'' \times 110''$  for Hera. The maps are resampled at  $0''.14/\text{pixel}$  and  $0''.11/\text{pixel}$  for Ares and Hera, corresponding to the resolution of the publicly available data.

#### 4.1.1. Lens Model Comparison

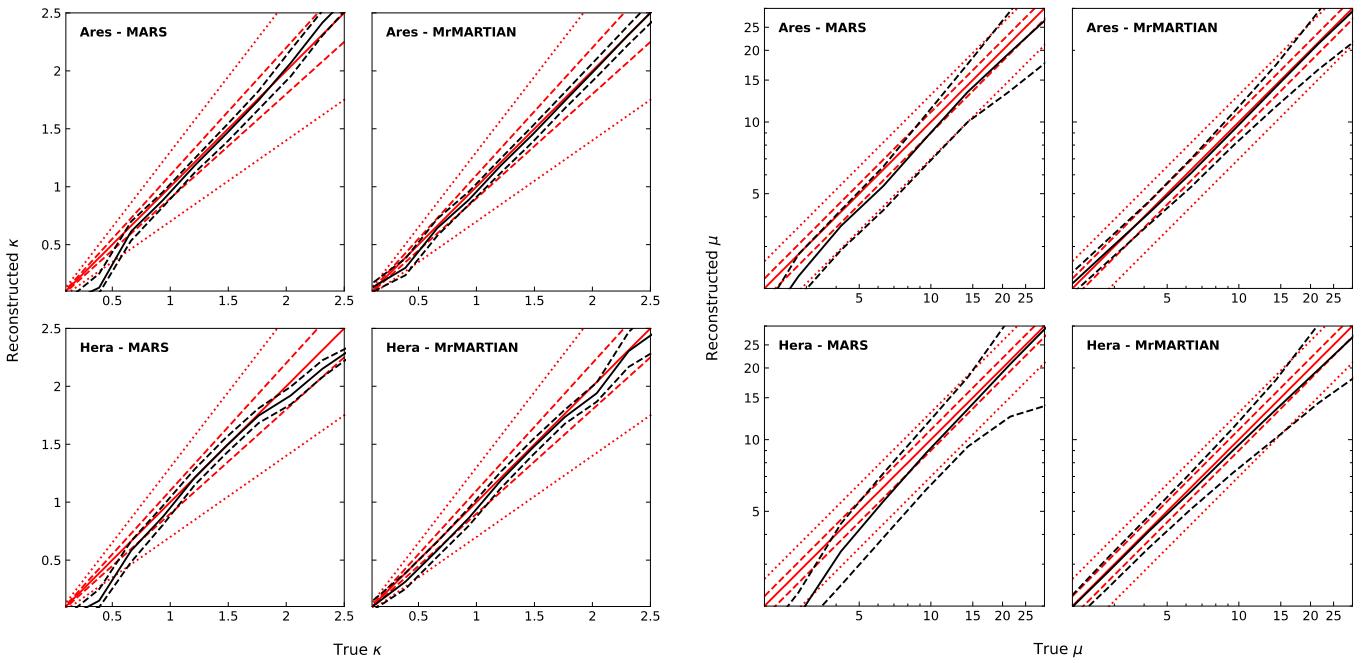
Figure 2 compares the relative differences between the true and reconstructed mass and magnification maps for both synthetic clusters using the full SL catalog. While the reconstructed mass maps from MARS and MrMARTIAN show small differences within the SL regime, significant discrepancies are observed in the cluster outskirts. Since there are no SL multiple images, the convergence  $\kappa$  from MARS mainly relies on its initial prior. By comparison, MrMARTIAN recovers the convergence well in both the outskirts and central regions, thanks to the inclusion of analytic halo profiles. Similar trends appear in the magnification maps, where MrMARTIAN more accurately reproduces the true values than MARS in outer regions, despite smaller discrepancies than in the convergence maps.

In addition to the qualitative comparison above, we present quantitative assessments by comparing the convergence and magnification values in the same regions (Figure 3). For convergence, the MARS result is clearly lower than the true values in the low- $\kappa$  ( $\lesssim 0.6$ ) regime, mainly due to the underestimation observed in the outskirts of the field (Figure 2). In the high- $\kappa$  ( $\gtrsim 2$ ) regime, the MARS result underestimates  $\kappa$  for Hera, likely caused by regularization-induced oversmoothing of the mass peaks. This bias is absent in the Ares result, possibly owing to denser SL constraints near the mass peaks

<sup>5</sup> We note that Cha & Jee (2022) uses different initial conditions used in this work: a field with  $\kappa = 0.5$ .



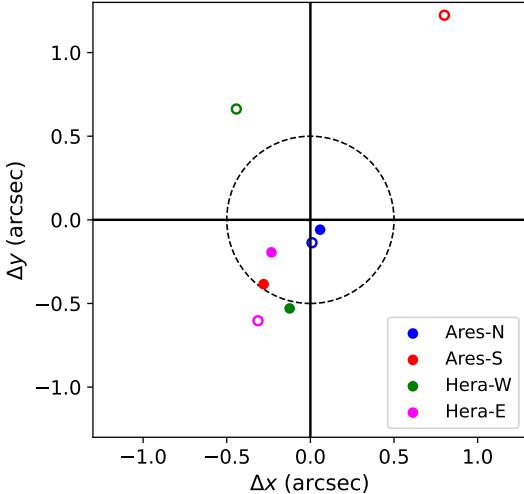
**Figure 2.** Relative difference between the true and reconstructed maps. The panels on the left (right) side indicate the residual maps of the convergence (magnification).



**Figure 3.** Convergence (left) and magnification (right) correlation between the true and reconstructed lens models. The black solid (dashed) lines correspond to the median (25th and 75th percentiles). The red solid lines show the perfect correlation. The red dashed (dotted) lines present indicate  $\pm 10\%$  ( $\pm 30\%$ ) deviations from the true values.

(see Figures 2 and 3 of Cha & Jee (2022)). Notably, MrMARTIAN yields more accurate results across the entire  $\kappa$  range, with median values deviating from the truth by  $\lesssim 10\%$ . The difference in magnification between the MARS and MrMARTIAN results is less pronounced than in convergence; nevertheless, the MrMARTIAN results remain superior in both precision and accuracy.

Another metric for assessing the quality of mass reconstruction is the accuracy of mass peak positions. We find that MrMARTIAN shows improved performance in this aspect. Figure 4 compares the offsets between the reconstructed and true mass peak positions for Ares and Hera. Of the four mass peaks, MrMARTIAN places three within  $0.^{\circ}5$  of the true positions and the remaining one within  $0.^{\circ}6$ , whereas MARS places only one within  $0.^{\circ}5$ , with the



**Figure 4.** Mass peak offsets between the true and reconstructed mass models. The filled (empty) circles indicate the offsets derived from MrMARTIAN (MARS). The black dashed line shows the  $0.^{\circ}5$  offsets from the true mass peaks.

largest offset reaching  $\gtrsim 1''$ . Although the comparison is based on a small sample of two, we expect the trend to hold more generally, as MrMARTIAN benefits from the assumption that highly concentrated mass peaks are located near the BCGs.

#### 4.1.2. Effect of Multiple-Image Sparsity

One advantage of integrating analytic nodes is the ability to achieve stable reconstructions even with a small number of multiple images. In general, pure free-form methods are more susceptible to the sparsity of multiple images than parametric ones, as the system is more underdetermined relative to the number of free parameters. We anticipate that the hybrid approach implemented in MrMARTIAN effectively mitigates this limitation.

