Layered Hodge Decomposition for Urban Transit Networks

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Abstract. Modeling the amount of passenger flow along any given line segment of an urban transit network such as the London Underground is a challenging problem due to the complexity of the system. In this paper, we embark on a characterization of these flows on the basis of a combination of (1) a layered decomposition of the origin-destination matrix, and (2) the Hodge decomposition, a discrete algebraic topology technique that partitions flows into gradient, solenoidal, and harmonic components. We apply our method to data from the London Underground. We find that the layered decomposition estimates the contribution of flow of each origin-destination pair on each network link, and that the solenoidal and harmonic flows are described by simple equations that bypass the need for complicated numerical solutions. This reduces much of the solution of the flow problem to determining gradient flows (i.e., flows that would occur if the transit system were a hydraulic or electric circuit). Our exploratory analysis suggests that it may be feasible to develop solution methods for the transit flow problem with a complexity equivalent to the solution of a hydraulic or electric circuit.

Keywords: urban transit networks, network flows, link flow analysis and predictions, Hodge decomposition, discrete algebraic topology

1 Introduction and Background

The prediction of patterns of passenger flow inside transit networks can be a challenging task due to the complexity of the network topology, its rules of functioning (time-tables, fares, loading capacities), and passengers' route preferences [1]. The origin of most of the difficulty stems from the feedback that emerges when individuals simultaneously crowd the network with trips going between a multiplicity of origins and destinations (so-called multi-commodity flows), generating congestion and a subsequent set of individual decisions that further affect said congestion. Over a long history of research [2], a multitude of ever-developing approaches have been applied to address this challenge, including network techniques [3–6], the theory of fluid flow [7], principles of time minimization [8, 9], optimization methods [10, 11], spatial coarse-graining [3], and agent-based modelling [12], among others. A useful review of the state of the literature can be found in [13]. Although algorithms in the literature are very powerful and well developed, limitations exist. Some algorithms generate spurious cyclic solutions [14], others capture certain behavior at the expense of others [15], and yet many results only address limit theorems. Fully analytical methods do not exist that can capture the realities of these transit systems.

In this paper, we develop an analytic decomposition of observed flows in a transit network based on a combination of a layered separation of flows and Hodge theory. First, each layer in our method is made up of the network structure together with a single source node from which flow is generated. By employing a route choice method, the source node distributes its flows all across the network to other destination nodes. Then, Hodge decomposition is used to produce a principled partition of the flow into gradient, solenoidal, and harmonic parts; each flow type travels the network in a distinct manner. We make the critical observation that both solenoidal and harmonic flows in each layer can be approximated well by simple analytic expressions that do not require complicated solution methods. This suggests that, starting from an origin-destination matrix (see below, also [1]) of passengers in a transit system, it is possible to solve only a set of gradient flows to obtain an approximation for the flow on each segment (link) of the network.

The process presented here is advantageous for studying the transit flow problem because (1) the data required (the origin-destination matrix and the network topology) to estimate the Hodge flows are generally easy to acquire; and (2) the solution method requiring little more than a solution of a circuit problem and the use of analytical approximations offer a robust and simple path to estimation of transit network flows. Although our results have been developed as a proof of principle and have a number of simplifying assumptions, they show considerable robustness, and further development is warranted in order to create an easily applicable method of approximation of transit network flows.

As a matter of providing background, Hodge decomposition has been applied in a variety of settings such as statistical ranking [16] and topological data analysis [17], to name a few. However, as far as the authors are aware, this method has not been applied to the study of traffic flows. Given its ability to tackle quite general types of flow in a network, we believe this offers a unique opportunity to explore its power when deployed in the context of transit flows.

The rest of this paper is organized as follows. The rest of this section describes in more detail the setting of our problem as well as the data we use to explore it empirically. Then, in Sect. 2 we provide a formal description of Hodge decomposition, defining the key equations that need to be used, and describe our application of the method to a layered decomposition of a transit network. Finally, we describe the results and findings from applying our method to London Underground data, as well as provide a discussion and conclusion in Sect. 3.

1.1 Transit Systems Flow and the Route Assignment Problem

In its simplest representation, a transit system can be seen as a network G, with n nodes and m links. Although such networks have directed links, it is common for each pair of nodes i and j in the system to be navigable in both directions. This fortunate bi-directionality is a necessary element of Hodge decomposition. Since this paper is concerned with the explanation of flows, our analysis can be done without reference to other complicating factors of transit systems such as fares, schedules, or vehicle capacities.

