COSC 3020
Algorithms
Spring 2025
Isomorphism-Nodes
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1. Prove that if two graphs A and B do not have the same number of nodes, they cannot be isomorphic.

   Answer

   Proof. Assume for the sake of contradiction that two graphs

   $$A = (V_1, E_1), \quad B = (V_2, E_2)$$

   are isomorphic, such that there exists some function

   $$\phi : V_1 \longrightarrow V_2$$

   that is one-to-one and onto, and

   $$(u, v) \in E_1 \iff (\phi(u), \phi(v)) \in E_2,$$

   and that $|V_1| \neq |V_2|$.

   Case 1: $|V_1| > |V_2|$. We now construct the following map:

   $$\phi : \begin{pmatrix} v_{1,1} & v_{1,2} & \cdots & v_{1,k} & \cdots & v_{1,n} \\ v_{2,1} & v_{2,2} & \cdots & v_{2,k} & & \end{pmatrix}$$

   where $n, k \in \mathbb{N},\ n > k$, and $v_{1,i} \in V_1,\ v_{2,i} \in V_2$.

   We now use the definition of an injective (one-to-one) function to derive a contradiction.

   Definition: A function $\phi : V_1 \to V_2$ is said to be injective if for all $v_1, v_2 \in V_1$,

   $$\phi(v_1) = \phi(v_2) \Rightarrow v_1 = v_2.$$

   To show that this doesn't work in this case, we want to find $v_1 \neq v_2$ such that $\phi(v_1) = \phi(v_2)$.

   Take $v_{1,k'}, v_{1,k'+1} \in V_1$ such that $k < k' < n$. Now note that both $v_{1,k'}, v_{1,k'+1}$ are different vertices so $v_{1,k'} \neq v_{1,k'+1}$ however when we map them it must be the case that we have $\phi(v_{1,k'}) = \phi(v_{1,k'+1})$ for some $v_{1,k'}, v_{1,k'+1} \in V_1$ since $k < k'$ and elements in V_2 only index up to k.

   In lamens terms this must be the case because there are more nodes in V_1 than there are in V_2, meaning at some point in the mapping more than one node from V_1 will get mapped to the same element in V_2.

   This is a contradiction because we assumed the map was bijective.

   Case 2: $|V_1| < |V_2|$. We now construct the following mapping:

   $$\phi : \begin{pmatrix} v_{1,1} & v_{1,2} & \cdots & v_{1,k} & & \\ v_{2,1} & v_{2,2} & \cdots & v_{2,k} & \cdots & v_{2,n} \end{pmatrix}$$

   Definition: A function $\phi : V_1 \to V_2$ is not surjective if there exists some $u \in V_2$ such that for all $v \in V_1$, $\phi(v) \neq u$.

   Choose $u = v_{2,k'}$ where $k' \in \mathbb{N}$ and $k < k' < n$ then because $k' > k\ \forall\ v_{1,k} \in V_1$ it must be the case that $\phi(v_{1,k}) \neq v_{2,k'}$

   In other words because there are more noddes in V_2 than V_1 there are simply some values you cannot map to at all. Thus we can see that this mapping cannot meet the criteria for a surjective function..

   Conclusion: Therefore it cannot be the case that $A \cong B$ if $|V_1| \neq |V_2|$.