To test this, we randomly select 54 (35) SL multiple images from the full catalog of 242 (65) images from Ares (Hera). Figure 5 shows the resulting mass reconstructions. While sparsity degrades the reconstruction quality in both Ares and Hera, the MARS results are more severely affected, exhibiting overly smoothed mass peaks. In some cases, the weak mass peaks are difficult to identify altogether.

## 4.2. MACSJ0416 Result

MrMARTIAN yields quasi-unique solutions across different initial conditions; however, some differences in fine details are inevitable. In this section, we present the result obtained using the following initial conditions, which we refer to as the reference model hereafter:  $(\kappa, c, r_s, \tau, i, \epsilon, z_{model}) = (0, 0.01, 1, 0.01, 0, 0, 3)$ ,

with the two analytic halos at the positions of BCG-N and BCG-S. As with both Ares and Hera, the multi-resolution approach is also employed for the reconstruction of MACSJ0416. For the final presentation, the entire mass reconstruction result is resampled using bicubic interpolation at  $0.^{\circ}02$  per pixel, matching the pixel scale of our JWST mosaic image.

### 4.2.1. Mass Model

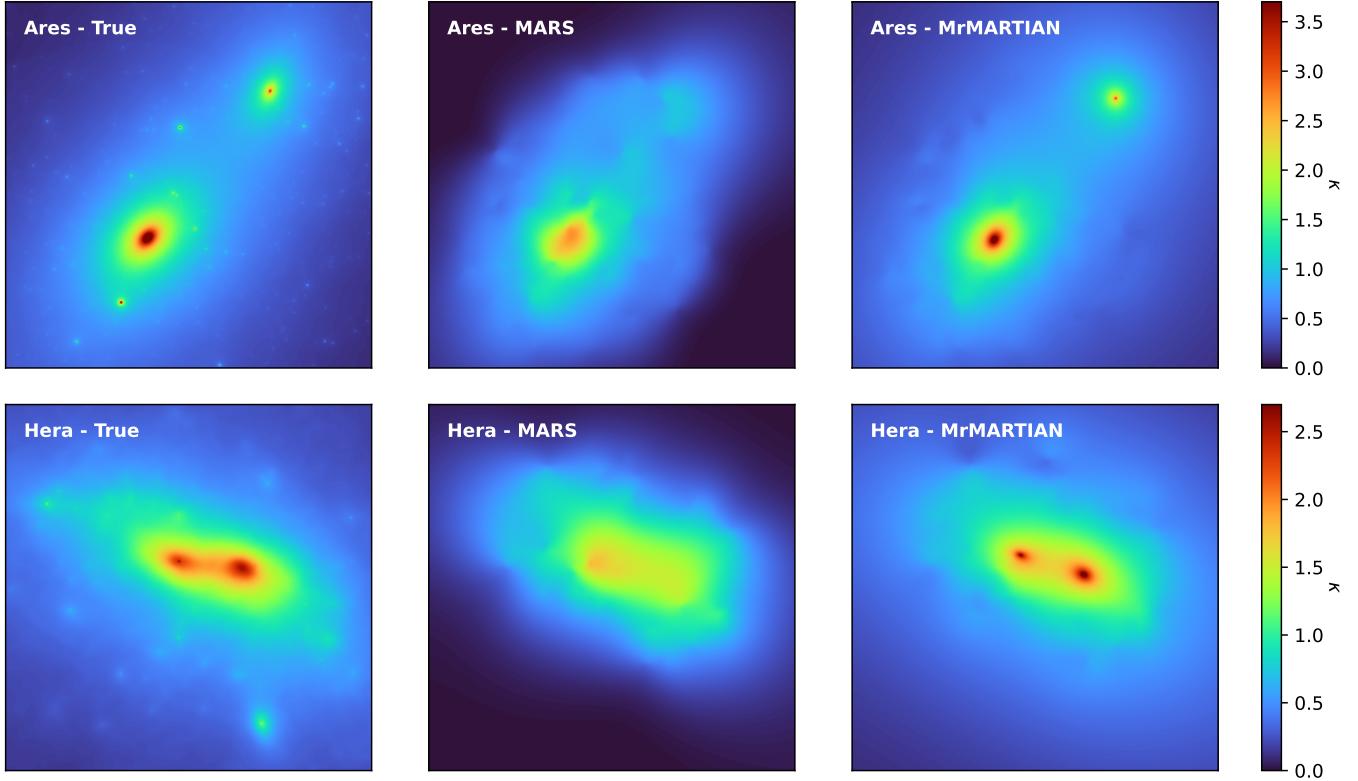
Figure 6 shows the reconstructed mass map of MACSJ0416. Our mass model reveals the elongated mass along the northeast-southwest direction, consistent with the result from MARS (see Figure 4 in Cha & Jee 2023) and other previous studies (e.g., Bergamini et al. 2023; Diego et al. 2024; Rihtarič et al. 2025; Perera et al. 2025). To assess the performance of our mass map, we evaluate the root-mean-square (rms) of the position differences between the observed and predicted multiple images:

$$\Delta_{\text{rms}} = \sqrt{\frac{1}{M} \sum_{m=1}^M |\theta_{\text{truth},m} - \theta_{\text{model},m}|^2}, \quad (11)$$

where  $M$  indicates the number of multiple images, while  $\theta_{\text{model},m}$  and  $\theta_{\text{truth},m}$  represent the locations of the predicted and observed multiple images for the  $m^{\text{th}}$  image, respectively.

The rms value of our mass model is  $0.^{\circ}11$ ,  $\sim 25\%$  larger than the rms value from MARS (Cha & Jee 2023,  $\Delta_{\text{rms}} = 0.^{\circ}084$ ). This increase is not surprising, as we use approximately twice as many multiple images (Cha & Jee 2023, 236 images), and some of the newly added images do not have secure redshift information. However, we note that our rms is still smaller than the values from other results using both parametric and free-form methods. Using a parametric method, Bergamini et al. (2023) achieved an rms value of  $\Delta_{\text{rms}} = 0.^{\circ}43$  based on 237 multiple images from HST observations. A slightly higher rms of  $0.^{\circ}53$  was reported by Rihtarič et al. (2025), who used 303 multiple images from JWST observations. Perera et al. (2025), using the free-form algorithm GRALE (Liesenborgs et al. 2006, 2007), report an rms scatter of  $0.^{\circ}19$  based on 237 spectroscopically confirmed multiple images, without incorporating JWST observations. The small rms value ( $0.^{\circ}11$ ) in our mass model demonstrates the ability of MrMARTIAN to reconstruct mass density profiles using analytic nodes while preserving the flexibility of MARS.

Figure 7 compares the cumulative mass and radial  $\kappa$  profiles for BCG-N and BCG-S among three different mass models: MARS, MrMARTIAN, and the model from Rihtarič et al. (2025), who make their results publicly



**Figure 5.** Reconstructed mass of Ares and Hera with sparsely distributed SL multiple images. We randomly select 54 (35) SL multiple images from 242 (65) multiple images of Ares (Hera). Left: true mass maps. Middle: mass maps reconstructed using MARS. Right: reconstructed mass maps from MrMARTIAN.

available. Rihtaršič et al. (2025) reconstructs the mass model of MACSJ0416 using `lenstool` (Jullo et al. 2007), based on 303 spectroscopically confirmed multiple images identified in JWST observations.

