Bayesian Optimization in Machine Learning

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Goals of this thesis

This Master's thesis has different and complimentary aims:

- Provide an introduction to both Gaussian Process regression and Bayesian optimization.
- Show that the Bayesian Optimization framework works in several real-world machine learning tasks.
- Write a complete software package (pyGPGO) for users to apply Bayesian Optimization in their research.

Organization of the work

Organized in 4 self-contained chapters.

- Chapter 2 focuses on an introduction to regression problems using Gaussian Processes. These are surrogate models we will use for Bayesian Optimization.
- **Chapter 3** covers the main topic in this work, Bayesian Optimization.
- Chapter 4 presents experiments using the Bayesian Optimization framework. Mostly mid-sized supervised-learning problems.
- **Chapter 5** provides technical explanations for pyGPGO, the software developed alongside this manual.

A brief introduction: setting the problem

Overall, Bayesian Optimization focuses on:

$$\max_{\mathbf{x} \in \mathcal{A}} f(\mathbf{x}) \tag{1}$$

We make almost no assumptions about f:

- f may not have a closed-form expression.
- Evaluations of f may be noisy.
- No gradient information needed.

An example arises when optimizing the *loss* function of a machine-learning model w.r.t. hyperparameters (e.g. log-loss):

$$\mathcal{L}(\boldsymbol{y}, \hat{\boldsymbol{y}}) = -\frac{1}{n} \sum_{i} \left(y_i \log(\hat{y}_i) + (1 - y_i) \log(1 - \hat{y}_i) \right) \tag{2}$$

We are interested in cases where f is expensive to evaluate.

A brief introduction: the big picture

Main idea

- We are going to use a Gaussian Process to model f.
- Our model will suggest new points to evaluate according to a GP dependant acquisition function α .
- Then iteratively update the model with new information.

Part I: Gaussian Process Regression

Gaussian Process Regression: basic definitions

Definition 1.

A Gaussian Process is a collection of random variables, any finite number of which have a joint Gaussian distribution. This process is defined by two functions. Its *mean function*:

$$m(\mathbf{x}) = \mathbb{E}\left[f(\mathbf{x})\right] \tag{3}$$

and its covariance function:

$$k(\mathbf{x}, \mathbf{x'}) = \mathbb{E}\left[\left(f(\mathbf{x}) - m(\mathbf{x})\right)\left(f(\mathbf{x'}) - m(\mathbf{x'})\right)\right] \tag{4}$$

We say that f is a Gaussian Process with mean m(x) and covariance function k(x, x') and write:

$$f(\mathbf{x}) \sim \mathcal{GP}(m(\mathbf{x}), k(\mathbf{x}, \mathbf{x'}))$$
 (5)

Gaussian Process Regression: basic definitions

Define then a covariance function, such as the *squared exponential* kernel:

$$k(\mathbf{x}, \mathbf{x}') = \exp\left(-\frac{1}{2}|\mathbf{x} - \mathbf{x}'|^2\right)$$
 (6)

where |.| denotes the standard L_2 norm. Drawing samples from a Gaussian Process, assuming zero mean, for given finite inputs X_* simplifies to sampling from:

$$\mathbf{f_*} \sim \mathcal{N}\left(\mathbf{0}, K(X_*, X_*)\right) \tag{7}$$

Gaussian Process Regression: prediction

Assume training data $\mathcal{D} = \{(\mathbf{x_i}, y_i) | i = 1, \dots, n\}$

Prediction using GP prior

Let \mathbf{y} and \mathbf{f}_* be jointly Gaussian:

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{f}_* \end{bmatrix} \sim \mathcal{N} \left(\mathbf{0}, \begin{bmatrix} K(X, X) + \sigma_n^2 I & K(X, X_*) \\ K(X_*, X) & K(X_*, X_*) \end{bmatrix} \right)$$
(8)

We want to condition f_* over y.

