Lecture: 4 - COMPUTATIONAL GEOMETRY I

Unit: 9 - STRING PROCESSING + COMPUTATIONAL GEOMETRY

Instructor: WESTON

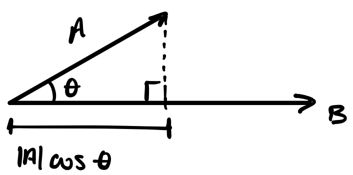
BACKGROUND : 1. POINTS

POINT → 2D / 3D → EUCLIDEAN DISTANCE

↳ $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

DOT PRODUCT

↳ PRODUCT OF THE LENGTH OF THE FIRST VECTOR BY THE LENGTH OF THE PROJECTION OF THE SECOND VECTOR ONTO THE FIRST ONE



$A \cdot B = |A| \cos\theta \, |B|$

$A \cdot B = \sum_{i=1}^{n} A_i \cdot B_i$

PROPERTIES :

1. $A \cdot B = B \cdot A$
2. $(\alpha \cdot A) \cdot B = \alpha \cdot (A \cdot B)$
3. $(A + B) \cdot C = A \cdot C + B \cdot C$

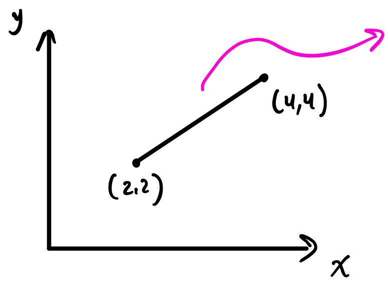
ADDITIONALLY :

1. NORM OF A → $|A|^2 = A \cdot A$
2. LENGTH OF A → $|A| = \sqrt{A \cdot A}$
3. PROJECTION OF A ONTO B → $\frac{A \cdot B}{|B|}$
4. ANGLE BETWEEN A AND B → $\theta = \arccos\left(\frac{A \cdot B}{|A| \cdot |B|}\right)$
   - → $A \cdot B > 0$ IF θ ACUTE
   - → $A \cdot B < 0$ IF θ OBTUSE
   - → $A \cdot B = 0$ IF θ RIGHT ∡

2. LINES

↳ SET OF POINTS IN AN EUCLIDEAN SPACE THAT SATISFY THE EQUATION:

$$Ax + By + C = 0$$



$$-x + y + 0 = 0$$
$$y = x$$

VERTICAL LINES : $B = 0$

NON VERTICAL LINES : $B = 1$

$$A = -\frac{Y_1 - Y_2}{X_1 - X_2}$$

$$C = -AX_1 - Y_1$$

→ TWO LINES ARE PARALLEL IF A AND B ARE THE SAME

→ IF NOT, THEY INTERSECT AT SOME POINT (X_0, Y_0)

