10/4/23

Having gone over this exercise last week, I wanted to tackle this problem while the content was still fresh in my mind yet not so new I would be able to call back exactly what you'd thrown up on the board. Let's begin with the provided definition from the README: $T(n) \in O(f(n)) \iff \exists c, n_0 : T(n) \leq c \cdot f(n) \; \forall n \geq n_0$. Recalling that two prove equivalency, we must show that anything belonging to the asymptotic complexity of $\log_5(n)$ also belongs to that of $\log_3(n)$ and vice-versa, we can get to work. Let's begin by subbing in our two $f(n)$ expressions:

$$T(n) \in O(\log_5(n)) \iff \exists c, n_0 : T(n) \leq c \cdot (\log_5(n)) \; \forall n \geq n_0$$

&

$$T(n) \in O(\log_3(n)) \iff \exists c, n_0 : T(n) \leq c \cdot (\log_3(n)) \; \forall n \geq n_0$$

This is a good start. Now, to prove equivalency, it makes sense to write our expressions in terms of each other. Fortunately the change of base formula is quite simple for logs: $\log_b(a) = \frac{\log_x(a)}{\log_x(b)}$ which can be rewritten as follows: $\frac{1}{\log_x(b)} \log_x(a)$. Let's take these results back to our expressions above to finish out this problem:

$$T(n) \in O(\log_3(n)) \iff \exists c, n_0 : T(n) \leq \frac{c}{\log_5(3)} \cdot \log_5(n) \; \forall n \geq n_0$$

&

$T(n) \in O(\log_3(n)) \iff \exists c, n_0 : T(n) \le d \cdot \log_5(n) \; \forall n \ge n_0$

where d is any arbitrary constant.

Furthermore we see something similar shake out for $O(\log_5(n))$.

$T(n) \in O(\log_5(n)) \iff \exists c, n_0 : T(n) \le \frac{c}{\log_3(5)} \cdot \log_3(n) \; \forall n \ge n_0$

&

$T(n) \in O(\log_5(n)) \iff \exists c, n_0 : T(n) \le d \cdot \log_3(n) \; \forall n \ge n_0$

where d is any arbitrary constant.

Essentially we have shown that $T(n) \in O \log_5(n)$ and $T(n) \in O \log_3(n)$. Expressed symbollically:

$$\frac{\forall T(n) \in O(\log_3(n)) \quad T(n) \in O(\log_5(n))}{\forall T(n) \in O(\log_5(n)) \quad T(n) \in O(\log_3(n))}$$

That wraps up this exercise: Feel free to provide corrections as there's a decent chance I messed something up without referencing the notes for Thursday.

Total time: ~2 hours