As individuals enter the system via some node i of G and travel to their destinations, they collectively generate link traffic flows. Data for these flows are generally found in two forms: the least detailed form is given by origin-destination (OD) matrices $\mathbf{O}$; the more specific form is given by values of link (or segment/edge) flows t_{ij} and t_{ji} which explicitly reference the directions of travel. In the first case, $\mathbf{O}_{ij}$ is equal to the number of trips that start at i and end at j over a specified period of time. In the second case, t_{ij} is the number of travelers along a specific link (i, j) of G, also over a specified time period, where the direction of travel is $i \to j$.

The fundamental connection between $\mathbf{O}$ and the set of segment flows t_{ij} emerges from passengers' choices on how they travel from their origin to destination. While transit authorities collect both kinds of data, OD matrices are much easier to obtain, especially with wide usage of electronic fare-collection cards like the Oyster card of Transport for London. On the other hand, the collection of segment traffic in detail is far less straightforward. In many cases, these data are generated on the basis of very detailed models highly tuned to the specific transit system. They are often solutions of the so-called route choice or assignment problem. This problem has generated a large body of research, but whose considerable advancements have led to increasing dependence on complicated computational models [13, 14]. These disparities in the ease with which these data can be obtained make it imperative to maximize the amount of information that can be extracted from OD matrices without the need to appeal to segment flow information, which is our goal in this study.

1.2 Data

Because a great deal of data has been made publicly available for the London Underground (LU) system, we choose to empirically explore our method using this system. We focus on LU's passenger flows on its line segments during a period of 1 day. To study this flow, we rely on a combination of data sources.

First, the network topology and spatial data is obtained from https://github.com/oobrien/vis. This data relies on OpenStreetMap and contains a list of LU stations, their latitude and longitude coordinates, and a list of segments connecting the stations and their coordinates. We convert the cartographic data into a network representation of the LU system that also includes information about the length of each segment (measured in meters according to its curvature).

Next, to construct link flow data, we use the Rolling Origin-Destination Survey (RODS) periodically published by Transport for London (http://crowding.data.tfl.gov.uk/). As part of RODS, TfL provides data on segment loading by train line and by time of the day in a typical weekday. This loading is the number of people moving between adjacent stations via a certain train line during each time interval. Because some segments lie on multiple train lines, we aggregate these to obtain total loading from all lines. This data will be used to evaluate the implementation of our approach.

Finally, TfL also provides OD matrices in the same RODS data set describing the flow from each of the LU stations to all other stations by time of day in a typical weekday. As explained above, an element of these matrices corresponds to the number of individuals entering the LU in a given station and exiting it at another station. In this study, we focus on average daily flows and therefore, both the segment loadings and the OD matrix data are aggregated to a full day. We use the 2017 data for both the segment loadings and OD matrices.

2 Methodology

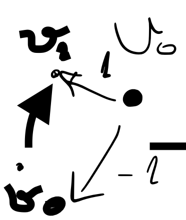
2.1 Hodge Decomposition

The Hodge decomposition is a discrete analogue of the Hemholtz decomposition which rests upon the fundamental theorem of vector calculus. Given a general flow vector on a discrete topology such as a network, we can represent the flow as a sum of gradient (curl-free), solenoidal (divergence-free), and harmonic flows (both curl- and divergence-free) [18]. To better illustrate the application of this decomposition to flow data, as well as its utility, we proceed to give a general introduction. For a more rigorous treatment of the topic, see [18, 19]. An enjoyable presentation of the topic with an applied flavor can be found in [20].

Let $G(V, E)$ represent a network of interest (for example, the LU network), where V is the set of n nodes and E the set of m undirected links. Depending on the exact set of links present, G may also exhibit triangles, i.e., triplets of nodes i, j, h forming a clique (all pairs of nodes of the triplet are connected by a link). In addition, we require the notion of orientation of the nodes, links, triangles, etc., which provides a convention over a network of what ordering the nodes must have. This orientation specifies a positive direction for flows. levi civita, permutations etc…

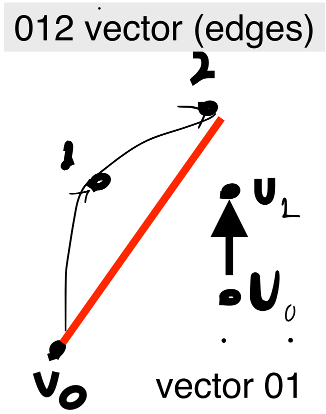
Hodge decomposition rests on operators that perform the equivalent functions of the continuous operators involved in Helmholtz's Theorem. In that context, one uses divergence (usually $\nabla\cdot$ of simply div), curl ($\nabla\times$ or curl), and the gradient operator (∇ or grad). A complete set of discrete equivalents to the continuous operators is available, which we now present.

