Part 3

Explaining optimization problems using logic cutting-planes

Explanations in CP

Satisfaction problems

- “Why has variable x value a in a/the solution?”
- “Why can variable x not have value b?”
- “Why is this problem UNSAT?”
- “Is there a solution where x = b?”
- “Are there any other solutions with other values for x?”
- …

Optimization problems

- “Why is x = a in a/the optimal solution?”
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Counterfactual explanations

Given a constraint optimization problem with optimal solution x^*

<u>User:</u> "Why not the solution $\bar{x}$ I thought of instead of optimal x^*?"

<u>Program:</u> "For $\bar{x}$ to be optimal the objective coefficients must change to d^*"

Ideally: changes to objective are as small as possible to ensure interpretability

Counterfactual explanations

Example

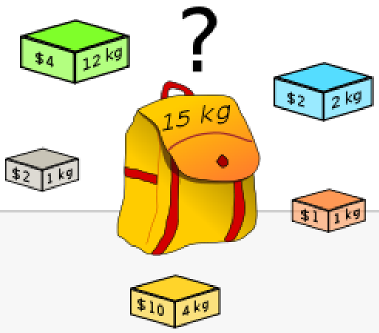
<u>User:</u>

“Why does the optimal not include gr?”

<u>Program:</u>

“Because to include gr into the knapsack, its value should change to at least 11 instead of 4”

Knapsack:

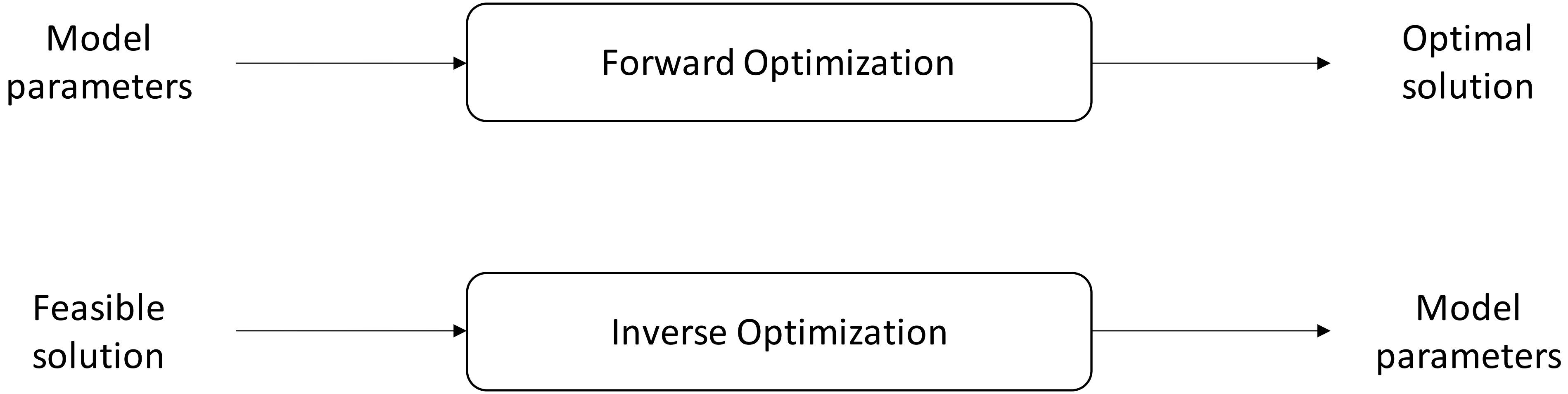



```
model = Model()

gr,bl,og,ye,gy = boolvar(shape=5)

model += (12*gr + 2*bl + 1*og + 4*ye + 1*gy <= 15)

model.maximize(4*gr + 2*bl + 1*og + 10*ye + 2*gy)


model.solve()
```

```
print(gr.value(), bl.value(), og.value(), ye.value(), gy.value())
```

```
0 1 1 1 1
```

(Inverse) Optimization



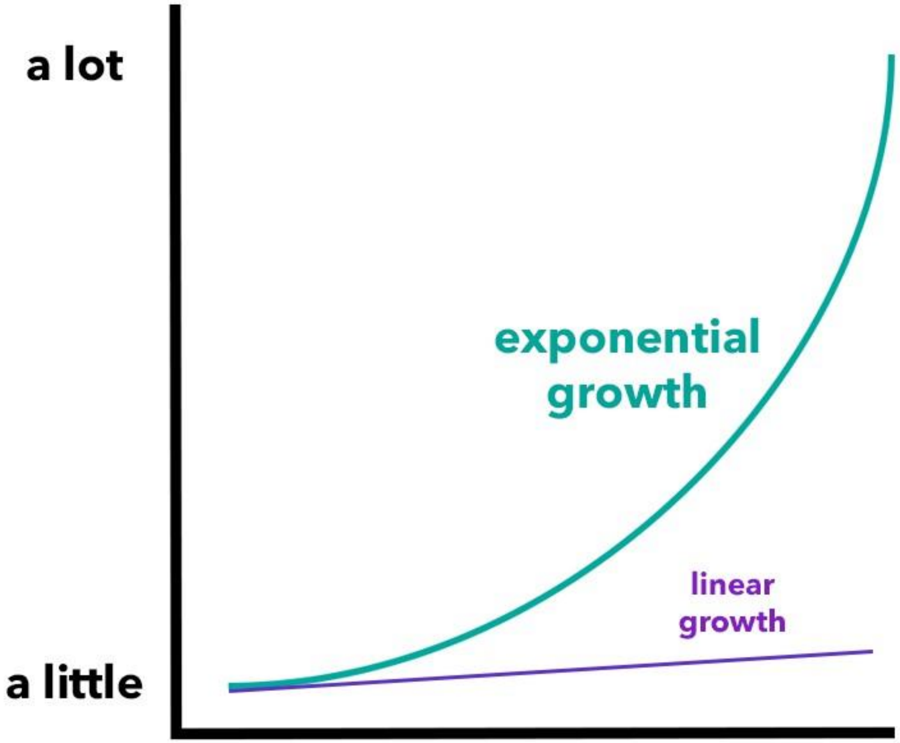
Inverse optimization and Counterfactual expl