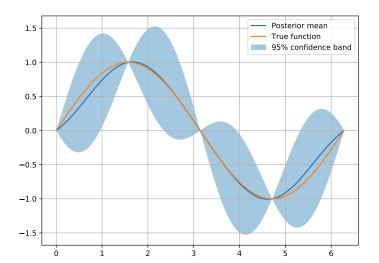
$$f_*|y \sim \mathcal{N}(\overline{f_*}, Cov(f_*))$$
 (9)

where:

$$\overline{\mathbf{f}_{*}} = K(X_{*}, X) \left(K(X, X) + \sigma_{n}^{2} I \right)^{-1} \mathbf{y}$$

$$Cov(\mathbf{f}_{*}) = K(X_{*}, X_{*}) - K(X_{*}, X) \left(K(X, X) + \sigma_{n}^{2} I \right)^{-1} K(X, X_{*})$$
(10)

Gaussian Process Regression: an example



Gaussian Process Regression: on covariance functions

Some common covariance function choices

$$k_{SE}(r) = \exp\left(-\frac{r^2}{2l^2}\right) \qquad k_{\text{Matèrn}}(r) = \frac{2^{1-\nu}}{\Gamma(\nu)} \left(\frac{\sqrt{2\nu}r}{l}\right)^{\nu} K_{\nu} \left(\frac{\sqrt{2\nu}r}{l}\right)$$
$$k_{\text{GE}}(r) = \exp\left(-\left(\frac{r}{l}\right)^{\gamma}\right) \qquad k_{RQ}(r) = \left(1 + \frac{r^2}{2\alpha l^2}\right)^{-\alpha} \tag{11}$$

where $r = |\mathbf{x} - \mathbf{x'}|$. Observations can be noisy:

$$k^{y}(\boldsymbol{x}_{p}, \boldsymbol{x}_{q}) = \sigma_{f}^{2} k(\boldsymbol{x}_{p}, \boldsymbol{x}_{q}) + \sigma_{n}^{2} \delta_{pq}$$
 (12)

■ There are different approaches towards hyperparameter treatment.

Gaussian Process Regression: Type II Maximum-Likelihood

An empirical Bayes approach to choosing hyperparameters. Noticing that $\mathbf{y} \sim \mathcal{N}(\mathbf{0}, K + \sigma_n^2 I)$

Marginal log-likelihood

$$\log p(\mathbf{y}|X) = -\frac{1}{2}\mathbf{y}^{T}(K + \sigma_{n}^{2}I)^{-1}\mathbf{y} - \frac{1}{2}\log|K + \sigma_{n}^{2}I| - \frac{n}{2}\log 2\pi \quad (13)$$

Derivative w.r.t hyperpameters

$$\frac{\partial}{\partial \theta_j} \log p(\mathbf{y}|X, \boldsymbol{\theta}) = \frac{1}{2} \mathbf{y}^T K^{-1} \frac{\partial K}{\partial \theta_j} K^{-1} \mathbf{y} - \frac{1}{2} \operatorname{tr} \left(K^{-1} \frac{\partial K}{\partial \theta_j} \right)$$
(14)

Gaussian Process Regression: incorporating diff. priors

We have assumed m(x) = 0 for simplicity. We can incorporate prior knowledge:

Different prior mean

$$f(\mathbf{x}) \sim \mathcal{GP}(m(\mathbf{x}), k(\mathbf{x}, \mathbf{x}^*))$$
 (15)

Posterior mean becomes:

$$\mathbf{f}_* = \mathbf{m}(X_*) + k(X_*, X)K^{-1}(\mathbf{y} - \mathbf{m}(X))$$
 (16)

Posterior variance remains unchanged.

Gaussian Process Regression: marginalizing over hyperparameters

The full Bayesian approach does not optimize the marginal likelihood, but integrates the uncertainty of hyperparameters θ into the model. either using MCMC or Variational Inference techniques.

