MATRICES

Definition:- A collection of numbers arranged in rows and columns is said to be an array, A matrix is a rectangular array of numbers closed in addition, substraction, multiplication, and division.

We represent a matrix by $A = [a_{ij}]_{m\times n}$, $i = 1(1)m$, $j = 1(1)n$.

i.e. $\therefore A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$

Diagonal matrix:-

$A = [a_{ij}]$ is such that $a_{ij} = 0 \ \forall i \neq j$

i.e. $A = diag[d_1, d_2, \ldots, d_n]$

Eg. $A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$ is a 3X3 diagonal mtx.

Scalar matrix:- $A = [a_{ij}] \ni a_{ij} = 0 \ \forall i \neq j$

$a_{ij} = k \ \forall i = j$, $k \in \mathbb{N}$.

Eg $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$

~~Upper~~ Triangular mtx:- If every element above or below the leading diagonal of a square matrix is zero, then the matrix is called a triangular matrix.

Upper triangular mtx:- $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$

Lower Triangular mtx:- $\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Equality of matrices:- Two matrices A and B of the same order are said to be equal, if and only if the corresponding elements are equal.

Multiplication of matrix by a scalar:- Matrix multiplication is associative and distributive but not commutative.

$A = [a_{ij}]_{m\times n}$

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & & & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix} = [ka_{ij}]_{m\times n}$$

Addition of Matrices:-

$A = [a_{ij}]_{m\times n}$ $B = [b_{ij}]_{m\times n}$

$A+B = [a_{ij} + b_{ij}]$

Matrix addition is commutative, associative and distributive.

Note:- Only a square matrix can have a determinant.

Transpose of a matrix:-

$A = [a_{ij}]_{m\times n}$

$A^T = [a_{ji}]_{n\times m}$

(1) $(A')' = A$.
(2) $(A+B)' = A' + B'$.
(3) $(AB)' = B'A'$.

Symmetric and Skew-symmetric matrices:-

For a symmetric mtx. A, $a_{ij} = a_{ji}\ \forall\ i,j$.

For a skew-symmetric mtx A, $a_{ij} = -a_{ji}\ \forall\ i \neq j$
$= 0\ \forall\ i = j$.

Ex.1. If A be any matrix, S.T. AA' and $A'A$ are symmetric.

Sol. $(AA')' = (A')'(A)' = AA'$

$(A'A)' = (A)'(A')' = A'A$

Hence AA' and $A'A$ both are symmetric.

Ex.2. If A and B are both symmetric, then AB is symmetric iff A and B commute.

Sol.

$A' = A$, $B' = B$

$(AB)' = B'A'$
$= BA$
$= AB$ iff A and B commute.

This shows $(AB)'$ is symmetric.

Ex.3. S.T. A^2 is symmetric, if A is either symmetric or skew-symmetric.

Sol. A^2 exists only if A is square mtx.

Let $A = [a_{ij}]$, $i,j = 1,2,\ldots,n$. Then

$A^2 = [c_{ij}]$, $i,j = 1,2,\ldots,n$, where,

$c_{ij} = \sum_{k=1}^{n} a_{ik}a_{kj}$

Case I :- When A is symmetric, then $a_{kj} = a_{jk}$.

$$\therefore c_{ij} = \sum_{k=1}^{n} a_{ik} a_{jk}$$

$\therefore c_{ji} = \sum a_{jk} a_{ik}$; on interchanging i and j.

$\therefore$ Clearly, $c_{ij} = c_{ji}$, $\therefore A^2$ is symmetric.

Case II :- When A is skew-symmetric, then $a_{kj} = -a_{jk}$.

$$\therefore c_{ij} = \sum_{k=1}^{n} a_{ik}(-a_{jk}) = -\sum_{k=1}^{n} a_{ik} a_{jk}$$

so that $c_{ji} = \sum a_{jk} a_{ik}$, on interchanging i and j.

Clearly, $c_{ij} = c_{ji}$

$\therefore$ Hence A^2 is again symmetric.

$\therefore A^2$ is symmetric if A is either symmetric or skew-symmetric.

Ex. 4. S.T. all positive integral powers of a symmetric matrix are symmetric.

Sol.
$$A' = A$$

$(A^n)' = (A A \ldots n \text{ times})'$, n be a positive integer

$$= A'A' \ldots n \text{ times}$$
$$= A A \ldots n \text{ times} \quad \text{as } A' = A$$
$$= A^n \text{; hence } A^n \text{ is symmetric.}$$

Ex. 5. S.T. all positive odd (even) integral powers of a skew-symmetric matrix are skew-symmetric (symmetric).

Sol.
$$A' = -A$$

$$(A^n)' = (A A \ldots n \text{ times})'$$
$$= A'A' \ldots n \text{ times}$$
$$= (-A)(-A) \ldots n \text{ times} \quad \text{as } A' = -A.$$
$$= (-1)^n A^n$$
$$= \begin{cases} A^n, & n = \text{even} \quad \therefore A^n \text{ is symmetric.} \\ -A^n, & n = \text{odd} \quad \therefore A^n \text{ is skew-symmetric.} \end{cases}$$

Ex. 6. If A is symmetric (skew-symmetric), show that $B'AB$ is symmetric (skew-symmetric).

Sol. Case I :- $A^T = A$
$$(B'AB)' = (B)'A'(B')'$$
$$= B'AB$$

Case II :- $A^T = -A$,
$$(B'AB)' = -B'AB$$

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Ex.7. If A and B are symmetric (skew-symmetric), S.T. A+B is symmetric (skew-symmetric).

Ex.8. If A be any sq. matrix, S.T. $A+A'$ is symmetric and $A-A'$ is skew-symmetric.

Sol.

$$(A+A')' = (A') + (A')' = A' + A = (A+A')$$

$$(A-A')' = (A') - (A')' = A' - A = -(A-A').$$

Ex.9. If A, B are symmetric, S.T. $AB+BA$ is symmetric and $AB-BA$ is skew-symmetric.

Ex.10. Show that every square matrix can be uniquely expressed as the sum of a symmetric and a skew-symmetric matrix.

Sol.

$$A = \frac{1}{2}(A+A') + \frac{1}{2}(A-A')$$

$$(A+A')' = A' + (A')' = A'+A$$

$$(A-A')' = A' - A = -(A-A')$$

$\therefore$ $A+A'$ is symmetric and $A-A'$ is skew-symmetric.

Conjugate and Triangulated of matrix:-

A matrix obtained by replacing each element of a mtx A by its complex conjugate is called the conjugate mtx of A and is denoted by $\overline{A}$.

A matrix is said to be real iff $\overline{A} = A$.

$(\overline{A})'$ is called triangulated matrix of A.

Hermitian and Skew-Hermitian matrices:-

$A^* = [a_{ij}]$ is Hermitian iff $a_{ij} = \overline{a}_{ij}$ $\forall$ i, j

$= \overline{a}_{ii}$ $\forall$ $i=j$

i.e. every diagonal element of a Hermitian mtx is real.

e.g. $\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$, $\begin{bmatrix} 4 & 1-i \\ 1+i & 2 \end{bmatrix}$ are the examples of Hermitian mtx.

$A = [a_{ij}]$ is skew-Hermitian iff $a_{ij} = -\overline{a}_{ji}$ $\forall$ i, j.

i.e. Every diagonal element of a skew-hermitian mtx is either purely imaginary or zero.

e.g. $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1-i \\ -1-i & 2i \end{bmatrix}$

Q. Show that $A = \begin{bmatrix} 3 & 1+2i \\ 1-2i & 2 \end{bmatrix}$ is Hermitian.

Sol. $\overline{A}$ = conjugate of $A = \begin{bmatrix} 3 & 1-2i \\ 1+2i & 2 \end{bmatrix}$

and $A^* = (\overline{A})'$ transpose of $\overline{A} = \begin{bmatrix} 3 & 1+2i \\ 1-2i & 2 \end{bmatrix}$

Clearly, $A^* = A$; hence A is Hermitian.

Note:- The above all results of symmetric and skew-symmetric matrices are true if we replace symmetric mtx by

VECTORS AND VECTOR SPACES

FIELD :→ Suppose there is a set F of objects $x, y, z, \ldots$ and two operations on the elements of F as follows. The first operation, called addition, associates with each pair of elements x, y in F, an element $(x+y)$, the second operation called multiplication, associates with each pair x, y, an element xy, and these two operations satisfy the following properties :

i) PROPERTIES OF ADDITION :→

(a) Closure : $x \in F,\ y \in F \Rightarrow x+y \in F$.

(b) Commutative : $x+y = y+x \Rightarrow x, y \in F$.

(c) Associative : $x+(y+z) = (x+y)+z,\ \forall\ x, y, z \in F$.

(d) Nutral element : There is an unique element zero (0) in F $\ni$

$$x+0 = x\ \forall\ x \in F.$$

(e) Inverse : To each $x \in F$, there corresponds an unique element $(-x)$ in F $\ni\ x+(-x) = 0$.

ii) PROPERTIES OF MULTIPLICATION :→

(a) Closure : $x \in F,\ y \in F, \Rightarrow xy \in F$.

(b) Commutative : $xy = yx \Rightarrow x, y \in F$.