- User provides part of new solution $\bar{x}$
  - We want an explanation in terms of variables in this partial assignment
- Find optimal objective for $\bar{x}$ and return as explanation

Inverse optimization

$$
\begin{aligned}
\mathbf{IO}(\mathbf{c}^0, \hat{\mathbf{x}}, \mathcal{X}): \ \underset{\mathbf{c}\in\mathcal{P}}{\text{minimize}} \quad & \left\|\mathbf{c} - \mathbf{c}^0\right\|_1 \\
\text{subject to} \quad & \mathbf{c}^\top \hat{\mathbf{x}} \leq \mathbf{c}^\top \mathbf{x}_j, \quad \forall \mathbf{x}_j \in \mathcal{E}(\mathcal{X})
\end{aligned}
$$

Inverse optimization



$$\forall \mathbf{x}_j \in \mathcal{E}(\mathcal{X})$$

Finding cutting planes

- Master problem:
  - Add cut to feasible region and find new optimal cost vector

$$\begin{aligned} \mathbf{MP}(\hat{\mathbf{x}}, \tilde{\mathcal{X}}): \quad \underset{\mathbf{c} \in \mathcal{P}}{\text{minimize}} \quad & \left\| \mathbf{c} - \mathbf{c}^0 \right\|_1 \\ \text{subject to} \quad & \mathbf{c}^\top (\hat{\mathbf{x}} - \mathbf{x}) \leq 0, \quad \forall \mathbf{x} \in \tilde{\mathcal{X}} \end{aligned}$$

- Sub problem:
  - Find extreme optimal point on convex hull given a cost vector c
  - I.e., solve the original forward problem with a new cost vector

$$\begin{aligned} \mathbf{FP}(\mathbf{c}, \mathcal{X}): \ \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & \mathbf{x} \in \mathcal{X} := \{\mathbf{A}\mathbf{x} \geq \mathbf{b}, \ \mathbf{x} \in \mathbb{Z}^{n-q} \times \mathbb{R}^q\}. \end{aligned}$$

Finding cutting planes

Iteratively add cuts → Incremental solver

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Finding cutting planes

Iteratively add cuts → Incremental MIP solver

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- Sub problem:
  - Find extreme optimal point on convex hull given a cost vector *c*
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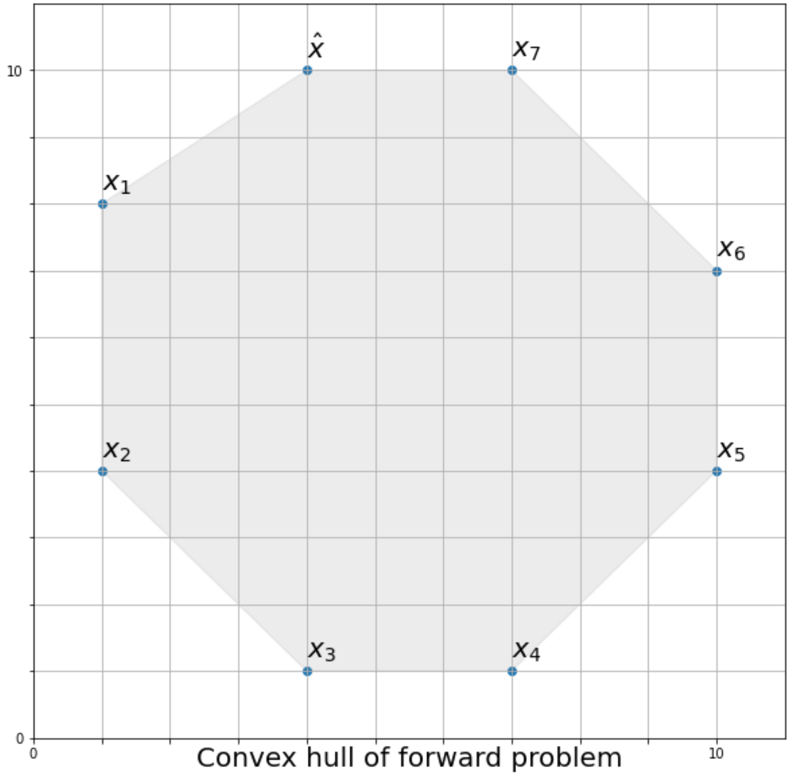
Original problem → problem specific solver