$$A_1 x_0 + B_1 y_0 + C_1 = 0$$
$$A_2 x_0 + B_2 y_0 + C_2 = 0$$

→

$$y_0 = -\frac{A_1}{B_1} x_0 + C_1$$
$$y_0 = -\frac{A_2}{B_2} x_0 + C_2$$

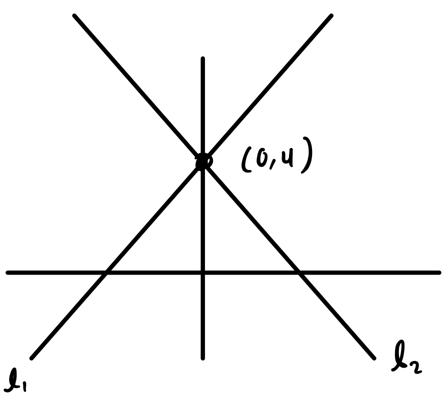
↓ ⋮

$$X_0 = \frac{B_2 C_1 - B_1 C_2}{A_2 B_1 - A_1 B_2}$$

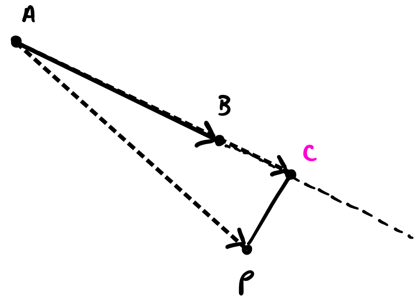
$$y_0 = -(A_1 X_0 + C_1)$$

l_1 : $-2x + y - 4 = 0$

l_2 : $2x + y - 4 = 0$



MINIMUM DISTANCE BETWEEN POINT P AND LINE l



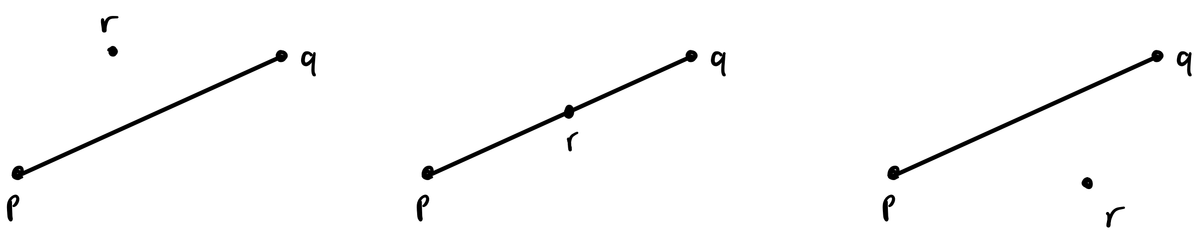
1) COMPUTE LOCATION OF POINT C
2) COMPUTE EUCLIDEAN DISTANCE BETWEEN P AND C

C CAN BE SEEN AS POINT A TRANSLATED BY A SCALE MAGNITUDE u OF VECTOR AB } $c = a + u \times AB$

u IS A SCALAR PROJECTION OF VECTOR AP ONTO VECTOR AB } $ac = u \times AB$

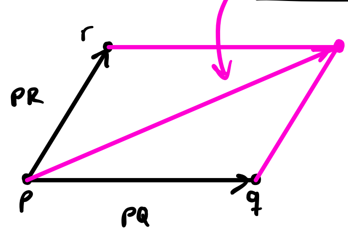
COLLINEARITY

GIVEN A LINE DEFINED BY TWO POINTS P, q DETERMINE IF POINT r IS ON THE LEFT / RIGHT HAND SIDE OF THE LINE OR IF P, q AND r ARE COLLINEAR.



THIS CAN BE DETERMINED USING THE CROSS PRODUCT

$|PR \times PQ|$ → MAGNITUDE OF VECTOR PR X PQ IS EQUAL TO THE AREA OF THE PARALLELOGRAM

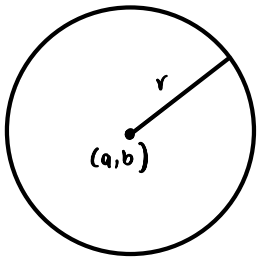


MAGNITUDE > 0 → r @ LEFT HS
MAGNITUDE < 0 → r @ RIGHT HS
MAGNITUDE = 0 → COLLINEAR

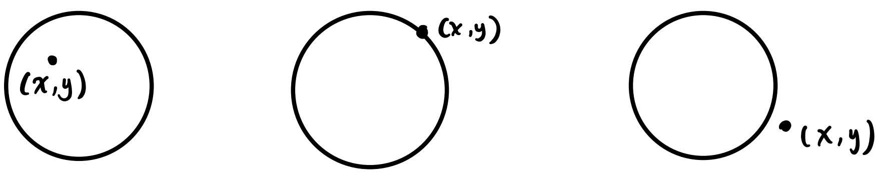
3. CIRCLES

↳ A CIRCLE CENTERED AT COORDINATE (a,b) IN A 2D EUCLIDEAN SPACE WITH RADIUS r IS THE SET OF ALL POINTS (x,y) SUCH THAT:

$$(x-a)^2+(y-b)^2=r^2$$



→ CHECK IF A POINT IS INSIDE / BORDER / OUTSIDE A CIRCLE:



$\Delta x = x-a$
$\Delta y = y-b$

$\Delta x^2+\Delta y^2 < r^2$ $\quad$ $\Delta x^2+\Delta y^2 = r^2$ $\quad$ $\Delta x^2+\Delta y^2 > r^2$

→ IF THE VALUE OF π IS NOT SPECIFIED, $\pi = \text{Acos}(-1.0)$ OR $\pi = 2\cdot\text{Acos}(0.0)$

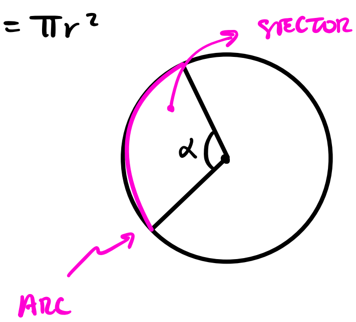
→ DIAMETER OF A CIRCLE $D = 2 \times r$

→ CIRCUMFERENCE OF CIRCLE $C = 2\pi \times r$

→ AREA OF CIRCLE $A = \pi r^2$

SECTOR AREA $= \dfrac{\alpha}{360} A$

→ $\boxed{\text{ARC}}$ OF A CIRCLE



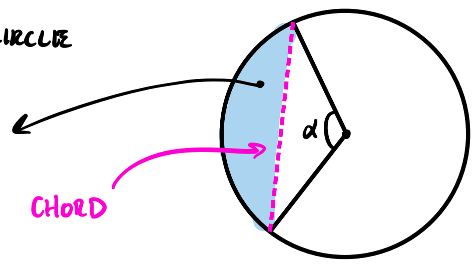
α : CENTRAL ANGLE (DEGREES)

