# Bayesian Optimization in Machine Learning

#### José Jiménez

Master's Thesis Master's degree in Statistics and Operations Research

supervised by Josep Ginebra 1 Department of Statistics and Operations Research. ETSEIB.





#### Goals of this thesis

This Master's thesis has different and complimentary aims:

- Provide an introduction to both Gaussian Process regression and Bayesian optimization.
- Show that the Bayesian Optimization framework works in several real-world machine learning tasks.
- Write a complete software package (pyGPGO) for users to apply Bayesian Optimization in their research.

### Organization of the work

Organized in 4 self-contained chapters.

- Chapter 2 focuses on an introduction to regression problems using Gaussian Processes. These are surrogate models we will use for Bayesian Optimization.
- **Chapter 3** covers the main topic in this work, Bayesian Optimization.
- **Chapter 4** presents experiments using the Bayesian Optimization framework. Mostly mid-sized supervised-learning problems.
- **Chapter 5** provides technical explanations for pyGPGO, the software developed alongside this manual.

#### A brief introduction

Overall, Bayesian Optimization focuses on:

$$\max_{\mathbf{x} \in \mathcal{A}} f(\mathbf{x}) \tag{1}$$

We make almost no assumptions about f:

- f may not have a closed-form expression.
- Evaluations of f may be noisy.
- No gradient information needed.

An example arises when optimizing the *loss* function of a machine-learning model, depending on its hyperparameters (e.g. log-loss):

$$\mathcal{L}(\boldsymbol{y}, \hat{\boldsymbol{y}}) = -\frac{1}{n} \sum_{i} (y_i \log(\hat{y}_i) + (1 - y_i) \log(1 - \hat{y}_i))$$
 (2)

### Gaussian Process Regression: basic definitions

#### Definition 1.

A Gaussian Process is a collection of random variables, any finite number of which have a joint Gaussian distribution. This process is defined by two functions. Its *mean function*:

$$m(\mathbf{x}) = \mathbb{E}\left[f(\mathbf{x})\right] \tag{3}$$

and its covariance function:

$$k(\mathbf{x}, \mathbf{x'}) = \mathbb{E}\left[\left(f(\mathbf{x}) - m(\mathbf{x})\right)\left(f(\mathbf{x'}) - m(\mathbf{x'})\right)\right] \tag{4}$$

We say that f is a Gaussian Process with mean m(x) and covariance function k(x, x') and write:

$$f(\mathbf{x}) \sim \mathcal{GP}(m(\mathbf{x}), k(\mathbf{x}, \mathbf{x'}))$$
 (5)

### Gaussian Process Regression: basic definitions

Define then a covariance function, such as the *squared exponential* kernel:

$$k(\mathbf{x}, \mathbf{x}') = \exp\left(-\frac{1}{2}|\mathbf{x} - \mathbf{x}'|^2\right)$$
 (6)

where |.| denotes the standard  $L_2$  norm. Drawing samples from a Gaussian Process, assuming zero mean, for given finite inputs  $X_*$  simplifies to sampling from:

$$\mathbf{f_*} \sim \mathcal{N}\left(\mathbf{0}, K(X_*, X_*)\right) \tag{7}$$

### Gaussian Process Regression: prediction

Assume training data  $\mathcal{D} = \{(\mathbf{x_i}, y_i) | i = 1, ..., n\}$ 

#### Prediction using GP prior

Let  $\mathbf{y}$  and  $\mathbf{f}_*$  be jointly Gaussian:

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{f}_* \end{bmatrix} \sim \mathcal{N} \left( \mathbf{0}, \begin{bmatrix} K(X, X) + \sigma_n^2 I & K(X, X_*) \\ K(X_*, X) & K(X_*, X_*) \end{bmatrix} \right)$$
(8)

We want to condition  $f_*$  over y.