First, the gradient operator is given by the $m \times n$ matrix $\mathbf{A}$ given by



$$\mathbf{A}_{[v_0,v_1],u_0} = \begin{cases} 1 & \text{if} \quad v_1 = u_0 \\ -1 & \text{if} \quad v_0 = u_0 \\ 0 & \quad \text{otherwise,} \end{cases}$$

where u_0, v_0, v_1 are nodes and $[v_0, v_1]$ an oriented link of G (where $[\cdot]$ is used to indicate orientation). If each node i is assigned some node potential value, say, η_i, the effect of $\mathbf{A}$ on the column vector $\boldsymbol{\eta}$ is to generate flows $\eta_i - \eta_j$ along a link $[i, j]$. These are gradient flows, contained in column vector $\boldsymbol{\phi}_g$.



Second, the transposed matrix $\mathbf{A}^T$ is perhaps more familiar than $\mathbf{A}$ in the networks literature. It is the (oriented) incidence matrix of G. It is not difficult show that $\mathbf{A}^T$ is equivalent to the $-$div (negative divergence) operator.

Third, the curl operator $\mathbf{B}$ is obtained in a similar way, and is the transpose of the oriented edge-face incidence matrix or the boundary-2 matrix:

$$\mathbf{B}_{[v_0,v_1,v_2],[u_0,u_1]} = \begin{cases} 1 & \text{if} \quad v_1 = u_0, v_2 = u_1 \quad \text{or} \quad v_0 = u_0, v_1 = u_1 \\ -1 & \text{if} \quad v_0 = u_0, v_2 = u_1 \\ 0 & \text{otherwise.} \end{cases}$$

The dimension of this matrix is $f \times m$, where f is the number of triangles or 2-simplices in the network. The curl effect can be seen from applying to $\mathbf{B}$ to a flow column vector, say $\boldsymbol{\phi}$, where each element is a flow along a link. The transpose matrix $\mathbf{B}^T$ has the effect of taking a column vector of network triangle potentials $\boldsymbol{\psi}$, and generating a column vector of link flows $\boldsymbol{\phi}_s$.

Theorem 1 (Cohomology). *The matrices* $\mathbf{A}^T$ *and* $\mathbf{B}^T$ *satisfy the cohomology property [18]*

$$\mathbf{A}^T\mathbf{B}^T = \mathbf{0}. \tag{1}$$

Similarly, through the definition of transpose of a matrix, the homology property is satisfied, i.e.

$$\mathbf{B}\mathbf{A} = \mathbf{0}. \tag{2}$$

This theorem tells us, in a more familiar calculus language, that div curl $= 0$ (i.e., curl is divergence free), and that curl grad $= 0$ (i.e., gradient is curl free).

To not lose sight of where we are headed, we must recall that the Helmholtz Theorem states that a vector field can be decomposed into a curl free divergent part and a divergence free rotational part. In discrete domains, the same basic notion applies. To construct these notions in discrete domains (i.e., networks), we take advantage of the linear algebra of the description.

Thus, for Hodge decomposition, our overall goal is to find a way to write any general flow $\boldsymbol{\phi}$ in an *organized* fashion. Let us define $\mathcal{F}_1$, the space of all possible flows $\boldsymbol{\phi}$ on the links of G. How could we decompose any flows in $\mathcal{F}_1$ into parts that are mutually exclusive, thus giving an unambiguous decomposition? If such a decomposition exists, $\mathcal{F}_1$ can be written as

$$\mathcal{F}_1 = \mathcal{F}_1^{(1)} \oplus \cdots \oplus \mathcal{F}_1^{(q)},$$

where it is assumed that $\mathcal{F}_1$ is decomposed into q mutually exclusive subsets of flows. Thus, any flow $\boldsymbol{\phi}$ can be constructed as a sum of elements of each of the subsets.