Full Bayesian approach

$$\theta \sim p_h(\theta)$$
 (17)

$$\mathbf{f} \sim \mathcal{N}(0, \Sigma_{\theta})$$
 (18)

$$\mathcal{L}(\mathbf{f}) = p(\operatorname{data}|\mathbf{f}) = p(\mathbf{y}|\mathbf{f})$$
(19)

We wish to sample from the joint posterior under unknowns:

$$p(\mathbf{f}, \theta | \text{data}) \propto \mathcal{L}(\mathbf{f}) p(\mathbf{f}) p_h(\theta)$$
 (20)

Several strategies for sampling from latent Gaussian models proposed in main text.

Part II: Bayesian Optimization

Bayesian Optimization: introduction

Again, assume:

$$\max_{\mathbf{x} \in \mathcal{A}} f(\mathbf{x}) \tag{21}$$

Assume that we have sampled our function f to optimize a small number of times n. Providing us with training data:

$$\mathcal{D}_{n} = \{ \mathbf{x}_{i}, y_{i}, i = 1, \dots, n \}.$$
 (22)

Our steps here are:

- **I** Fit a Gaussian Process regression model on \mathcal{D}_n .
- **2** Choose the next point to sample, according to an *acquisition* function, depending on GP.
- 3 Evaluate said point. Augment training data. Repeat.

Bayesian optimization: algorithm

Algorithm 1 Bayesian optimization framework

- 1: Sample a small number of points $\mathbf{x} \in \mathcal{A}$. Evaluate $f(\mathbf{x})$ to get \mathcal{D}_n
- 2: **for** n = 1, 2, ... **do**
- 3: Fit a GP regression model on \mathcal{D}_n
- 4: $\mathbf{x}_{n+1} \leftarrow \operatorname{arg\,max}_{\mathbf{x}} \alpha(\mathbf{x}, \mathcal{D}_n)$
- 5: Evaluate $f(x_{n+1}) = y_{n+1}$
- 6: Augment data $\mathcal{D}_{n+1} = \{\mathcal{D}_n, (\mathbf{x}_{n+1}, y_{n+1})\}$
- 7: end for

Part II: Bayesian Optimization

Bayesian optimization: example

Bayesian optimization: on acquisition functions

Three popular categories: improvement-based, optimistic, and information-based policies.

Improvement-based

Denote τ our best evaluation so far.

Probability of Improvement:

$$\alpha_{\mathrm{PI}}(\mathbf{x}, \mathcal{D}_n) = P(\nu > \tau) = \Phi\left(\frac{\mu_n(\mathbf{x}) - \tau}{\sigma_n(\mathbf{x})}\right)$$
 (23)

Expected Improvement:

$$\alpha_{\rm EI}(\mathbf{x}, \mathcal{D}_n) = (\mu_n(\mathbf{x}) - \tau) \Phi\left(\frac{\mu_n(\mathbf{x}) - \tau}{\sigma_n(\mathbf{x})}\right) + \sigma_n(\mathbf{x}) \phi\left(\frac{\mu_n(\mathbf{x}) - \tau}{\sigma_n(\mathbf{x})}\right)$$
(24)

Bayesian optimization: on acquisition functions

Optimistic acquisitions

Upper Confidence Bound:

$$\alpha_{\text{UCB}}(\mathbf{x}, \mathcal{D}_n) = \mu_n(\mathbf{x}) + \beta_n \sigma_n(\mathbf{x})$$
 (25)

Information-based

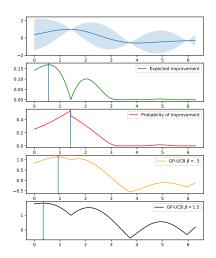
Entropy-based:

$$\alpha_{\mathrm{ES}}(\mathbf{x}|\mathcal{D}_n) = H(\mathbf{x}^*|\mathcal{D}_n) - \mathbb{E}_{\mathbf{y}|\mathcal{D}_n,\mathbf{x}}\left[H(\mathbf{x}^*|\mathcal{D}_n \cup \{(\mathbf{x},\mathbf{y})\})\right]$$
(26)