The radial profiles from MARS are consistent with MrMARTIAN within the SL regime ( $\lesssim 150$  kpc). Beyond this radius, as the density of multiple images decreases, the convergence  $\kappa$  drops rapidly. As discussed in §4.1.1, this decline results from the lack of SL constraints, causing MARS to default to the flat initial prior. When compared with the lens model from Rihtaršič et al. (2025), MrMARTIAN yields slightly higher  $\kappa$  values in both the BCG-N and BCG-S regions. Given the differences in the selection of multiple images and the mass modeling algorithm, this mass offset is small, corresponding to  $\sim 3\%$  discrepancy in the projected mass at  $r = 200$  kpc.

#### 4.2.2. Magnification

In Figure 8, we present the magnification map derived from our lens model. It features three major loops: two inner loops surrounding each BCG and one outer loop. While the overall morphology agrees with results in the literature, some details differ. When compared with the result from Rihtaršič et al. (2025) (see right panel), our

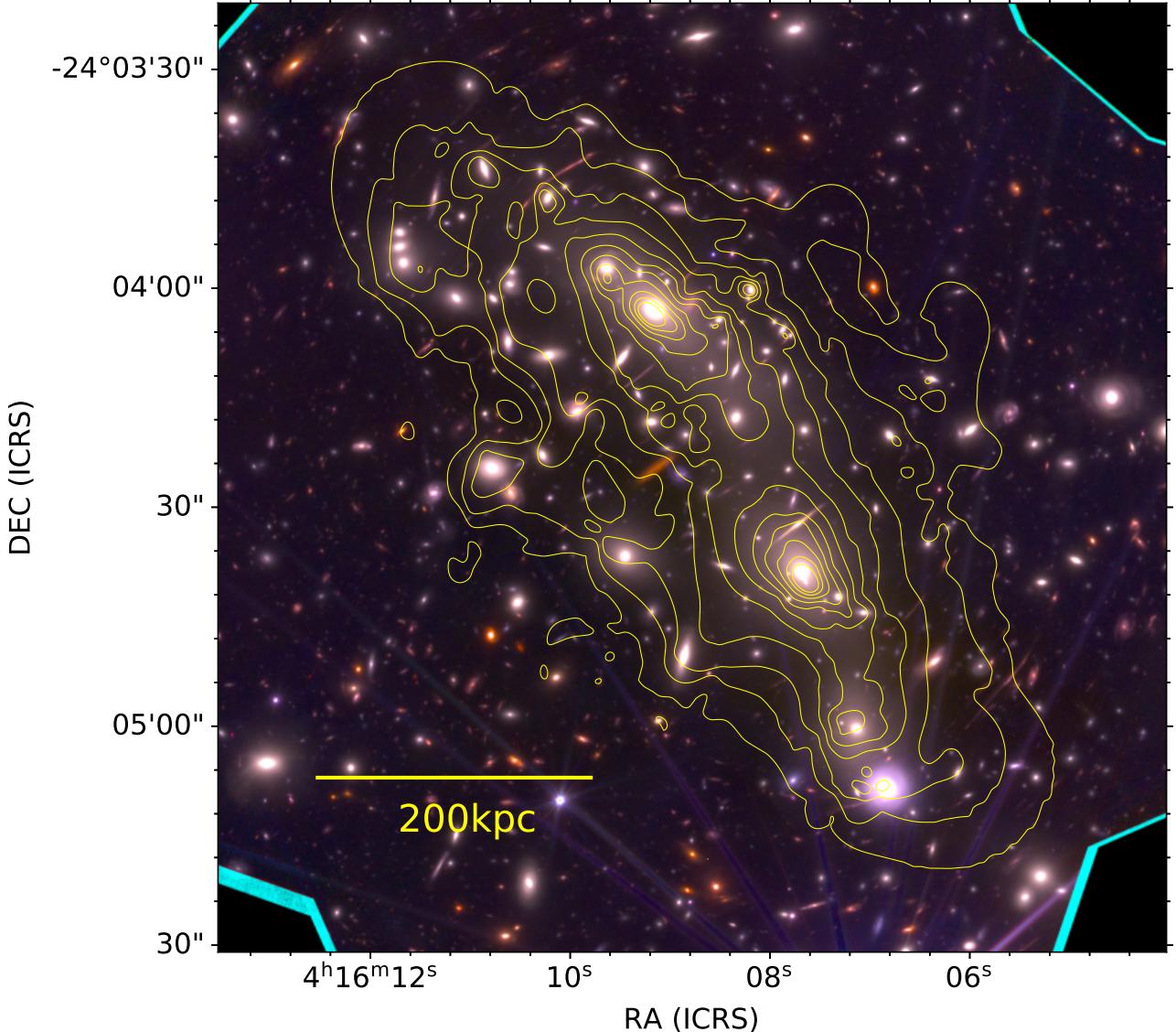
magnification map lacks many small loops around the cluster galaxies because we assign analytic halos only to the two BCGs, whereas Rihtaršič et al. (2025) assign them to every cluster member. Furthermore, their critical curve extends farther to the southwest than in our model, owing to the inclusion of galaxies in that region in their modeling.

## 5. DISCUSSION

### 5.1. Effect of Multi-resolution

As described in §3.4, MrMARTIAN adopts a multi-resolution approach for reconstruction. While this enhances the modeling efficiency, the use of multiple resolutions can introduce biases in the reconstructed mass maps due to resolution mixing. To assess the potential bias of the multi-resolution approach, we compare reconstructions obtained with uniform and multi-resolution grids. The resolution of the uniform grid model is set to match the highest resolution used in the multi-resolution model. Any significant deviation of the multi-resolution result from the uniform grid result would indicate the presence of biases.

Figure 9 compares the mass maps with the uniform grid and multi-resolution models. The two results are