(c) Associative : $x(yz) = (xy)z \Rightarrow x, y, z \in F$

(d) Nutral element : There is an unique non-zero element 1 in F

$$\ni\ x \times 1 = x\ \forall\ x \in F.$$

(e) Inverse : To each non-zero $x \in F$, there corresponds an unique element x^{-1} (or, $\frac{1}{x}$) in F $\ni\ xx^{-1} = 1$.

iii) PROPERTIES OF ADDITION & MULTIPLICATION :→
(DISTRIBUTIVITY)

Multiplication distributes over addition, i.e.,

$$x \times (y+z) = xy + xz\ \forall\ x, y, z \in F$$

The set F together with these two operations is called a *field*.

Ex:→ $F = \{0, 1\}$.

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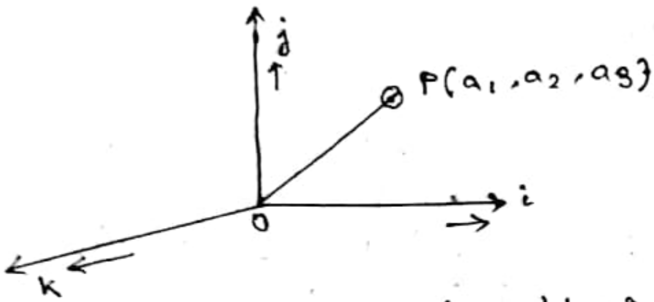


A vector in elementary physics, is a physical quantity having both magnitude and direction.

$$\overline{OP} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}.$$

Instead of characterising a vector by magnitude and direction, an equally satisfactory description could be achieved by the terminal point of vector originated from the origin.

Hence, we write $\rightarrow$ $\underset{\sim}{a} = (a_1, a_2, a_3)$, where a_i's the ith component.

An ordered array of numbers :$\rightarrow$ An array of numbers $(a_1, a_2, a_3, \ldots, a_n)$ is said to be an ordered array of numbers if $(a_{i_1}, a_{i_2}, a_{i_3}, \ldots, a_{i_n})$ is not the same or equivalent to $(a_{j_1}, a_{j_2}, a_{j_3}, \ldots, a_{j_n})$; where $(i_1, i_2, \ldots, i_n)$ and $(j_1, j_2, \ldots, j_n)$ are two different permutation of $(1, 2, \ldots, n)$.

An ordered array of n-numbers $(a_1, a_2, \ldots, a_n)$ will be called an ordered n-tuple.

Definition of Vector :$\rightarrow$

1) An n-component vector $\underset{\sim}{a}$ is an ordered n-tuple written as a row $(\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_n)$ or written as a column

$$\begin{pmatrix} a_1 \\ \underset{\sim}{a}_2 \\ \vdots \\ \underset{\sim}{a}_n \end{pmatrix}.$$

2)

** An ordered n-tuple of real numbers specifies a point in an n-dimensional space is called an n-component vector.

3) An ordered set of elements of a field is called a vector; the elements are called components. A vector of n components is called an 'n-component vector' or simply an 'n-vector'. An n-vector can be expressed in a horizontal or vertical line and in accordance, a row or column vector will appear.

TYPES :→

(a) Null Vector :

$$\underset{\sim}{0} = (0, 0, \ldots 0)'$$

all of those elements are zero.

(b) Unit Vector :

$$\underset{\sim}{e}_1 = (1, 0, 0, \ldots\ldots, 0)$$

$$\underset{\sim}{e}_2 = (0, 1, 0, \ldots\ldots, 0)$$

$$\vdots$$

(ith element)

$$\underset{\sim}{e}_i = (0, 0, 0, \ldots, 1, 0, \ldots, 0) \qquad [\bullet\ a_i' \underset{\sim}{e}_i = a_i \quad \forall\ i = 1(1)n.]$$

$$\vdots$$

$$e_n = (0, 0, 0, \ldots\ldots, 0, 1)$$

—are called the unit vectors.

(c) Sum Vector :

$$\underset{\sim}{1} = (1, 1, \ldots\ldots 1)'$$

all of ~~the~~ whose components are unity. $[\bullet\ \underset{\sim}{1}.\underset{\sim}{x} = \sum_{i=1}^{n} x_i]$

VECTOR OPERATIONS :→

Let $\underset{\sim}{a} = (a_1, a_2, \ldots, a_n)'$ and $\underset{\sim}{b} = (b_1, b_2, \ldots, b_n)'$ be two n-component vectors.

(a) Equality : Then $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are said to be equal iff.

$\underset{\sim}{a}_i = \underset{\sim}{b}_i \quad \forall\ i = 1(1)n$. Then we can say, $\underset{\sim}{a} = \underset{\sim}{b}$.

NOTE : The vectors $(1, 2)$ and $(1, 2, 0)$ are not equal. Two vectors can't be equal unless they have the same number of components.

(b) Addition : The sum of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is defined as

$$\underset{\sim}{a} + \underset{\sim}{b} = (a_1 + b_1, a_2 + b_2, \ldots\ldots, a_n + b_n)'$$

NOTE : This definition is appied only to the vectors which have equal number of components.

(c) Scalar Multiplication : The product of a scalar λ (constant real no.) and a vector $\underset{\sim}{a}$ is defined as

$$\lambda\underset{\sim}{a} = (\lambda a_1, \lambda a_2, \ldots\ldots, \lambda a_n)'$$

(d) Subtraction :

$$\underset{\sim}{a} - \underset{\sim}{b} = \underset{\sim}{a} + (-1)\underset{\sim}{b}$$

$$= (a_1, a_2, \ldots\ldots, a_n)' + (-b_1, -b_2, \ldots, -b_n)'$$

$$= (a_1 - b_1, a_2 - b_2, \ldots\ldots, a_n - b_n)'$$

Some Geometrical Concepts :

(a) Scalar Product : The scalar product of two vectors

$\underset{\sim}{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ is defined to be scalar if

$$\underset{\sim}{a}'\underset{\sim}{b} = \sum_{i=1}^{n} a_i b_i .$$

PROPERTIES :

1) $\underset{\sim}{a}'\underset{\sim}{b} = \underset{\sim}{b}'\underset{\sim}{a}$
2) $(\lambda \underset{\sim}{a}) \cdot \underset{\sim}{b} = \lambda (\underset{\sim}{a} \cdot \underset{\sim}{b})$
3) $\underset{\sim}{a}(\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c}$

(b) Distance : The distance of a vector (or a point) $\underset{\sim}{a}$ to the vector (or the point) $\underset{\sim}{b}$ is defined as the scalar,

$$|\underset{\sim}{a} - \underset{\sim}{b}| = \sqrt{(a-b)'(a-b)}$$
$$= \sqrt{\sum_{i=1}^{n} (a_i - b_i)^2}$$

PROPERTIES :

1) $|\underset{\sim}{a} - \underset{\sim}{b}| = |\underset{\sim}{b} - \underset{\sim}{a}|$
2) $|\underset{\sim}{a} - \underset{\sim}{b}| \geq 0$
3) $|\underset{\sim}{a} - \underset{\sim}{b}| + |\underset{\sim}{b} - \underset{\sim}{c}| \geq |\underset{\sim}{c} - \underset{\sim}{a}|$

RESULT : Prove that for any two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$,
$(\underset{\sim}{a} \cdot \underset{\sim}{b})^2 \leq |\underset{\sim}{a}|^2 \cdot |\underset{\sim}{b}|^2$. [Cauchy - Schwartz Inequality]

Proof : For any scalar λ,

$$|\lambda \underset{\sim}{a} + \underset{\sim}{b}| \geq 0$$
$$\Leftrightarrow |\lambda \underset{\sim}{a} + \underset{\sim}{b}|^2 \geq 0 \qquad \left[\because \underset{\sim}{a}'\underset{\sim}{a} = \sum_{i=1}^{n} a_i^2 = |\underset{\sim}{a}|^2\right]$$
$$\Leftrightarrow (\lambda \underset{\sim}{a} + \underset{\sim}{b})(\lambda \underset{\sim}{a} + \underset{\sim}{b}) \geq 0$$
$$\Leftrightarrow \lambda^2 |\underset{\sim}{a}|^2 + 2\lambda (\underset{\sim}{a} \cdot \underset{\sim}{b}) + |\underset{\sim}{b}|^2 \geq 0$$
$$\Leftrightarrow |\underset{\sim}{a}|^2 \left\{ \lambda^2 + 2\lambda \cdot \frac{(\underset{\sim}{a} \cdot \underset{\sim}{b})}{|\underset{\sim}{a}|^2} \right\} + |\underset{\sim}{b}|^2 \geq 0$$
$$\Leftrightarrow |\underset{\sim}{a}|^2 \left\{ \lambda + \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}|^2} \right\}^2 + |\underset{\sim}{b}|^2 - \frac{(\underset{\sim}{a} \cdot \underset{\sim}{b})^2}{|\underset{\sim}{a}|^2} \geq 0$$
$$\Leftrightarrow |\underset{\sim}{a}|^2 \left\{ \lambda + \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}|^2} \right\}^2 + \frac{|\underset{\sim}{a}|^2 |\underset{\sim}{b}|^2 - (\underset{\sim}{a} \cdot \underset{\sim}{b})^2}{|\underset{\sim}{a}|^2} \geq 0$$

For, $\lambda = -\frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}|^2}$, then $\frac{|\underset{\sim}{a}|^2 |\underset{\sim}{b}|^2 - (\underset{\sim}{a} \cdot \underset{\sim}{b})^2}{|\underset{\sim}{a}|^2} \geq 0$

$$\Rightarrow \boxed{(\underset{\sim}{a} \cdot \underset{\sim}{b})^2 \leq |\underset{\sim}{a}|^2 \cdot |\underset{\sim}{b}|^2}$$ provided $\underset{\sim}{a}$ & $\underset{\sim}{b}$ have finite length.