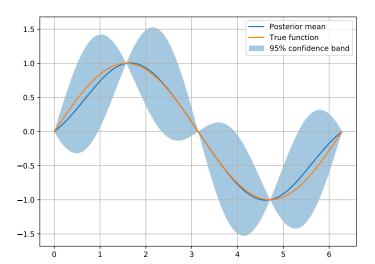
$$f_*|y \sim \mathcal{N}(\overline{f_*}, Cov(f_*))$$
 (9)

where:

$$\overline{\mathbf{f_*}} = K(X_*, X) \left( K(X, X) + \sigma_n^2 I \right)^{-1} \mathbf{y}$$

$$Cov(\mathbf{f_*}) = K(X_*, X_*) - K(X_*, X) \left( K(X, X) + \sigma_n^2 I \right)^{-1} K(X, X_*)$$
(10)

### Gaussian Process Regression: an example



### Gaussian Process Regression: on covariance functions

#### Some common covariance function choices

$$k_{SE}(r) = \exp\left(-\frac{r^2}{2l^2}\right) \qquad k_{\text{Matèrn}}(r) = \frac{2^{1-\nu}}{\Gamma(\nu)} \left(\frac{\sqrt{2\nu}r}{l}\right)^{\nu} K_{\nu} \left(\frac{\sqrt{2\nu}r}{l}\right)$$
$$k_{\text{GE}}(r) = \exp\left(-\left(\frac{r}{l}\right)^{\gamma}\right) \qquad k_{RQ}(r) = \left(1 + \frac{r^2}{2\alpha l^2}\right)^{-\alpha} \tag{11}$$

where  $r = |\mathbf{x} - \mathbf{x'}|$ . Observations are noisy:

$$k^{y}(\mathbf{x}_{p}, \mathbf{x}_{q}) = \sigma_{f}^{2} k(\mathbf{x}_{p}, \mathbf{x}_{q}) + \sigma_{n}^{2} \delta_{pq}$$
 (12)

# Gaussian Process Regression: Type II Maximum-Likelihood

An empirical Bayes approach to choosing hyperparameters. Noticing that  $\mathbf{y} \sim \mathcal{N}(\mathbf{0}, K + \sigma_n^2 I)$ 

#### Marginal log-likelihood

$$\log p(\mathbf{y}|X) = -\frac{1}{2}\mathbf{y}^{T}(K + \sigma_{n}^{2}I)^{-1}\mathbf{y} - \frac{1}{2}\log|K + \sigma_{n}^{2}I| - \frac{n}{2}\log 2\pi \quad (13)$$

#### Derivative w.r.t hyperpameters

$$\frac{\partial}{\partial \theta_j} \log p(\mathbf{y}|X, \boldsymbol{\theta}) = \frac{1}{2} \mathbf{y}^T K^{-1} \frac{\partial K}{\partial \theta_j} K^{-1} \mathbf{y} - \frac{1}{2} \operatorname{tr} \left( K^{-1} \frac{\partial K}{\partial \theta_j} \right)$$
(14)

# Gaussian Process Regression: incorporating diff. priors

We have assumed m(x) = 0 for simplicity. If we dispose of prior knowledge, we can use it

#### Different prior mean

$$f(\mathbf{x}) \sim \mathcal{GP}(m(\mathbf{x}), k(\mathbf{x}, \mathbf{x}^*))$$
 (15)

Posterior mean becomes:

$$\mathbf{f}_* = \mathbf{m}(X_*) + k(X_*, X)K^{-1}(\mathbf{y} - \mathbf{m}(X))$$
 (16)

Posterior variance remains unchanged.

# Gaussian Process Regression: marginalizing over hyperparameters

The full Bayesian approach does not optimize the marginal likelihood, but integrates the uncertainty of hyperparameters  $\theta$  into the model, either using MCMC or Variational Inference techniques.

