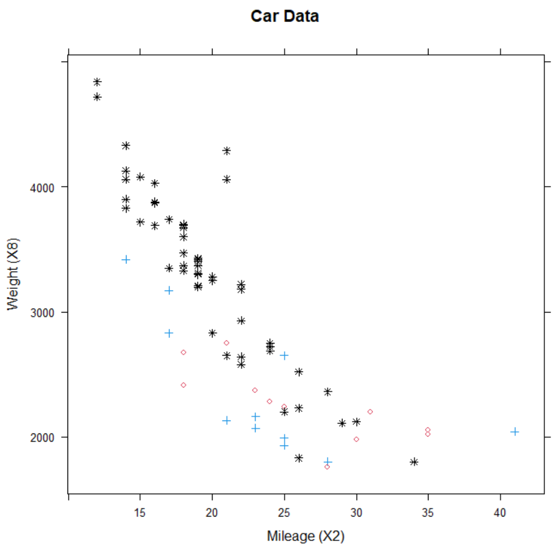
Exercise 3.4 Compute the covariance between the variables
*X*2 = miles per gallon,
*X*8 = weight
from the car data set (Sect. 22.3). What sign do you expect the covariance to have?
Sol.

$$Cov(X_2, X_8) = \mathbf{-3732.025}$$



After calculate by R, it can be seen that the covariance of X_2 and X_8 is less than zero, it means that X_2 and X_8 have an inverse relationship. In other words, the higher weight of the car, the less mileage it can run (at the same gallon).
我們也可以藉由散佈圖驗證，當三種車款的X_2有較大值時，對應X_8會有較小值，形成負斜率的散佈圖，符合負的共變異數有負斜率的散佈圖這個特徵。

Exercise 3.25 Compute the covariance matrix *S* = Cov*(X)* where *X* denotes the matrix of observations on the counterfeit bank notes. Make a Jordan decomposition of *S*. Why are all of the eigenvalues positive?
Sol.

$$S = Cov(X) = \begin{bmatrix} 0.124 & 0.032 & 0.024 & -0.101 & 0.019 & 0.012 \\ 0.032 & 0.065 & 0.047 & -0.024 & -0.012 & -0.005 \\ 0.024 & 0.047 & 0.089 & -0.019 & 0.0001 & 0.034 \\ -0.101 & -0.024 & -0.019 & 1.281 & -0.490 & 0.238 \\ 0.019 & -0.012 & 0.0001 & -0.490 & 0.404 & -0.022 \\ 0.012 & -0.005 & 0.034 & 0.238 & -0.022 & 0.311 \end{bmatrix}$$

Eigenvalues:
$\lambda_1 = 1.54932311$ $\lambda_2 = 0.31919416$ $\lambda_3 = 0.19370655$ $\lambda_4 = 0.10385594$
$\lambda_5 = 0.08437643$ $\lambda_6 = 0.02452663$
Eigenvectors:

$$\vec{\gamma} = \begin{bmatrix} -0.068 & 0.088 & -0.530 & -0.294 & 0.774 & 0.146 \\ -0.014 & -0.010 & -0.363 & -0.418 & -0.258 & -0.791 \\ -0.009 & 0.130 & -0.400 & -0.415 & -0.557 & 0.583 \\ 0.900 & 0.051 & 0.240 & -0.343 & 0.103 & 0.021 \\ -0.390 & 0.493 & 0.563 & -0.527 & 0.097 & -0.011 \\ 0.180 & 0.854 & -0.229 & 0.413 & -0.060 & -0.108 \end{bmatrix}$$

Jordan decomposition of S:

$$J=\vec{\gamma}\lambda\vec{\gamma}^T=\begin{bmatrix}-0.068 & 0.088 & -0.530 & -0.294 & 0.774 & 0.146\\ -0.014 & -0.010 & -0.363 & -0.418 & -0.258 & -0.791\\ -0.009 & 0.130 & -0.400 & -0.415 & -0.557 & 0.583\\ 0.900 & 0.051 & 0.240 & -0.343 & 0.103 & 0.021\\ -0.390 & 0.493 & 0.563 & -0.527 & 0.097 & -0.011\\ 0.180 & 0.854 & -0.229 & 0.413 & -0.060 & -0.108\end{bmatrix}$$

$$*\begin{bmatrix}1.549 & 0 & 0 & 0 & 0 & 0\\ 0 & 0.319 & 0 & 0 & 0 & 0\\ 0 & 0 & 0.194 & 0 & 0 & 0\\ 0 & 0 & 0 & 0.104 & 0 & 0\\ 0 & 0 & 0 & 0 & 0.084 & 0\\ 0 & 0 & 0 & 0 & 0 & 0.025\end{bmatrix}$$

