THE NUMBER OF CIRCLES COVERING A SET.*

By Richard Kershner.

It is well known that the "best" way to cover a given area with circles of a given radius ϵ, or to pack such circles within a given region is to place the centers of the circles on an equilateral triangle network, i. e., to circumscribe (inscribe) the circles about the hexagons of a regular hexagon network or honeycomb. This, of course, is not a precise statement, and, in fact, it is difficult to make a precise statement in this direction that is true. Roughly, the statement becomes more true as ϵ is taken smaller in relation to the area of the given region.

The most usual [1] precise statement of this fact is that the densest plane Punktgitter is that of the equilateral triangle. This statement avoids the difficulties caused by the boundedness of the bounded region but is less general than might be desired in that permissible packings or coverings are limited to those in which the centers of the circles form a Punktgitter.

The object of the present paper is to give a new and elementary proof of a precise statement in this direction; a statement involving no restriction on the nature of permissible coverings or on the nature of the given region. Specifically the statement to be proved is the following:

Theorem. *Let M denote a bounded plane point set and let $N(\epsilon)$ be the minimum number of circles of radius ϵ which can cover M. Then*

$$(\alpha) \qquad \lim_{\epsilon \to 0} \pi\epsilon^2 N(\epsilon) = (2\pi\sqrt{3}/9) \text{ meas } \bar{M}$$

where $\bar{M}$ denotes the closure of M.

Since the left side of (α) is simply the total area of the circles covering M, the constant $(2\pi\sqrt{3}/9) = 1.209 \cdots$ may be thought of as measuring the proportion of unavoidable overlapping.

In the proof of the theorem there will be needed a number of lemmas, several of which have an independent interest. For instance, Lemma 5 and Lemma 6 together constitute, in one sense, a considerable refinement of (α) as providing an estimate for the error term implied by the limiting relation

* Received December 13, 1938.

[1] Cf., e. g., Hilbert, Cohn-Vossen, *Anschauliche Geometrie*, p. 32.

(α); but, of course, this refinement is obtained only by the sacrifice of the generality of the region M.

A proof of the known[2] Lemma 1 has been included for the sake of completeness.

LEMMA 1. *Let Γ denote a bounded plane network[3] consisting of a finite number of finite polygons. Suppose that each vertex of Γ is on at least three edges. Then the average number of sides of the polygons of Γ is < 6.*

Proof. Let V, E, F denote the number of vertices, edges and faces respectively, in Γ. Then the Euler relation for Γ is

$$V - E + F = 1. \tag{1}$$

The fact that each vertex is on at least three edges may be expressed, in view of the fact that each edge is on exactly two vertices, by the relation

$$3V \leqq 2E. \tag{2}$$

Elimination of V between (1) and (2) gives

$$E \leqq 3F - 3. \tag{3}$$

Now let the faces of Γ be numbered and let e_i $(i = 1, 2, \cdots, F)$ denote the number of edges on the face numbered i. Then, since some, but not all, edges are on two faces,

$$\sum_{i=1}^{F} e_i < 2E \leqq 6F - 6.$$

Thus,

$$(1/F) \sum_{i=1}^{F} e_i < 6 - 6/F < 6.$$

This completes the proof of Lemma 1.

The next two lemmas that are needed are of an extremely elementary nature

LEMMA 2. *Let σ be a fixed circle and let A_k denote the area of a regular polygon of k sides inscribed in σ. Then*

$$0 < A_{k+1} - A_k < A_k - A_{k-1} \qquad (k \geqq 3).$$

Proof. The first of these inequalities is well known. To show that the second inequality of Lemma 2 holds, let

[2] Cf., e. g., M. Goldberg, "The isoperimetric problem for polyhedra," *Tohoku Mathematical Journal*, vol. 40 (1934), pp. 228-229.

[3] By a network is meant the figure consisting of a set of non-overlapping polygons which together cover a simply connected domain.

$$f(k) = A_{k+1} - A_k = \frac{k+1}{2} r^2 \sin \frac{2\pi}{k+1} - \frac{k}{2} r^2 \sin \frac{2\pi}{k}.$$

Then

$$\frac{df(k)}{dk} = \frac{r^2}{2}\left[\sin \frac{2\pi}{k+1} - \frac{2\pi}{k+1} \cos \frac{2\pi}{k+1}\right] - \frac{r^2}{2}\left[\sin \frac{2\pi}{k} - \frac{2\pi}{k} \cos \frac{2\pi}{k}\right].$$

But this expression is negative for $k \geqq 3$ since $\sin x - x \cos x$ is an increasing function of x in $(0, \pi)$. Thus $f(k)$ is a decreasing function of k for $k \geqq 3$ which completes the proof.

