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Week 5- Tutorial

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Conditions for a Binomial Experiment.

- A binomial experiment must satisfy the following four conditions.

1. There are **n** identical trials under identical conditions, ie; the given experiment is repeated **n** times, where **n** is a positive integer

2. Each trial has two and only two outcomes. These outcomes are usually called a success and a failure, respectively.

3. The probability of success is denoted by **p** and that of failure by **q**, and **p+q =1**. The probabilities p and q remain constant for each trial.

4. The trials are independent, the outcome of one trial does not affect the outcome of another trial.

Describing the Binomial distribution

If X follows a binomial distribution: X ∽ B(n, p)

$$P(x) = nC_x p^x\, q^{n-x}$$

$n \;= total\ number\ of\ trials$
$p \;= probability\ of\ success$
$q \;=\; 1-p \quad = probability\ of\ failure$
$x \;= number\ of\ successes\ in\ n\ trials$
$n-x = \; number\ of\ failures\ in\ n\ trials$

where $nC_x = \dfrac{n!}{x!(n-x)!}$

Mean= $\mu \;=\; np$

Variance= $\sigma^2 \;=\; npq$

Standard deviation=$\sigma \;=\; \sqrt{npq}$.

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Question 1: Binomial distribution

- A company is considering drilling four oil wells. The probability of success for each well is 0.40, independent of the results for any other well. The cost of each well is $200,000. Each well that is successful will be worth $600,000.

a) What is the probability that one or more wells will be successful?

b) What is the expected number of successes?

c) What is the expected profit for the successful wells?

d) What is the standard deviation of the number of successes

Solution : Question 1

- Which pdf can we use ?

1) There is a fixed number of trials denoted as n.

2) There are only two possible outcomes called "success" and "failure," for each trial………. p + q = 1.

3)The n trials are independent and are repeated using identical conditions.

Use Binomial distribution with n = 4, p = 0.40.

Solution.. 1a)

Given:

$n = 4, \; p = 0.40, \; q = 0.60$

a) What is the probability that one or more wells will be successful?

$\text{P}[one\ or\ more\ successful\ wells] = 1 - \text{P}[no\ successful\ wells]$

$\text{P}(\text{X} = \text{r}) = {}_nC_r\, p^r q^{(n-r)}$

$\text{P}(\text{X} = 0) = {}_4C_0(\,0.4)^0(0.6)^4$

$\text{P}(\text{X} = 0) = (1)(1)\,(0.6)^4$

$\text{P}(\text{X} = 0) = 0.1296$

$\text{Pr}[one\ or\ more\ successful\ wells] = 1 - 0.1296 = 0.8704$

Solution.. 1b)

b) What is the expected number of successes?

Given:

$n = 4, \; p = 0.40, \; q = 0.60$

b) What is the expected number of successes?

$$E(X) = np$$
$$E(X) = 1.6$$

Solution ..1 c)

c) What is the expected profit for the successful wells?

Given:

$n = 4, \; p = 0.40, \; q = 0.60$

$Expected\ gain = Expected\ success \times Gain - Total\ cost$

$Expected\ gain = (1.6)(\$600{,}000) - (4)(\$200{,}000) = \$160{,}000.$

d) standard deviation of the number of successes

Given:

$n = 4, \; p = 0.40, \; q = 0.60$

$std(X) = \sqrt{npq} = \sqrt{4 \times 0.4 \times 0.6} = 0.98$

Notations for the Poisson distribution

X ~ P(μ) where μ is the mean number of occurrences per fixed interval.

Expected Mean = $E\ (x)\ = \mu\ =\ \lambda$

Standard deviation $\sigma = \sqrt{\mu}$

Variance $V(x) =\ \sigma^2 = \mu = \lambda$

$$P(x) = \frac{\lambda^x . e^{-\lambda}}{x!}$$

Where λ=the mean number of occurrences in that interval or the Poisson parameter

Question 2: Poisson distribution

- The number of meteors found by a radar system in any 30- second interval under specified conditions averages 1.81. Assume the meteors appear randomly and independently.
- a) What is the probability that no meteors are found in a one minute interval?
- b) What is the probability of observing at least five but not more than eight meteors in two minutes of observation

Solution ..2 a)

a) What is the probability that no meteors are found in a one-minute interval

λ= 1.81 per 30sec

= 3.62 per min

X=0

Using $P(x) = \frac{\lambda^x . e^{-\lambda}}{x!}$

$= \frac{3.62_0 \times e_{-3.62}}{0!}$

= 0.0268

Solution..2b)

b) What is the probability of observing at least five but not more than eight meteors in two minutes of observation?

