Lecture: 9.1 – STRING MATCHING I

Unit: 9 STRING PROCESSING + COMPUTATIONAL GEOMETRY

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9.1.1. KNUTH-MORRIS-PRATT'S ALGORITHM

MOTIVATING PROBLEM: FIND ALL OCURRENCES OF A SUBSTRING P IN A LONG STRING T

- $|P| = m$
- $|T| = n$

→ [PREFIX FUNCTION] (STRING S, $n = |S|$)

↳ ARRAY π OF LENGTH n. $\pi[i]$: LENGTH OF THE LONGEST PROPER PREFIX OF SUBSTRING $S[0, \dots, i]$ WHICH IS ALSO A SUFFIX OF THIS SUBSTRING

- $\pi[0] = 0$

BY DEFINITION ← PREFIX ≠ O.G. STRING

- $\pi[i] = \max_{k=0,\dots,i} \{k : S[0, \dots, k-1] = S[i-(k-1), \dots, i]\}$

EXAMPLE:

S = "ABCABCD"
$n = 7$
$\pi = [0, 0, 0, 1, 2, 3, 0]$
1 2 3 4 5 6 7

S = "AABAAAB"
$n = 7$
$\pi = [0, 1, 0, 1, 2, 2, 3]$
1 2 3 4 5 6 7

* GO TO TRIVIAL ALGORITHM IN C++

↳ $O(n^3)$

↑

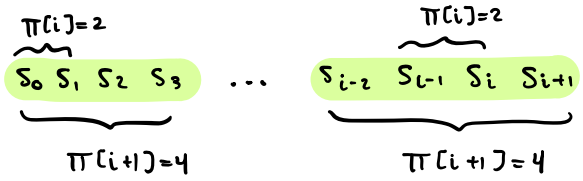
LET'S DO BETTER

① FIRST OPTIMIZATION

- THE VALUES OF π CAN ONLY INCREASE BY ONE AT MOST
- WHEN MOVING TO $\pi[i+1]$ ITS VALUE +1, -1 OR STAY THE SAME

PROOF: IF $\pi[i+1] > \pi[i]+1$ THEN WE CAN TAKE THE SUFFIX ENDING AT POSITION $i+1$ WITH LENGTH $\pi[i+1]$ AND REMOVE THE LAST CHARACTER FROM IT. WE END UP WITH A SUFFIX ENDING AT POSITION i WITH LENGTH $\pi[i+1]-1$ WHICH IS BETTER THAN $\pi[i]$ ▽

$\pi[i]$ SHOULD BE THE LONGEST

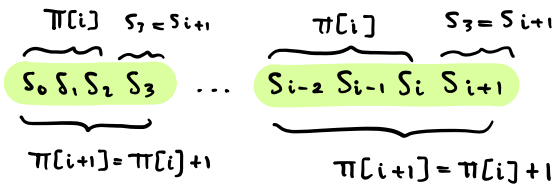


$S_0 S_1 S_2 S_3 = S_{i-2} S_{i-1} S_i S_{i+1}$ $\quad \pi[i+1]=4$

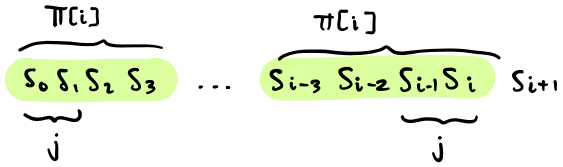
$S_0 S_1 S_2 = S_{i-2} S_{i-1} S_i$ $\quad \pi[i]=3 \longrightarrow \pi[i]=2$ ▽

② SECOND OPTIMIZATION

- IF $S[i+1] = S[\pi[i]] \longrightarrow \pi[i+1] = \pi[i]+1$



- IF $S[i+1] \neq S[\pi[i]] \longrightarrow$ TRY A SHORTER STRING j : $j < \pi[i]$ AND $S[0,\dots,j-1] = S[i-j+1,\dots,i]$



IF WE FIND j, WE COMPARE $S[i+1]$ WITH $S[j]$. IF $S[i+1] = S[j] \longrightarrow S[i+1] = j+1$, OTHERWISE, WE KEEP DECREASING j UNTIL $j=0$:

IF $S[i+1] = S[0] \longrightarrow \pi[i+1]=1$

OTW. $\pi[i+1]=0$

KMP ALGORITHM $O(n) \longrightarrow$ DOES NOT PERFORM ANY STRING COMPARISONS

1) COMPUTE $\pi[i]$ ITERATING FROM $i=1 \longrightarrow n-1$ $\quad (\pi[0]=0)$ $\quad$ [ZERO - BASED]

2) TO GUESS $\pi[i]$ WE SET $j = \pi[i-1]$, j DENOTES THE LENGTH OF THE BEST SUFFIX FOR $i-1$

3) TEST IF SUFFIX OF LENGTH $j+1$ IS ALSO A PREFIX BY COMPARING $s[j]$ AND $s[i]$

↳ IF $s[j] = s[i] \rightarrow \pi[i] = j+1$
OTW. $j = \pi[j-1]$ AND REPEAT

[KMP IS AN ONLINE ALGORITHM]

4) IF $j = 0 \rightarrow \pi[i] = 0$

* GO TO KMP.CPP

APPLICATIONS:

