Follow the programming?

- Open the VM
- ./login
- cd rfun/
- git pull
- make

I made the repository public, just for you.

RFun: a reversible functional programming language

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Overview

- Development of reversible programs
    - Embedding a function
- Introduction to RFun
- Development in RFun
    - Arithmetic
    - List functions
    - Equality / duplication

Tutorial can be found at:
http://topps.diku.dk/pirc/rfundocs/

Embeddings

Thinking reversibly

$$2 + 3 = 5$$

$$5 = A + B$$

Reversible embeddings:

$$+(2,3) = ([(2,3),(3,2),(4,1),(5,0)],5)$$

$$+(2,3) = ((2,3),5)$$

Embedded addition

$$+(x,y) = (x, y + x)$$

Reversible embeddings

- How to get a reversible program from a known irreversible program?
- Can we do it automatically?
  - I.e. using program transformation

Embeddings

- Landauer embedding
- Bennett embedding
- (Incremental check-pointing)

Landauer embedding

$$+(x, y) = ([...trace...], x + y)$$

$$+(2, 3) = ([(2, 3), (3, 2), (4, 1), (5, 0)], 5)$$

- Also called trace embedding
  - First (indirectly) prosed by Landauer
- Store a needed amount of information every time you make a irreversible choice
- Will have size equivalent to run-time

Bennett embedding

$$+(x, y) = ((x, y), x + y)$$

$$+(2, 3) = ((2, 3), 5)$$

- Input-output embedding
- First proposed by Bennett
- Can be implemented by Bennett's compute-copy-uncompute method
- Size is limited by input/output
  - but computational space is at the size of Landauer embedding
- Cannot be cascaded

Reversible implementation

$$+(x, y) = (x, x + y)$$

If function injective

- Know reversible implementation exist
  - McCarthy's generate-and-test approach
- Can still be hard to find

If function non-injective

- We have to redefine out problem

Considering your semantics

$$+(x, y) = ([...trace...], x + y)$$

$$+(x, y) = ((x, y), x + y)$$

$$+(x, y) = (x, x + y)$$

Garbage

- Semantically undesired values

Ancillae

- Values that are *guaranteed* unchanged over a computation

RFun

Fibonacci

```
procedure fib(int x1, int x2, int n)
    if n = 0 then
        x1 += 1
        x2 += 1
    else
        n -= 1
        call fib(x1, x2, n)
        x1 += x2
        x1 <=> x2
    fi x1 = x2
```

Fibonacci

```
procedure fib(int x1, int x2, int n)
    if n = 0 then
        x1 += 1
        x2 += 1
    else
        n -= 1
        call fib(x1, x2, n)
        x1 += x2
        x1 <=> x2
    fi x1 = x2
```

Comparison from Janus

```
fib :: Nat <-> (Nat, Nat)
fib Z      = ((S Z),(S Z))
fib (S m) =
  let (x,y) = fib m
      y' = plus x y
  in (y',x)
```

RFun

- A history-free functional reversible language
- Implements (often) injective partial functions
  - I.e. we are usually not implementing in bijections

Background

- First formalised in 2012 by Tetsuo Yokoyama, Robert Glück, and Holger Bock Axelsen [1]
  - Untyped, first-order language based on constructor terms
- Implemented and explored by Michael Kirkedal Thomsen and Holger Bock Axelsen [2]
  - Extension to (somewhat) second-order language with syntactical support for tuples, lists, and natural numbers
- New version designed for this training school by Michael Kirkedal Thomsen
  - Updated syntax and extended with type system

Important concepts

- Linearity
- Ancillae
- First-match policy

Important concepts - Linear typing

- Stems from linear logic
- We **must** use a resource exactly once

Examples of usage

- Handling of memory resources
- Seen in many modern language
    - most recently Rust

Linearity - explained

```
fib :: Nat <-> (Nat, Nat)
fib Z     = ((S Z),(S Z))
fib (S m) =
  let (x,y) = fib m
      y'  = plus x y
  in (y',x)
```

- Variable `m`
  - introduced on the left-hand-side
  - used exactly in the recursive call to `fib`.
- Similarly for `y` and `y'`
  - also consider the return of `y'` as a usage.

Important concepts - Ancillae type

- Variables for which we can *guarantee* that the value is unchanged over a function call.
- The *guarantee* is important
  - using a conservative approach.