#### Full Bayesian approach

$$\theta \sim p_h(\theta)$$
 (17)

$$\mathbf{f} \sim \mathcal{N}(0, \Sigma_{\theta})$$
 (18)

$$\mathcal{L}(\mathbf{f}) = p(\operatorname{data}|\mathbf{f}) = p(\mathbf{y}|\mathbf{f})$$
(19)

We wish to sample from the joint posterior under unknowns:

$$p(\mathbf{f}, \theta | \text{data}) \propto \mathcal{L}(\mathbf{f}) p(\mathbf{f}) p_h(\theta)$$
 (20)

Several strategies for sampling from latent Gaussian models proposed in main text.

### Bayesian Optimization: introduction

Again, assume:

$$\max_{\mathbf{x} \in \mathcal{A}} f(\mathbf{x}) \tag{21}$$

Assume that we have sampled our function f to optimize a small number of times n. Providing us with training data:

$$\mathcal{D}_{n} = \{ \mathbf{x}_{i}, y_{i}, i = 1, \dots, n \}.$$
 (22)

Our steps here are:

- **1** Fit a Gaussian Process regression model on  $\mathcal{D}_n$ .
- **2** Choose the next point to sample, according to an *acquisition* function, depending on GP.
- 3 Evaluate said point. Augment training data. Repeat.

### Bayesian optimization: algorithm

#### Algorithm 1 Bayesian optimization framework

- 1: Sample a small number of points  $\mathbf{x} \in \mathcal{A}$ . Evaluate  $f(\mathbf{x})$  to get  $\mathcal{D}_n$
- 2: **for** n = 1, 2, ... **do**
- 3: Fit a GP regression model on  $\mathcal{D}_n$
- 4:  $\mathbf{x}_{n+1} \leftarrow \operatorname{arg\,max}_{\mathbf{x}} \alpha(\mathbf{x}, \mathcal{D}_n)$
- 5: Evaluate  $f(\boldsymbol{x}_{n+1}) = y_{n+1}$
- 6: Augment data  $\mathcal{D}_{n+1} = \{\mathcal{D}_n, (\mathbf{x}_{n+1}, y_{n+1})\}$
- 7: end for

# Bayesian optimization: example

#### Bayesian optimization: on acquisition functions

Three popular categories: improvement-based, optimistic, and information-based policies.

#### Improvement-based

 $\tau$  denotes our best evaluation so far.

Probability of improvement:

$$\alpha_{\mathrm{PI}}(\mathbf{x}, \mathcal{D}_n) = P(\nu > \tau) = \Phi\left(\frac{\mu_n(\mathbf{x}) - \tau}{\sigma_n(\mathbf{x})}\right)$$
 (23)

Expected improvement:

$$\alpha_{\rm EI}(\mathbf{x}, \mathcal{D}_n) = (\mu_n(\mathbf{x}) - \tau) \Phi\left(\frac{\mu_n(\mathbf{x}) - \tau}{\sigma_n(\mathbf{x})}\right) + \sigma_n(\mathbf{x}) \phi\left(\frac{\mu_n(\mathbf{x}) - \tau}{\sigma_n(\mathbf{x})}\right)$$
(24)

#### Bayesian optimization: on acquisition functions

#### Optimistic acquisitions

Upper confidence bound:

$$\alpha_{\text{UCB}}(\mathbf{x}, \mathcal{D}_n) = \mu_n(\mathbf{x}) + \beta_n \sigma_n(\mathbf{x})$$
 (25)

#### Information-based

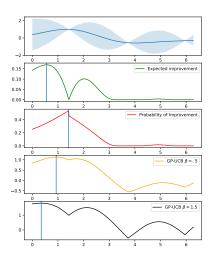
Entropy-based:

$$\alpha_{\mathrm{ES}}(\mathbf{x}|\mathcal{D}_n) = H(\mathbf{x}^*|\mathcal{D}_n) - \mathbb{E}_{\mathbf{y}|\mathcal{D}_n,\mathbf{x}}\left[H(\mathbf{x}^*|\mathcal{D}_n \cup \{(\mathbf{x},\mathbf{y})\})\right]$$
(26)