- [SEARCH FOR A SUBSTRING IN A STRING]

  - GIVEN A TEXT t AND A STRING s, DISPLAY ALL POSITIONS OF OCCURRENCES OF s IN t

    - $n = |s|$
    - $m = |t|$

    $\Big\}\ m > n$

  - CONCATENATE STRING $s\#t$ WHERE $\#$ IS A SEPARATOR $\# \notin s$, $\# \notin t$
  - CALCULATE PREFIX FUNCTION FOR $s\#t$
  - $\pi[i] \leq n \quad \forall i \in [0, \dots, n+m]$ DUE TO THE SEPARATOR

    ↳ IF $\pi[i] = n \rightarrow$ s APPEARS COMPLETELY AT POSITION i AND APPEARS AT POSITION $\underbrace{i-2n}$ IN t

    $\downarrow$

    $i \in [0, \dots, n+m]$ $\qquad i - (n+1) - n + 1 = i - 2n$

EXTRA: STRING HASHING

→ MOTIVATING PROBLEM: COMPARE STRINGS EFFICIENTLY

- BRUTE FORCE: $O(\min(s_1, s_2))$
- EFFICIENTLY: $O(1)$ → CONVERT s_1 AND s_2 TO NUMBERS AND COMPARE BOTH NUMBERS

  HASHES ↙

[IF TWO STRINGS ARE EQUAL, ALSO THEIR HASHES SHOULD BE EQUAL]

CONDITION ↗ HAS TO HOLD

↳ HOWEVER IF HASH(S) = HASH(T) THEN S AND T DO NOT NECESSARILY ARE EQUAL DUE TO COLLISIONS

HASH STRING : (OF LENGTH m)

$$\text{HASH}(s) = s[0] + s[1]p + s[2]p^2 + \cdots + s[n-1]p^{n-1} \bmod m$$

$$= \sum_{i=0}^{n-1} s[i]p^i \bmod m$$

- p AND m ARE CHOSEN, POSITIVE INTEGERS
- HASH(S) IS A POLYNOMIAL ROLLING HASH FUNCTION
- p SHOULD BE A PRIME NUMBER ≈ # OF CHARS IN THE INPUT ALPHABET
  ↳ p = 31 → ONLY LOWERCASE
  p = 53 → AND UPPERCASE
- $m = 10^9 + 9$

* GO TO STRING_HASHING.CPP

APPLICATIONS :

- [SEARCH FOR DUPLICATE STRINGS IN AN ARRAY OF STRINGS]
  - n = |ARRAY| AND EACH STRING IS AT MOST m CHARACTERS LONG
  - TRIVIAL ALGORITHM: SORT ARRAY AND COMPARE PAIR OF STRINGS → O(nm log n) SINCE EACH COMPARISON TAKES O(m)
  - BETTER ALGORITHM: SORT ARRAY AND PERFORM COMPARISONS IN O(1) USING HASHES → O(nm + n log n)

    O(nm) TO COMPUTE HASHES

A = ["ABC", "AA", "BC", "ABC", "AAA", "AA"]

- [FAST HASH CALCULATION OF SUBSTRINGS OF GIVEN STRING]
  - FIND HASH(s[i ... j])
  - BY DEFINITION : $\text{HASH}(s[i \ldots j]) = \sum_{k=i}^{j} s[k] \cdot p^{k-i} \bmod m$
  - MULTIPLYING BY p^i : $\text{HASH}(s[i \ldots j])\, p^i = \sum_{k=i}^{j} s[k] \cdot p^k \bmod m$

$p^k = p^{k-i} p^i = p^{k-i+i}$

$= \text{HASH}(s[0 \dots j]) - \text{HASH}(s[0 \dots i-1]) \bmod m$

- $\text{HASH}(s[0]) = 0$
- $\text{HASH}(s[0 \dots i]) = (\text{HASH}(s[0 \dots i-1]) + s[i]p^i) \bmod m$

EXTRA: RABIN-KARP ALGORITHM (STRING MATCHING) $O(|s|+|t|)$

→ MOTIVATING PROBLEM: GIVEN TWO STRINGS (PATTERN s AND TEXT t) DETERMINE IF s APPEARS IN t. IF IT DOES, ENUMERATE ALL ITS OCCURRENCES

EXAMPLE:
- PATTERN $s =$ "ABC" $n = |s|$
- TEXT $t =$ "AACCBCABCCCABCBCABCACC" $m = |t|$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21
AACCBCABCCCABCBCABCACC
6 16

TRIVIAL ALGORITHM: ITERATE OVER EACH CHAR. IN t, IF $t[i] = s[0]$, THEN ITERATE OVER EACH CHAR j IN s AND INCREMENTE COUNTER IF $s_0 s_1 \dots s_n$ IS EQUAL TO $t_i t_{i+1} \dots t_{i+n}$

$O(nm)$

BETTER ALGORITHM: RABIN-KARP:

$O(n+m)$

1) CALCULATE HASH FOR s
2) CALCULATE HASH VALUES FOR ALL PREFIXES IN t
3) COMPARE EACH SUBSTRING OF LENGTH $|s|$ WITH PATTERN s