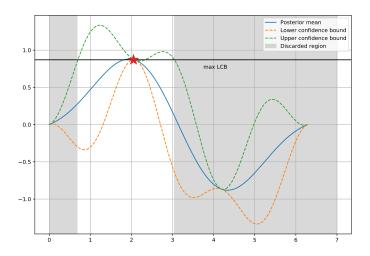
Predictive Entropy Search:

$$\alpha_{\text{PES}}(\mathbf{x}, \mathcal{D}_n) = H(y|\mathcal{D}_n, \mathbf{x}) - \mathbb{E}_{\mathbf{x}^*|\mathcal{D}_n}[H(y|\mathcal{D}_n, \mathbf{x}, \mathbf{x}^*)]$$
(27)

Bayesian optimization: acquisition function behaviour



Bayesian optimization: why does it work?



Bayesian optimization: role of hyperparameters

Gaussian Process hyperparameters θ cause uncertainty in the optimization procedure:

$$\alpha(\mathbf{x}) = \mathbb{E}_{\theta|\mathcal{D}_n} \left[\alpha(\mathbf{x}, \theta) \right] = \int_{\Theta} \alpha(\mathbf{x}|\theta) p(\theta|\mathcal{D}_n) d\theta. \tag{28}$$

Approach 1. MAP estimate $\hat{ heta}_{\mathrm{MAP}}$

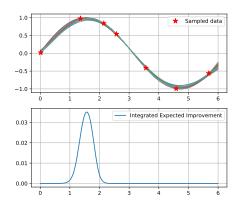
$$\hat{\alpha}(\mathbf{x}) = \alpha(\mathbf{x}, \hat{\theta}) \tag{29}$$

Approach 2. Sample posterior $p(\theta|\mathcal{D}_n)$. Integrated acquisition functions

$$\mathbb{E}_{\theta|\mathcal{D}_n}\left[\alpha(\mathbf{x},\theta)\right] \approx \frac{1}{M} \sum_{i=1}^{M} \alpha(\mathbf{x},\theta_n^{(i)})$$
(30)

Bayesian optimization: Integrated acquisition example

Figure: Means of 200 posterior predictive distributions, taken from sampled posterior $p(\theta|\mathcal{D}_n)$



Bayesian optimization: optimizing acquisitions

We have assumed that the acquisition function is easily optimized. This is not necessarily the case:

- It is often multimodal.
- Or non-convex.
- Furthermore theoretical convergence is guaranteed when finding global maximum **x***.

One may be thinking that we have switched optimization problems! (f by α). However, α is usually very cheap to compute.

In practice α is optimized using multi-start quasi-Newton methods (e.g. L-BFGS-B) or evolutionary approaches (e.g. CMA-ES).

Bayesian optimization: Other surrogates

The Bayesian Optimization framework is not limited to a Gaussian Process surrogate:

- Random Forests: empirical estimate of variance provided by predictions of trees.
- Gradient Boosting Machines: using quantile regression for variance estimate.
- Sparse pseudo-input GPs
- Student-t Processes

Student-t Process example

$$\mathbf{f}^*|\mathbf{f} \sim \mathcal{T}\left(\nu + n, \phi_2, \frac{\nu + \beta_1 - 2}{\nu + n - 2}\tilde{K}_{**}\right)$$
 (31)

Part III: Experiments and benchmarking

Experiments

The aim is to demonstrate that Bayesian Optimization works in real hyperparameter optimization problems.

- Mid-sized supervised learning problems. Both regression and classification considered
- Two examples explained in detail: datasets related to relevant computational chemistry problems.
- Other examples drawn from other areas, such as medicine. Available from other sources (UCI ML repository).