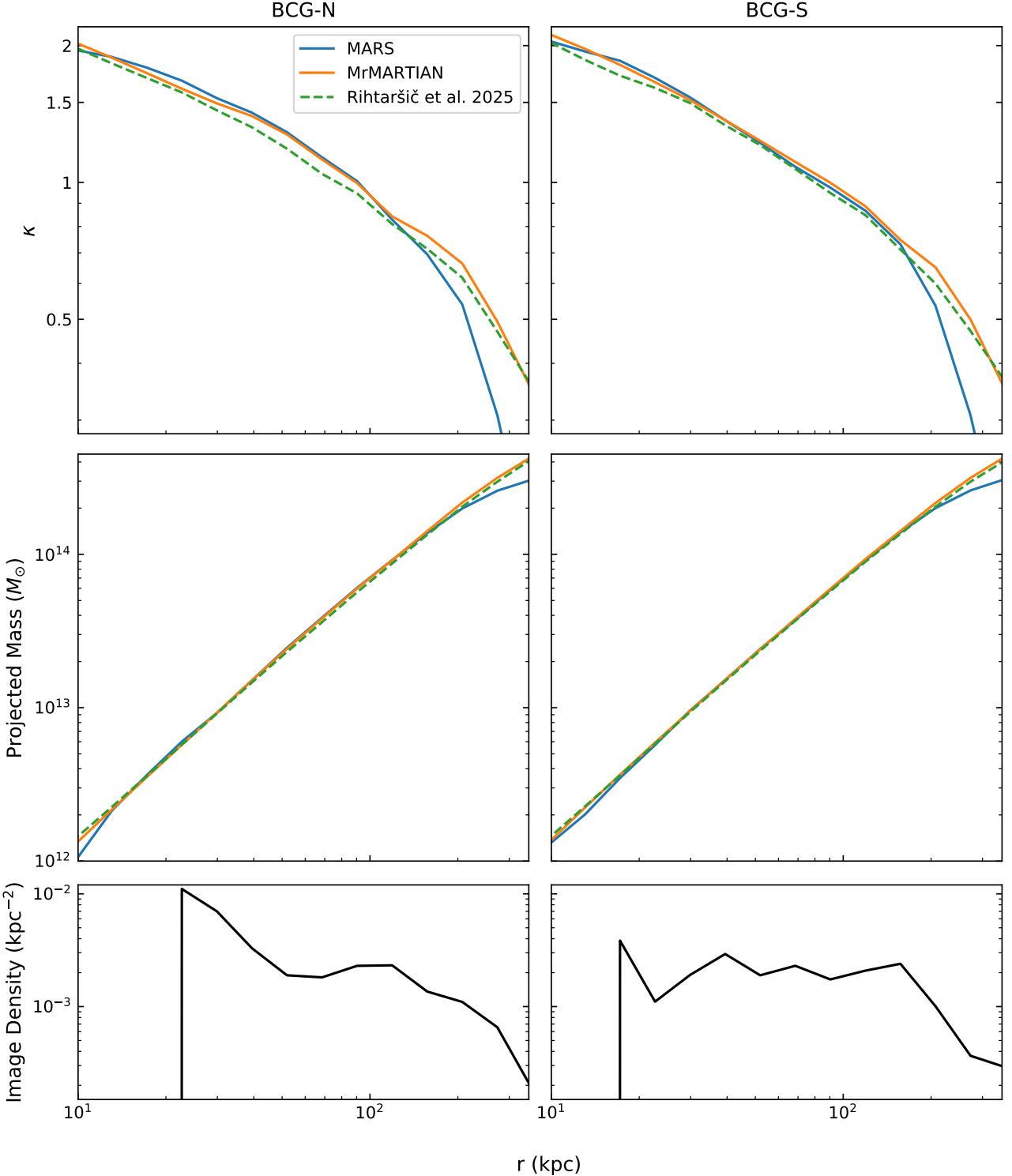
**Figure 6.** Mass contours of MACSJ0416 overlaid on the color-composite image. The yellow contours show the convergence  $\kappa$  and range from 0.6 to 2.1 with intervals of 0.15. The color-composite image is generated using the F090W + F115W filters for blue, the F150W + F200W + F277W filters for green, and the F356W + F444W filters for red.

in excellent agreement across the entire field, with no significant discrepancies apparent upon visual inspection. The residual map (right panel) shows low-contrast ( $\Delta\kappa \sim 0.01$ ) grid patterns in the cluster outskirts, where the multi-resolution model employs low-resolution grids. This is inevitable, as differences in grid cell size produce distinct pixelation effects. In addition, some large-scale residual patterns perpendicular to the elongation of the MACSJ0416 system are visible. The maximum amplitude of these residuals is  $\sim 0.025$  which is comparable to the mass reconstruction uncertainties (see Figure 10). However, since the average difference lies well below the uncertainty level, we conclude that both reconstructions are highly consistent. Their radial  $\kappa$  profiles are also

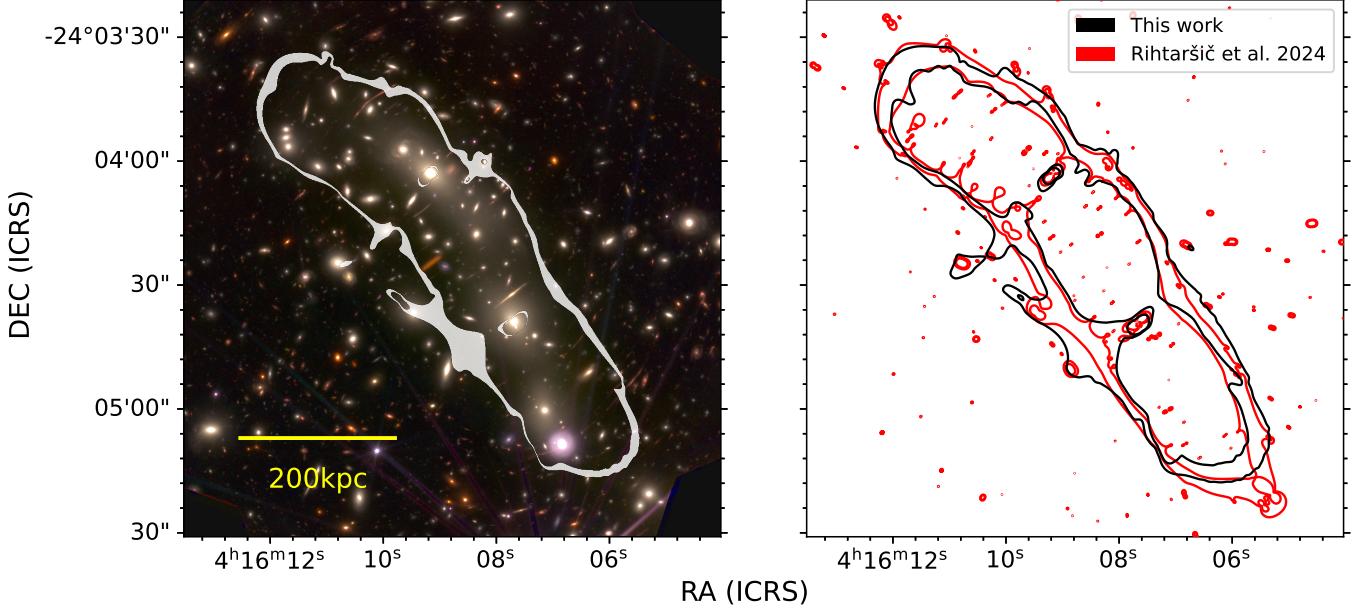
nearly indistinguishable. Finally, we note that the large differences at the BCG locations should not be interpreted as evidence of bias, as the positions of the analytic halos can vary between different mass reconstruction runs.

The above test shows that MrMARTIAN maintains reconstruction quality while using only  $\sim 13\%$  of free parameters required by a high-resolution uniform grid. This capability enables faster and more efficient computation of both the lens model and its associated uncertainties than our previous uniform-resolution approach.

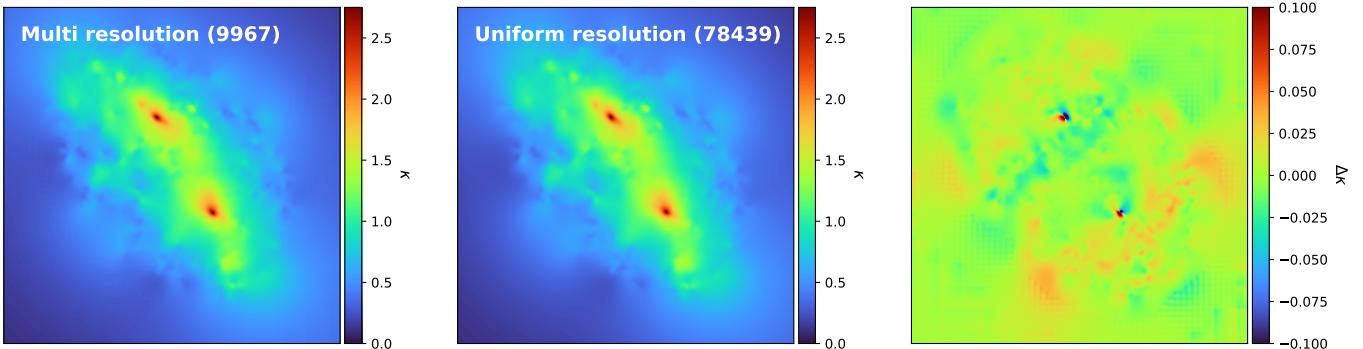
## 5.2. Mass Reconstruction Uncertainty



**Figure 7.** Radial  $\kappa$  (top), projected mass (middle), and SL multiple image density (bottom) profile comparison. We display radial profiles of BCG-N and BCG-S. The solid blue and dashed orange lines indicate the profiles derived from the **MARS** and **MrMARTIAN**, respectively. The solid green line represents the profiles derived from the [Rihtaršič et al. \(2025\)](#). The solid black lines in the bottom panels show the SL multiple image density.