$$*\begin{bmatrix}-0.068 & -0.014 & -0.009 & 0.900 & -0.390 & 0.180\\ 0.088 & -0.010 & 0.130 & 0.051 & 0.493 & 0.854\\ -0.530 & -0.363 & -0.400 & 0.240 & 0.563 & -0.229\\ -0.294 & -0.418 & -0.415 & -0.343 & -0.527 & 0.413\\ 0.774 & -0.258 & -0.557 & 0.103 & 0.097 & -0.060\\ 0.146 & -0.791 & 0.583 & 0.021 & -0.011 & -0.108\end{bmatrix}$$

$$=\begin{bmatrix}0.124 & 0.032 & 0.024 & -0.101 & 0.019 & 0.012\\ 0.032 & 0.065 & 0.047 & -0.024 & -0.012 & -0.005\\ 0.024 & 0.047 & 0.089 & -0.019 & 0.0001 & 0.034\\ -0.101 & -0.024 & -0.019 & 1.281 & -0.490 & 0.238\\ 0.019 & -0.012 & 0.0001 & -0.490 & 0.404 & -0.022\\ 0.012 & -0.005 & 0.034 & 0.238 & -0.022 & 0.311\end{bmatrix}$$

$$Q(\vec{\gamma})=\vec{\gamma}^TS\vec{\gamma}$$

$$=[1.549 \quad 0.319 \quad 0.194 \quad 0.104 \quad 0.084 \quad 0.025]$$

$$*\begin{bmatrix}0.124 & 0.032 & 0.024 & -0.101 & 0.019 & 0.012\\ 0.032 & 0.065 & 0.047 & -0.024 & -0.012 & -0.005\\ 0.024 & 0.047 & 0.089 & -0.019 & 0.0001 & 0.034\\ -0.101 & -0.024 & -0.019 & 1.281 & -0.490 & 0.238\\ 0.019 & -0.012 & 0.0001 & -0.490 & 0.404 & -0.022\\ 0.012 & -0.005 & 0.034 & 0.238 & -0.022 & 0.311\end{bmatrix}\begin{bmatrix}1.549\\ 0.319\\ 0.194\\ 0.104\\ 0.084\\ 0.025\end{bmatrix}\approx \mathbf{0.339}>0$$

Because the quadratic form of matrix S is positive, so it is positive definite matrix, it means that the eigenvalues will also be positive.

Exercise 3.26 Compute the covariance of the counterfeit notes after they are linearly transformed by the vector $a = (1,1,1,1,1,1)^T$.

Sol.

$Var(Y) = Var(a^T X) = a^T S a$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.124 & 0.032 & 0.024 & -0.101 & 0.019 & 0.012 \\ 0.032 & 0.065 & 0.047 & -0.024 & -0.012 & -0.005 \\ 0.024 & 0.047 & 0.089 & -0.019 & 0.0001 & 0.034 \\ -0.101 & -0.024 & -0.019 & 1.281 & -0.490 & 0.238 \\ 0.019 & -0.012 & 0.0001 & -0.490 & 0.404 & -0.022 \\ 0.012 & -0.005 & 0.034 & 0.238 & -0.022 & 0.311 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

≈ 1.742

Linear transformation of the matrix X leads to $Var(Y)$ equal 1.742.

R code:

```
# clear all variables
rm(list = ls(all = TRUE))
graphics.off()

# 3.4
# cov
car <- read.table("C:/Users/user/Desktop/多變量 11101/HW3_1019/MVA-ToDo-master/QID-1532-MVAcareffect/carc.dat")
cov(car$V3,car$V9)
# install and load packages
libraries = c("lattice")
lapply(libraries, function(x) if (!(x %in% installed.packages())) {
  install.packages(x)
})
lapply(libraries, library, quietly = TRUE, character.only = TRUE)
# scat
# load data
M = car[, 3]
W = car[, 9]
C = car[, 14]
# point definition
D = C
D[car[, 14] == 2] = 1
D[car[, 14] == 1] = 8
# color definition
P = C
P[car[, 14] == 3] = 4
```

```
P[car[, 14] == 2] = 2
P[car[, 14] == 1] = 1
leg = c(8, 1, 3)
# plot
xyplot(W ~ M, pch = D, col = P, xlab = "Mileage (X2)", ylab = "Weight (X8)", main =
"Car Data")

# 3.25
bank <- read.table("C:/Users/user/Desktop/多變量 11101/HW3_1019/MVA-ToDo-
master/QID-948-MVApcabankr/bank2.dat")
S <- cov(bank[101:200,])
S
Avalues <- eigen(S)$values
Avectors <- eigen(S)$vectors
Avalues
Avectors
t(Avectors)
J <- Avectors%*%diag(Avalues)%*%t(Avectors)
J
Q <- t(matrix(Avalues))%*%S%*%matrix(Avalues)
Q

# 3.26
a <- matrix(c(1, 1, 1, 1, 1, 1))
a
Sy <- t(a)%*%S%*%a
Sy
```