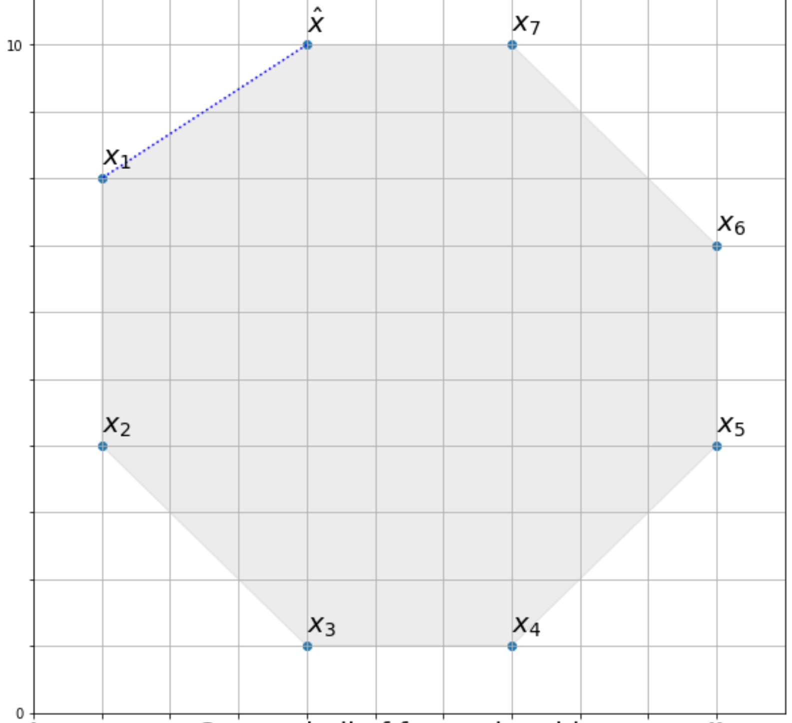
Cutting planes for Inverse Optimization



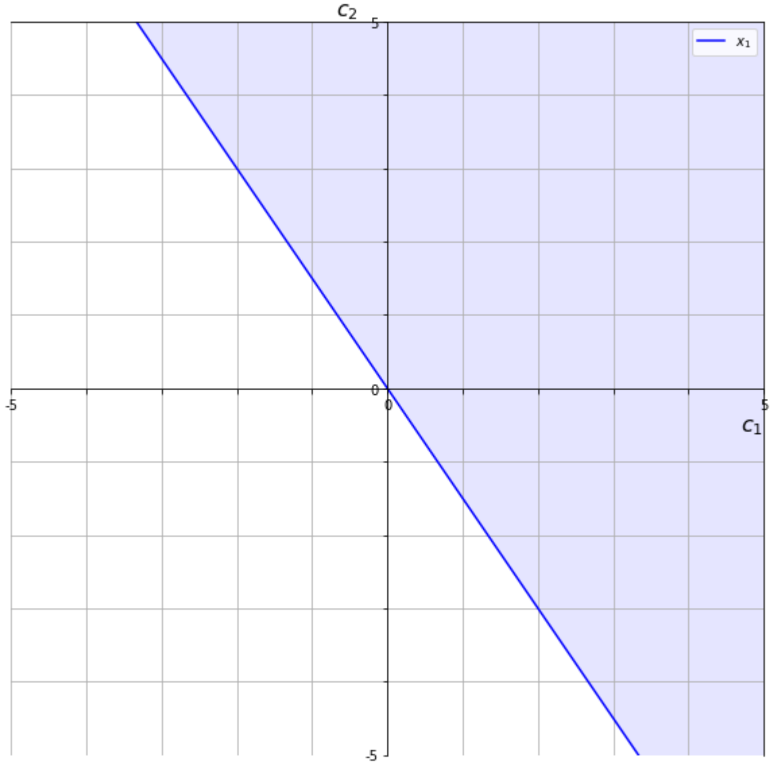
Every extreme point on the convex hull corresponds to an optimal solution for *some* cost vector c

Task: find the cost vector making $\hat{x}$ optimal
while minimizing changes to origninal cost vector

Cutting planes for Inverse Optimization



Convex hull of forward problem



Half spaces induced by cutting planes

Bodur, Merve, Timothy CY Chan, and Ian Yihang Zhu. "Inverse Mixed Integer Optimization: Polyhedral Insights and Trust Region Methods." INFORMS Journal on Computing (2022).

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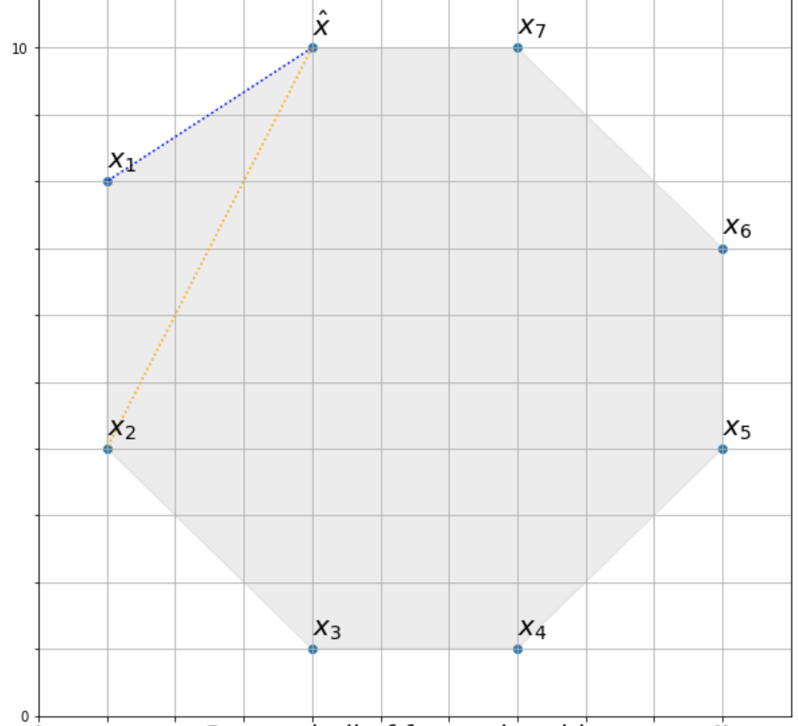