LIMIT

Definition of limit:- $\underset{x\to a}{Lt} f(x) = l$ if for every $\epsilon > 0$, $\exists$ a 'δ'(>0) such that $|f(x) - l| < \epsilon$ whenever $|x-a| < \delta$.

Theorem:- $\underset{x\to a}{Lt} f(x) = l$ iff for every sequence $\{x_n\}$ converges to 'a', i.e. $\underset{n\to\infty}{lt} f(x_n) = l$.

~~If for two sequences $\{a_n\}$ and $\{b_n\}$ converge to 'a', then~~

Remark:- Two sequences $\{a_n\}$ and $\{b_n\}$ are said to be equivalent if $\underset{n\to\infty}{lt} \frac{a_n}{b_n} = l \neq 0$.

Similarly, two functions $f(x)$ and $g(x)$ are said to be equivalent if for large x, $\underset{x\to\infty}{lt} \frac{f(x)}{g(x)} = l \neq 0$.

Examples:-

(i) $\frac{\sqrt{n}+n}{n+1} \sim n\sqrt{n}$

(ii) $\sin\frac{1}{n} \sim \frac{1}{n}$.

(iii) $a^{1/n} - 1 \sim \frac{1}{n}$

(iv) $\sqrt{n+1} - \sqrt{n} \sim \frac{1}{\sqrt{n+1}+\sqrt{n}}$

(v) $\sqrt{n+1} - \sqrt{n} \sim \frac{1}{\sqrt{n}}$

(vi) $(n+1)^2 - n^2 \sim n$.

Note:- $\lim_{x\to a} f(x) = l \Leftrightarrow$ for given $\epsilon > 0$, $\exists$ a $\delta > 0 \ni$

$$0 < |x-a| < \delta \Rightarrow |f(x) - l| < \epsilon$$
$$\Leftrightarrow a-\delta < x < a+\delta \Rightarrow l-\epsilon < f(x) < l+\epsilon$$
$$\Leftrightarrow x \in (a-\delta, a+\delta) \Rightarrow f(x) \in (l-\epsilon, l+\epsilon).$$

(except possibly at $x=a$)

In other words, a real number l is a limit of the function f as x approaches to a if for every nbd of l $\exists$ a nbd of 'a' $\ni$ for every x in nbd of 'a', $f(x)$ is in nbd of l.

Ex. If f is given by $f(x) = \begin{cases} \frac{x^2-a^2}{x^2+a^2}, & x \neq a \\ 0, & ow \end{cases}$

then show that $\lim_{x\to a} f(x) = 2a$.

Sol.

$$|f(x) - 2a| < \epsilon$$
$$\Rightarrow \left|\frac{x^2-a^2}{x-a} - 2a\right| < \epsilon$$
$$\Rightarrow |x-a| < \epsilon$$

Now if we choose a number $\delta \ni 0 \leq \delta \leq \epsilon$, then $|f(x) - 2a| < \epsilon$ whenever $|x-a| < \delta \Rightarrow \lim_{x\to a} f(x) = 2a$.

Right-hand limit :- A real no. l is said to be a limit of a function f as x tends to 'a' from above (from the right) iff for every given $\epsilon > 0$, $\exists$ a +ve $\delta \ni$

$$|f(x) - l| < \epsilon \quad \text{whenever} \quad a < x < a+\delta.$$

We write $\lim_{x \to a+0} f(x) = l = f(a+0)$

Here $a < x < a+\delta \Rightarrow l - \epsilon < f(x) < l + \epsilon$

Left hand limit :- A real number l is said to be a limit of a function f as x tends to 'a' from below (from the left) iff for every given $\epsilon > 0$, $\exists$ a +ve number $\delta \ni$

$$|f(x) - l| < \epsilon \quad \text{whenever} \quad a - \delta < x < a.$$

We write $\lim_{x \to a-0} f(x) = l = f(a-0)$.

Here $a - \delta < x < a \Rightarrow l - \epsilon < f(x) < l + \epsilon$.

Ques :- A function $f(x)$ is said to have a limit l iff both the RHL and LHL exist and are equal to l.

Sol. By definition of limit,

$\lim_{x \to a} f(x) = l \Leftrightarrow$ for a given $\epsilon > 0$, $\exists$ a $\delta > 0 \ni$

$|f(x) - l| < \epsilon$ whenever $0 < |x - a| < \delta$

$|f(x) - l| < \epsilon$ whenever $a - \delta < x < a + \delta$

$\Leftrightarrow |f(x) - l| < \epsilon$ whenever $a - \delta < x < a$ and

$\Leftrightarrow |f(x) - l| < \epsilon$ whenever $a < x < a + \delta$

$\Leftrightarrow \lim_{x \to a+} f(x) = l$ and $\lim_{x \to a-0} f(x) = l$.

Hence the result.

Ex. If a function f is defined as $f(x) = [1 - x]$ $\forall x \in \mathbb{R}$. Then S.T. $\lim_{x \to 0+} f(x) \neq \lim_{x \to 0-} f(x)$.

sol. If $x \in (0,1) \Rightarrow 0 < x < 1$

$\Rightarrow 0 < 1 - x < 1$

$\Rightarrow f(x) = 0 \ \forall x \in (0,1)$

So for any $\epsilon > 0$ and for any $\delta > 0$ but less than 1, we have

$|f(x) - 0| < \epsilon$ whenever $x \in (0, \delta)$

$\Rightarrow \lim_{x \to 0+0} f(x) = 0$

$x \in (-1, 0)$, $\Rightarrow -1 < x < 0$

$\Rightarrow 1 < 1 - x < 2$

$\Rightarrow f(x) = 1 \ \forall x \in (-1, 0)$

So for any $\epsilon > 0$ and for $\delta > 0$ but less than 1, we have

$|f(x) - 1| < \epsilon$ whenever $x \in (-\delta, 0)$

$\Rightarrow \lim_{x \to 0-0} f(x) = 1$.

$\therefore f(0+0) \neq f(0-0)$

Rule I:- To evaluate $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)}$ divide the numerator and denominator by the highest power of x involved in $f(x)$ and $g(x)$.

Examples:- 1.

(i) $\lim\limits_{x\to\infty} \left\{\sqrt{x+1} - \sqrt{x}\right\}$

Sol. $\lim\limits_{x\to\infty} \left\{\sqrt{x+1} - \sqrt{x}\right\}$ Now, $\sqrt{x+1} - \sqrt{x} \sim \frac{1}{\sqrt{x}}$

$= \lim\limits_{x\to\infty} \frac{1}{\sqrt{x}}$

$= 0.$

(ii) $\lim\limits_{x\to\infty} \left\{\sqrt{x+\sqrt{x+\sqrt{x}}} - \sqrt{x}\right\}$

Sol. $\sqrt{x+\sqrt{x+\sqrt{x}}} - \sqrt{x}$

$$= \frac{x+\sqrt{x+\sqrt{x}} - x}{\sqrt{x+\sqrt{x+\sqrt{x}}} + \sqrt{x}}$$

$$= \frac{\sqrt{1+\frac{1}{\sqrt{x}}}}{\sqrt{1+\sqrt{\frac{1}{x}+\frac{1}{x^{3/2}}}} + 1}$$

$$\longrightarrow \frac{\sqrt{1+0}}{\sqrt{1+0}+1} = \frac{1}{2} \text{ as } x\to\infty.$$

(iii) $\lim\limits_{x\to\infty} x^3 \left\{\sqrt{x^2+\sqrt{1+x^4}} - x\sqrt{2}\right\}$

Sol. $$\lim_{x\to\infty} \frac{x^3\left\{x^2+\sqrt{1+x^4} - 2x^2\right\}}{\left\{\sqrt{x^2+\sqrt{1+x^4}} + x\sqrt{2}\right\}}$$

$$= \lim_{x\to\infty} \frac{x^3\left\{\sqrt{1+x^4} - x^2\right\}}{\left\{\sqrt{x^2+\sqrt{1+x^4}} + x\sqrt{2}\right\}}$$

$$= \lim_{x\to\infty} \frac{x^3\left\{1+\cancel{x^4}-\cancel{x^4}\right\}}{\left\{\sqrt{x^2+\sqrt{1+x^4}} + x\sqrt{2}\right\}\left\{\sqrt{1+x^4}+x^2\right\}}$$

$$= \lim_{x\to\infty} \frac{1}{\left\{\sqrt{1+\sqrt{1+\frac{1}{x^4}}} + \sqrt{2}\right\}\left\{\sqrt{\frac{1}{x^4}+1} + 1\right\}}$$

$$= \frac{1}{\left\{\sqrt{1+\sqrt{1+0}} + \sqrt{2}\right\}\left\{\sqrt{1+0}+1\right\}}$$

$$= \frac{1}{4\sqrt{2}}.$$

2. (i) $\lim\limits_{x\to\infty} \dfrac{ax^{p}+bx^{p-2}+c}{dx^{p}+cx^{p-2}+b}$, $p>0$

Sol.

$$= \lim_{x\to\infty} \frac{a+\frac{b}{x^2}+\frac{c}{x^p}}{d+\frac{c}{x^2}+\frac{b}{x^p}}, \quad p>0$$

$$= \frac{a}{d}.$$

(ii) $\lim\limits_{x\to-\infty} \dfrac{x^4 \sin\frac{1}{x}+x^2}{1+|x|^3}$

Sol.

$$= \lim_{x\to-\infty} \frac{x\sin\frac{1}{x}+\frac{1}{x}}{\frac{1}{x^3}-1} \qquad \left[\text{Here } \lim_{x\to-\infty} x\sin\frac{1}{x}=1\right].$$

$$= \frac{1+0}{0-1}$$

$$= -1.$$

3. (i) If $\lim\limits_{n\to\infty}\left(an-\dfrac{1+n^2}{1+n}\right)=b$, then find the values of a and b.

