ALGOMANIACS

MATHS

Prime Numbers

Using brute force, we can check if a number is prime in $O(\sqrt{n})$ time. Using an algorithm called Sieve of Eratosthenes, we can cut that down to constant time with a $O(n\log n)$ preprocessing step.

```cpp
int n;
vector<bool> is_prime(n+1, true);
is_prime[0] = is_prime[1] = false;
for (int i = 2; i <= n; i++) {
    if (is_prime[i] && (long long)i * i <= n) {
        for (int j = i * i; j <= n; j += i)
            is_prime[j] = false;
    }
}
```

By slightly modifying this code, we can also get all the primes up till n and the smallest prime factor of each number in the same time complexity.

Deepak’s Dilemma

Deepak is afraid of prime numbers, but they are not afraid of all prime numbers. They were afraid of only a special kind of prime numbers. He is afraid of the prime numbers (without the digit zero, they love all the primes which have digits 0 in them) that remain prime no matter how many of the leading digits are omitted. For example, he is afraid of 4632647 because it doesn't have the digit 0 and each of its truncations (632647, 32647, 2647, 647, 47, and 7) are primes.

You are given a simple task, given a number of N, find out the number of primes not greater that N, that Deepak is afraid of. There are T testcases.

$T \le 10^5$
$1 \le N < 10^6$

Exponentiation and ModInv

$a^b = a^{b/2} * a^{b/2} * a^{b\%2}$

This recursive formula means that we can calculate a^b in $O(\log_2 n)$

```
int pow(int a, int b) {
    int res = 1;
    while (b > 0) {
        if (b & 1)
            res = res * a % mod;
        a = a * a % mod;
        b >>= 1;
    }
    return res;
}
```

Fermat's Little Theorem - $a^{p-1} = 1 \pmod p$

Therefore $a^{-1} = a^{p-2} \pmod p$

Prime pair connection

Consider the consecutive primes p_1 = 19 and p_2 = 23. It can be verified that 1219 is the smallest number such that the last digits are formed by p_1 whilst also being divisible by p_2.

In fact, with the exception of 3 and 5, for every pair of consecutive primes p_1, p_2, there exist values of n for which the last digits are formed by p_1 and is divisible by p_2. Let S be the smallest of these values of n.

Given L and R, find ΣS for every pair of consecutive primes with $L \le p_1 \le R$.

$T \le 10$
$5 \le L \le R < 10^9$
$|R-L| \le 10^6$

Number Theory: Important facts

$$\gcd(a,b) = \begin{cases} a, & \text{if } b = 0 \\ \gcd(b, a \bmod b), & \text{otherwise.} \end{cases}$$

gcd(a,b) = gcd(a-b,b) (provided a-b > 0)

lcm(a,b) = a*b/gcd(a,b)

The number of primes < n is approximately n/log(n)

Euler Totient Function : The number of numbers < n coprime to n $\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1).$

See also - Extended euclidean algorithm

Tanmay and Caltech

Tanmay wants to go to Caltech, but in the entrance exam he came across a very difficult task: Given an integer n, it is required to calculate $\sum lcm(c, gcd(a,b))$, for all triples of positive integers (a,b,c), where $a+b+c=n$.

$3 \le n \le 10^5$

Too many primes?

Given *N,L,R*, you need to compute the number of integers *x* in the interval *[L,R]* such that *x* is coprime with *N*.
There are *T* testcases.

$T \le 100$
$N \le 10^9$
$1 \le L \le R < 10^{15}$

Gainz

Shobhit has *k* minutes to spend on his workout routine. He can choose from *n* different exercises, but if he spends more than *m* minutes on any single exercise, he risks injury. Fortunately, he can also choose to skip some exercises if needed.
Shobhit wants to use up all *k* minutes without overdoing any single exercise. In how many ways can he plan his workout?

Note:Each workout must have an integer number of minutes

$1 \le n,m,k < 10^5$

Trippy Adventure

Shreyans has just returned from an exciting "adventure" and now finds himself at the entrance of a peculiar $n \times n$ maze at position *(0,0)(top left corner)*. The maze is divided into two regions, with lava covering the upper-right half (i.e., all positions (i, j) where $j > i$). Shreyans must navigate through the maze without stepping into the lava taking only down and right steps. In how many different ways can he successfully cross the maze?

Note: Position (i,j) is the *i*th row from the top and *j*th row from the left.

$1 \leq n < 10^6$

Apple

Given a number n, find largest x such that $x^2 \leq n$

$n < 10^{100000}$

Assume you can store the number in $O(n)$ space and do basic operations (+,-,*,/) in $O(log(n))$ time and other operations accordingly.(using python or java or custom c++ class)

Additional Resources

Number theory other important topics -
https://cp-algorithms.com/algebra/extended-euclid-algorithm.html
https://cp-algorithms.com/algebra/linear-diophantine-equation.html
https://cp-algorithms.com/algebra/fibonacci-numbers.html
https://cp-algorithms.com/algebra/chinese-remainder-theorem.html
https://forthright48.com/category/cpps/number-theory/

PNC other important topics- https://usaco.guide/gold/combo?lang=cpp#binomial-coefficients
https://cp-algorithms.com/combinatorics/burnside.html

Pseudo-Random Based Prime Fact. in O($n^{1/4}$)-https://cp-algorithms.com/algebra/factorization.html#pollards-rho-algorithm

Info/Questions on Inclusion/Exclusion - https://codeforces.com/blog/entry/64625

Newton’s method tc -
https://en.citizendium.org/wiki/Newton%27s_method#Computational_complexity

Maths other imp stuff - Probability, Matrices, FFT, Geometry(Very Rare)

Thank you