**Figure 8.** Magnification maps of MACSJ0416 at the reference redshift  $z_f = 9$ . The white curves in the left panel indicate regions where the magnification is larger than 50. The black (red) contours in the right panel represent the magnification from this study (from Rihtaršič et al. 2025). The magnification contours indicate  $|\mu| = 20$  at the reference redshift  $z_f = 9$ . The color-composite image is the same as in Figure 6.

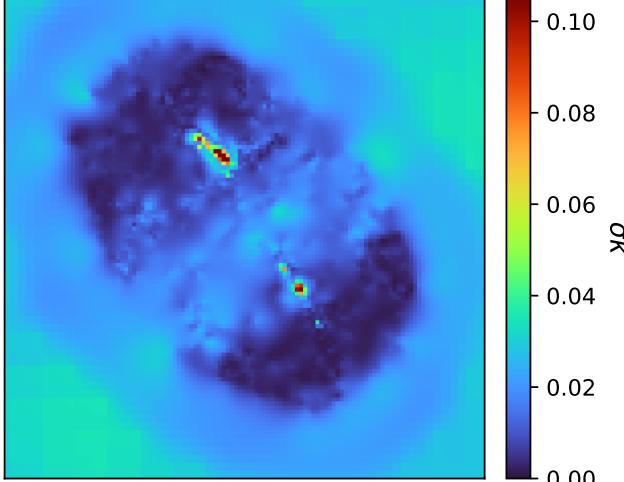


**Figure 9.** Comparison between the uniform and multi-resolution. The left (middle) column shows the mass reconstruction with (without) the multi-resolution approach. The number on the upper left in each panel presents the number of free parameters for lens modeling. The right column represents the residual map between the mass maps with and without a multi-resolution approach.

**Table 3.** Uncertainties of the two analytic nodes.

Analytic Node	$x^1$ ('')	$y^1$ ('')	$c$	$r_s$ (kpc)	$\tau$	$i$ (degree)	$\epsilon$
BCG-N <sup>2</sup>	$-0.13^{+0.27}_{-0.63}$	$-0.34^{+0.54}_{-0.34}$	$4.11^{+0.38}_{-0.42}$	$350^{+56}_{-40}$	$19.87^{+8.83}_{-11.64}$	$138.64^{+6.26}_{-5.18}$	$0.248^{+0.001}_{-0.042}$
BCG-S <sup>2</sup>	$0.48^{+0.13}_{-0.12}$	$-0.37^{+0.14}_{-0.16}$	$4.19^{+0.32}_{-0.41}$	$359^{+47}_{-40}$	$17.90^{+10.79}_{-9.45}$	$130.13^{+4.63}_{-2.80}$	$0.249^{+0.001}_{-0.002}$

NOTE—<sup>1</sup>Offsets relative to each BCG. <sup>2</sup> Median (center), 16th (lower), and 84th (upper) percentiles from the 200 realization. We note that the median values are not derived from the reference model.



**Figure 10.** The standard deviation of the 200 realizations. The resolution of the  $\kappa$  pixels is  $1''.1/\text{pixel}$ .

In Figure 10, we present the standard deviation from the 200 realizations using randomly generated initial conditions. Across the entire FOV, the error of the mass map is  $\sigma\kappa < 0.05$ , except around the BCGs. The relatively larger uncertainties around the BCGs ( $\sigma\kappa \sim 0.1$ ) are mainly caused by the dispersion in the positions of the analytic nodes. Table 3 presents the uncertainties in the parameters of the two analytic nodes derived from the 200 realizations. We note that the median values in Table 3 are not those of the reference model. Although the halo positions are allowed to move freely, their uncertainties remain below  $\sim 0.5''$ . All parameters except for the truncation ratio are consistent within  $1\sigma$  between the two analytic nodes. The truncation ratios show large uncertainties, which span most of the prior range. This is not surprising, as the truncation ratio affects the outskirts of the analytic profiles where there are no SL constraints. Despite this, the overall mass distributions across the 200 realizations remain consistent, supporting that the large uncertainties in the truncation ratios have a limited impact on the mass reconstruction within the SL regime.

## 6. CONCLUSION

By combining grid cells and analytic nodes, we have presented a new hybrid SL modeling algorithm **MrMARTIAN**. The grid component is regularized through maximum cross-entropy, as in **MARS**, preserving flexibility and smoothness in mass map reconstruction. **MrMARTIAN** also supports negative  $\kappa$  values for lens modeling. To address the limitations of **MARS**, which tends to oversmooth mass peaks and has difficulty reproducing the global mass density profile beyond the SL regime, we

introduce the pseudo-elliptical truncated NFW profile as an analytic node in this study.

We validate **MrMARTIAN** using the publicly available simulated galaxy clusters Ares and Hera. **MrMARTIAN** reconstructs global mass density profiles better than **MARS** beyond SL regimes, where no multiple images exist. For magnification, **MrMARTIAN** shows good agreement with true values while **MARS** exhibits large deviations. In scenarios with sparsely distributed multiple images, **MrMARTIAN** shows better performance than **MARS**, effectively recovering compact core structures.

In addition, we apply **MrMARTIAN** to MACSJ0416, using 412 multiple images. Our resulting mass map reveals two prominent density peaks at the two BCGs with flexible mass distributions indicated by the small scatters in the image plane ( $\sim 0''.11$ ). The critical curve from our lens model differs in the southwest regions from the previous parametric lens models. We attribute this discrepancy to the absence of multiple images and different analytic halo implementations between lens modeling algorithms.

We assess the robustness of the multi-resolution implementation in **MrMARTIAN**. The deviations are observed between the uniform and multi-resolution near the boundaries of different resolution levels, with differences comparable to the uncertainties in convergence  $\kappa$ . However, despite these discrepancies, the radial convergence  $\kappa$  and cumulative projected mass remain consistent with the uniform-resolution result. In the MACSJ0416 case, the multi-resolution approach reduces free parameters to  $\sim 13\%$  of those required for uniform resolution, enabling faster and more efficient computation without losing information.

In this study, we demonstrate that **MrMARTIAN** can reconstruct flexible mass distributions with robust density profiles, using unprecedentedly large SL multiple images. This capability enables us to reveal detailed mass distributions, the nature of DM, and the evolution of high-redshift galaxies in the JWST era.