Solution:—

$$b = \lim_{n\to\infty}\left\{an-\frac{1+n^2}{1+n}\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{an+an^2-1-n^2}{1+n}\right\}$$

$$= \lim_{n\to\infty}\frac{(a-1)n^2+(an-1)}{(1+n)}$$

If $(a-1)\neq 0$ then $\lim\limits_{n\to\infty}\dfrac{(a-1)n^2+(an-1)}{(1+n)}=+\infty$ but it is given that the limitting value is b (finite).

$\therefore (a-1)=0 \Rightarrow a=1.$ $\therefore$ $b=\lim\limits_{n\to\infty}\dfrac{n-1}{n+1}=1.$

(ii) If $\underset{x\to\infty}{\text{lt}}\left\{\sqrt{x^4+x^2+1}-ax^2-b\right\}=0$, then find a and b?

Solution:—

$$0=\underset{x\to\infty}{\text{lt}}\left\{\sqrt{x^4\mp x^2+1}-ax^2-b\right\}$$

$$=\underset{x\to\infty}{\text{lt}}\frac{(x^4\mp x^2+1)-(ax^2+b)^2}{\sqrt{x^4\mp x^2+1}+ax^2+b}$$

$$=\underset{x\to\infty}{\text{lt}}\frac{(1-a^2)x^4-(1+2ab)x^2+(1-b^2)}{\sqrt{x^4\mp x^2+1}+ax^2+b}$$

Hence $1-a^2=0 \Rightarrow a=\pm 1.$

If $a=1$, then $-(1+2b)=0$
$\Rightarrow b=-1/2$

If $a=-1$, then $(1-2b)=0$
$\Rightarrow b=1/2.$

• Useful Formulae on Real Numbers •

Absolute Value of a Real Numbers: $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

Thus, we already have $|x| \geq 0$, also by definition
$|x| = |-x|$.

Note:-
1. $|x| = \max(x, -x)$
2. $|-x| = \max(-x, -(-x)) = \max(-x, x) = |x|$
3. $-|x| = \min(x, -x)$
4. $|x|^2 = |-x|^2 = x^2$
5. $|xy| = |x| \cdot |y|$
6. $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, provided $y \neq 0$.

Remark:- If a, b are real numbers, then show that
$\max(a, b) = \frac{a+b+|a-b|}{2}$ and $\min(a, b) = \frac{a+b-|a-b|}{2}$.

Triangle Inequalities:- For all real numbers x, y show that
(i) $|x+y| \leq |x| + |y|$, and
(ii) $|x-y| \geq \big||x| - |y|\big|$.

Proof:- (i) $|x+y|^2 = (x+y)^2 = x^2 + y^2 + 2xy$
$\leq |x|^2 + |y|^2 + 2|x| \cdot |y|$ $[\because xy \leq |xy| = |x| \cdot |y|]$
$= (|x| + |y|)^2$

since the quantities are non-negative, so taking the +ve sq. root,
$|x+y| \leq |x| + |y|$

(ii). Similar method.

Ex.1. For real nos. $x, a, \epsilon > 0$, show that
(i) $|x| < \epsilon \Leftrightarrow -\epsilon < x < \epsilon$,
(ii) $|x-a| < \epsilon \Leftrightarrow a - \epsilon < x < a + \epsilon$.

Ex.2. If $a, b \in \mathbb{R}$ be $\ni$ $a < b + \epsilon$ for each $\epsilon > 0$, then $a \leq b$.

Ex.3. If $a, b \in \mathbb{R}$, show that if $a \leq b + \frac{1}{n}$, $\forall\, n \in \mathbb{N}$, then $a \leq b$.

Ex.4. If for any $\epsilon > 0$, $|b-a| < \epsilon$, then $b = a$.

Ex.5. If $a, b \in \mathbb{R}$ and $a < c$ for each $c > b$, then $a \leq b$.

Sol. Do yourself.

Ex.6 $[a+b] \geq [a] + [b]$ for all real numbers a, b.

Ex.7. $[a] + [-a] = \begin{cases} 0, & a \text{ is an integer} \\ -1, & \text{o/w} \end{cases}$

• SEQUENCE OF REAL NUMBERS •

The word "Sequence" is used to convey the idea that the things are arranged in order. Before introductioing the concept of sequence in $\mathbb{R}$, we define function or mapping or transformation between two sets A and B.

Let $f: A \to B$ is a mapping or function if for every $x \in A$, there exists $(\exists)$ a unique value of $y \in B$. Then the rule f is called a mapping or a function of A into B.
Here we write $y = f(x)$, where $x \in A$ and $y \in B$.

Note that, $y = \pm x$ is not a function, it's a relation.

• **DEFINITION:—** A sequence of real numbers (or, a sequence in $\mathbb{R}$) is a function defined on the set $\mathbb{N}$ of natural numbers whose range is a subset of the set $\mathbb{R}$ of real numbers; i.e. if for every $n \in \mathbb{N}$ (a set of natural numbers), $\exists$ a real number a_n, then the order set

$$a_1, a_2, \cdots, a_n, \cdots$$

is said to define a set of real numbers.

Notation:—

(1) If a_n is the n^{th} term of a sequence, then we write $a_1, a_2, \cdots\cdots, a_n, \cdots\cdots$ to describe the sequence.

(2) $f: \mathbb{N} \to \mathbb{R}$ is a sequence.

(3) A sequence f is generally denoted by the symbol $\{f(n)\}$.
e.g. $\{a_n\}$, $\{b_n\}$, $\{x_n\}$, $\{y_n\}$.

Examples:—

(a) If $b \in \mathbb{R}$, the sequence $B = \{b, b, \cdots\cdots\}$, all of whose terms are equal b, is called the constant sequence $\{b\}$.

(b) Let $f: \mathbb{N} \to \mathbb{R}$ is defined by $f(n) = \frac{n}{n+1}$, $n \in \mathbb{N}$. The sequence is $\left\{\frac{n}{n+1}\right\}$. It is also denoted by $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots\cdots\right\}$

(c) Let $f: \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = (-1)^n$, $n \in \mathbb{N}$. The sequence is $\{(-1)^n\}$. It is also denoted by $\{-1, 1, -1, 1, \cdots\cdots\}$. The range of the sequence is $\{-1, 1\}$.

(d) The celebrated Fibonacci sequence $F = (f_n)$ is given by the inductive definition

$$f_1 = 1, \; f_2 = 1, \; f_{n+1} = f_{n-1} + f_n \quad (n \geq 2)$$

Thus each term past the second is the sum of its two immediate predecessors. The sequence is $\{1, 1, 2, 3, 5, 8, 13, 21, 34, \cdots\cdots\}$.

(e) Null Sequence: A null sequence is one whose terms approach zero, i.e. $\lim a_n = 0 \Rightarrow \{a_n\}$ be a null sequence.

Example:- $\left\{\frac{1}{n}\right\}$ is a null sequence.

Note that, If $\{a_n\}$ be a null sequence then $\{|a_n|\}$ is a null sequence and conversely.

The Limit of a Sequence:

- **DEFINITION:** A sequence $\{a_n\}$ is said to have a limit $l \in \mathbb{R}$ if for every $\epsilon > 0$, $\exists$ a natural number $N(\epsilon)$ $\ni$ for all $n \geq N(\epsilon)$, the terms $\{a_n\}$ satisfy $|a_n - l| < \epsilon$.

Ex.(1): Show that the sequence $\left\{\frac{1}{n}\right\}$ has the limit 0.

Solution: We show that $\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$.

Now, $\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \epsilon$, whenever $n > \frac{1}{\epsilon}$

We choose $n_0 = \left[\frac{1}{\epsilon}\right] + 1$ $\therefore$ $\left|\frac{1}{n} - 0\right| < \epsilon$ holds when $n \geq n_0$.