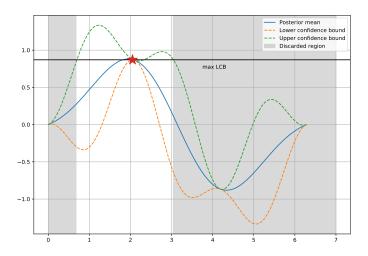
Predictive entropy search:

$$\alpha_{\text{PES}}(\mathbf{x}, \mathcal{D}_n) = H(y|\mathcal{D}_n, \mathbf{x}) - \mathbb{E}_{\mathbf{x}^*|\mathcal{D}_n}[H(y|\mathcal{D}_n, \mathbf{x}, \mathbf{x}^*)]$$
(27)

# Bayesian optimization: acquisition function behaviour



### Bayesian optimization: why does it work?



#### Bayesian optimization: role of hyperparameters

GP hyperparameters  $\theta$  cause uncertainty in the optimization procedure:

$$\alpha(\mathbf{x}) = \mathbb{E}_{\theta|\mathcal{D}_n} \left[ \alpha(\mathbf{x}, \theta) \right] = \int_{\Theta} \alpha(\mathbf{x}|\theta) p(\theta|\mathcal{D}_n) d\theta.$$
 (28)

Approach 1. MAP estimate  $\hat{ heta}_{\mathrm{MAP}}$ 

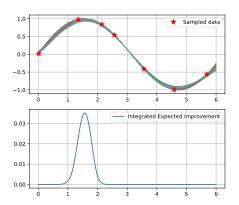
$$\hat{\alpha}(\mathbf{x}) = \alpha(\mathbf{x}, \hat{\theta}) \tag{29}$$

Approach 2. Sample posterior  $p(\theta|\mathcal{D}_n)$ . Integrated acquisition functions

$$\mathbb{E}_{\theta|\mathcal{D}_n}\left[\alpha(\mathbf{x},\theta)\right] \approx \frac{1}{M} \sum_{i=1}^{M} \alpha(\mathbf{x},\theta_n^{(i)})$$
 (30)

### Bayesian optimization: Integrated acquisition example

Figure: Means of 200 posterior predictive distributions, taken from sampled posterior  $p(\theta|\mathcal{D}_n)$ 



### Bayesian optimization: optimizing acquisitions

We have assumed that the acquisition function is easily optimized. This is not necessarily the case:

- It is often multimodal.
- Or non-convex.
- Furthermore theoretical convergence is guaranteed when finding true maximum x\*.

One may be thinking that we have switched optimization problems! (f by  $\alpha$ ). However,  $\alpha$  is very cheap to compute.

In practice  $\alpha$  is optimized using multi-start quasi-Newton methods (e.g. L-BFGS-B) or evolutionary approaches (CMA-ES).

### Bayesian optimization: Other surrogates

- Random Forests: empirical estimate of variance provided by predictions of trees.
- Gradient Boosting Machines: using quantile regression for variance estimate.
- Sparse pseudo-input GPs
- Student-t Processes
- **...**

#### Student-t Process example

$$\mathbf{f}^*|\mathbf{f} \sim \mathcal{T}\left(\nu + n, \phi_2, \frac{\nu + \beta_1 - 2}{\nu + n - 2}\tilde{K}_{**}\right)$$
 (31)

### Experiments

The aim is to demonstrate that Bayesian Optimization works in real hyperparameter optimization problems.

- Mid-sized supervised learning problems. Both regression and classification considered.
- Two examples explained in detail: datasets retrieved from computational chemistry problems.
- Other examples drawn from other areas, such as medicine. Available from other sources (UCI ML repository).