Loss functions used:

Binary classification

Logarithmic loss:

$$\mathcal{L}(\boldsymbol{y}, \hat{\boldsymbol{y}}) = -\frac{1}{n} \sum_{i} \left(y_i \log(\hat{y}_i) + (1 - y_i) \log(1 - \hat{y}_i) \right)$$
(32)

Regression

Mean Squared Error (MSE)

$$\mathcal{L}(\mathbf{y}, \hat{\mathbf{y}}) = \frac{1}{n} \sum_{i} (y_i - \hat{y}_i)^2$$
(33)

- Shuffled k = 5 fold cross-validation scheme for loss evaluation.
- Features scaled: zero mean, unit variance.
- = n = 50 (+3 to account GP fitting) total function evaluations.
- Gaussian Process surrogate, with squared exponential covariance function.

- Type II ML estimation of all hyperparameters at each stage $(I, \sigma_n^2, \sigma_f^2)$.
- **E**xpected Improvement and GP-UCB ($\beta = 0.5$, $\beta = 1.5$) acquisitions.
- Random search is considered as baseline for all experiments.
- Machine-learning models used:
 - Support Vector Machines
 - K-Nearest-Neighbours
 - Gradient Boosting Machines
 - One-Hidden Layer Neural Networks

Table: SVM parameters.

Parameter	Туре	Bounds
C	\mathbb{R}^+	$[10^{-5}, 10^5]$ (log-scaled)
γ	\mathbb{R}^+	$\begin{bmatrix} 10^{-5}, 10^{5} \\ 10^{-5}, 10^{5} \end{bmatrix}$ (log-scaled)

Table: KNN parameters.

Parameter	Туре	Bounds
k	Integer	$\{10,\ldots,50\}$

Table: GBM parameters.

Parameter	Type	Bounds
learning_rate	\mathbb{R}^+	$10^{-5}, 10^{-2}$
n_{-} estimators	Integer	$\{10, \dots, 100\}$
max_depth	Integer	$\{2, \dots, 100\}$
$min_samples_split$	Integer	$\{2, \dots, 100\}$

Table: MLP parameters.

Parameter	Туре	Bounds
hidden_layer_size	Integer	[5, 50]
alpha	\mathbb{R}^+	[0, 0.9]

Experiment I: Binding affinity prediction dataset

■ Problem is to predict some binding/inhibition constant between a protein and a ligand (K_d, K_i) . Target variable is defined as $y = -\log_{10} K$

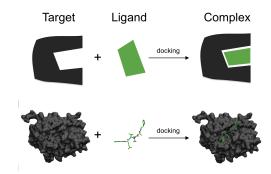


Figure: Illustration of the protein-ligand binding scenario.

Experiment I: Binding affinity prediction dataset

Set of features used

Pair of atom-type counts from protein and ligand in the binding site.

$${P_{j}}_{j=1}^{9} = {C, N, O, F, P, S, CI, Br, I}$$

$${L_{i}}_{i=1}^{9} = {C, N, O, F, P, S, CI, Br, I}$$

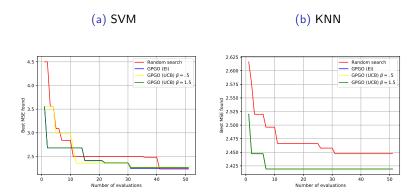
$$\mathbf{x}(Z(P_j), Z(L_i)) = \sum_{k=1}^{P_j} \sum_{l=1}^{L_i} \Theta(d_{\text{cutoff}} - d_{kl})$$
 (34)

Data

- PDBbind v.2015 database of affinity and kinetic biological data.
- A refined set of n = 3623 of protein-ligand pairs for training, and another m = 195 structurally diverse for testing.

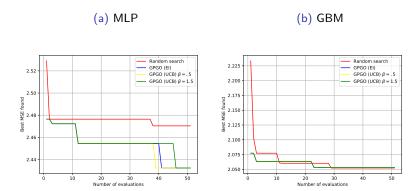
Experiment I: Binding affinity prediction dataset

Figure: Benchmarking results for the binding affinity dataset (1/2).