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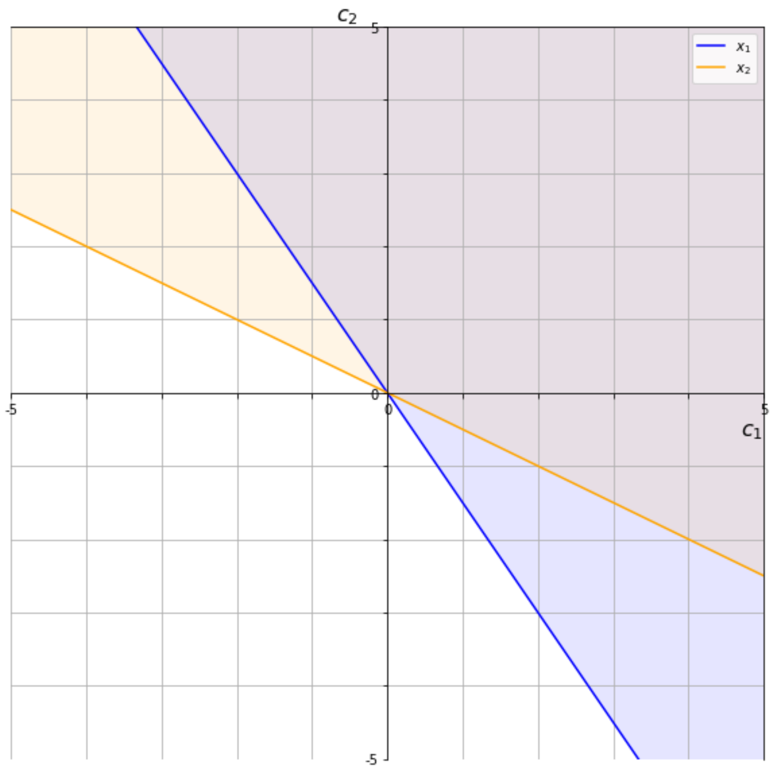

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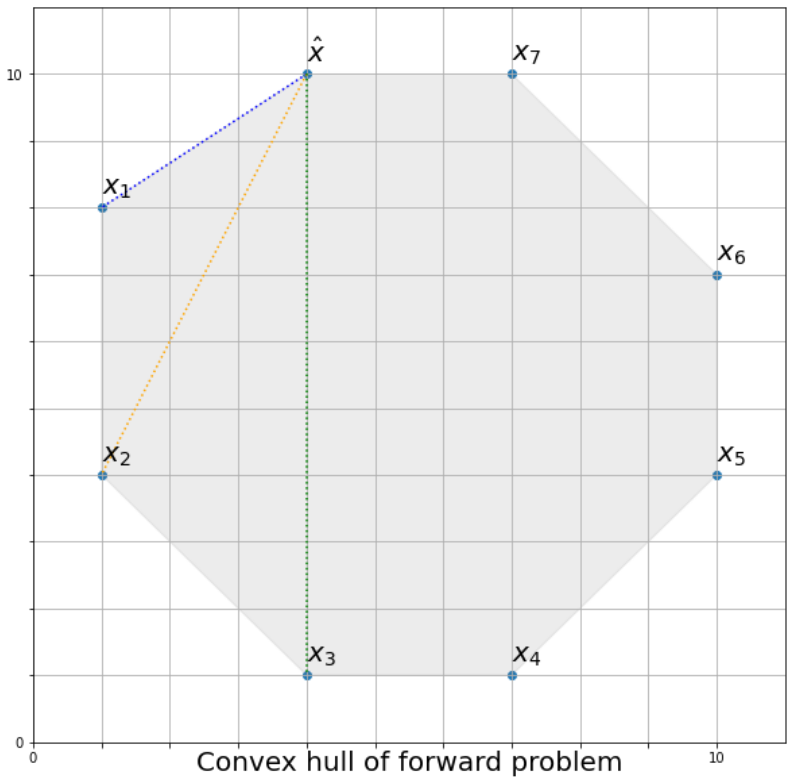

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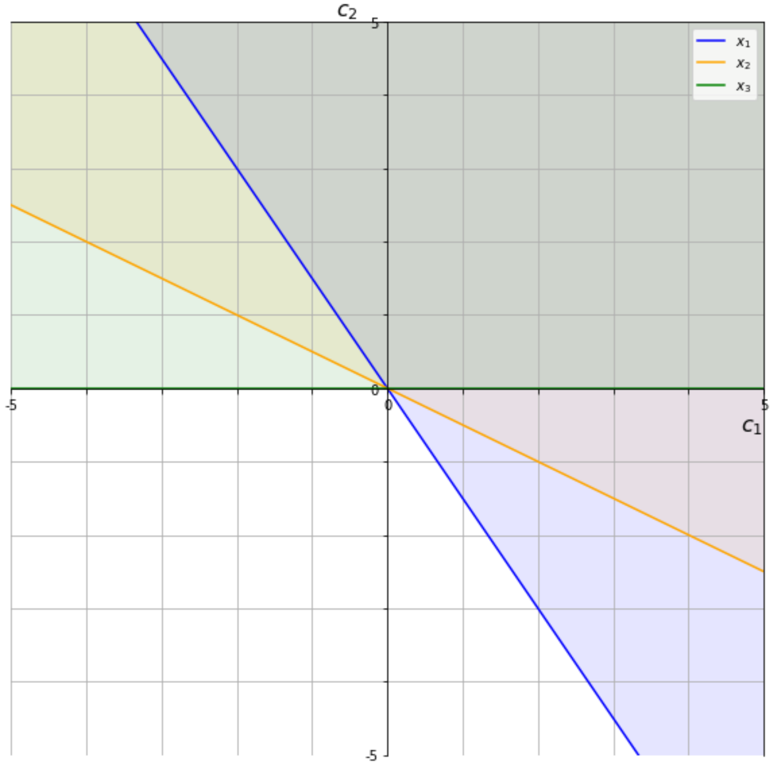