If λ= 1.81 for 30sec

λ=1.81$\times$4=7.24 for 2 mins

$$P(x) = \frac{\lambda^{x}.e^{-\lambda}}{x!}$$

$$P(5 \leq X \leq 8) = \{ P(X = 5) + P(X = 6) + P(X = 7) + P(X = 8)\}$$

$$P(X = 5) = \frac{7.24^{5}e^{-7.24}}{5!} = 0.1189$$

$$P(X = 6) = \frac{7.24^{6}e^{-7.24}}{6!} = 0.1435$$

$$P(X = 7) = \frac{7.24^{7}e^{-7.24}}{7!} = 0.1484$$

$$P(X = 8) = \frac{7.24^{8}e^{-7.24}}{8!} = 0.1343$$

$P(at\ least\ 5\ but\ not\ more\ than\ 8) = 0.1189 + 0.1435 + 0.1484 + 0.1343$

= 0.545

Relationship between Binomial and Poisson distributions

- The Binomial distribution tends towards the Poisson distribution as $n \to \infty$, $p \to 0$.
- The Poisson distribution with $\lambda = n$p closely approximates the binomial distribution if n is large and p is small.
- In general, this is when $n \geq 20$ and $p \leq 0.05$

Question 3

A manufacturer produces lightbulbs that are packed into boxes of 100. If quality control studies indicate that 0.5% of the lightbulbs produced are defective, what percentage of the boxes will contain 2 or more defectives?

* **Use Binomial approximation to Poisson's distribution,**

Solution: Question 3

As n is large and p, the P(defective bulb), is small, use the Poisson approximation to the binomial probability distribution.

If X = number of defective bulbs in a box, then

$$X \sim P(\mu) where \mu = n \times p = 100 \times 0.005 = 0.5$$

Therefore λ= 0.5

$$P(X \geq 2) = 1 - [P(X = 0) + P(X = 1)]$$

P(X=0) = $\frac{e^{-0.5} \times 0.5^{0}}{0!}$ =0.6065

P(X=1) =$\frac{e^{-0.5} \times 0.5^{1}}{1!}$ = 0.3033

$P(X \geq 2) = 1 - (0.6065) + (0.3033) = \mathbf{0.0902}$ or **9%**

Probability Distributions for continuous Random Variable

- The **curve** is called the **probability density function** (abbreviated as **pdf**) and the symbol **f(x)** is used to represent the curve.
- Area under the curve is given by a different function called the **cumulative distribution function** (abbreviated as **cdf**).
- The cumulative distribution function is used to evaluate probability as area.

Probability Distributions for continuous Random Variable

- The entire area under the curve and above the x-axis is equal to one.
- $P(a < x < b)$ is the probability that the random variable X is in the interval between the values a and b.
- $P(a < x < b)$ is the same as $P(a \leq x \leq b)$ because probability is equal to area.
- $P(x = a) = 0$. The probability that x takes on any single individual value is zero
- $P(a < x < b)$ is the area under the curve,

Notation and Properties of the Uniform Distribution

- X ~ U(a, b) where a = lowest value of x and b = highest value of x.
- The probability density function is $f(x) = \frac{1}{(b-a)}$
- The mean of a uniform probability distribution, $\mu=\frac{a+b}{2}$
- Variance of a uniform probability distribution, $\sigma^2 = \frac{(b-a)^2}{12}$
- The standard deviation, $\sigma = \frac{b-a}{\sqrt{12}}$
- Probability = Area = Base · height

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Question 4: Uniform distribution

The time (in minutes) until the next bus departs a major bus depot follows a distribution with $f(x) = \frac{1}{20}$ where x goes from 25 to 45 minutes.

(a) Calculate the mean and standard deviation.

(b) Find the probability that the time is at most 30 minutes.

(c) Find the probability that the time is between 30 and 40 minutes.

Solution ..4a)

(a) Calculate the mean and standard deviation.

Given: X~U(25, 45)

$$\mu = \frac{A+B}{2} = \frac{70}{2} = 35 \text{ minutes}$$

$$\sigma = \sqrt{\frac{(B-A)^2}{12}} = \sqrt{\frac{20^2}{12}} = 5.77 \text{ minutes}$$

Mean = 35 minutes

Standard deviation = 5.77 minutes

Solution.. 4b)

- Find the probability that the time is at most 30 minutes.

$$P(X \leq x) = \frac{x - A}{B - A}$$

$$P(X \leq 30) = \frac{30 - 25}{45 - 25}$$

$$= 0.25$$

P(X≤30)=0.25

Solution ..4c)

(c) Find the probability that the time is between 30 and 40 minutes.

$$P(c \le X \le d) = \frac{d - c}{B - A}$$

$$P(30 \le X \le 40) = \frac{40 - 30}{45 - 25}$$

$$= 0.5$$

$$P(30 \le x \le 40) = \ 0.5$$

Notation and Properties of the Exponential Distribution

- X ~ Exp(λ) where λ = is the rate of decay (λ > 0)
- The probability density function is $f(x) = \lambda e^{-\lambda x}$
- The mean of an exponential probability distribution, $\mu = \frac{1}{\lambda}$
- The standard deviation, σ = µ
- P(x > k) = $e^{-\lambda x}$
- P(x < k) = 1 - $e^{-\lambda x}$
- $kth\ percentile = \frac{In\ (area\ to\ right)}{-\lambda}$

Question 5: Exponential distribution

On the average, a certain computer part lasts ten years. The length of time the computer part lasts is exponentially distributed.

a. What is the probability that a computer part lasts more than 7 years?

b. Eighty percent of computer parts last at most how long?

c. What is the probability that a computer part lasts between nine and 11 years?