Loss functions used:

#### Binary classification

Logarithmic loss:

$$\mathcal{L}(\boldsymbol{y}, \hat{\boldsymbol{y}}) = -\frac{1}{n} \sum_{i} \left( y_i \log(\hat{y}_i) + (1 - y_i) \log(1 - \hat{y}_i) \right)$$
(32)

#### Regression

Mean Squared Error (MSE)

$$\mathcal{L}(\boldsymbol{y}, \hat{\boldsymbol{y}}) = \frac{1}{n} \sum_{i} (y_i - \hat{y}_i)^2$$
 (33)

- Shuffled k = 5 fold cross-validation scheme for loss evaluation.
- Features scaled: zero mean, unit variance.
- n = 50 (+3 to account GP fitting) total function evaluations.
- Gaussian Process surrogate, with squared exponential covariance function.

- Type II ML estimation of all hyperparameters at each stage (I,  $\sigma_n^2$ ,  $\sigma_f^2$ ).
- Expected Improvement and GP-UCB ( $\beta=0.5,\ \beta=1.5$ ) acquisitions.
- Machine-learning models used: Support Vector Machines,
   K-Nearest-Neighbours, Gradient Boosting Machines and One-Hidden Layer Neural Networks.
- Random search is considered as baseline for all experiments.

Table: SVM parameters.

Parameter	Туре	Bounds
С	$\mathbb{R}^+$	$[10^{-5}, 10^5]$ (log-scaled)
$\gamma$	$\mathbb{R}^+$	$egin{bmatrix} [10^{-5},10^5] & ( ext{log-scaled}) \ [10^{-5},10^5] & ( ext{log-scaled}) \end{bmatrix}$

Table: KNN parameters.

Parameter	Туре	Bounds
k	Integer	$\{10,\ldots,50\}$

Table: GBM parameters.

Parameter	Туре	Bounds
learning_rate	$\mathbb{R}^+$	$[10^{-5}, 10^{-2}]$
$n_{-}$ estimators	Integer	$\{10,\ldots,100\}$
$\max_{d} depth$	Integer	$\{2,\ldots,100\}$
$min\_samples\_split$	Integer	$\{2,\ldots,100\}$

Table: MLP parameters.

Parameter	Туре	Bounds
hidden_layer_size	Integer	[5, 50]
alpha	$\mathbb{R}^+$	[0, 0.9]

■ Problem is to predict some binding/inhibition constant between a protein and a ligand  $(K_d, K_i)$ . Target variable is defined as  $y = -\log_{10} K$ 

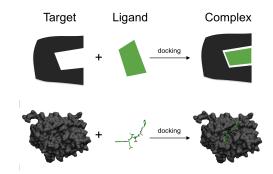


Figure: Illustration of the protein-ligand binding scenario.

#### Set of features used

Pair of atom-type counts from protein and ligand in the binding site.

$${P_{j}}_{j=1}^{9} = {C, N, O, F, P, S, CI, Br, I}$$
  
$${L_{i}}_{i=1}^{9} = {C, N, O, F, P, S, CI, Br, I}$$

$$x(Z(P_j), Z(L_i)) = \sum_{k=1}^{P_j} \sum_{l=1}^{L_i} \Theta(d_{\text{cutoff}} - d_{kl})$$
 (34)

#### Data

- PDBbind v.2015 database of affinity and kinetic biological data.
- A refined set of n = 3623 of protein-ligand pairs for training, and another m = 195 structurally diverse for testing.

Figure: Benchmarking results for the binding affinity dataset (1/2).

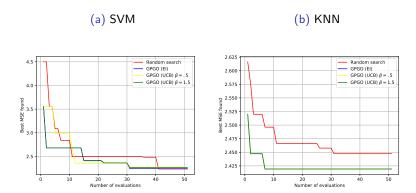
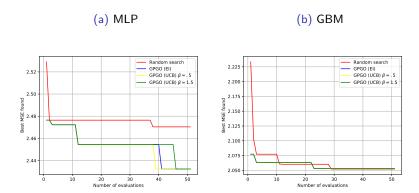


Figure: Benchmarking results for the binding affinity dataset (2/2).