Consider the set of flows $\mathcal{F}_1^{(g)}$ that can be obtained from applying $\mathbf{A}$ to a node potential $\boldsymbol{\eta}$. This set is formally equal to the image set of $\mathbf{A}$, and thus can be written as $\mathcal{F}_1^{(g)} = \text{Image}(\mathbf{A})$. Let us denote by $\mathcal{F}_1^{(g^\perp)}$ the complement set of $\mathcal{F}_1^{(g)}$ with respect to $\mathcal{F}_1$, which includes all flows that cannot be generated from the application of $\mathbf{A}$ to a node potential. From this perspective, $\mathcal{F}_1 = \mathcal{F}_1^{(g)} \oplus \mathcal{F}_1^{(g^\perp)}$.

Another set of flows mentioned above were those given by the application of $\mathbf{B}^T$ to a triangle potential $\boldsymbol{\psi}$. Let us label this set as $\mathcal{F}_1^{(s)}$, which is formally given by $\mathcal{F}_1^{(s)} = \text{Image}(\mathbf{B}^T)$. Once again, it is also possible to define the complement set of $\mathcal{F}_1^{(s)}$, $\mathcal{F}_1^{(s^\perp)}$, and note that $\mathcal{F}_1 = \mathcal{F}_1^{(s)} \oplus \mathcal{F}_1^{(s^\perp)}$. The cohomology or homology relations shown before (Eqs. 1 and 2) also led to the realization that if $\boldsymbol{\phi} \in \mathcal{F}_1^{(g)}$, then it is not in $\mathcal{F}_1^{(s)}$. Similarly, if $\boldsymbol{\phi} \in \mathcal{F}_1^{(s)}$, then it is not in $\mathcal{F}_1^{(g)}$. These relations imply that $\mathcal{F}_1^{(g)} \cap \mathcal{F}_1^{(s)} = \emptyset$. Therefore, $\mathcal{F}_1$ can be decomposed as

$$\mathcal{F}_1 = \mathcal{F}_1^{(g)} \oplus \mathcal{F}_1^{(s)} \oplus \mathcal{F}_1^{(gs^\perp)}.$$

This implies that a general flow $\boldsymbol{\phi}$ can be written as a sum

$$\boldsymbol{\phi} = \boldsymbol{\phi}_g + \boldsymbol{\phi}_s + \boldsymbol{\phi}_h, \tag{3}$$

where $\boldsymbol{\phi}_g$ is a purely gradient flow, $\boldsymbol{\phi}_s$ is purely *solenoidal*, and $\boldsymbol{\phi}_h$ is called *harmonic* flow. Written in another way,

$$\boldsymbol{\phi} = \mathbf{A}\boldsymbol{\eta} + \mathbf{B}^T\boldsymbol{\psi} + \boldsymbol{\phi}_h, \tag{4}$$

where η_i and ψ_τ are called the "Hodge potential" of node i and triangle $\tau = (h, j, k)$, respectively.

To solve for the potentials of the nodes, note that

$$\boxed{\begin{aligned} -\mathbf{A}^T\boldsymbol{\phi} &= -\mathbf{A}^T\boldsymbol{\phi}_g - \mathbf{A}^T(\boldsymbol{\phi}_s + \boldsymbol{\phi}_h) \\ &= -\mathbf{A}^T\boldsymbol{\phi}_g = -\mathbf{A}^T\mathbf{A}\boldsymbol{\eta} = -\boldsymbol{\mathcal{L}}\boldsymbol{\eta}, \end{aligned}}$$

where $\boldsymbol{\mathcal{L}}$ is the Laplacian matrix of G. Matrix $\boldsymbol{\mathcal{L}}$ is singular (all its rows/columns add to 0), so only up to $n-1$ potentials can be uniquely determined. Arbitrarily assuming that one of the nodes has $\eta = 0$ eliminates the corresponding row and column in the problem and generates a reduce matrix $\boldsymbol{\mathcal{L}'}$ that can be used to obtain the remaining η. Specifically, we solve

$$\boldsymbol{\eta} = -\boldsymbol{\mathcal{L}'}^{-1}(\text{div } \boldsymbol{\phi}), \tag{5}$$

where it is understood that only the unknown potentials and the respective flows that contribute to them are included. This is in essence the gradient problem solved by the Poisson equation ($-\mathbf{A}^T\boldsymbol{\phi}$ provides the flow sources and sinks).