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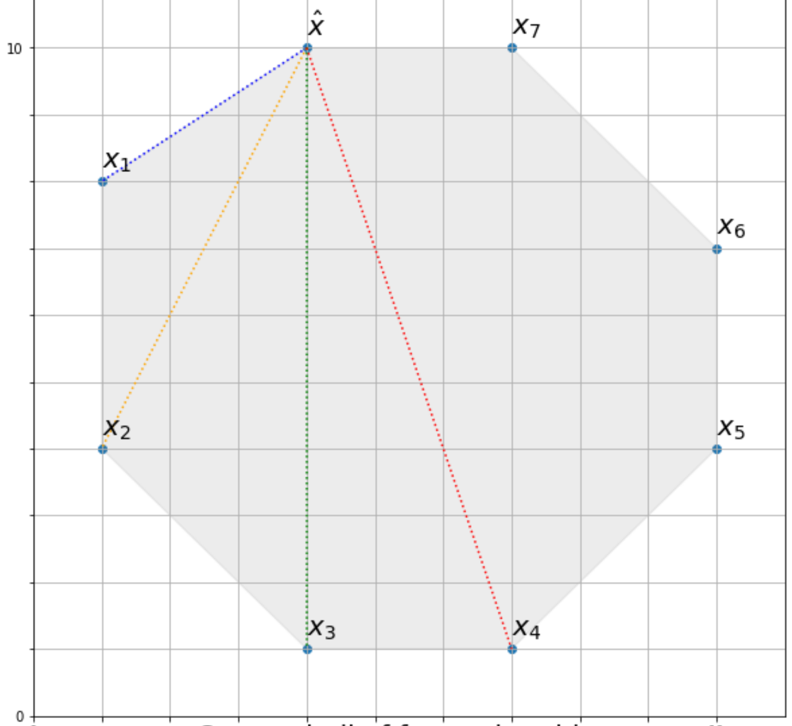

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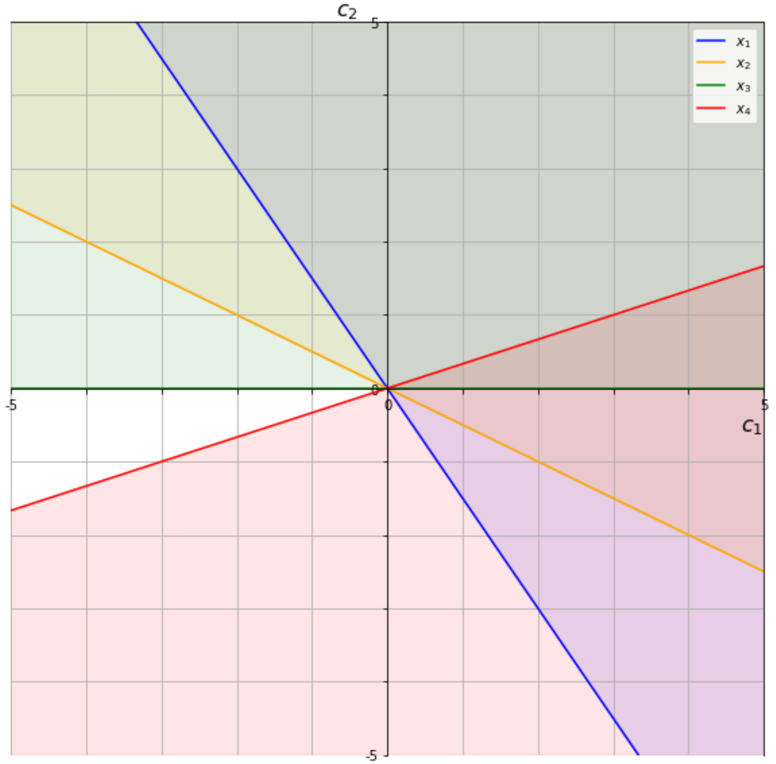

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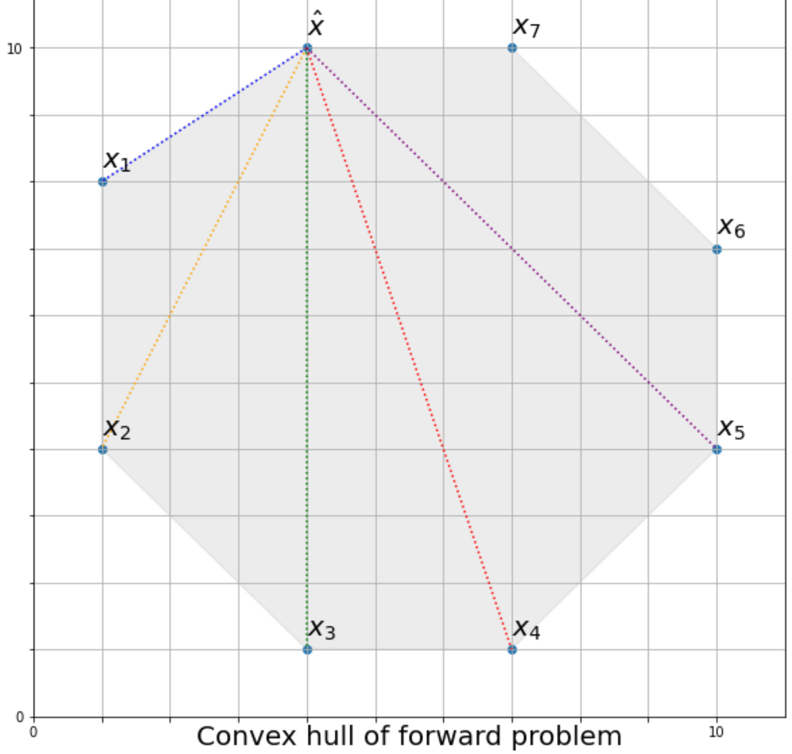