We express our gratitude to Hyungjin Joo for providing the JWST NIRCam image of MACSJ0416. This work is based on observations created with NASA/ESA/CSA JWST and downloaded from the Mikulski Archive for Space Telescope (MAST) at the Space Telescope Science Institute (STScI). The specific observations analyzed can be accessed via [doi: 10.17909/mw1y-gb32](https://doi.org/10.17909/mw1y-gb32) and [doi: 10.17909/ph4n-6n76](https://doi.org/10.17909/ph4n-6n76). M. J. Jee acknowledges support for the current research from the National Research Foundation (NRF) of Korea under the programs 2022R1A2C1003130 and RS-2023-00219959. SC acknowledges this research was supported by Basic Science Research Program through the NRF

funded by the Ministry of Education (No. RS-2024-00413036).

(Harris et al. 2020), PyTorch (Paszke et al. 2019), SciPy (Virtanen et al. 2020)

*Software:* Astropy (Astropy Collaboration et al. 2013, 2018, 2022), Matplotlib (Hunter 2007), NumPy

## APPENDIX

### A. TRUNCATED PSEUDO-ELLIPTICAL NFW PROFILE

#### A.1. Truncated NFW Profile

In this section, we briefly explain the truncated NFW profile. For more details, we refer readers to Baltz et al. (2009). The NFW profile (Navarro et al. 1996) is one of the most widely used universal profiles to describe DM distributions and is formulated by the following:

$$\rho(r) = \frac{\delta_c \rho_c}{(r/r_s)(1+r/r_s)^2}, \quad (\text{A1})$$

where  $\rho_c$  and  $r_s$  are the critical density and the scale radius, respectively. The characteristic overdensity  $\delta_c$  is defined as:

$$\delta_c = \frac{\Delta}{3} \frac{c^3}{\ln(1+c) - c/(1+c)}, \quad (\text{A2})$$

where  $c$  is the concentration and  $\Delta$  indicates the density contrast. In this study, we use  $\Delta = 200$ .

The truncated NFW profile is designed to implement a smooth truncation after the tidal radius of the DM halo. The smoothed truncation of the NFW profile is computed as follows:

$$\rho_T(x) = \frac{\delta_c \rho_c}{x(1+x)^2} \left( \frac{\tau^2}{\tau^2 + x^2} \right)^n, \quad (\text{A3})$$

where  $n$  controls the degree of truncation while  $x$  indicates  $r/r_s$ .  $\tau$  represents  $r_t/r_s$ , where  $r_t$  is the tidal radius. In this study, we set  $n = 1$  to implement the truncation of the NFW profile.

#### A.2. Pseudo-Elliptical Profile

We refer readers to Golse & Kneib (2002); Dímet-Montoya et al. (2012) for more details. In this paper, we adopt the following description to implement the pseudo-elliptical profile:

$$\begin{aligned} a_{1\epsilon} &= 1 - \epsilon, \\ a_{2\epsilon} &= 1 + \epsilon, \end{aligned} \quad (\text{A4})$$

where the elliptical parameter  $\epsilon$  is in range  $0 \leq \epsilon < 1$  (Blandford & Kochanek 1987). We set the upper limit of the elliptical parameter to 0.25 to avoid the unphysical peanut or boxy shapes of the mass distribution in this study (Golse & Kneib 2002; Dímet-Montoya et al. 2012). The ellipticity of the lensing potential,  $\epsilon_\varphi$  is defined as:

$$\epsilon_\varphi = 1 - \sqrt{\frac{a_{1\epsilon}}{a_{2\epsilon}}} = 1 - \sqrt{\frac{1-\epsilon}{1+\epsilon}}. \quad (\text{A5})$$

For dimensionless coordinates, we define  $\mathbf{x} = \mathbf{r}/r_s$  where  $\mathbf{r}$  is the distance from the center of a profile in the lens plane. An elliptical coordinate system can be expressed by the following:

$$\begin{cases} x_{1\epsilon} &= \sqrt{a_{1\epsilon}} x_1 \\ x_{2\epsilon} &= \sqrt{a_{2\epsilon}} x_2 \\ x_\epsilon &= \sqrt{x_{1\epsilon}^2 + x_{2\epsilon}^2} \\ \phi_\epsilon &= \arctan(x_{2\epsilon}/x_{1\epsilon}) \end{cases} \quad (\text{A6})$$

where  $x_{1(2)}$  is the first (second) component of  $\mathbf{x}$ . From the above expressions,  $\kappa_\epsilon$ ,  $\alpha_\epsilon$ ,  $\gamma_\epsilon$ , and  $\varphi_\epsilon$  derived from a pseudo-elliptical NFW profile are

$$\begin{cases} \kappa_\epsilon(\mathbf{x}) &= \kappa(x_\epsilon) + \epsilon \cos 2\phi_\epsilon \gamma(x_\epsilon) \\ \alpha_{1\epsilon}(\mathbf{x}) &= \alpha(x_\epsilon) \sqrt{a_{1\epsilon}} \cos \phi_\epsilon \\ \alpha_{2\epsilon}(\mathbf{x}) &= \alpha(x_\epsilon) \sqrt{a_{2\epsilon}} \sin \phi_\epsilon \\ \gamma_{1\epsilon}(\mathbf{x}) &= -\epsilon \kappa(x_\epsilon) - \gamma(x_\epsilon) \cos 2\phi_\epsilon \\ \gamma_{2\epsilon}(\mathbf{x}) &= -\sqrt{1-\epsilon^2} \gamma(x_\epsilon) \sin 2\phi_\epsilon \\ \varphi_\epsilon(\mathbf{x}) &= \varphi(x_\epsilon) \end{cases} \quad (\text{A7})$$

where  $\kappa$ ,  $\alpha$ ,  $\gamma$ , and  $\varphi$  are the expressions of convergence, deflection angle, shear, and lensing potential in the simple NFW halo profile in the dimensionless coordinates (Bartelmann 1996).