Solution 5a)

a)What is the probability that a computer part lasts more than 7 years?

Solution

- Let X = the amount of time (in years) a computer part lasts.

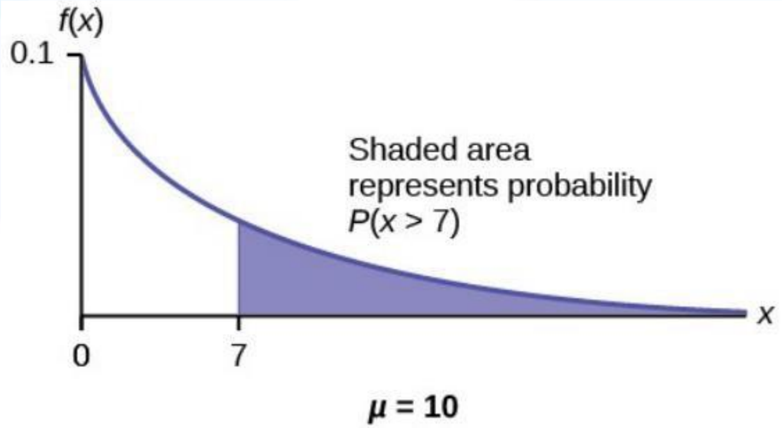
μ =10 and $\mu=\frac{1}{\lambda}$ therefore λ=0.1

P(x>7)=?

P(x > k) = $e^{-\lambda x}$

P(x>7)= $e^{-0.1\times 7}$

= 0.4966.



Solution 5b)- Percentiles for exponential functions

- Find the 80th percentile. Let k = the 80th percentile.

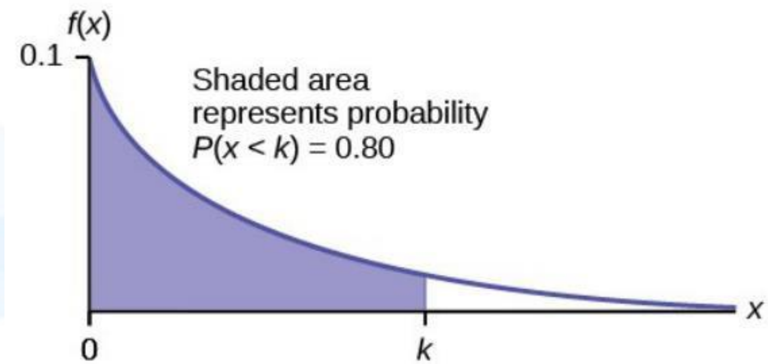
Solve for k:

80% is below k and 20% is above k.

If **cdf** =1- $e^{-\lambda x}$=Area under the graph

$P(x < k) = 0.8$ of the area

$$0.8 = 1 - e^{-\lambda x}$$



Given λ=0.1 from part a)

$$0.8 = 1 - e^{-0.1x}$$
$$-0.1x = \ln 0.2$$
$$x = 16.1$$

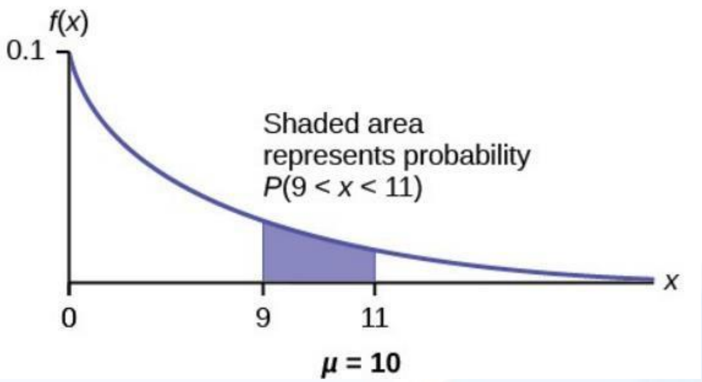
$$\text{kth percentile} = \frac{\ln(\textit{area to right})}{-\lambda}$$

$$k = \frac{\ln(.2)}{-.1} = 16.1 \text{ years}$$

- Eighty percent of the computer parts last at most 16.1 years.

Solution 5c)

- Find P(9 < x < 11).



P(9 < x < 11) = P(x < 11) – P(x < 9)

P(x < 11) = 1 – $e^{(-0.1)(11)}$ = 0.6671

P(x < 9) = 1 – $e^{(-0.1)(9)}$ = 0.5934

P(9 < x < 11) =0.6671 – 0.5934 = 0.0737.

The probability that a computer part lasts between nine and 11 years is 7.37%

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Thank you!

Happy studying ☺

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