### Experiments: Protein-protein interface prediction

- Predict whether a particular residue is in contact with a chain of another protein.
- Interfaces are **not** considered target-specific.

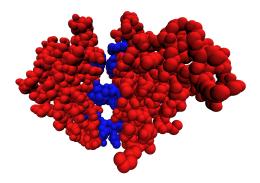


Figure: An interface between two chains of a dimer. Residues considered interficial are marked in blue.

# Experiments: Protein-protein interface prediction

#### Features used

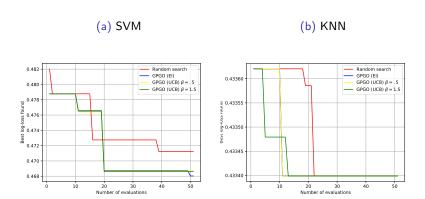
Descriptors resemble those detailed in (Jiménez *et al.* 2017). Average pharmacophoric-like features computed in a particular neighbourhood (20Å<sup>3</sup>) of a residue.

#### Data

PIFACE database of clustered protein interface. Over 22k structurally unique interfaces extracted from the Protein Data Bank.

### Experiments: Protein-protein interface prediction

Figure: Benchmarking results for the protein-protein interface dataset (1/2).



### Experiments: Protein-protein interface prediction

Figure: Benchmarking results for the protein-protein interface dataset (2/2).

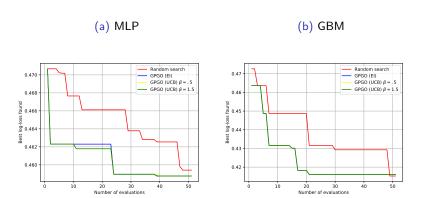


Figure: Benchmarking results for the LSVT Voice Rehabilitation dataset.

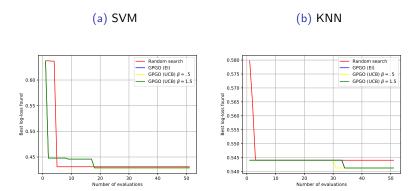


Figure: Benchmarking results for the LSVT Voice Rehabilitation dataset.

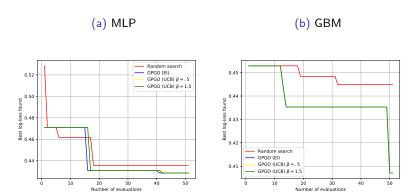


Figure: Benchmarking results for the Parkinson's dataset.

(a) SVM

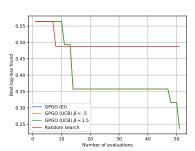


Figure: Benchmarking results for the Parkinson's dataset.

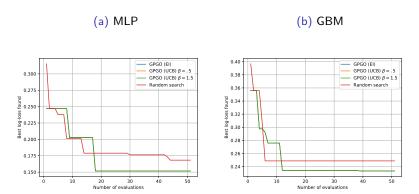


Figure: Benchmarking results for the Breast cancer dataset.

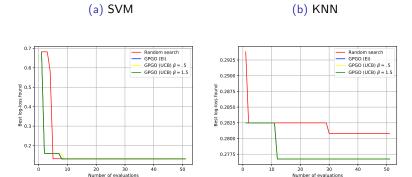
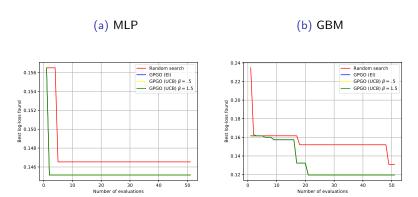


Figure: Benchmarking results for the Breast cancer dataset.