Also seen in

- Reversible logic
- Restore model

Ancillae - explained

```
fib :: Nat <-> (Nat, Nat)
fib Z     = ((S Z),(S Z))
fib (S m) =
  let (x,y) = fib m
      y'  = plus x y
  in (y',x)
```

- Variable `x`
  - introduced by recursive call to `fib`
  - used by the `plus` function
  - returned by the `fib` function.
- Here `plus` is using `x` as an ancillae

Important concepts - First-match policy

- important to guarantee injectivity of the functions
- conceptually, function must not return value that can be the result of any previous branches
  - knowing nothing about the possible content of variables.

Two sides to the coin

- Pattern matching for clauses in inverse interpretation
- Alternative to Janus assertions

First-match policy - detailed

$$
\begin{array}{l}
\textbf{case } l \textbf{ of} \\
\quad l_1 \rightarrow \ \cdots \ \textbf{in } l'_1 \\
\qquad \vdots \\
\quad l_i \rightarrow \ \cdots \ \textbf{in } l'_i \\
\qquad \vdots \\
\quad l_n \rightarrow \ \cdots \ \textbf{in } l'_n
\end{array}
$$

- The value v of l is matched against the left-hand side of each branch $(l_1, l_2, \ldots)$ until the first successful match l_i.
- The right-hand side of the i-th branch is then evaluated in σ_i and a value v' is returned by l'_i.
- Now, for symmetry, v' must not match any of the preceding $l'_1, \ldots, l'_{i-1}$
- Otherwise, the case-expression is undefined.

First-match policy - explained

```
procedure fib(int x1, int x2, int n)
    if n = 0 then
        x1 += 1
        x2 += 1
    else
        n -= 1
        call fib(x1, x2, n)
        x1 += x2
        x1 <=> x2
    fi x1 = x2
```

```
fib :: Nat <-> (Nat, Nat)
fib Z     = ((S Z),(S Z))
fib (S m) =
  let (x,y) = fib m
      y' = plus x y
  in (y',x)
```

- Base-clause matches the non-recursive branch

First-match policy - explained

```
fib :: Nat <-> (Nat, Nat)
fib Z     = ((S Z),(S Z))
fib (S m) =
  let (x,y) = fib m
      y'  = plus x y
  in (y',x)
```

- FMP can often be check statically
  - based on the type definitions
  - This is not yet implemented in RFun
- FMP is in general impossible to check statically.

E.g. `fib` **cannot** be statically checked

Implementation in RFun

Natual number arithmetic

Define Peano numbers

```
data Nat = Z | (S Nat)
```

- Read: a natural number (`Nat`) is either zero (`Z`) or the successor of a natural number (`(S Nat)`).

Type for simple incremental function

```
inc :: Nat <-> Nat
```

- Read: increment (`inc`) is a (reversible) function that transforms a `Nat` to a `Nat`

Natual number arithmetic

```
data Nat = Z | (S Nat)
inc :: Nat <-> Nat
```

Implementation if `inc`

Natual number arithmetic

```
data Nat = Z | (S Nat)
inc :: Nat <-> Nat
```

Implementation if `inc`

```
inc n = (S n)
```

- Read: `inc` given a natural number `n` returns the successor of `n`.

Natual number arithmetic

```
data Nat = Z | (S Nat)
inc :: Nat <-> Nat
inc n = (S n)
```

Decremental can be implemented as

```
dec :: Nat <-> Nat
```

Natual number arithmetic

```
data Nat = Z | (S Nat)
inc :: Nat <-> Nat
inc n = (S n)
```

Decremental can be implemented as

```
dec :: Nat <-> Nat
dec (S n) = n
```

Natual number arithmetic

```
data Nat = Z | (S Nat)
inc :: Nat <-> Nat
inc n = (S n)
```

Decremental can be implemented as

```
dec :: Nat <-> Nat
dec (S n) = n
```

or even better

```
dec :: Nat <-> Nat
dec n = inc! n
```

- `!` at the end of the function specifies reverse execution.

Reversible Addition

$$+(x, y) = (x, y + x)$$

Type for addition?

Reversible Addition

$$+(x, y) = (x, y + x)$$

Type for addition

```
plus' :: (Nat, Nat) <-> (Nat, Nat)
```

Reversible Addition

$$+(x, y) = (x, y + x)$$

Type for addition

```
plus' :: (Nat, Nat) <-> (Nat, Nat)
```

Considering that x is an ancilla value

```
plus :: Nat -> Nat <-> Nat
```

- Read: `plus` is a function that given a natural number (`Nat`), will transform one natural number (`Nat`) to another natural number (`Nat`).