LEMMA 3. *With the notations of Lemma 2 and $k > j \geqq 3$,*

$$(k - j)(A_k - A_{k-1}) \leqq A_k - A_j \leqq (k - j)(A_{j+1} - A_j).$$

Proof. This is an immediate corollary to Lemma 2 since Lemma 2 states that the interval from A_j to A_k consists of $(k - j)$ subintervals of which the longest is $A_{j+1} - A_j$ and the shortest $A_k - A_{k-1}$.

LEMMA 4. *Let Γ denote a bounded plane network consisting of F finite polygons. Suppose that each vertex of Γ is on at least three edges. Suppose, finally, that each polygon of Γ can be covered by a circle σ of fixed radius r. Then the total area of Γ is $< FA_6$, where $A_6 = (3\sqrt{3}/2)r^2$ is the area of a regular hexagon inscribed in σ.*

Proof. Let n_i $(i = 3, 4, \cdots, m)$ denote the number of polygons in Γ with i edges, so that $n_3 + n_4 + \cdots + n_m = F$. Then, according to Lemma 1, the average number of sides of the polygons of Γ is < 6, i. e.,

$$\sum_{i=3}^{m} i n_i < 6 \sum_{i=3}^{m} n_i$$

which may also be written

$$\sum_{i=7}^{m} (i - 6) n_i < \sum_{i=3}^{5} (6 - i) n_i. \tag{4}$$

By Lemma 2, if the left side of (4) be multiplied by $A_7 - A_6$ and the right by $A_6 - A_5$ the inequality will be strengthened. Thus

$$\sum_{i=7}^{m} (i - 6)(A_7 - A_6) n_i < \sum_{i=3}^{5} (6 - i)(A_6 - A_5) n_i. \tag{5}$$

Now, by the first inequality of Lemma 3,

$$(6 - i)(A_6 - A_5) \leqq A_6 - A_i \qquad (i < 6); \tag{6}$$

and by the second,

$$A_i - A_6 \leqq (i - 6)(A_7 - A_6) \qquad (i > 6). \tag{7}$$

Then from (5), (6) and (7),

$$\sum_{i=7}^{m} (A_i - A_6) n_i < \sum_{i=3}^{5} (A_6 - A_i) n_i$$

or

$$\sum_{i=3}^{m} A_i n_i < A_6 \sum_{i=3}^{m} n_i = F A_6. \tag{8}$$

Since the left member of the inequality (8) is $\geqq$ the total area of Γ the proof of Lemma 4 is complete.

LEMMA 5. *Let ρ denote a rectangle in the plane with area R. Let $N(\epsilon)$ denote the minimum number of circles of radius ϵ which can cover ρ. Then*

$$\pi\epsilon^2 N(\epsilon) > (2\pi\sqrt{3}/9)(R - 2\pi\epsilon^2).$$

Proof. Let σ_i $(i = 1, 2, \cdots, N(\epsilon))$ be $N(\epsilon)$ circles of radius ϵ which cover ρ and let C_i denote the center of σ_i. If the circles σ_i and σ_j intersect let c_{ij} denote their common chord. If the circles σ_i and σ_j are disjoint or tangent it will be said that c_{ij} does not exist.

Now, for every $i = 1, \cdots, N(\epsilon)$, let π_i be defined as the set of points Q in $\rho \cdot \sigma_i$ satisfying

$$d(Q, C_i) \leqq d(Q, C_j) \qquad \text{for any } j \neq i; \tag{9}$$

where $d(O, Q)$ is the distance between the points O and Q. The set π_i defined by (9) is clearly a closed subset of σ_i for every i. Furthermore, π_i is non-empty for every i. This is clear if C_i is in ρ, but in any case there must exist some point Q in $\rho \cdot \sigma_i$ which is in no σ_j with $j \neq i$; otherwise the $N(\epsilon) - 1$ circles σ_j, $j \neq i$, would cover ρ, contradicting the assumption that $N(\epsilon)$ was the minimum number of circles which could cover ρ.

Let B denote a boundary point of π_i. Then either B is a boundary point of ρ or

$$d(B, C_i) = d(B, C_j) \qquad \text{for some } j \neq i.$$

Thus B is either on the boundary of ρ or on some c_{ij}, so that π_i is a polygon bounded by subsegments of the c_{ij} and subsegments of the boundary of ρ.