In a similar fashion, the triangle potentials can be determined from

$$\boldsymbol{\psi} = (\mathbf{B}\mathbf{B}^T)^{-1}\mathbf{B}\boldsymbol{\phi} \tag{6}$$

which provides a solution based on the measured curl values of the flow, i.e. $\mathbf{B}\boldsymbol{\phi}$. Finally, harmonic flows satisfy the homogeneous generalized Laplace equation

$$(\mathbf{A}\mathbf{A}^T + \mathbf{B}^T\mathbf{B})\boldsymbol{\phi}_h = \mathbf{0}. \tag{7}$$

2.2 Layered Hodge Decomposition and Flow Prediction

Our goal is to describe and reconstruct link (segment) flows in the London Underground system, or any other transit system, from just the knowledge of the network structure and an OD matrix. Note that the Hodge decomposition as described above could not achieve this since the input it requires are all the measured flows on the network. In other words, from the stated inputs, Hodge decomposition can describe the flows but not predict them. To overcome this limitation, we explore an adapted version of the Hodge decomposition which we refer to as the *layered Hodge decomposition* (LHD) method.

In the LHD method, the problem is broken up into n layers, one for each network node. In layer L_i, there is only one source node, i, and the remaining nodes can only be sinks. The i^{th} row sum $\|\mathbf{O}_i\|$ of the OD matrix tells us how many units of flow enter the system through i, and $\mathbf{O}_{ij}$ tells us how many of those are absorbed by a sink j. This layer approach allows us to use $\mathbf{O}$ in a way that is able to address two important effects: (1) the contribution of flow along a link due to travellers departing a particular node i, and (2) directionality of flows. For each layer, we can apply a method of choice to solve the flows from an origin i to all other destinations. Since in this paper we are addressing a proof of concept, we apply the "all-or-nothing" solution method of shortest paths (weighted by segment lengths), but this choice is not necessary for the validity of our results.

To provide further intuition for our approach, let us represent the link flows that result from solving the route assignment problem as $\boldsymbol{\varphi}^{(L_i)}$, where $\boldsymbol{\varphi}_{uv}^{(L_i)}$ is the flow on the oriented basis link $[u, v]$. The ability of our layered approach to address the contribution by a given origin can be seen intuitively: any non-zero flow $\boldsymbol{\varphi}_{uv}^{(L_i)}$ will increase monotonically with $\|\mathbf{O}_i\|$. To see why the approach addresses the directionality problem, consider that if the solution method leads to $|\boldsymbol{\varphi}_{uv}^{(L_i)}| > 0$, formally $\boldsymbol{\varphi}_{vu}^{(L_i)} = 0$, i.e., a link in the solution of a given layer is not traversed in both directions, $u \to v$ and $v \to u$. If this were the case, the method would not be optimizing flows. Violation of this assumption may be allowed in some methods with traveller uncertainty or error but this would only occur in rare cases. Thus, we neglect this effect.

Proceeding with our approach, each $\boldsymbol{\varphi}^{(L_i)}$ can be decomposed into the gradient, solenoidal, and harmonic flows that emerge from trips that start at node i and end everywhere else in the network. Critically, the application of Hodge decomposition to $\boldsymbol{\varphi}^{(L_i)}$ gives us a set parameters (the Hodge potentials $\boldsymbol{\eta}, \boldsymbol{\psi}$, and the harmonic flow $\widetilde{\boldsymbol{\phi}}_h$) in each layer that can be recorded and whose relationships with the network structure and flow characterizations can be analyzed and modeled. If such relationships can be understood and well-represented functionally across all of the layers, then to estimate link flows we would only require this functional knowledge and discard the rest of the information. In particular, suppose that

$$\widetilde{\boldsymbol{\eta}} = \mathbf{f}(G, \mathbf{O}), \quad \widetilde{\boldsymbol{\psi}} = \mathbf{g}(G, \mathbf{O}), \quad \text{and} \quad \widetilde{\boldsymbol{\phi}}_h = \mathbf{e}(G, \mathbf{O})$$

are our global approximations (in the sense that they are valid in all of the layers) of these Hodge potentials and the harmonic flows, and that $\mathbf{f}(\cdot), \mathbf{g}(\cdot)$, and $\mathbf{e}(\cdot)$ are some functions of only the network structure and the OD matrix. Then following Eq. 4 in the L_i layer we can estimate the link flows as

$$\begin{aligned}\boldsymbol{\phi}^{(L_i)} &\approx \mathbf{A}\widetilde{\boldsymbol{\eta}}^{(L_i)} + \mathbf{B}^T\widetilde{\boldsymbol{\psi}}^{(L_i)} + \widetilde{\boldsymbol{\phi}}_h^{(L_i)} \\ &= \mathbf{A}\mathbf{f}(G,\mathbf{O}_i) + \mathbf{B}^T\mathbf{g}(G,\mathbf{O}_i) + \mathbf{e}(G,\mathbf{O}_i),\end{aligned} \tag{8}$$