## REFERENCES

- Acebron, A., Jullo, E., Limousin, M., et al. 2017, MNRAS, 470, 1809, doi: [10.1093/mnras/stx1330](https://doi.org/10.1093/mnras/stx1330)
- Astropy Collaboration, Robitaille, T. P., Tollerud, E. J., et al. 2013, A&A, 558, A33, doi: [10.1051/0004-6361/201322068](https://doi.org/10.1051/0004-6361/201322068)
- Astropy Collaboration, Price-Whelan, A. M., Sipőcz, B. M., et al. 2018, AJ, 156, 123, doi: [10.3847/1538-3881/aabc4f](https://doi.org/10.3847/1538-3881/aabc4f)
- Astropy Collaboration, Price-Whelan, A. M., Lim, P. L., et al. 2022, ApJ, 935, 167, doi: [10.3847/1538-4357/ac7c74](https://doi.org/10.3847/1538-4357/ac7c74)
- Bagley, M. B., Finkelstein, S. L., Koekemoer, A. M., et al. 2023, ApJL, 946, L12, doi: [10.3847/2041-8213/acbb08](https://doi.org/10.3847/2041-8213/acbb08)
- Baltz, E. A., Marshall, P., & Oguri, M. 2009, JCAP, 2009, 015, doi: [10.1088/1475-7516/2009/01/015](https://doi.org/10.1088/1475-7516/2009/01/015)
- Bartelmann, M. 1996, A&A, 313, 697, doi: [10.48550/arXiv.astro-ph/9602053](https://doi.org/10.48550/arXiv.astro-ph/9602053)
- Bartelmann, M., & Schneider, P. 2001, PhR, 340, 291, doi: [10.1016/S0370-1573\(00\)00082-X](https://doi.org/10.1016/S0370-1573(00)00082-X)
- Baxter, E., Chang, C., Jain, B., et al. 2017, ApJ, 841, 18, doi: [10.3847/1538-4357/aa6ff0](https://doi.org/10.3847/1538-4357/aa6ff0)
- Bergamini, P., Rosati, P., Mercurio, A., et al. 2019, A&A, 631, A130, doi: [10.1051/0004-6361/201935974](https://doi.org/10.1051/0004-6361/201935974)
- Bergamini, P., Rosati, P., Vanzella, E., et al. 2021, A&A, 645, A140, doi: [10.1051/0004-6361/202039564](https://doi.org/10.1051/0004-6361/202039564)
- Bergamini, P., Grillo, C., Rosati, P., et al. 2023, A&A, 674, A79, doi: [10.1051/0004-6361/202244834](https://doi.org/10.1051/0004-6361/202244834)
- Blandford, R. D., & Kochanek, C. S. 1987, ApJ, 321, 658, doi: [10.1086/165660](https://doi.org/10.1086/165660)
- Bridle, S. L., Hobson, M. P., Lasenby, A. N., & Saunders, R. 1998, MNRAS, 299, 895, doi: [10.1046/j.1365-8711.1998.01877.x](https://doi.org/10.1046/j.1365-8711.1998.01877.x)
- Bushouse, H., Eisenhamer, J., Dencheva, N., et al. 2024, JWST Calibration Pipeline, 1.16.0, doi: [10.5281/zenodo.7038885](https://doi.org/10.5281/zenodo.7038885)
- Caminha, G. B., Suyu, S. H., Grillo, C., & Rosati, P. 2022, A&A, 657, A83, doi: [10.1051/0004-6361/202141994](https://doi.org/10.1051/0004-6361/202141994)
- Cha, S., Cho, B. Y., Joo, H., et al. 2025, ApJL, 987, L15, doi: [10.3847/2041-8213/add2f0](https://doi.org/10.3847/2041-8213/add2f0)
- Cha, S., HyeongHan, K., Scofield, Z. P., Joo, H., & Jee, M. J. 2024, ApJ, 961, 186, doi: [10.3847/1538-4357/ad0cbf](https://doi.org/10.3847/1538-4357/ad0cbf)
- Cha, S., & Jee, M. J. 2022, ApJ, 931, 127, doi: [10.3847/1538-4357/ac69df](https://doi.org/10.3847/1538-4357/ac69df)
- . 2023, ApJ, 951, 140, doi: [10.3847/1538-4357/acd111](https://doi.org/10.3847/1538-4357/acd111)
- Coe, D., Bradley, L., & Zitrin, A. 2015, ApJ, 800, 84, doi: [10.1088/0004-637X/800/2/84](https://doi.org/10.1088/0004-637X/800/2/84)
- Diego, J. M., Tegmark, M., Protopapas, P., & Sandvik, H. B. 2007, MNRAS, 375, 958, doi: [10.1111/j.1365-2966.2007.11380.x](https://doi.org/10.1111/j.1365-2966.2007.11380.x)
- Diego, J. M., Sun, B., Yan, H., et al. 2023, A&A, 679, A31, doi: [10.1051/0004-6361/202347556](https://doi.org/10.1051/0004-6361/202347556)
- Diego, J. M., Li, S. K., Meena, A. K., et al. 2024, A&A, 681, A124, doi: [10.1051/0004-6361/202346761](https://doi.org/10.1051/0004-6361/202346761)
- Dúmet-Montoya, H. S., Caminha, G. B., & Makler, M. 2012, A&A, 544, A83, doi: [10.1051/0004-6361/201118485](https://doi.org/10.1051/0004-6361/201118485)
- Eckert, D., Jauzac, M., Shan, H., et al. 2015, Nature, 528, 105, doi: [10.1038/nature16058](https://doi.org/10.1038/nature16058)
- Elíasdóttir, Á., Limousin, M., Richard, J., et al. 2007, arXiv e-prints, arXiv:0710.5636, doi: [10.48550/arXiv.0710.5636](https://doi.org/10.48550/arXiv.0710.5636)
- Finner, K., Jee, M. J., Golovich, N., et al. 2017, ApJ, 851, 46, doi: [10.3847/1538-4357/aa998c](https://doi.org/10.3847/1538-4357/aa998c)
- Furtak, L. J., Labbé, I., Zitrin, A., et al. 2024, Nature, 628, 57, doi: [10.1038/s41586-024-07184-8](https://doi.org/10.1038/s41586-024-07184-8)
- Ghosh, A., Williams, L. L. R., Liesenborgs, J., et al. 2021, MNRAS, 506, 6144, doi: [10.1093/mnras/stab1196](https://doi.org/10.1093/mnras/stab1196)
- Giocoli, C., Meneghetti, M., Bartelmann, M., Moscardini, L., & Boldrin, M. 2012, MNRAS, 421, 3343, doi: [10.1111/j.1365-2966.2012.20558.x](https://doi.org/10.1111/j.1365-2966.2012.20558.x)
- Golse, G., & Kneib, J. P. 2002, A&A, 390, 821, doi: [10.1051/0004-6361:20020639](https://doi.org/10.1051/0004-6361:20020639)
- Harris, C. R., Millman, K. J., van der Walt, S. J., et al. 2020, Nature, 585, 357, doi: [10.1038/s41586-020-2649-2](https://doi.org/10.1038/s41586-020-2649-2)