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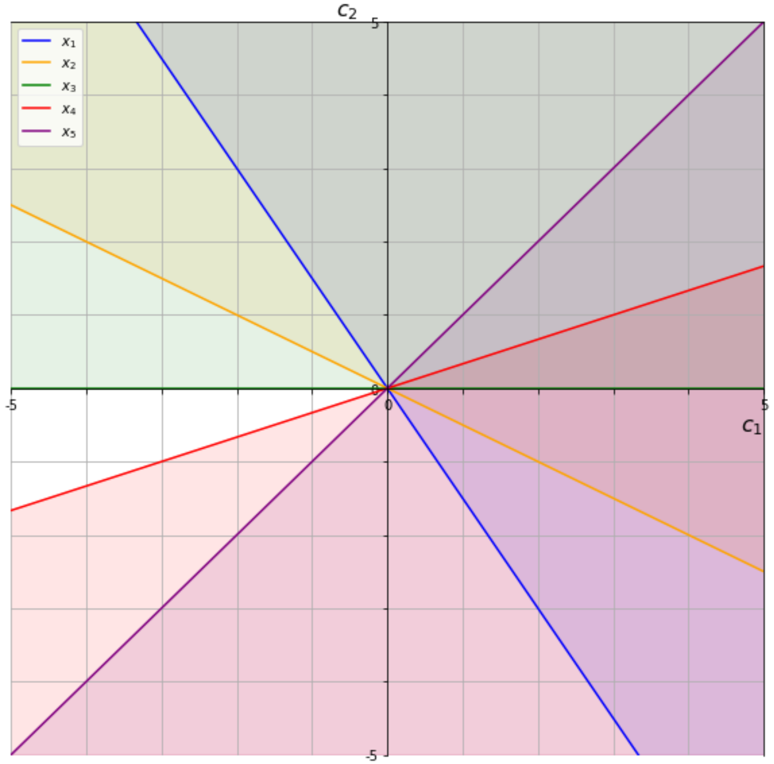

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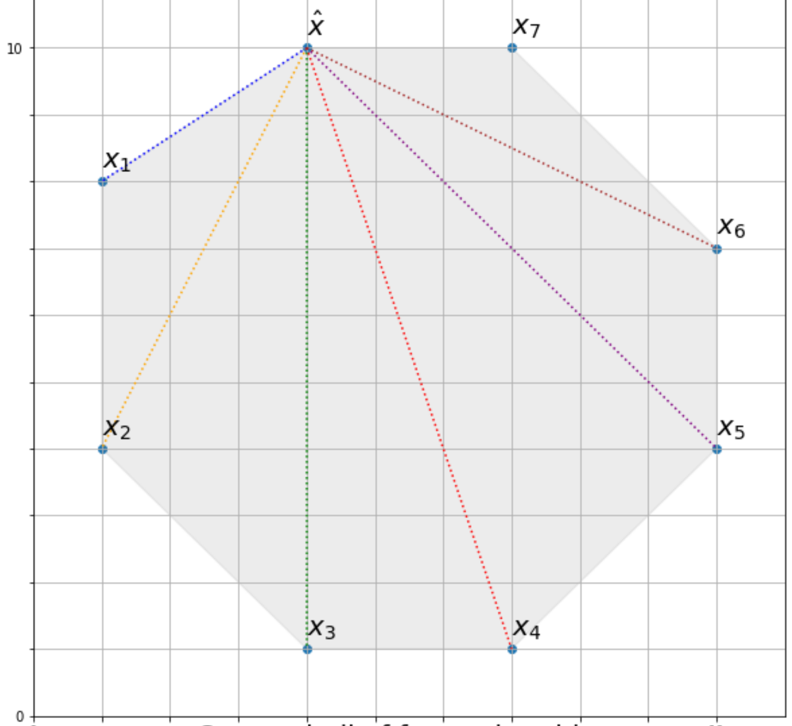

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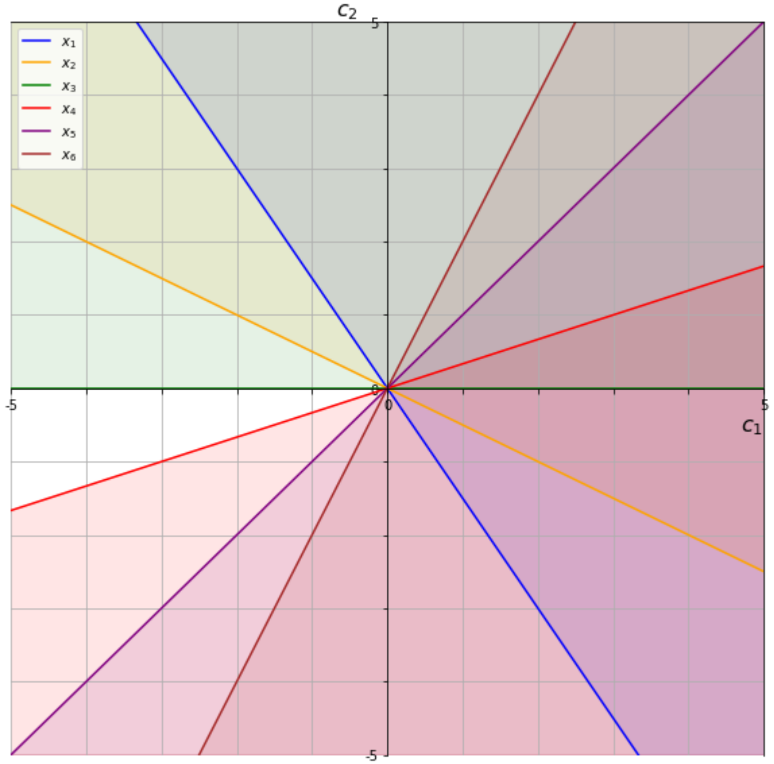