C : CIRCUMFERENCE

↳ LENGTH OF ARC $= \dfrac{\alpha}{360} C$

→ $\boxed{\text{CHORD}}$ OF A CIRCLE

$\boxed{\text{SEGMENT}}$



LENGTH OF CHORD $= \sqrt{2r^2(1-\cos\alpha)}$

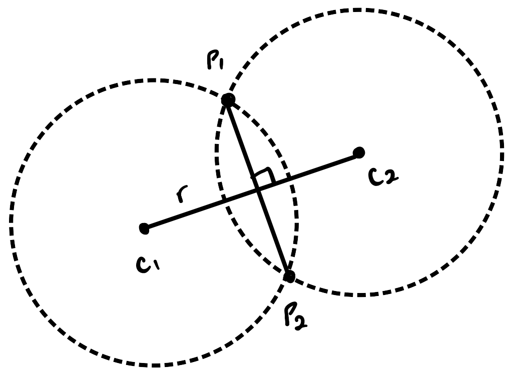
$= 2r\sin\left(\frac{\alpha}{2}\right)$

↓

SEGMENT AREA = AREA OF SECTOR − $\boxed{\frac{BH}{2}}$

↳ AREA OF ISOSCELES TRIANGLE WITH SIDES r, r AND CHORD LENGTH

→ GIVEN TWO POINTS ON THE CIRCLE P_1, P_2 AND RADIUS r OF THE CORRESPONDING CIRCLE, DETERMINE THE LOCATION OF THE CENTERS C_1 AND C_2



4. TRIANGLES

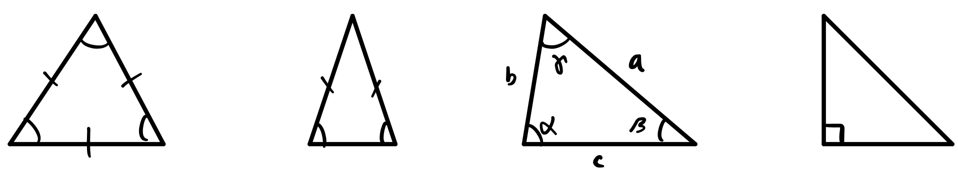
↳ POLYGON WITH 3 VERTICES AND 3 EDGES

- EQUILATERAL : EQUAL LENGTH EDGES AND INTERIOR ANGLES = 60°
- ISOSCELES : TWO EDGES HAVE EQUAL LENGTH AND TWO INTERIOR ANGLES ARE EQUAL
- SCALENE : ALL EDGES HAVE DIFFERENT LENGTH
- RIGHT : ONE OF ITS INTERIOR ANGLE IS 90°

→ AREA $A = \frac{BH}{2}$

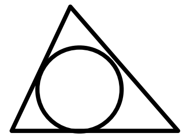
→ PERIMETER $P = \overbrace{A+B+C}^{\text{SIDES}}$ SEMIPERIMETER $S = \frac{P}{2}$

→ HERON'S FORMULA $A = \sqrt{s(s-A)(s-B)(s-C)}$

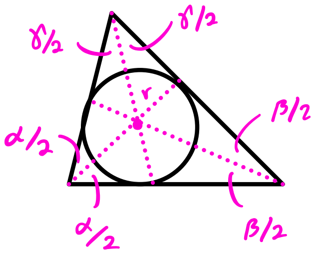


EQUILATERAL ISOSCELES SCALENE RIGHT

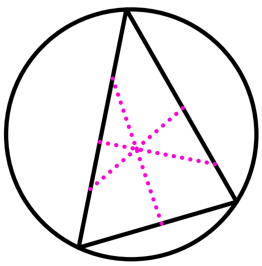
→ TRIANGLE WITH AREA A AND SEMI-PERIMETER S HAS AN INSCRIBED CIRCLE WITH RADIUS $r = \frac{A}{S}$



→ THE CENTER OF THE INCIRCLE IS THE MEETING POINT BETWEEN THE TRIANGLE'S ANGLE BISECTORS. IF WE HAVE TWO ANGLE BISECTORS WE CAN GET THE CENTER IF WE FIND ITS INTERSECTION



→ A TRIANGLE WITH SIDES A, B, C AND AREA A HAS A CIRCUMSCRIBED CIRCLE WITH RADIUS $R = \frac{ABC}{4A}$



→ THE CENTER OF THE CIRCUMSCRIBED CIRCLE IS THE MEETING POINT BETWEEN THE TRIANGLE'S PERPENDICULAR BISECTORS

→ CHECK IF THREE LINE SEGMENTS FORM A TRIANGLE:

$$(A+B > C) \text{ AND } (A+C > B) \text{ AND } (B+C > A)$$

IF A, B, C ARE SORTED THEN WE CHECK $(A+B > C)$

→ LAW OF COSINES $\gamma = \text{ACOS}\left(\frac{A^2+B^2-C^2}{2AB}\right)$

→ LAW OF SINES $\frac{A}{\sin(\alpha)} = \frac{B}{\sin(\beta)} = \frac{C}{\sin(\gamma)} = 2R$

→ PYTHAGOREAN THEOREM $C^2 = A^2 + B^2$

→ PYTHAGOREAN TRIPLE IF $A^2+B^2=C^2$ IS A PYTHAGOREAN TRIPLE, THEN SO IS

(KA, KB, KC) FOR ANY POSITIVE K

[REFERENCES : HALIM S., HALIM F.. COMPETITIVE PROGRAMMING 3]