Experiment I: Binding affinity prediction dataset

Figure: Benchmarking results for the binding affinity dataset (2/2).



- Predict whether a particular residue is in contact with a chain of another protein.
- Interfaces are not considered target-specific.

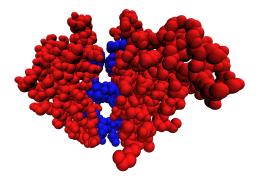


Figure: An interface between two chains of a dimer. Residues considered interficial are marked in blue.

Features used

Descriptors resemble those detailed in (Jiménez et al. 2017). Average pharmacophoric-like features computed in a particular neighbourhood (20Å³) of a residue.

Data

PIFACE database of clustered protein interface. Over 22k structurally unique interfaces extracted from the Protein Data Bank.

Figure: Benchmarking results for the protein-protein interface dataset (1/2).

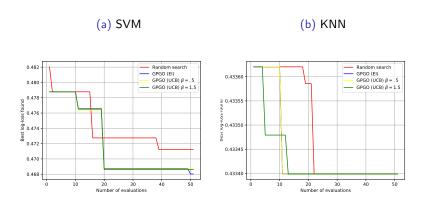
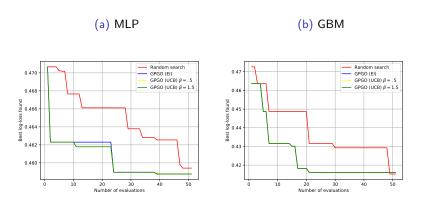
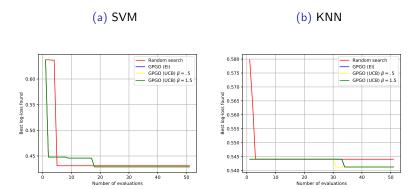


Figure: Benchmarking results for the protein-protein interface dataset (2/2).



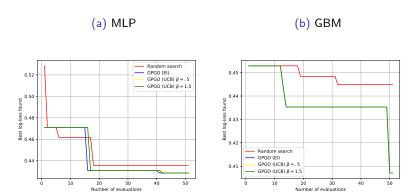
Experiments: other datasets (1/3)

Figure: Benchmarking results for the LSVT Voice Rehabilitation dataset.



Experiments: other datasets (1/3)

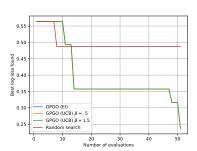
Figure: Benchmarking results for the LSVT Voice Rehabilitation dataset.



Experiments: other datasets (2/3)

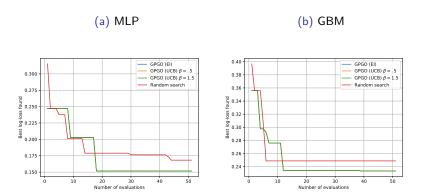
Figure: Benchmarking results for the Parkinson's dataset.

(a) SVM



Experiments: other datasets (2/3)

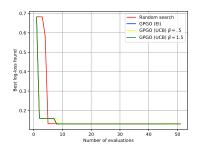
Figure: Benchmarking results for the Parkinson's dataset.

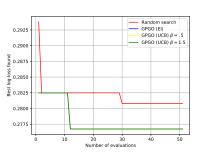


Experiments: other datasets (3/3)

Figure: Benchmarking results for the Breast cancer dataset.

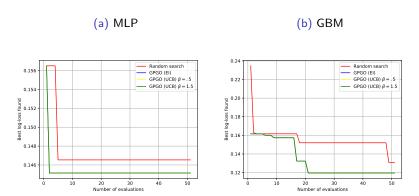
(a) SVM (b) KNN





Experiments: other datasets (3/3)

Figure: Benchmarking results for the Breast cancer dataset.