By the above definition, $\lim_{n\to\infty} \frac{1}{n} = 0$. [Here $\left[\frac{1}{\epsilon}\right]$ denotes the integral part of $\frac{1}{\epsilon}$]

In other words, $\left\{\frac{1}{n}\right\}_n$ converges to 0.

Note: Let $\{f(n)\}$ be a real sequence. A real number l is said to be a limit of the sequence $\{f(n)\}$ if corresponding to a pre-assigned positive quantity ϵ $\exists$ a natural number K (depending on ϵ) such that

$$\boxed{|f(n) - l| < \epsilon \quad \forall\ n \geq K}$$

i.e. $l - \epsilon < f(n) < l + \epsilon \ \forall\ n \geq K$.

To be explicit, a real number l is said to be a limit of the sequence $\{f(n)\}$, if for a pre-assigned positive ϵ $\exists$ a natural number K $\ni$ all elements of the sequence, excepting the first $K-1$ at most, lie in the ϵ-neighbourhood of l.

Theorem: A sequence can have at most one limit.

Proof: If possible, let a sequence $\{f(n)\}$ have two distinct limits l_1 and l_2, where $l_1 < l_2$.

Let $\epsilon = \frac{1}{2}(l_2 - l_1)$ Then $\epsilon > 0$ and $l_1 + \epsilon = l_2 - \epsilon$. Therefore the neighbourhoods $(l_1 - \epsilon, l_1 + \epsilon)$ and $(l_2 - \epsilon, l_2 + \epsilon)$ are disjoint.

But, as l_1 is a limit of a sequence, then

$$l_1 - \epsilon < f(n) < l_1 + \epsilon \quad \forall\ n \geq K_1$$

Similarly, $$l_2 - \epsilon < f(n) < l_2 + \epsilon \quad \forall\ n \geq K_2.$$

Let, $K = \max\{K_1, K_2\}$

Then, $l_1 - \epsilon < f(n) < l_1 + \epsilon$ and $l_2 - \epsilon < f(n) < l_2 + \epsilon$ for all $n \geq K$.

This can't happen since the neighbourhoods $N(l_1, \epsilon)$ and $N(l_2, \epsilon)$ are disjoint. Therefore our assumption $l_1 \neq l_2$ is wrong.

Hence, $l_1 = l_2$ and this proves the theorem.

The Range: The Range or the Range Set is the set consisting of all distinct elements of a ~~set of~~ sequence, without repetition and without regard to the position of a term. Thus, the range may be a finite or an infinite set, without ever being the null set.

Bounds of a Sequence:

- **DEFINITION:** – A sequence $\{a_n\}$ is said to be bounded if $\exists$ two real numbers m and $M \ni m \leq a_n \leq M \ \forall \ n \in \mathbb{N}$.
Here, M is called the upper bound of the sequence and m is called the lower bound of the sequence.

If a sequence is not bounded then it is called unbounded sequence.

Ex:- (1) The sequence $\{\frac{1}{n}\}$ is a bounded sequence; as $0 < \frac{1}{n} \leq 1 \ \forall n$; 0 and 1 are the lower and upper bound respectively.

(2) $\{(-1)^n+1\}$ is a bounded sequence as

$$(-1)^n + 1 = \begin{cases} 0 & \text{if } n \text{ is odd.} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

(3) $\{(-1)^n+n\}$ is not a bounded sequence as $\nexists$ an $M \in \mathbb{R}$ such that $a_n \leq M$.

(4) The sequence $\{n^2\}$, $\{n^2+n\}$ are unbounded.

(5) Let $f(n) = (-1)^n n$, $n \in \mathbb{N}$, the sequence $\{f(n)\}$ is unbounded above and unbounded below. The sequence is $\{-1, 2, -3, 4, \dots\}$
Here $\sup\{f(n)\} = \infty$ and $\inf\{f(n)\} = -\infty$.

Note:- Therefore a real sequence is bounded if and only if it is bounded above as well as bounded below. In this case the range of the sequence is a bounded set.

The least upper bound of a real sequence $\{f(n)\}$ is a real number M (denoted by $\sup\{f(n)\}$) satisfy the following conditions:

(i) $f(n) \leq M$ for all $n \in \mathbb{N}$,
(ii) for each pre-assigned positive ϵ, there exists a natural number k such that $f(k) > M - \epsilon$.

The greatest lower bound of a real sequence $\{f(n)\}$ is a real number m (denoted by $\inf\{f(n)\}$) satisfy the following conditions:

(i) $f(n) \geq m$ for all $n \in \mathbb{N}$,
(ii) for each pre-assigned positive ϵ, $\exists$ a natural number k such that $f(k) < m + \epsilon$.

For a real sequence $\{f(n)\}$ is said to be unbounded above if $\sup\{f(n)\} = \infty$.

For a real sequence $\{f(n)\}$ is said to be unbounded below if $\inf\{f(n)\} = -\infty$.

Extrema of Functions of several variables :

If $D \subseteq \mathbb{R}^n$ be the domain of a function 'f', we shall denote a point in D as $\underset{\sim}{x}$ and the value of the function at $\underset{\sim}{x}$ by $f(\underset{\sim}{x}) = f(x_1, x_2, \cdots, x_n)$.

Definition:→ A function 'f' has a local (or, relative) maximum or minimum at x_0 if there is some neighbourhood $N_\delta(x_0) = \{x : |x - x_0| < \delta\}$ such that, —

$$f(\underset{\sim}{x}) \leq f(\underset{\sim}{x_0}) \text{ or, } f(\underset{\sim}{x}) \geq f(\underset{\sim}{x_0}) \quad \forall \underset{\sim}{x} \in N_\delta(\underset{\sim}{x_0}).$$

A necessary condition for an extemum of a differentiable function 'f' at x_0 is

$$\left(f_{x_1}(\underset{\sim}{x_0}), f_{x_2}(\underset{\sim}{x_0}), \cdots\cdots, f_{x_n}(\underset{\sim}{x_0})\right) = \underset{\sim}{0}.$$

$\boxed{\text{OR}}$ $\nabla f(\underset{\sim}{x_0}) = \underset{\sim}{0}$ or, $f_{x_i}(\underset{\sim}{x_0}) = 0 \quad \forall i = 1(1)n$,

[The partial derivative of 'f' w.r.t. x_i is $\dfrac{\partial f(\underset{\sim}{x})}{\partial x_i}$ or

$$f_{x_i}(\underset{\sim}{x}) = \lim_{h \to 0} \frac{\{f(x_1, \cdots, x_i + h, \cdots x_n) - f(x_1, \cdots, x_i, \cdots, x_n)\}}{h}$$

Notation: $f_i(\underset{\sim}{x})$, $\dfrac{\partial f}{\partial x_i}$, etc.]

$\boxed{A}$ Functions of two variables :

Theorem: 1. If the 2nd order partial derivatives of 'f' are continuous at each point in D is an open origin in $\mathbb{R}^2$ and if $(x_0, y_0) \in D$ such that $\nabla f(x_0, y_0) = \underset{\sim}{0}$, i.e. $f_x(x_0, y_0) = f_y(x_0, y_0)$, then

i) $f(x_0, y_0)$ is a local maximum if

$$D = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}_{(x,y) = (x_0, y_0)}$$ is negative definite.

i.e. $f_{xx}(x_0, y_0) < 0$ and $f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0) > 0$

ii) $f(x_0, y_0)$ is a local minimum if D is positive definite i.e. $f_{xx}(x_0, y_0) > 0$ and

$$f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f^2_{xy}(x_0, y_0) > 0$$

iii) $f(x_0, y_0)$ is neither a maximum nor a minimum if D is indefinite, i.e.

$$f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f^2_{xy}(x_0, y_0) < 0.$$

☆ **EXAMPLE :** → 1. Find all the local maximum and minimum of $f(x,y) = 2x^2 - xy + 2y^2 - 20x$.

Soln. → $f_x(x,y) = 4x - y - 20$

$f_y(x,y) = -x + 4y$

For points of extremum,

$$\left.\begin{aligned} f_x(x,y) &= 0 \\ f_y(x,y) &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} 4x - y &= 20 \\ x &= 4y \end{aligned} \Rightarrow \begin{cases} x = 16/3 \\ y = 4/3 \end{cases}$$

Now, $f_{xx}(x,y) = 4$

$f_{yy}(x,y) = 4$ and $f_{xy}(x,y) = -1$

Now,

$$D = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}_{(x,y) = \left(\frac{16}{3}, \frac{4}{3}\right)} = \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}$$

Note that,

$$f_{xx}\left(\frac{16}{3}, \frac{4}{3}\right) = 4 > 0$$

and

$$|D| = f_{xx}\left(\frac{16}{3}, \frac{4}{3}\right) f_{yy}\left(\frac{16}{3}, \frac{4}{3}\right) - f^2_{xy}\left(\frac{16}{3}, \frac{4}{3}\right)$$

$$= 16 - 1$$

$$= 15 > 0$$

Hence, D is positive definite.

Therefore, $f\left(\frac{16}{3}, \frac{4}{3}\right)$ is the minimum value.