where again we assume that each link (u,v) should be traversed only in one direction and that should be reflected by the sign of $\phi_{uv}^{(L_i)}$. Finally, we can provide estimates $\tilde{t}_{uv}$ and $\tilde{t}_{vu}$ of directed link flows in the system at study by adding up our estimations across all of the layers, or

$$\begin{aligned}\tilde{t}_{uv} &= \textstyle\sum_i \phi_{uv}^{(L_i)} \times \theta(\phi_{uv}^{(L_i)}), \\ \tilde{t}_{vu} &= \textstyle\sum_i -\phi_{uv}^{(L_i)} \times \theta(-\phi_{uv}^{(L_i)}),\end{aligned} \tag{9}$$

where $\theta(\cdot)$ is the Heaviside step function.

Equation 9 provides a road map of our method. We need to find $\mathbf{f}, \mathbf{g}$, and $\mathbf{e}$. The first of these, $\mathbf{f}$, is actually quite straightforward and formally a function of $\mathbf{O}$ and G. That is because $\mathbf{f}$ is the solution to the classic gradient problem of hydraulic or electric circuits, where $-\boldsymbol{\eta}$ corresponds to hydrostatic pressure or electric potential. The challenge left is to determine the contributions from the last two terms and, as we show below, this can be estimated from knowledge of $\mathbf{O}$ and G.

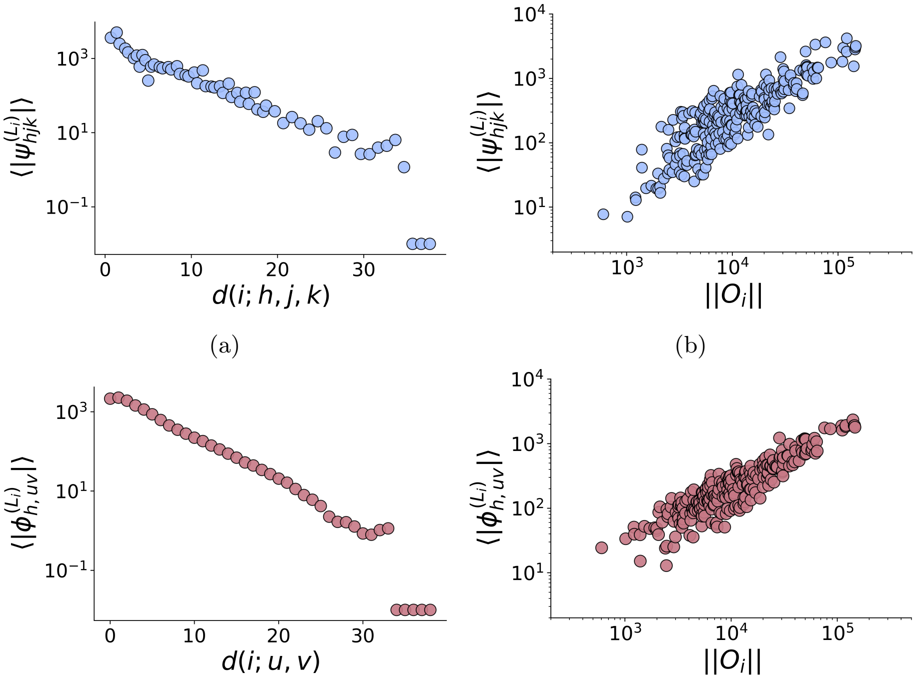
It is important to remind the reader that the estimated flows $\tilde{t}_{uv}$ depend on the specific route choice algorithm used to solve each layer. Since we are attempting to illustrate a proof of concept, it is reasonable to start from a simple shortest path algorithm, which approaches the desired rider behavior. A more detailed analysis may suggest a better choice, but the principles of our findings should be robust to this change.

3 Exploratory Analysis Using London Underground Data

We now apply LHD to describe the flows in the LU using the network structure and OD matrix of trips between LU stations in a typical day in 2017 (see Sect. 1.2 for more details about the data).