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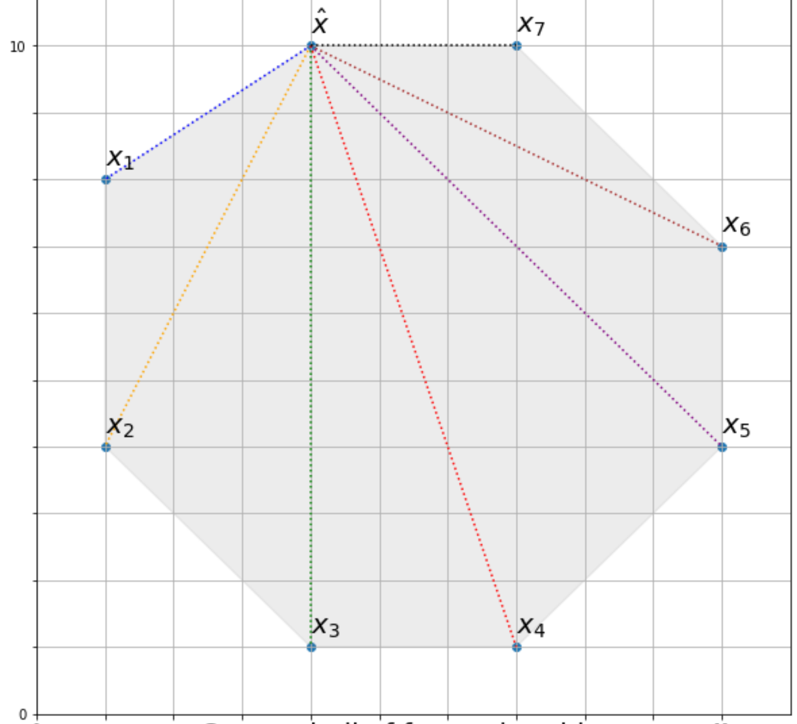

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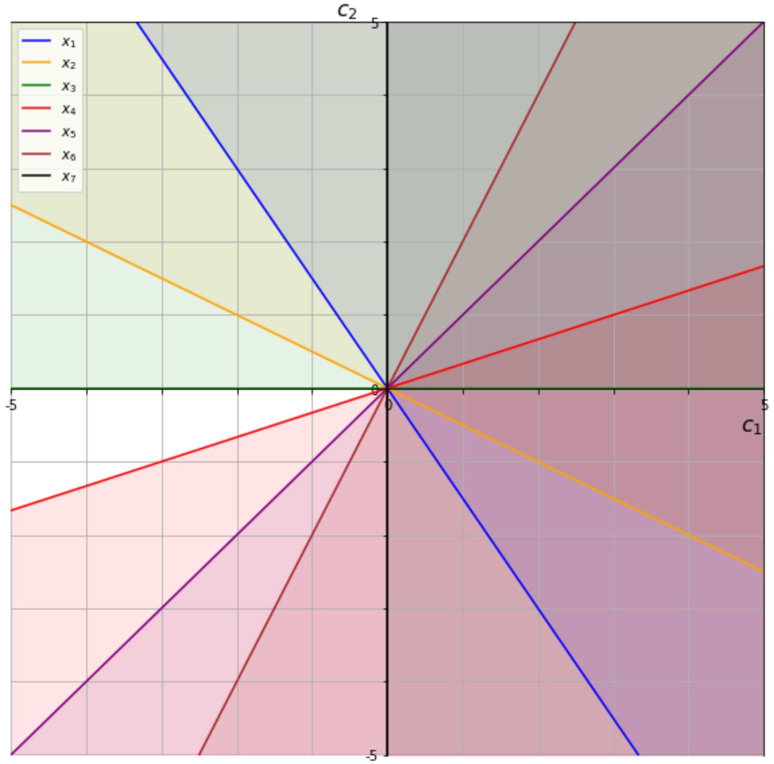

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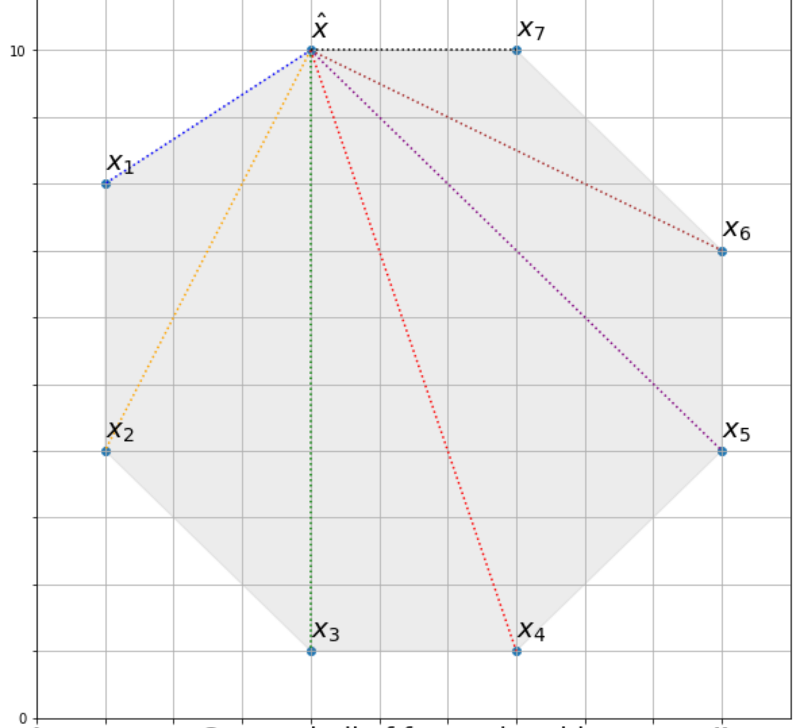