Proof of the theorem1. →

$$f(x_0+h, y_0+k) = f(x_0, y_0) + h \cdot f_x(x_0, y_0) + k \cdot f_y(x_0, y_0) + \frac{1}{\sqrt{2}}\{h^2 f_{xx}(x_0+\theta h, y_0+\theta k)$$
$$+ k^2 f_{yy}(x_0+\theta h, y_0+k\theta) + 2hk f_{xy}(x_0+\theta h, y_0+\theta k)\}$$

$$= f(x_0, y_0) + \frac{1}{\lfloor 2}(h, k)\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}\begin{pmatrix} h \\ k \end{pmatrix}.$$

as $f_x(x_0, y_0) = 0 = f_y(x_0, y_0)$

Now,

$$f(x_0+h, y_0+k) - f(x_0, y_0) = \frac{1}{2}(h, k)\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}\begin{pmatrix} h \\ k \end{pmatrix}$$

If $D = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}_{(x_0, y_0)}$ is n.d.,

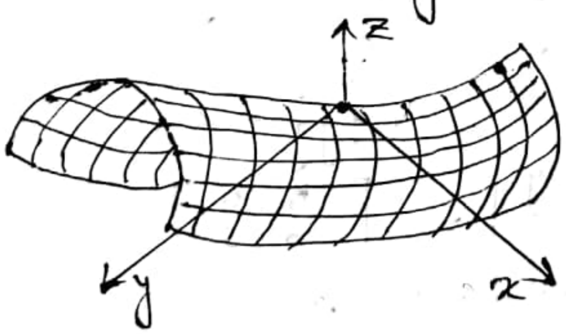
then $f(x_0+h, y_0+k) \leq f(x_0, y_0)$

i.e. $f(x_0, y_0)$ is the maximum.

Remark: →

1) A function $f(x_0, y_0)$ has a saddle point if
$f_x(x_0, y_0) = 0 = f_y(x_0, y_0)$ and
$f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f^2_{xy}(x_0, y_0) < 0$
i.e. it is neither a maximum, nor a minimum.

i.e. $f(x_0+h, y_0+k) - f(x_0, y_0)$ can take +ve or -ve values in every neighbourhood of (x_0, y_0).
The curve $z = -xy$ has a saddle point at $(0, 0)$



size z is -ve in 1st and 3rd quadrants and +ve in the 2nd and 4th quadrant.

2) If $f_x(x_0, y_0) = 0 = f_y(x_0, y_0)$ and $|D| = 0$
i.e. $f_{xx}(x_0, y_0) \cdot f_{yy}(x_0, y_0) - f^2_{xy}(x_0, y_0) = 0$.
Then it is a doubtful case and required further investigation.

Example 2. Give n points (x_i, y_i), where x_i's are distinct, find a and b such that the function
$$f(a,b) = \sum_{i=1}^{n} (ax_i + b - y_i)^2$$ is minimum.

Soln. → Here, $$f_a = \sum_{i=1}^{n} 2(ax_i + b - y_i)\, x_i$$
$$f_b = \sum_{i=1}^{n} 2(ax_i + b - y_i) \cdot 1$$

$$f_{aa} = 2\sum x_i^2$$
$$f_{bb} = 2 \quad \text{and} \quad f_{ab} = 2\sum x_i$$

For points of extreme $\left.\begin{array}{l} f_a = 0 \\ f_b = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} a = \dfrac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} = \hat{a} \\ b = \bar{y} - \hat{a}\bar{x} = \hat{b}. \end{array}\right.$

Now, $$D = \begin{bmatrix} f_{aa} & f_{ab} \\ f_{ab} & f_{bb} \end{bmatrix}_{(\hat{a}, \hat{b})} = 2\begin{bmatrix} \sum x_i^2 & \sum x_i \\ \sum x_i & n \end{bmatrix}$$

Here, $f_{aa}(\hat{a}, \hat{b}) = 2\sum x_i^2 > 0$

and $|D| = 4\left\{ n\sum x_i^2 - \left(\sum x_i\right)^2 \right\} = 4n\sum (x_i - \bar{x})^2 > 0$

Hence, $f(\hat{a}, \hat{b})$ is the minimum value of $f(a,b)$.

Ex. 3. Show that $f(x,y) = y^2 + x^2y + x^4$ has a minimum at $(0,0)$.

Soln. → For points of extremum,
$$0 = f_x = 2xy + 4x^3$$
$$0 = f_y = 2y + x^2 \qquad \text{Again,} \left\{\begin{array}{l} f_{xx} = 2y + 12x^2 \\ f_{xy} = 2x \\ f_{yy} = 2. \end{array}\right.$$
$$\Rightarrow x = 0 = y$$

Note that, $$D = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}_{(x,y)=(0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

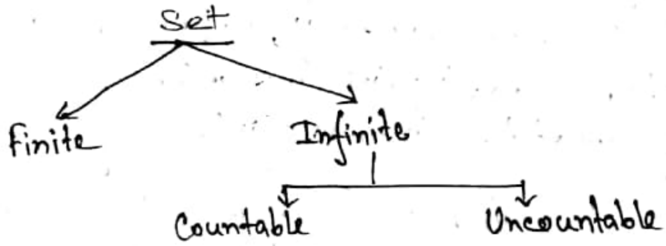
$$\therefore |D| = f_{xx}(0,0) f_{yy}(0,0) - f_{xy}^2(0,0) = 0$$

Hence, it is a doubtful case and requires further investigation.

The concept of a SET :> A set is a collection of some elements which are its members. And the members are called the elements of the set. Synonyms for set are class, aggregate and collection. A set can be defined by actually listing its elements or, if this is not possible, by describing some property held by all members and by no nonmembers. The first is called the roster method and the second is called the property method.

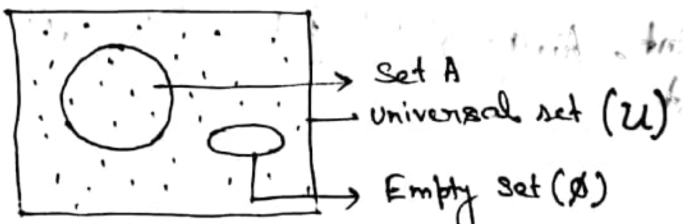
Ex. 1. The set of all vowels in the english alphabet can be defined by the roster method as {a, e, i, o, u} or by property method as {x | x is a vowel}, the vertical line | is read "such that" or "given that".

Ex. 2. The set {x | x is a triangular in a plane} is the set of all triangles in a plane. Note that the rooster method can't be used here.



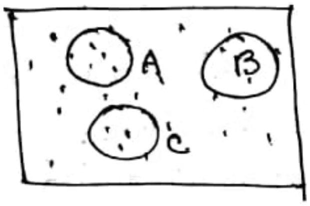
Depending on how many elements it has, a set may be finite or infinite. The set $M = \{1, 2, \ldots, 50\}$ is finite and contains 50 elements. The set of all natural numbers $N_1 = \{1, 2, \ldots, n, \ldots\}$ is infinite. The set of all even numbers $N_2 = \{2, 4, \ldots, 2n, \ldots\}$ is also infinite. An infinite is said to be countable if all its elements can be enumerated. Both the set N_1, N_2 above are countable. The set C of all points with in or on a circle of radious $r > 0$, $C = \{(x, y) : x^2 + y^2 \le r^2\}$ is infinite and uncountable. Its elements can't be enumerated.

Universal set and Null set :> Empty set is a set containing no elements, It's a member of all other sets.



The diagram representing a set is called a Vann diagram.

Disjoint Sets :→ Two sets A, B are disjoint if they have no common element, Similarly sets $A_1, A_2, \ldots, A_m$ are mutually disjoint if no two them have any common element. Here the sets A, B, C are mutually exclusive.



The set containing all the possible points representing the elementary events of a random expt. i.e. the universal set is called the 'sample space'. It is represented as S. Thus in tossing a coin once, $S = \{H, T\}$. In tossing a coin twice, $S = \{HH, HT, TH, TT\}$. In throwing two dice, $S = \{(1,1), (1,2), \ldots, (6,6)\}$.

Ex. 1. A coin is tossed until a head appears. Here the sample space consists of elementary events H, TH, TTH, TTTH, ..., these points are countable and infinite in number. The sample space consists of countably infinite number of cases.

SUBSETS :→ If each element of a set A also belongs to a set B we call A, a subset of B, written $A \subset B$ or $B \supset A$ and read "A is contained in B" or "B contains A" respectively. If $A \subset B$ but $A \neq B$ we call A a proper subset of B.

Ex. 1. $\{a, i, u\}$ is a proper subset of $\{a, e, i, o, u\}$.

Some elements of set theory :→

(a) Union → If the sets $A_1, \ldots, A_m$ are mutually disjoint, then we shall sometime write $\bigcup_{1}^{m} A_i = A_1 + A_2 + \cdots + A_m$. $\bigcup_{i=1}^{m} A_i$ represents the event that at least one of $A_1, \ldots, A_m$ occurs. $\bigcup_{i=1}^{m} A_i$ is called a total event.

(b) Intersection → The set of all elements which belong to both A and B is called the intersection. If the sets $A_1, \ldots, A_m$ are mutually disjoint, $A_i \cap A_j = \emptyset \ \forall\ i \neq j = 1, \ldots, m$. $\bigcap_{i=1}^{m} A_i$ is called a compound event.