The set of polygons π_i form a non-overlapping collection of polygons which exactly cover ρ. Unfortunately this collection of polygons does not quite satisfy the conditions of Lemma 4 since the four vertices of ρ are vertices of some π_i where only two edges meet. If these four vertices are "straightened out" a new collection of polygons π'_i results which "almost" covers ρ and, in particular, having a total area $> R - 2\pi\epsilon^2$. It will now be shown that the network consisting of the π'_i satisfies the conditions of Lemma

4 with $r = \epsilon$. To do this it is clearly sufficient to show that the network consisting of the π_i satisfy these conditions save for the four exceptional vertices mentioned. Since the second hypothesis of Lemma 4 is obviously satisfied, all that is required is to show that each vertex of any π_i, not a vertex of ρ, is on at least three edges. To this end let V be a vertex of π_k and let it be supposed first that the two edges of π_k intersecting at V are both subsegments of chords c_{ki} and c_{kj}. Then, by the definition of the c_{kh} together with the definition (9) of π_k,

$$d(V, C_k) = d(V, C_i) = d(V, C_j) \leqq d(V, C_h) \quad \text{for any } h = 1, 2, \cdots, N(\epsilon).$$

But these same relations imply that V is a point of π_i and a point of π_j so that at least three polygons, and, consequently, at least three edges, meet at V. The other case to be considered is the case that one of the edges of π_k through V is a segment of the boundary of ρ, the other edge being, of course, a subsegment of some c_{ki}. In this case, only two polygons can be shown by the above reasoning to meet at V, namely, π_k and π_i. But since the exterior of ρ also has V as a limit point there are still at least three regions about V and three edges (two of which will be segments of the boundary of ρ) meeting at V. Thus the assumptions of Lemma 4 are satisfied by the network of the π'_i.

Applying Lemma 4 it is seen that

$$R - 2\pi\epsilon^2 < N(\epsilon)(3\sqrt{3}/2)\epsilon^2$$

or

$$\pi\epsilon^2 N(\epsilon) > (2\pi\sqrt{3}/9)(R - 2\pi\epsilon^2).$$

This completes the proof of Lemma 5.

LEMMA 6. *Let ρ denote a rectangle in the plane with area R and perimeter p. Let $N(\epsilon)$ denote the minimum number of circles of radius ϵ which cover ρ. Then*

$$\pi\epsilon^2 N(\epsilon) < (2\pi\sqrt{3}/9)(R + 2p\epsilon + 16\epsilon^2).$$

Proof. Let $\tau(\epsilon)$ denote the regular hexagon inscribable in a circle of radius ϵ. Let the plane be "paved" with regular hexagons $\tau_i(\epsilon)$ and let $\tau_1(\epsilon), \tau_2(\epsilon), \cdots, \tau_N(\epsilon)$ be the hexagons of this "paving" which have a point in common with ρ. Then the set of hexagons $\tau_i(\epsilon)$ $(i = 1, 2, \cdots, N)$, cover ρ and are covered by the rectangle obtained by adjoining a linear strip of width 2ϵ all around ρ. This latter rectangle has area $R + 2p\epsilon + 16\epsilon^2$, so that

$$(3\sqrt{3}/2)\epsilon^2 N < R + 2p\epsilon + 16\epsilon^2. \tag{10}$$

But since $\tau_1(\epsilon) + \tau_2(\epsilon) + \cdots + \tau_N(\epsilon)$ covers ρ, the N circles of radius ϵ which circumscribe these $\tau_i(\epsilon)$ cover ρ, i. e.,

$$N(\epsilon) \leqq N. \tag{11}$$

Combining (10) and (11) completes the proof of Lemma 6.

Proof of the theorem. Let M be an arbitrary bounded plane point set and let $N(\epsilon)$ denote the minimum number of circles of radius ϵ which can cover M. Let $\bar{M}$ denote the closure of M.

It is well known that since $\bar{M}$ is a closed, bounded, plane point set, there is for every $\eta > 0$, a finite number $K = K(\eta)$ of rectangles ρ_i of area R_i and perimeter p_i $(i = 1, 2, \cdots, K)$ with the following properties:

$$\rho_1 + \rho_2 + \cdots + \rho_K \supset \bar{M} \tag{12}$$

$$\text{meas } \bar{M} \leqq R_1 + R_2 + \cdots + R_K \leqq \text{meas } \bar{M} + \eta. \tag{13}$$

Let $N_i(\epsilon)$ $(i = 1, 2, \cdots, K)$ be the minimum number of circles of radius ϵ covering ρ_i. Then, by (12),

$$N(\epsilon) \leqq N_1(\epsilon) + N_2(\epsilon) + \cdots + N_K(\epsilon).$$

Thus, by Lemma 6,

$$\pi\epsilon^2 N(\epsilon) < (2\pi\sqrt{3}/9)\left[\sum_{i=1}^{k} R_i + 2\epsilon \sum_{i=1}^{k} p_i + 16K\epsilon^2\right]$$

or, by (13),

$$\pi\epsilon^2 N(\epsilon) < (2\pi\sqrt{3}/9)\left[\text{meas } \bar{M} + \eta + 2\epsilon P(\eta) + 16\epsilon^2 K(\eta)\right]$$

where $P(\eta) = p_1 + p_2 + \cdots + p_K$. Thus

$$\limsup_{\epsilon \to 0} \pi\epsilon^2 N(\epsilon) \leqq (2\pi\sqrt{3}/9)(\text{meas } \bar{M} + \eta).$$