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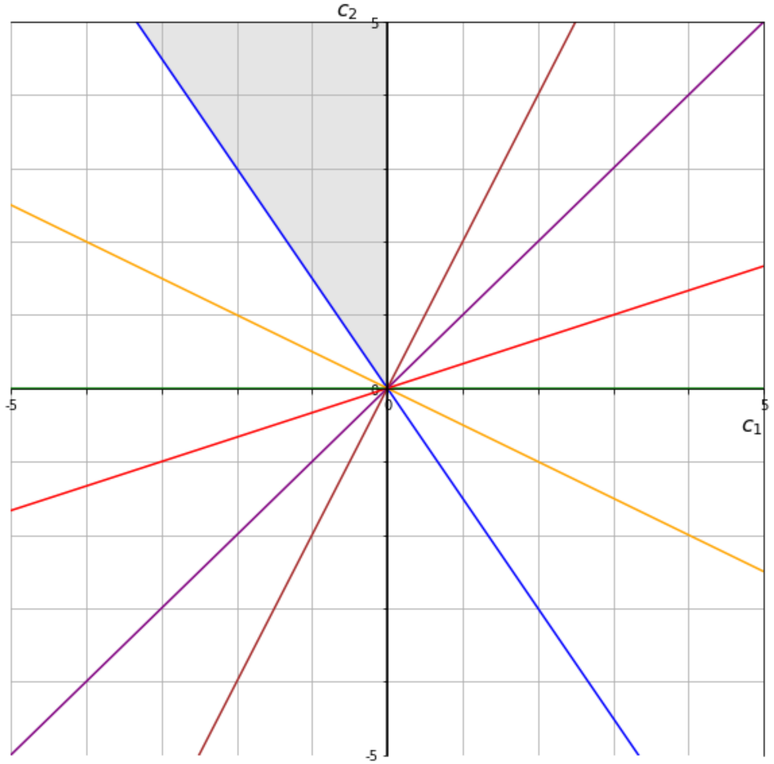

Intersection of all half planes

Bodur, Merve, Timothy CY Chan, and Ian Yihang Zhu. "Inverse Mixed Integer Optimization: Polyhedral Insights and Trust Region Methods." INFORMS Journal on Computing (2022).

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Inverse optimization in CPMpy

```python
def inverse_optimize(SP, c, x, x_d, keep_static=None):

    # Decision variable for new parameter vector
    d = intvar(0,INFTY, shape=len(x_d), name="d")

    # create the master problem
    MP = SolverLookup.get("gurobi")
    MP.minimize(norm(c-d,1))
    MP += SP.constraints
    while MP.solve():
        # find new cost vector
        new_d = d.value()
        print(f"New costvector = {new_d}")

        # find point on convex hull corresponding to new_d
        SP.maximize(sum(new_d * x))
        SP.solve()

        if sum(new_d * x_d) >= sum(new_d * x.value()):
            # solution is optimal
            break

        # add new cut to MP
        MP += sum(d * x_d) >= sum(d * x.value())

    return new_d, x.value()
```

Minimizing L1 norm

Solving the forward problem

Iteratively building cut constraints

Example: Knapsack problem

- Solver finds optimal solution to given problem:

```
Objective value: 32
Used capacity: 31
values = array([5, 0, 3, 3, 7, 9, 3, 5])
weights = array([2, 4, 7, 6, 8, 8, 1, 6])
capacity = 35
array([ True, False, False,  True,  True,  True,  True,  True])
```

Example: knapsack problem

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```

```
# User query
# "I want my solution to really contain item 1 and 2"
model += all(items[[1,2]])
assert model.solve()

x_d = items.value()
print("Objective value:",model.objective_value())
print("Used capacity:", sum(x_d * weights))

x_d
```

Find new solution satisfying extra constraints

```
Objective value: 29
Used capacity: 35

array([ True,  True,  True, False,  True,  True, False,  True])
```

Example: knapsack problem

```
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weights = array([2, 4, 7, 6, 8, 8, 1, 6])
capacity = 35
array([ True, False, False,  True,  True,  True,  Tru
```

New objective only changes for items 1 and 2!

```
Original values: [5 0 3 3 7 9 3 5]
new costvector = [5 0 3 3 7 9 3 5] New cut = [ True False False  True  True  True  True  True]
new costvector = [5 0 6 3 7 9 3 5] New cut = [ True False  True False  True  True  True  True]
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Conclusions

- Several applications in CP require repeated solver calls
  - Use CP-solvers as an oracle
  - Big efficieny gains using incremental solving
  - Use complementary strengths of several solvers
- Many applications in explanation generation:
  - MUS enumeration/extraction
  - OUS/SMUS Implicit hitting set algorithms
  - OCUS Implicit hitting set algorithms with constraints
  - Inverse optimization with cutting planes
  - …

Conclusions

- CPMpy is the right tool for the job:
  - Easy prototyping in Python/Numpy
  - Supports many solver paradigms
  - Incremental solving
  - Open source
    - We welcome all contributions/feedback!

https://github.com/CPMpy/cpmpy