- Hobson, M. P., & Lasenby, A. N. 1998, MNRAS, 298, 905, doi: [10.1046/j.1365-8711.1998.01707.x](https://doi.org/10.1046/j.1365-8711.1998.01707.x)
- Hoekstra, H., Bartelmann, M., Dahle, H., et al. 2013, SSRv, 177, 75, doi: [10.1007/s11214-013-9978-5](https://doi.org/10.1007/s11214-013-9978-5)
- Hsiao, T. Y.-Y., Coe, D., Abdurro'uf, et al. 2023, ApJL, 949, L34, doi: [10.3847/2041-8213/acc94b](https://doi.org/10.3847/2041-8213/acc94b)
- Hunter, J. D. 2007, Computing in Science & Engineering, 9, 90, doi: [10.1109/MCSE.2007.55](https://doi.org/10.1109/MCSE.2007.55)
- HyeongHan, K., Jee, M. J., Cha, S., & Cho, H. 2024, Nature Astronomy, doi: [10.1038/s41550-023-02164-w](https://doi.org/10.1038/s41550-023-02164-w)
- Jee, M. J., Ford, H. C., Illingworth, G. D., et al. 2007, ApJ, 661, 728, doi: [10.1086/517498](https://doi.org/10.1086/517498)
- Jullo, E., & Kneib, J. P. 2009, MNRAS, 395, 1319, doi: [10.1111/j.1365-2966.2009.14654.x](https://doi.org/10.1111/j.1365-2966.2009.14654.x)
- Jullo, E., Kneib, J. P., Limousin, M., et al. 2007, New Journal of Physics, 9, 447, doi: [10.1088/1367-2630/9/12/447](https://doi.org/10.1088/1367-2630/9/12/447)
- Kassiola, A., & Kovner, I. 1993, ApJ, 417, 450, doi: [10.1086/173325](https://doi.org/10.1086/173325)
- Kawamata, R., Oguri, M., Ishigaki, M., Shimasaku, K., & Ouchi, M. 2016, ApJ, 819, 114, doi: [10.3847/0004-637X/819/2/114](https://doi.org/10.3847/0004-637X/819/2/114)
- Kelly, P. L., Rodney, S. A., Treu, T., et al. 2015, Science, 347, 1123, doi: [10.1126/science.aaa3350](https://doi.org/10.1126/science.aaa3350)
- Kim, J., Jee, M. J., Hughes, J. P., et al. 2021, ApJ, 923, 101, doi: [10.3847/1538-4357/ac294f](https://doi.org/10.3847/1538-4357/ac294f)
- Kneib, J.-P., & Natarajan, P. 2011, A&A Rv, 19, 47, doi: [10.1007/s00159-011-0047-3](https://doi.org/10.1007/s00159-011-0047-3)
- Kochanek, C. S. 2006, Strong Gravitational Lensing (Berlin, Heidelberg: Springer Berlin Heidelberg), 91–268, doi: [10.1007/978-3-540-30310-7\\_2](https://doi.org/10.1007/978-3-540-30310-7_2)
- Kuchner, U., Aragón-Salamanca, A., Pearce, F. R., et al. 2020, MNRAS, 494, 5473, doi: [10.1093/mnras/staa1083](https://doi.org/10.1093/mnras/staa1083)
- Lam, D., Broadhurst, T., Diego, J. M., et al. 2014, ApJ, 797, 98, doi: [10.1088/0004-637X/797/2/98](https://doi.org/10.1088/0004-637X/797/2/98)
- Liesenborgs, J., De Rijcke, S., & Dejonghe, H. 2006, MNRAS, 367, 1209, doi: [10.1111/j.1365-2966.2006.10040.x](https://doi.org/10.1111/j.1365-2966.2006.10040.x)
- Liesenborgs, J., de Rijcke, S., Dejonghe, H., & Bekaert, P. 2007, MNRAS, 380, 1729, doi: [10.1111/j.1365-2966.2007.12236.x](https://doi.org/10.1111/j.1365-2966.2007.12236.x)
- Limousin, M., Kneib, J. P., Bardeau, S., et al. 2007, A&A, 461, 881, doi: [10.1051/0004-6361:20065543](https://doi.org/10.1051/0004-6361:20065543)
- Limousin, M., Sommer-Larsen, J., Natarajan, P., & Milvang-Jensen, B. 2009, ApJ, 696, 1771, doi: [10.1088/0004-637X/696/2/1771](https://doi.org/10.1088/0004-637X/696/2/1771)
- Lotz, J. M., Koekemoer, A., Coe, D., et al. 2017, ApJ, 837, 97, doi: [10.3847/1538-4357/837/1/97](https://doi.org/10.3847/1538-4357/837/1/97)
- Magaña, J., Acebrón, A., Motta, V., et al. 2018, ApJ, 865, 122, doi: [10.3847/1538-4357/aada7d](https://doi.org/10.3847/1538-4357/aada7d)
- Markevitch, M., Gonzalez, A. H., Clowe, D., et al. 2004, ApJ, 606, 819, doi: [10.1086/383178](https://doi.org/10.1086/383178)
- Meneghetti, M., Natarajan, P., Coe, D., et al. 2017, MNRAS, 472, 3177, doi: [10.1093/mnras/stx2064](https://doi.org/10.1093/mnras/stx2064)
- Natarajan, P., Williams, L. L. R., Bradač, M., et al. 2024, SSRv, 220, 19, doi: [10.1007/s11214-024-01051-8](https://doi.org/10.1007/s11214-024-01051-8)
- Navarro, J. F., Frenk, C. S., & White, S. D. M. 1996, ApJ, 462, 563, doi: [10.1086/177173](https://doi.org/10.1086/177173)
- Oguri, M., & Hamana, T. 2011, MNRAS, 414, 1851, doi: [10.1111/j.1365-2966.2011.18481.x](https://doi.org/10.1111/j.1365-2966.2011.18481.x)
- Pascale, M., Frye, B. L., Pierel, J. D. R., et al. 2025, ApJ, 979, 13, doi: [10.3847/1538-4357/ad9928](https://doi.org/10.3847/1538-4357/ad9928)
- Paszke, A., Gross, S., Massa, F., et al. 2019, in Advances in Neural Information Processing Systems, ed. H. Wallach, H. Larochelle, A. Beygelzimer, F. d'Alché-Buc, E. Fox, & R. Garnett, Vol. 32 (Curran Associates, Inc.). [https://proceedings.neurips.cc/paper\\_files/paper/2019/file/bdbca288fee7f92f2bfa9f7012727740-Paper.pdf](https://proceedings.neurips.cc/paper_files/paper/2019/file/bdbca288fee7f92f2bfa9f7012727740-Paper.pdf)
- Perera, D., Williams, L. L. R., Liesenborgs, J., et al. 2025, MNRAS, 536, 2690, doi: [10.1093/mnras/stae2753](https://doi.org/10.1093/mnras/stae2753)
- Planelles, S., Borgani, S., Fabjan, D., et al. 2014, MNRAS, 438, 195, doi: [10.1093/mnras/stt2141](https://doi.org/10.1093/mnras/stt2141)
- Ragagnin, A., Meneghetti, M., Calura, F., et al. 2024, A&A, 687, A270, doi: [10.1051/0004-6361/202449872](https://doi.org/10.1051/0004-6361/202449872)
- Richard, J., Claeysens, A., Lagattuta, D., et al. 2021, A&A, 646, A83, doi: [10.1051/0004-6361/202039462](https://doi.org/10.1051/0004-6361/202039462)
- Rihtaršić, G., Bradač, M., Despres, G., et al. 2025, A&A, 696, A15, doi: [10.1051/0004-6361/202451117](https://doi.org/10.1051/0004-6361/202451117)
- Robertson, A., Harvey, D., Massey, R., et al. 2019, MNRAS, 488, 3646, doi: [10.1093/mnras/stz1815](https://doi.org/10.1093/mnras/stz1815)
- Sebesta, K., Williams, L. L. R., Mohammed, I., Saha, P., & Liesenborgs, J. 2016, MNRAS, 461, 2126, doi: [10.1093/mnras/stw1433](https://doi.org/10.1093/mnras/stw1433)
- Virtanen, P., Gommers, R., Oliphant, T. E., et al. 2020, Nature Methods, 17, 261, doi: <https://doi.org/10.1038/s41592-019-0686-2>
- Welch, B., Coe, D., Diego, J. M., et al. 2022, Nature, 603, 815, doi: [10.1038/s41586-022-04449-y](https://doi.org/10.1038/s41586-022-04449-y)
- Williams, L. L. R., & Liesenborgs, J. 2019, MNRAS, 482, 5666, doi: [10.1093/mnras/sty3113](https://doi.org/10.1093/mnras/sty3113)
- Willott, C. J., Doyon, R., Albert, L., et al. 2022, PASP, 134, 025002, doi: [10.1088/1538-3873/ac5158](https://doi.org/10.1088/1538-3873/ac5158)
- Windhorst, R. A., Cohen, S. H., Jansen, R. A., et al. 2023, AJ, 165, 13, doi: [10.3847/1538-3881/aca163](https://doi.org/10.3847/1538-3881/aca163)
- Wong, K. C., Suyu, S. H., Chen, G. C. F., et al. 2020, MNRAS, 498, 1420, doi: [10.1093/mnras/stz3094](https://doi.org/10.1093/mnras/stz3094)
- Zitrin, A., Meneghetti, M., Umetsu, K., et al. 2013, ApJL, 762, L30, doi: [10.1088/2041-8205/762/2/L30](https://doi.org/10.1088/2041-8205/762/2/L30)