For matrix A=$\begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix}$.

Q1. Write the procedure and a program for calculating the Jordan decomposition of the matrix A.

$$\det(A-\lambda I)=\begin{vmatrix} 25-\lambda & -2 & 4 \\ -2 & 4-\lambda & 1 \\ 4 & 1 & 9-\lambda \end{vmatrix}=0$$

$\rightarrow(25-\lambda)(4-\lambda)(9-\lambda)+(-8)-8-16(4-\lambda)-4(9-\lambda)-(25-\lambda)=0$

$\rightarrow-\lambda^3+29\lambda^2-100\lambda+9\lambda^2-261\lambda+900-16-64+16\lambda-36+4\lambda-25+\lambda=0$

$\rightarrow-\lambda^3+38\lambda^2-340\lambda+759=0$

$\rightarrow\lambda^3-38\lambda^2+340\lambda-759=0$

令 $y=\lambda-\frac{38}{3}$，則$y^3+py+q=0$，

其中$p=\frac{3ac-b^2}{3a^2}$, $q=\frac{2b^3-9abc+27a^2d}{27a^3}$, $a=1, b=-38, c=340, d=-759$

$\rightarrow p=\frac{3*330-38^2}{3}=\frac{-424}{3}\approx-141.33$,

$$q=\frac{2*(-38)^3-9(-38)340+27(-759)}{27}=\frac{-13957}{27}\approx-516.925$$

$\rightarrow(u+v)^3=u^3+3u^2v+3uv^2+v^3\ \rightarrow(u+v)^3-3uv(u+v)-(u^3+v^3)=0$

$\rightarrow$設想$u+v$即為$y^3+py+q=0$的一個解，故兩式細數相同，得

$p=-3uv, q=-u^3-v^3\ \rightarrow u^3+v^3=-q, u^3v^3=-\frac{p^3}{27}$，利用根與係數的關係，

可知u^3和v^3為下列二次方程式的解: $z^2+qz-\frac{p^3}{27}=0$

令ρ_1和ρ_2代表此二次方程的解:

$\rho_1=-\frac{q}{2}+\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}$, $\rho_2=-\frac{q}{2}-\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}$

$$\to \rho_1 = \frac{13957}{54} + \sqrt{\left(\frac{13957}{108}\right)^2 + \left(\frac{-424}{81}\right)^3} \approx 258.46 + 128.675i$$

$$\to \rho_2 = \frac{13957}{54} - \sqrt{\left(\frac{13957}{108}\right)^2 + \left(\frac{-424}{81}\right)^3} \approx 258.46 - 128.675i$$

故得$u^3 = \rho_1$&$v^3 = \rho_2$，

$u = \sqrt[3]{\rho_1}\ or\ \omega\sqrt[3]{\rho_1}\ or\ \omega^2\sqrt[3]{\rho_1},\quad p = \sqrt[3]{\rho_2}\ or\ \omega\sqrt[3]{\rho_2}\ or\ \omega^2\sqrt[3]{\rho_2}$

其中，$\omega^3 = 1$，$\omega^2 + \omega + 1 = 0 \to \omega = \frac{-1+\sqrt{3}i}{2}$。

$\because uv = -\frac{p}{3}$是實數，$\therefore$ *解為下列三組*:

$y_1 = \sqrt[3]{\rho_1} + \sqrt[3]{\rho_2}$
$y_2 = \omega\sqrt[3]{\rho_1} + \omega^2\sqrt[3]{\rho_2}$
$y_3 = \omega^2\sqrt[3]{\rho_1} + \omega\sqrt[3]{\rho_2}$

$\to y_1 \approx \sqrt[3]{258.46 + 128.68i} + \sqrt[3]{258.46 - 128.68i} \approx \sqrt{43} + i + \sqrt{43} - i \approx 13.15$

$\to y_2 \approx \frac{-1+\sqrt{3}i}{2}\sqrt[3]{258.46 + 128.68i} + \frac{-1-\sqrt{3}i}{2}\sqrt[3]{258.46 - 128.68i} \approx$