(c) Complementation → The complement of set A, denoted as A^c or A' or $\bar{A}$ is the set of all elements not contained in A. Hence $A^c = \mathcal{U} - A$.
The set A^c represents the event then A does not occur.
Clearly, $\mathcal{U}^c = \phi$, $\phi^c = \mathcal{U}$, $(A^c)^c = A$, $P(A^c) = 1 - P(A)$.

(d) Difference :→ The difference $A-B$ is the set of all elements contained in A but not in B.

SOME THEOREMS INVOLVING SETS :→

1) Commutativity : $A \cup B = B \cup A$; $A \cap B = B \cap A$.

2) Associativity : $A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C$;
$A \cap (B \cap C) = (A \cap B) \cap C = A \cap B \cap C$.

3) Distributivity : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

4) Idempotency : $A \cup A = A$; $A \cap A = A$.

5) $A - B = A \cap B'$

6) If $A \subset B$, then $A' \supset B'$ or $B' \subset A'$.

7) $A \cup \phi = A$, $A \cap \phi = \phi$

8) $A \cup \mathcal{U} = \mathcal{U}$, $A \cap \mathcal{U} = A$.

9) For any sets A and B, $A = (A \cap B) \cup (A \cap B^c)$

10) De Morgan's 1st law :→

$$(A \cup B)^c = A^c \cap B^c$$

For m sets, A_i, $i = 1, \ldots, m$.

$$\left(\bigcup_{i=1}^{m} A_i\right)^c = \bigcap_{i=1}^{m} A_i^c.$$

11) De Morgan's 2nd law :→

$$(A \cap B)^c = A^c \cup B^c$$

For m sets, A_i, $i = 1, \ldots, m$.

$$\left(\bigcap_{i=1}^{m} A_i\right)^c = \bigcup_{i=1}^{m} A_i^c.$$

Examples :→

1) Prove that $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$

Proof :→ We've → $A\cap(B\cup C)=\{x \mid x\in A, x\in B\cup C\}$

$=\{x \mid x\in A, x\in B \text{ or } x\in C\}$

$=\{x \mid x\in A, x\in B \text{ or } x\in A, x\in C\}$

$=\{x \mid x\in A\cap B \text{ or } x\in A\cap C\}$

$=(A\cap B)\cup(A\cap C)$

2) $(A\cup B)'=A'\cap B'$.

Proof :→ We have, $(A\cup B)'=\{x \mid x\notin A\cup B\}$

$=\{x \mid x\notin A, x\notin B\}$

$=\{x \mid x\in A', x\in B'\}$

$=A'\cap B'$.

3) Prove that for any sets A and B we have $A=(A\cap B)\cup(A\cap B')$.

Proof :→ Method-1. $A=\{x \mid x\in A\}$

$=\{x \mid x\in A\cap B \text{ or } x\in A\cap B^c\}$

$=(A\cap B)\cup(A\cap B^c)$

Method-2. let $C=B'$.

from ①, we know, $A\cap(B\cup B')=(A\cap B)\cup(A\cap B')$

$A\cap \mathcal{U}=(A\cap B)\cup(A\cap B')$

& $A=(A\cap B)\cup(A\cap B')$

4) Prove that if $A\subset B$ and $B\subset C$, then $A\subset C$.

Proof :→ Let x be ~~the~~ any element of A, i.e. $x\in A$. Then since $A\subset B$, i.e. every element of A is in B. we have $x\in B$. Also since $B\subset C$, we have $x\in C$. Thus every element of A is an element of C and so $A\subset C$.

Measure of Central Tendency

C.U

⇨ Define Central Tendency? or, What do you mean by Central Tendency of a frequency distribution?

Ans:→ A set of observations shows a tendency or motive to have a value (generally centrally located) by which they may be replaced. This character is termed as central Tendency.

For any frequency distribution we find a tendency of the variate values to cluster around a central value; in other words, most of the values lies in a small interval about a central value. this characteristic is called the central tendency of a frequency distribution. In relation to a frequency distribution, an average is also termed as a measure of location, because it helps to locate the position of the distribution on the axis of the variable.

⇨ Measures of Central Tendency.

Ans:→ Central tendency is measured by —

i) Mean,
ii) Median,
iii) Mode,
iv) Quartile,
v) Decile,
etc......

⇨ Arithmatic Mean.

For Non Frequency or raw data.

The arithmatic mean of a variable is derived by dividing the sum of its values by the no. of values. If x denotes the variable under consideration and its values namely $x_1, x_2, \ldots, x_n$ are given, then the arithmatic mean of x, denoted by $\bar{x}$, is given by

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i .$$

Note:→ The computation of the arithmatic mean, in some cases, is simplified by subtracting a suitable factor c, say, from each observation.

Suppose $y_i = x_i - c$, for each i,

or, $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} (x_i - c)$, where n denotes number of given values.

or, $\sum_{i=1}^{n} y_i/n = \sum_{i=1}^{n} x_i/n - nc/n$

or, $\bar{y} = \bar{x} - c$

Then, $\bar{x} = \bar{y} + c$.

For frequency Data.

For Discrete variable

If the values of a discrete variable are exhibited alongwith their corresponding frequencies, then the mean can be obtained in the following way:

$$\bar{x} = \frac{x_1 f_1 + x_2 f_2 + \cdots + x_n f_n}{f_1 + f_2 + \cdots + f_n}$$

$$= \sum_{i=1}^{n} x_i f_i / N \quad \text{, where } N = \sum_{i=1}^{n} f_i \text{ the total frequency.}$$

where $x_1, x_2, \ldots, x_n$ denote the distinct values of the variable x and $f_1, f_2, \ldots, f_n$ indicate their respective frequencies.

Note:→ Data: $x_1, x_2, \ldots, x_n$

New data: $\bar{x}, \bar{x}, \ldots, \bar{x}$, where $\bar{x} = \frac{1}{n}\sum x_i$

Error: $x_1 - \bar{x}, x_2 - \bar{x}, \ldots, x_n - \bar{x}$.

$$\begin{aligned}
\text{Total Error} &= \sum (x_i - \bar{x}) \\
&= \sum x_i - \sum \bar{x} \\
&= \sum x_i - n\bar{x} \\
&= \sum x_i - \sum x_i \\
&= 0.
\end{aligned}$$

So, if we replace each observation by its mean, we are not doing any error, i.e. if observations are replaced by its mean, the observation remained unaffected.

For continuous Variable.

Again, for a continuous variable, the data are summarised in a frequency table showing the various class intervals and their corresponding class frequencies. In this case, the class-mark of a class-interval is supposed to represent the interval and on the basis of this assumption, an approximate value of the mean may be obtained. Hence the mean ($\bar{x}$) is expressed in the form

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i f_i}{\sum_{i=1}^{n} f_i} = \frac{\sum_{i=1}^{n} x_i f_i}{N}$$

In the case of equal width of the class intervals, calculation of the mean may be facilitated through a change of origin (or base) and scale. We are to subtract c from each class-mark and then divide the resultant by d, where c is the chosen origin, usually a class-mark near the middle of the range and d, the scale, is the common width. If y_i be the new value corresponding to x_i, then

$$y_i = \frac{x_i - c}{d}$$

or, $x_i = c + d y_i$, for each i

or, $x_i f_i = c f_i + d y_i f_i$, for each i

or, $\sum_i x_i f_i = c \sum_i f_i + d \sum_i y_i f_i$

or, $\frac{1}{n} \sum_i x_i f_i = c + \frac{d}{n} \sum_i y_i f_i$, where $n = \sum_i f_i$

or, $\bar{x} = c + d\bar{y}$.

Calculation of Mean :→

Class Boundaries	Frequency f_i	Class mark x_i	$y_i = \frac{x_i - 55.5}{10}$	$y_i f_i$
30.5–40.5	6	35.5	−2	−12
40.5–50.5	14	45.5	−1	−14
50.5–60.5	20	55.5	0	0
60.5–70.5	7	65.5	1	7
70.5–80.5	3	75.5	2	6
Total =	N=50	—	—	−13

Here, $\bar{y} = \sum_i y_i f_i / N$, where $N = \sum f_i$

$= \frac{-13}{50} = -0.26.$

Since $x_i = 55.5 + 10 y_i$,

$\bar{x} = 55.5 + 10\bar{y}$

$= 55.5 + 10(-0.26)$

$= 52.9.$

Some important properties of AM :→

(a) If the observed values of a variable are all equal, then their mean will be the common value.

Suppose we are given n values $x_1, x_2, \ldots, x_n$ of a variable x, where $x_i = c$, for each i.

Then $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} c = nc$.

Hence $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{n} \times nc = c.$

(b) The sum of the deviations of the values of a variable from its mean is zero.

Case I : Suppose a variable x assuming n values $x_1, x_2, \ldots, x_n$ has mean $\bar{x}$, where

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\Rightarrow \sum_{i=1}^{n} x_i = n\bar{x}$$

Now, $\sum_{i=1}^{n} (x_i - \bar{x}) = \sum_{i=1}^{n} x_i - n\bar{x}$

$= \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i = 0$

Case II: For discrete frequency distribution, we get —

$\bar{x} = \frac{1}{N}\sum_{i=1}^{n} x_i f_i$, where $N = \sum_{i=1}^{n} f_i$

or, $\sum_{i=1}^{n} x_i f_i = N\bar{x}$.