But since $\eta > 0$ was arbitrary, this is equivalent to

$$\limsup_{\epsilon \to 0} \pi\epsilon^2 N(\epsilon) \leqq (2\pi\sqrt{3}/9) \ \text{meas } \bar{M}. \tag{14}$$

Now let $\sigma_i(\epsilon)$ $(i = 1, 2, \cdots, N(\epsilon))$ be $N(\epsilon)$ circles of radius ϵ which cover M, and consequently $\bar{M}$. Let T be the set of all points of $\rho_1 + \rho_2 + \cdots + \rho_K$ which are not points of $\sigma_1(\epsilon) + \sigma_2(\epsilon) + \cdots + \sigma_{N(\epsilon)}(\epsilon)$. This set T, which by (13) has measure at most η, is clearly composed of a finite number of connected open regions. Thus there are a finite number $L = L(\eta)$ of rectangles ρ'_i $(i = 1, 2, \cdots, L)$ of total area 2η which cover T. Now let U be the set of points of $\rho_1 + \rho_2 + \cdots + \rho_K$ which are not points of $\rho'_1 + \rho'_2 + \cdots + \rho'_L$. This set U is again composed of a finite number $J = J(\eta)$ of non-overlapping rectangles ρ_i'' with areas R_i'' and perimeters p_i'' $(i = 1, 2, \cdots, J)$. These ρ_i'' have the following properties:

$$(15) \qquad \rho_1'' + \rho_2'' + \cdots + \rho_J'' \subset \sigma_1(\epsilon) + \sigma_2(\epsilon) + \cdots + \sigma_{N(\epsilon)}(\epsilon)$$

$$(16) \qquad R_1'' + R_2'' + \cdots + R_J'' \geqq \text{meas } \bar{M} - 2\eta.$$

Let ρ_i''' $(i = 1, 2, \cdots, J)$ denote the rectangle of area R_i''' obtained by removing a linear strip of width 2ϵ [4] all around ρ_i'', so that

$$(17) \qquad R_i''' = R_i'' - 2\epsilon p_i'' + 16\epsilon^2 > R_i'' - 2\epsilon p_i'' + 2\pi\epsilon^2.$$

Let $N_i''(\epsilon)$ denote the number of circles $\sigma_j(\epsilon)$ $(j = 1, 2, \cdots, N(\epsilon))$ which are completely within ρ_i''. The $N_i''(\epsilon)$ circles $\sigma_j(\epsilon)$ within ρ_i'' obviously cover ρ_i''', in view of (15), so that, by Lemma 5,

$$\pi\epsilon^2 N_i''(\epsilon) > (2\pi\sqrt{3}/9)(R_i''' - 2\pi\epsilon^2)$$

or, by (17),

$$(18) \qquad \pi\epsilon^2 N_i''(\epsilon) > (2\pi\sqrt{3}/9)(R_i'' - 2\epsilon p_i'').$$

But, by the definition of $N_i''(\epsilon)$,

$$N(\epsilon) \geqq N_1''(\epsilon) + N_2''(\epsilon) + \cdots + N_J''(\epsilon).$$

Thus, (18) implies

$$\pi\epsilon^2 N(\epsilon) > (2\pi\sqrt{3}/9)[\sum_{i=1} R_i'' - 2\epsilon \sum_{i=1}^{J} p_i'']$$

or, by (16),

$$\pi\epsilon^2 N(\epsilon) > (2\pi\sqrt{3}/9)[\text{meas } \bar{M} - 2\eta - 2\epsilon P''(\eta)]$$

where $P''(\eta) = p_1'' + p_2'' + \cdots + p_J''$. Thus

$$\liminf_{\epsilon \to 0} \pi\epsilon^2 N(\epsilon) \geqq (2\pi\sqrt{3}/9)(\text{meas } \bar{M} - 2\eta).$$

But, since $\eta > 0$ was arbitrary this is equivalent to

$$(19) \qquad \liminf_{\epsilon \to 0} \pi\epsilon^2 N(\epsilon) \geqq (2\pi\sqrt{3}/9) \text{ meas } \bar{M}.$$

The two inequalities (14) and (19) together imply the desired equality (α).

THE UNIVERSITY OF WISCONSIN.

[4] The ρ_i'' being fixed it is supposed that ϵ is chosen to be sufficiently small for this to be possible, for all $i = 1, 2, \cdots, J = J(\eta)$.