$\frac{1}{2}\Big[-\sqrt[3]{258.46 + 128.68i} + \sqrt{3}i\big(\sqrt[3]{258.46 + 128.68i}\big) + \Big(-\sqrt[3]{258.46 - 128.68i} -$

$\sqrt{3}i\big(\sqrt[3]{258.46 - 128.68i}\big)\Big)\Big] \approx \frac{1}{2}\big[-\sqrt{43} + \sqrt{129} + \big(\sqrt{129} - 2 - \sqrt{43}\big)i\big] \approx 2.4 +$

$1.4i$

$\to y_3 \approx \frac{-1-\sqrt{3}i}{2}\sqrt[3]{258.46 + 128.68i} + \frac{-1+\sqrt{3}i}{2}\sqrt[3]{258.46 - 128.68i} \approx \frac{1}{2}\big[\big(-\sqrt{43} -$

$\sqrt{129}\big) + \big(-\sqrt{129} - 2 - \sqrt{43}\big)i\big] \approx -8.9 - 9.95i$

$\because y_k = \lambda_k - \dfrac{38}{3}, k = 1, 2, 3.$

$\therefore y_1 = \lambda_1 - \dfrac{38}{3} \to \lambda_1 \approx 13.15 + 12.67 \approx 26.07$

$\lambda_2 \approx 2.4 + 1.4i + 12.67 \approx 8.49,\ \lambda_3 \approx -8.9 - 9.95i + 12.67 \approx 3.42$

For $\lambda_1 \approx 26.07,\ A - \lambda_1 I \approx \begin{bmatrix} -1.07 & -2 & 4 \\ -2 & -22.07 & 1 \\ 4 & 1 & -17.07 \end{bmatrix}$

$$\rightarrow R_1 \div (-1.07) \Rightarrow \begin{bmatrix} 1 & 1.86 & -3.71 \\ -2 & -22.07 & 1 \\ 4 & 1 & -17.07 \end{bmatrix}$$

$$\rightarrow R_2 - (-2)R_1 \Rightarrow \begin{bmatrix} 1 & 1.86 & -3.71 \\ 0 & -18.369 & -6.418 \\ 4 & 1 & -17.07 \end{bmatrix}$$

$$\rightarrow R_3 - 4R_1 \Rightarrow \begin{bmatrix} 1 & 1.86 & -3.71 \\ 0 & -18.369 & -6.418 \\ 0 & -6.418 & -2.242 \end{bmatrix}$$

$$\rightarrow R_2/(-18.369) \Rightarrow \begin{bmatrix} 1 & 1.86 & -3.71 \\ 0 & 1 & 0.349 \\ 0 & -6.418 & -2.242 \end{bmatrix}$$

$$\rightarrow R_3 + 6.418R_2 \Rightarrow \begin{bmatrix} 1 & 1.86 & -3.71 \\ 0 & 1 & 0.349 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow R_1 - 1.86R_2 \Rightarrow \begin{bmatrix} 1 & 0 & -4.36 \\ 0 & 1 & 0.349 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{cases} x_1 - 4.36x_3 = 0 \\ x_2 + 0.349x_3 = 0 \end{cases} \rightarrow \begin{cases} x_1 = 4.36x_3 \\ x_2 = -0.349x_3 \\ x_3 = x_3 \end{cases} \rightarrow v_1 = \begin{bmatrix} 4.357 \\ -0.349 \\ 1 \end{bmatrix}$$

→(r code) $\overrightarrow{\gamma_1} = \begin{bmatrix} 0.9717 \\ -0.0779 \\ 0.2230 \end{bmatrix}$

For $\lambda_2 \approx 8.4957$, $A - \lambda_2 I \approx \begin{bmatrix} 16.504 & -2 & 4 \\ -2 & -4.496 & 1 \\ 4 & 1 & 0.504 \end{bmatrix}$

$$\rightarrow \overrightarrow{\gamma_2} = \begin{bmatrix} -0.1914 \\ 0.2935 \\ 0.9366 \end{bmatrix}$$