Reversible Addition

Implementation

```
plus :: Nat -> Nat <-> Nat
plus Z     x = x
plus (S y) x =
  let x' = plus y x
  in  (S x')
```

Reversible Addition

Implementation

```
plus :: Nat -> Nat <-> Nat
plus Z     x = x
plus (S y) x =
  let x' = plus y x
  in  (S x')
```

- Why is the first argument guaranteed to be unchanged?
- Why is the second argument linear?
- Why is the first-match policy upheld?

Reversible Addition

```
plus :: Nat -> Nat <-> Nat
plus Z     x = x
plus (S y) x =
  let x' = plus y x
  in  (S x')
```

- Why is the first argument guaranteed to be unchanged?
  - `y` is only used as an ancilla in the recursive call
- Why is the second argument linear?
  - Both `x` and `x'` is first introduced then used
- Why is the first-match policy upheld?
  - First (ancilla) argument of two clauses are disjoint.

Addition - FMP explained

```
plus :: Nat -> Nat <-> Nat
plus Z     x = x
plus (S y) x =
  let x' = plus y x
  in  (S x')
```

We transform `plus` into

```
plusP :: (Nat, Nat) <-> (Nat, Nat)
plusP (Z,    x) = (Z, x)
plusP (S(y), x) =
  let (y, x') = plusP (y, x)
  in  (S(y), S(x'))
```

- Here is it clear that the first element of the branch tuples are disjoint, making the entire tuple disjoint.

Addition transformation explained

```
plusP :: (Nat, Nat) <-> (Nat, Nat)
plusP (Z,    x) = (Z, x)
plusP (S(y), x) =
  let (y, x') = plusP (y, x)
  in  (S(y), S(x'))
```

How do we generate `plusP`?

- wrap our input into a tuple,
- add the ancillae arguments to all output leaves,
- wrap all function calls into tuples,
- add the ancillae inputs to function calls to the output.

Transformation of ancilla functions

With the above method, we can always transform

```
f :: a -> b <-> c
```

into

```
f :: (a, b) <-> (a, c)
```

- This does not (in general) work in the opposite direction.

List functions

- List is a predefined type in RFun
- Use standard notation
  - `[` and `]` for list specification
  - `(l : ls)` for list construction

Examples

- List of 5 elements and empty list
  - `[1,2,3,4,5]` and `[]`
- A list construction
  - `(1:2:[3,4,5])`

The lenght of a list

The type of this is

```
length :: [a] <-> Nat
```

!!!!

The lenght of a list

The type of this is

```
length_wrong :: [a] <-> Nat
```

- Length should be treated as a property of a list
  - I.e. given a list we need to extract the information

```
length :: [a] -> () <-> Nat
```

- Read: `length` is a function that given a list, transforms nothing to a natural number.
  - Empty tuple does not contain any information
- I.e. we are making a Bennett embedding of the normal length function.

Implementation of length

We can then implement length as the standard

```
length :: [a] -> () <-> Nat
length []       () = Z
length (x : xs) () =
  let  n = length xs ()
  in   (S n)
```

- Why is the input list guaranteed to be ancillae?
- How is linear typing upheld?
- How is FMP upheld?

Implementation of length

We can then implement length as the standard

```
length :: [a] -> () <-> Nat
length []       () = Z
length (x : xs) () =
  let  n = length xs ()
  in   (S n)
```

- Why is the input list guaranteed to be ancillae?
  - `xs` is only used for ancilla to `length`
- How is linear typing upheld?
  - The introduced variable `n` is used again
- How is FMP upheld?
  - `Z` is disjoint from `(S Nat)`

Mapping functon over list

```
map :: (a <-> b) -> [a] <-> [b]
```

- Read: given a function that transforms `a`'s to `b`'s, `map` will transform a list of `a`'s to a list of `b`'s.
  - This is exactly how we consider the normal map-function.

Mapping functon over list

Implementation

```
map :: (a <-> b) -> [a] <-> [b]
map fun     []   = []
map fun (x : xs) =
  let x'  = fun x
      xs' = map fun xs
  in  (x' : xs')
```

- Ancilla of the mapped function?
- Linearity of the lists?
- Is FMP upheld?

List reversal

```
reverse :: [a] <-> [a]
```

Two different approaches

- Appending a first element of a list to the end of a reversed list
  - Squared to the length of the list run-time
- Using an accumulator
  - Linear to the length of the list run-time
  - Helper function moves elements from one list to the other

We will go for the fast version.