To develop some intuition about the nature of the solution of the route choice algorithm used here, we first study the relation between t_{uv} measured for the LU, and $\sum_i \boldsymbol{\varphi}_{uv}^{(L_i)}$ obtained from Dijsktra's shortest path algorithm (with segment lengths as weights) on each layer. An OLS regression with a zero y-intercept gives regression coefficient of 0.75 (s.e. 0.02) with $R^2 = 0.70$. This provides a basic idea of the representative power of this algorithm with respect to the actual flow data. However, since our immediate goal is not to strictly reproduce the flow for the LU, we do not dwell on this point any further.

To obtain the node potentials $\widetilde{\boldsymbol{\eta}}^{(L_i)}$, the only inputs needed are the ith row of $\mathbf{O}$ and G, which are then used in Eq. 5 to recover the desired solution. Note that any algorithm used to create flows along links layer by layer (the route choice



(a) (b) (c) (d)

Fig. 1: Panels (a) and (c): Expectations of the solenoidal potential $|\boldsymbol{\psi}_{hjk}^{(L_i)}|$ and harmonic flow $|\boldsymbol{\phi}_{h,uv}^{(L_i)}|$. The first is conditioned on the average hop-count from the source node i to the nodes of the triangle (h, j, k), and the second, from the source node i to link (u, v), as measured by the hop-count from i to either u or v, whichever is closest. Panels (b) and (d): conditional expectations of $|\boldsymbol{\psi}_{hjk}^{(L_i)}|$ and $|\boldsymbol{\phi}_{h,uv}^{(L_i)}|$, respectively, given the magnitude of flow originating from the source node.

algorithm) still needs to satisfy the same $\mathbf{O}$ matrix, and therefore, $\widetilde{\boldsymbol{\eta}}^{(L_i)}$ for all the layers is in fact *independent of the route choice algorithm*!

Going beyond the gradient flow, our analysis of the set of solenoidal potentials $\boldsymbol{\psi}$ in each layer leads to an approximate functional representation of $\boldsymbol{\psi}$ that can be inserted into Eq. 8. Concretely, we find that the magnitude of triangle potential hjk in each layer, $|\boldsymbol{\psi}_{hjk}^{(L_i)}|$, is a function of both the (hop-count) distance d from the source node to that triangle and the magnitude of input flow, $\|\mathbf{O}_i\|$, from the source node in that layer. The relationship with d is well-described by a log-linear function (Fig. 1a); with $\|\mathbf{O}_i\|$, the relation is linear (Fig. 1b). An equivalent functional form applies for $\widetilde{\boldsymbol{\phi}}_{h,uv}^{(L_i)}$, with linear dependence on $\|\mathbf{O}_i\|$ and an exponential decay due to distance between the source node and link (u, v).