For $\lambda_3 \approx 3.4258$, $A - \lambda_3 I \approx \begin{bmatrix} 21.574 & -2 & 4 \\ -2 & 0.574 & 1 \\ 4 & 1 & 5.574 \end{bmatrix}$

$$\rightarrow \overrightarrow{\gamma_3} = \begin{bmatrix} -0.1384 \\ -0.9528 \\ 0.2703 \end{bmatrix}$$

$J = \overrightarrow{\gamma_1}\lambda_1\overrightarrow{\gamma_1}^T$

$$= \begin{bmatrix} 0.9717 & -0.1914 & -0.1384 \\ -0.0779 & 0.2935 & -0.9528 \\ 0.2230 & 0.9366 & 0.2703 \end{bmatrix} \begin{bmatrix} 26.07 & 0 & 0 \\ 0 & 8.49 & 0 \\ 0 & 0 & 3.42 \end{bmatrix} \begin{bmatrix} 0.9717 & -0.0779 & 0.2230 \\ -0.1914 & 0.2935 & 0.9366 \\ -0.1384 & -0.9528 & 0.2703 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix}$$

Q2. Use the answer of Q1 to check whether matrix A is positive definite. Explain your reason.

$$Q(\vec{\gamma}) = \overrightarrow{\gamma_J}^T A \overrightarrow{\gamma_J} = [26.07 \quad 8.49 \quad 3.42] \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix} \begin{bmatrix} 26.07 \\ 8.49 \\ 3.42 \end{bmatrix} = 17283.16 > 0$$

$$\Longrightarrow \text{ matrix A is positive definite}$$

Q3. Use the answer of Q1 to calculate $A^{-1/2}$.

$$A^{\frac{1}{2}} = \sum_{J=1}^{3} \sqrt{\lambda_j \overrightarrow{\gamma_J \gamma_J}^T}$$

$\rightarrow A^{\frac{1}{2}} =$

$$\begin{bmatrix} 5.11 & 0 & 0 \\ 0 & 2.91 & 0 \\ 0 & 0 & 1.85 \end{bmatrix} \begin{bmatrix} 0.9717 & -0.1914 & -0.1384 \\ -0.0779 & 0.2935 & -0.9528 \\ 0.2230 & 0.9366 & 0.2703 \end{bmatrix} \begin{bmatrix} 0.9717 & -0.0779 & 0.2230 \\ -0.1914 & 0.2935 & 0.9366 \\ -0.1384 & -0.9528 & 0.2703 \end{bmatrix} =$$

$$\begin{bmatrix} 4.96 & -0.31 & 0.52 \\ -0.31 & 1.96 & 0.24 \\ 0.51 & 0.24 & 2.95 \end{bmatrix}$$

$$\rightarrow A^{-\frac{1}{2}} = \left(A^{\frac{1}{2}}\right)^{-1} = \begin{bmatrix} 0.21 & 0.04 & -0.04 \\ 0.04 & 0.52 & -0.05 \\ -0.04 & -0.05 & 0.35 \end{bmatrix}$$

Q4. Is matrix A an orthogonal matrix? Explain your answer.

$$if\ AA^T = A^T A = I, matrix\ A\ is\ a\ orthogonal\ matrix$$

$$AA^T = \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix} \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 645 & -54 & 134 \\ -54 & 21 & 5 \\ 134 & 5 & 98 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence, matrix A isn't an orthogonal matrix.

R Code:

```
A <- matrix(c(25,-2,4,-2,4,1,4,1,9),3,3)
A
Avalues <- eigen(A)$values
Avectors <- eigen(A)$vectors
Avalues
Avectors
J <- Avectors%*%diag(Avalues)%*%t(Avectors)
J
Q <- t(matrix(Avalues))%*%A%*%matrix(Avalues)
Q
K <- Avectors%*%sqrt(diag(Avalues))%*%t(Avectors)
K
solve(K)
```