Part IV: Software development

pyGPGO: Bayesian Optimization for Python

There are many possible choices in the Bayesian Optimization framework:

- Choice of surrogate model.
- Covariance function to use
- Acquisition behaviour
- Hyperparameter treatment

They naturally motivate a modular design. Most of the available software focuses on a particular implementation of the Bayesian optimization algorithm.

pyGPGO is a package I wrote to cover as many of these scenarios as possible.

pyGPGO: features

pyGPGO features:

- A completely modular and customizable design. Easy to setup and minimal dependencies.
- A wide range of surrogate models: Gaussian Processes, Student-t Processes, Random Forests, Extra Random Forests and Gradient Boosting Machines.
- Most of the usual covariance functions, as well as its derivatives: squared exponential, Matérn, γ -exponential, rational quadratic, exponential sine, and dot product.
- Many acquisition function behaviours: probability of improvement, expected improvement, upper confidence bound and entropy-based, as well as their integrated versions.
- Type II Maximum-Likelihood of covariance function hyperparameters.
- MCMC sampling for full-Bayesian treatment of hyperparameters (via pyMC3).

pyGPGO: a simple example

```
1 import numpy as np
2 from pyGPGO.covfunc import matern32
3 from pyGPGO.acquisition import Acquisition
4 from pyGPGO.surrogates.GaussianProcess import GaussianProcess
5 from pyGPGO.GPGO import GPGO
6
  def f(x, y):
      # Franke's function (https://www.mathworks.com/help/curvefit/franke.html)
      one = 0.75 * np.exp(-(9 * x - 2) ** 2 / 4 - (9 * y - 2) ** 2 / 4)
10
      two = 0.75 * np.exp(-(9 * x + 1) ** 2/49 - (9 * y + 1) / 10)
11
      three = 0.5 * np.exp(-(9 * x - 7) ** 2 / 4 - (9 * y - 3) ** 2 / 4)
12
      four = 0.25 * np.exp(-(9 * x - 4) ** 2 - (9 * y - 7) ** 2)
13
      return one + two + three - four
14
```

pyGPGO: a simple example

pyGPGO: optimizing SVM hyperparameters

```
1 import numpy as np
2 from sklearn.datasets import make_moons
3 from sklearn.svm import SVC
  from sklearn.model selection import cross val score
5
  from pyGPGO.GPGO import GPGO
  from pyGPGO.surrogates.GaussianProcess import GaussianProcess
  from pyGPGO.acquisition import Acquisition
  from pvGPGO.covfunc import squaredExponential
11
12
  def evaluateModel(C, gamma):
      clf = SVC(C=10**C, gamma=10**gamma)
14
      return np.average(cross_val_score(clf, X, y))
15
```

pyGPGO: optimizing SVM hyperparameters

pyGPGO: Bayesian Optimization for Python

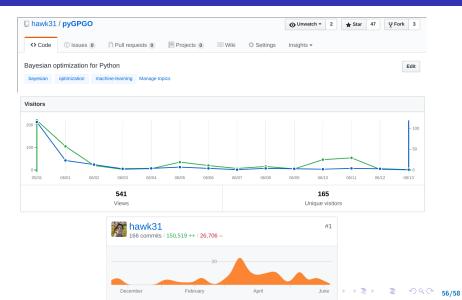
- The software is MIT licensed and can be downloaded from its GitHub repository @ https://github.com/hawk31/pyGPGO
- Extensive documentation can be found in ReadTheDocs.
 - pygpgo.readthedocs.io/en/latest
- Anyone can contribute to the project! (Implement new features, report bugs, improve docs...)
- All benchmarking, examples and thesis code is also available.







pyGPGO in numbers



pyGPGO: Future work

Like any other piece of software, there is always stuff to do!

- Constrained optimization. $P(C(x)) \ge 1 \delta$
- Better support for non-stationary covariance functions.
- GPU support for non-MCMC samplers (BBVI).
- A parallel interface for experiment running.
- More acquisitions, surrogates...

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