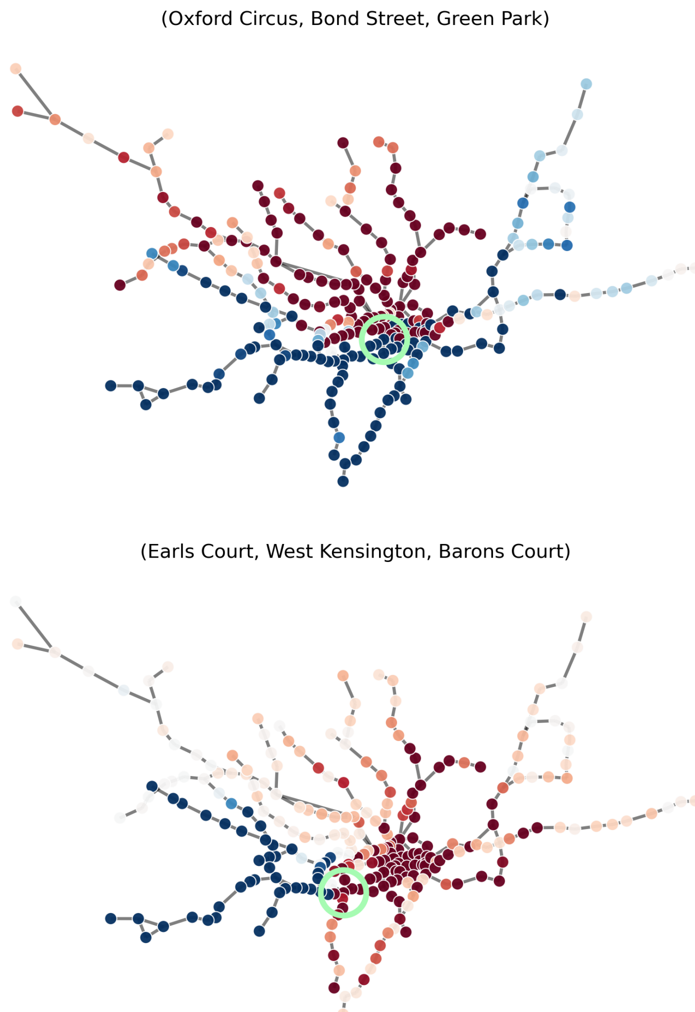


Fig. 2: A depiction of the value of the solenoidal potential $\boldsymbol{\psi}$ on two of the triangles in the LU network. When a node is colored red (blue), this means that in the layer where it is a source, the solenoidal potential on the triangle labeled in the title obtained from solving the LHD problem in that layer is negative (positive). When the color is grey, the potential value is 0. The approximate locations of the triangles are indicated via the green circles. It is clear to see that the sign of $\boldsymbol{\psi}_{hjk}^{(L_i)}$ has some dependence on the network structure and the geographic location of the source node i relative to the triangle (h, j, k). Similar behaviors are observed with the rest of the triangles.

These empirical regularities allow us to approximate functional representations for $|\boldsymbol{\psi}|$ and $|\boldsymbol{\phi}_h|$, given by

$$|\widetilde{\boldsymbol{\psi}}_{hjk}^{(L_i)}| \approx \mathbf{g}'(G, \mathbf{O}_i) = \kappa_1 \|\mathbf{O}_i\| \exp[-\kappa_2 d(i; h, j, k)], \tag{10}$$

$$|\widetilde{\boldsymbol{\phi}}_{h,uv}^{(L_i)}| \approx \mathbf{e}'(G, \mathbf{O}_i) = \gamma_1 \|\mathbf{O}_i\| \exp[-\gamma_2 d(i; u, v)], \tag{11}$$

where the numerical constants are given by $\kappa_1 \approx 0.1, \kappa_2 \approx 0.1, \gamma_1 \approx 0.07$, and $\gamma_2 \approx 0.1$. Equations 10 and 11 only describe the magnitudes of the solenoidal potentials and harmonic flows, but not the signs. In fact, both $\boldsymbol{\psi}_{hjk}$ or $\boldsymbol{\phi}_{h,uv}$ can take on positive or negative values, or even 0. Focusing on $\boldsymbol{\psi}_{hjk}$, we find a very interesting pattern in which the way $\boldsymbol{\psi}_{hjk}$ takes on a positive or negative sign depends highly on the geographic location of the source node relative to the triangle. This dependence is clear to see from the maps we have plotted in Fig. 2, suggesting convincingly that $\boldsymbol{\psi}$ can likely be written as a function solely of G and $\mathbf{O}$. Similar ideas apply for $\boldsymbol{\psi}_{hjk}$. While we do not work out the exact specification of this sign function in this exploratory analysis, these geographic regularities provide a path forward for the immediate future work.

As a final step, we need to sum all the flows and potentials through Eq. 8 and then apply those into Eq. 9.

3.1 Discussion and Conclusion

Our work suggests that the non-trivial flows observed in a transit network such as the London Underground do not generally require a full solution of the Hodge decomposition to be predicted. Instead, Eqs. 10 and 11 provide a way to approximate the solenoidal and harmonic flows using only the network structure and the OD matrix, both of which are trivially necessary for any solution of the problem. Moreover, an improvement on Eqs. 10 and 11 would make it possible to reduce the need to employ difficult algorithms to try and predict flows along network links. In addition, given that some methods for solution of flows are stochastic in nature, reasonable approximations to $\widetilde{\boldsymbol{\psi}}$ and $\widetilde{\boldsymbol{\phi}}_h$ may provide as much information as some algorithms.

In order to move the Hodge decomposition description of network flows into a method, we would require a systematic study of the variations of $\widetilde{\boldsymbol{\psi}}$ and $\widetilde{\boldsymbol{\phi}}_h$ at short time intervals in order to use this quantity to be able to handle faster flow adjustments happening within the time frame of a day or even within-day. Our analysis should be put to the test in other transit systems in order to test the robustness of our finding more thoroughly. However, the observation that many urban transit networks, especially in monocentric cities, are structurally similar to the LU network (hub-and-spoke designs) gives us some optimism. On the whole, our exploratory analysis here should provide a convincing case for the viability of the novel method we have described and more efforts in this direction are thus warranted.

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