List reversal

Function from Haskell

```
reverse_haskell l =  rev l []
  where
    rev []     a = a
    rev (x:xs) a = rev xs (x:a)
```

List reversal

Function from Haskell

```
reverse_haskell l =  rev l []
  where
    rev []      a = a
    rev (x:xs) a = rev xs (x:a)
```

Reversible implementation of `rev`

```
rev :: ([a], [a]) <-> ([a], [a])
rev (   []   , l) = ([], l)
rev ((x:xs), l) = move (xs, (x:l))
```

- Is linearity guarenteed?
- Is FMP upheld?

Moving elements

Instead lets make a function that moves some elements

```
move :: Nat -> ([a], [a]) <-> ([a], [a])
move  Z     (    x   , l) = (x, l)
move (S s) ((x:xs), l) = move s (xs,(x:l))
```

- Is Ancilla types guaranteed?
- Is linearity guaranteed?
- Is FMP upheld?

List reversal

Final reversal of list

```
reverse :: [a] <-> [a]
reverse xs =
  let xs_s = length xs ()
      ([], ys) = move xs_s (xs, [])
      () = length! ys xs_s
  in  ys
```

- How is Ancilla types guaranteed?
- Is linearity guaranteed?
- Is FMP upheld?
- Is this linear run-time to the size of the list?

Equality and Duplication

Equality (and duplication) have a special place in FRun.

- Predefined type

```
data EQ = Eq | Neq a
```

- This is equivalent to the definition of Maybe monad from Haskell
    - We will not use it as a monad...

Predefine function of type

```
eq :: a -> a <-> EQ
```

Equality and Duplication

```
data EQ = Eq | Neq a
eq :: a -> a <-> EQ
```

Given `eq x y`,

- first argument (`x`) is ancillae
- second argument (`y`), will be transformed into a `EQ` type, where the result is
  - `Eq` if `x` is equal to `y`
  - `Neq y` if `x` is different from `y`.

Note, equality can remove one copy of the two values.

- No implementation of `eq` is possible in RFun.

Equality and Duplication

```
data EQ = Eq | Neq a
eq :: a -> a <-> EQ
```

Based on `eq` we can then make a duplication function by inverse execution

```
dup :: a -> () <-> a
dup x () =  eq! x Eq
```

Run-length encoding

```
pack :: [a] <-> [(a, Nat)]
pack [] = []
pack (c1 : r) =
  case (pack r) of
    [] -> [(c1, 1)]
    ((c2, n) : t) ->
      case (eq c1 c2) of
        (Neq c2p) -> ((c1, 1) : (c2p, n) : t)
        (Eq) -> ((c1, (S n)) : t)
```

- Notice usage of `eq`

Example of Running it

```
pack [1,1,3,2,2,2] = [(1,2),(3,1),(2,3)]
```

- Why is FMP upheld? Can we statically check it?

Summing of RFun

RFun

- A history-free functional reversible language
- Implements (often) injective partial functions
  - I.e. we are usually not implementing in bijections
- Looks like a functional language
- First steps toward higher-order language
- Type system with linear and ancilla types
- Support for tuples and lists

Future work and extensions

- Int type
- Guards
- Static check of FMP
- Support for kinds; especially with usage for `eq`
- Partial applications and higher-order functions

Arithmetic Exercises

- Implement a `minus` function
- Implement a function that multiplies by 2
- Implement a multiplication function **(hard)**
  - What embedding makes sense?
- Implement a function `even`, that checks if a natural number is even
  - What embedding of `even` makes sense?

Arithmetic List

- Implement a function that checks if all elements of a list is even
- Implement `splitAt` that splits a list in two after a given lenght
  - What is a good embedding?
  - You can find the Haskell definition here http://hackage.haskell.org/package/base-4.10.0.0/docs/Prelude.html#v:splitAt
- Implement an `append` function
  - Again, the embedding is not clear?

Arithmetic List - continued

-
  - Implement `scanl` and `scanr` **(hard)**
  - You can find the Haskell definition here http://hackage.haskell.org/package/base-4.10.0.0/docs/Prelude.html#v:scanl
- Implement `foldl` and `foldr` **(even hard)**
  - What is a resonable embedding?
  - How does this differ from the scan's
- Continue with `interleave`, `intercalate`, `permutations`, `group`, `subsequences`, `transpose`, ...

References

[1] T. Yokoyama and H. B. Axelsen and R. Gluck, Towards a reversible functional language, Reversible Computation, RC '11, 7165 14--29 (2012)

[2] M. K. Thomsen and H. B. Axelsen, Interpretation and Programming of the Reversible Functional Language, Proceedings of the 27th Symposium on the Implementation and Application of Functional Programming Languages, 8:1--8:13 (2016)

[3] M. K. Thomsen, RFun tutorial, http://topps.diku.dk/pirc/rfundocs/

Thank you!