# pyGPGO: Bayesian Optimization for Python

There are many possible choices in the Bayesian Optimization framework:

- Choice of surrogate model.
- Covariance function to use
- Acquisition behaviour
- Hyperparameter treatment

They naturally motivate a modular design. Most of the available software focuses on a particular implementation of the Bayesian optimization algorithm.

pyGPGO is a package I wrote to cover as many of these scenarios as possible.

### pyGPGO: features

#### pyGPGO features:

- A completely modular and customizable design. Easy to setup and minimal dependencies.
- A wide range of surrogate models: Gaussian Processes, Student-t Processes, Random Forests, Extra Random Forests and Gradient Boosting Machines.
- Most of the usual covariance functions, as well as its derivatives: squared exponential, Matérn, γ-exponential, rational quadratic, exponential sine, and dot product.
- Many acquisition function behaviours: probability of improvement, expected improvement, upper confidence bound and entropy-based, as well as their integrated versions.
- Type II Maximum-Likelihood of covariance function hyperparameters.
- MCMC sampling for full-Bayesian treatment of hyperparameters (via pyMC3).

# pyGPGO: a simple example

```
1 import numpy as np
2 from pyGPGO.covfunc import matern32
3 from pyGPGO.acquisition import Acquisition
4 from pyGPGO.surrogates.GaussianProcess import GaussianProcess
5 from pyGPGO.GPGO import GPGO
6
8 \text{ def } f(x, y):
       # Franke's function (https://www.mathworks.com/help/curvefit/franke.html)
       one = 0.75 * \text{np.exp}(-(9 * x - 2) ** 2 / 4 - (9 * y - 2) ** 2 / 4)
10
      two = 0.75 * np.exp(-(9 * x + 1) ** 2/49 - (9 * y + 1) / 10)
11
      three = 0.5 * np.exp(-(9 * x - 7) ** 2 / 4 - (9 * y - 3) ** 2 / 4)
12
      four = 0.25 * np.exp(-(9 * x - 4) ** 2 - (9 * y - 7) ** 2)
13
      return one + two + three - four
14
```

# pyGPGO: a simple example

# pyGPGO: optimizing SVM hyperparameters

```
1 import numpy as np
2 from sklearn.datasets import make_moons
3 from sklearn.svm import SVC
  from sklearn.model selection import cross val score
5
  from pyGPGO.GPGO import GPGO
  from pyGPGO.surrogates.GaussianProcess import GaussianProcess
  from pyGPGO.acquisition import Acquisition
  from pvGPGO.covfunc import squaredExponential
11
12
13 def evaluateModel(C, gamma):
      clf = SVC(C=10**C, gamma=10**gamma)
14
      return np.average(cross_val_score(clf, X, y))
15
```

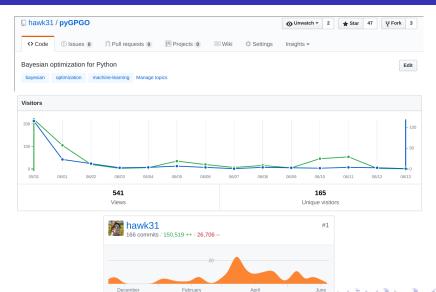
# pyGPGO: optimizing SVM hyperparameters

# pyGPGO: Bayesian Optimization for Python

- The software is MIT licensed and can be downloaded from its GitHub repository @ https://github.com/hawk31/pyGPGO
- Extensive documentation can be found in ReadTheDocs.
  - pygpgo.readthedocs.io/en/latest
- Anyone can contribute to the project! (Implement new features, report bugs, improve docs...)
- All benchmarking, examples and thesis code is also available.



# pyGPGO in numbers



# pyGPGO: Future work

Like any other piece of software, there is always stuff to do!

- Constrained optimization.  $P(C(x)) \ge 1 \delta$
- Better support for non-stationary covariance functions.
- GPU support for non-MCMC samplers (BBVI).
- A parallel interface for experiment running.
- More acquisitions, surrogates...

# Thank you!