Now, $\sum_{i=1}^{n} f_i(x_i - \bar{x}) = \sum_{i=1}^{n} f_i x_i - \bar{x}\sum_{i=1}^{n} f_i$

$= N\bar{x} - N\bar{x}$

$= 0.$

THEORY OF PROBABILITY

Meanings of Probability :- It's a measure of chance of occurance of a phenomenon.

⟨1⟩ The word 'Probability' may be used to mean 'the degree of belief' of a person making a statement or proposition. It is used in the sense when we say that a certain football team will be the champion in a league or we say that the 'Mahabharat' is very probably the work of several authors.

⟨2⟩ On the other hand, the word has a different meaning, when we use it in the context of an experiment that can be repeated any no. of times under identical conditions. By the probability of any outcome of the experiment we shall now mean the long run relative frequency of any particular outcome of the experiment. We use the probability in this sense when we say that the probability of getting a 'head' in tossing a coin is 3/4 or the probability that an article produced by a machine will defective is negligable. In statistics, we generally use the term in 2nd sense.

In probability and statistics, we concern ourselves to same special type of experiment.

(1) **Random Experiment :-**

A random experiment or statistical experiment is an experiment in which-

⟨i⟩ all possible outcomes of the experiment are known in advance.

⟨ii⟩ any performance of the experiment results in an outcome that is not known in advance.

⟨iii⟩ The experiment can be repeated under identical or similar condition.

Ex : Consider an experiment of 'tossing a coin'. If the coin does not stand on the side there are two possible outcomes : Head (H), Tail (T). On any performance of the experiment, one does not know what the result will be. coin can be tossed as many times as desired under identical or similar condition. Hence, tossing of one is a random experiment.

(2) Sample Space :– The collection or set of all possible outcomes of a random experiment is called the sample space of the random experiment. It's noted by Ω (or S). The elements of the sample space (Ω) are called the 'Sample Point'.

Ex: (1) Consider a random experiment of 'tossing a coin' twice. Write down the sample space?

Sol. The sample space is – $\Omega = \{HH, HT, TH, TT\}$

The sample points are – HH, HT, TH, TT.

Ex: (2)

In each of the following experiment. What is the sample space?

i) a coin is tossed thrice.

ii) a die is rolled twice.

iii) a coin is tossed until a head appear.

Sol. i) $\Omega = \{HHH, HTH, THT, HHT, TTH, HTT, THH, TTT\}$

ii) $\Omega = \{(i,j) : i,j = 1(1)6\}$ [arithmatic progression[a(d)l]]

iii) $\Omega = \{H, TH, TTH, TTTH, \ldots\}$

Ex: (3) In each of the following experiments, what is the sample space?

i) In a survey of families with 3 children, the genders of the childrens are recorded in increasing of their age.

Sol.

$\Omega = \{BBB, BBG, BGG, GBG, GGB, GBB, BGB, GGG\}$

ii) The experiment consists of selecting four items from a manufacturers output and observing whether or not each item is defective.

Sol.

$\Omega = \{(a,b,c,d) : a, b, c, d$ is either defective or non-defective, consisting of 16 sample points$\}$

iii) Two cards are drawn from an ordinary deck of cards (a) with replacement ; (b) without replacements

Sol. (a) $\Omega = \{(x,y) : x, y = 1(1)52\}$ [cosisting 52^2 sample points]

(b) $\Omega = \{(x,y) : x, y = 1(1)52$ but $x \neq y\}$ [consisting 52×51 sample points]

Ex. (4) In each of the following experiments what is the sample space?

(i) Noting the lifetime of an electronic bulb.

(ii) A point is selected from a rod of unit length.

Sol. (i) $\Omega = \{x : 0 < x < \infty\}$ [continuous sample space]

(ii) $\Omega = \{x : 0 \leq x \leq 1\}$ [Here x is the distance of the selected point from the origin]

(3) **Trial:-** A trial refers to a special type of experiment in which there are two possible outcomes — 'success' and 'failure' with varying probability of success.

(4) **Outcome:-** Result of an experiment.

(5) **Sample:-** It is a part of the population and is supposed to represent the characteristic of the population.

(6) **Event:-** An event is a subset of sample space

(i) Elementary Event:- If an event contains only one sample point, it's known as an elementary event.

(ii) Composite Event:- If an event contain more than one sample points, it's known as a composite event.

Ex. (1). Consider the random experiment of 'tossing a fair coin twice'. Identify elementary & composite events.

Sol. $\Omega = \{HH, HT, TH, TT\}$

The event (i) 'at least one head' is $A = \{HH, HT, TH\}$, is called a composite event.

(ii) 'no head' is $B = \{TT\}$, is called an elementary event.

Ex: 2: A club has 5 members A, B, C, D, E. It's required to select a chairman and a secretary. Assuming that 1 member can't occupy both positions. Write the sample space associated with this section. What's the event that member A is an officeholder.

Sol^n sample space is, $\Omega = \{(x, y) : x, y = A, B, C, D, E \text{ but } x \neq y\}$

Here x stands for chairman and y stands for secretary.

Event is, $P = \{AB, BA, AC, AD, AE, CA, DA, EA\}$

$= \{(x, y) : \text{If } x = A \text{ then } y = B, C, D, E. \text{ If } y = A \text{ then } x = B, C, D, E\}$.

Mutually Exclusive Events :→ Several events $A_1, A_2, \ldots, A_n$ in relation to a random experiment are said to be mutually exclusive (or disjoint) if any two of them can't occur simultaneously, everytime the experiment is performed is $A_i \cap A_j = \phi, \forall (i \neq j)$/ $i<j = 1(1)n$.

Exhaustive Events :→ Several events $A_1, A_2, \ldots, A_n$ in relation to a random experiment are said to be exhaustive events if any of them must necessarily occur, everytime the experiment is performed that is $\bigcup_{i=1}^{n} A_i = \Omega$.

Equally Likely Cases (or events) :→ Several cases $A_1, A_2, A_3, \ldots$ are said to be equally likely if, after taking into consideration all relevant evidance, there is no reason to believe that one is more likely than the other.

Ex :→ For a random experiment of 'tossing a coin twice', the sample space is — $\Omega = \{HH, HT, TH, TT\}$.

Let A be the event of getting at least one head and B be the event of getting at most one head.

Then — $A = \{HT, TH, HH\}$

$B = \{HT, TH, TT\}$

$A \cup B = \Omega$ and $A \cap B \neq \phi$.

Hence, the event A and B are exhaustive but not mutually exclusive.

Let C be the event of getting 'no head', then $C = \{TT\}$, $A \cup C = \Omega$, $A \cap C = \phi$,

Hence, the event A and C are exhaustive and mutually exclusive too.

The Classical Definition of Probability :→ If a random experiment can result in N (finite) mutually exclusive, exhaustive and equally likely cases and N(A) of them are favorable to the occurance of the event A, then the probability of occurance of A is —

$$P[A] = \frac{N(A)}{N}.$$

PROBABILITY

"It is a measure of chance of occurrance of a Phenomenon."

NAME :- TANUJIT CHAKRABORTY.
TOPIC :- PROBABILITY THEORY 2.
HONS :- STATISTICS
CLG :- BIDHANNAGAR COLLEGE
YEAR :- B.SC 2ND YEAR

The problem in Probability is —
"Given a stochastic model, what we can say about the outcome".

"STATISTICS"

Sores of Information → $\boxed{\text{Statistical Method}}$ → A valid Decision.

PROBABILITY THEORY 2.

Generating Function :- the generating function of a random variable X, is a function of the form $E(\psi(t, X))$; where t is a non-random variable.

P.g.f. (Probability Generating Function) :- This is meant for a discrete random variable whose mass points are non-negative integers or some subsets of the whole set of non-negative integers. Here $\psi(t, x)$ is of the form t^X. Note that $E(t^X)$ necessarily exists for $|t| = 1$. Hence, because of the comparison test we find that the series $p_0 + t.p_1 + t^2 p_2 + \dots\dots$ is also absolutely convergent for $|t| < 1$. As such the p.g.f. of a non-negative integer valued random variable necessarily exists. It is denoted by $P_X(t)$.

Example:

1. Binomial Distribution :- (with parameter n, p)

$$f(x) = \binom{n}{x} p^x q^{n-x} \quad ; \quad x \geq 0$$

$$P(t) = E(t^X)$$
$$= \sum_{x=0}^{n} t^x \binom{n}{x} p^x q^{n-x}$$
$$= \sum_{x=0}^{n} \binom{n}{x} (pt)^x q^{n-x}$$
$$= (q + pt)^n \text{, defined for all real } t.$$

2. Poisson Distribution :- (with parameter λ)

$$f(x) = e^{-\lambda} \cdot \frac{\lambda^x}{x!} \quad , x \geq 0 ,$$

$$P(t) = E(t^X) = \sum_{x=0}^{\infty} t^x e^{-\lambda} \cdot \frac{\lambda^x}{x!} = \sum_{x=0}^{\infty} e^{-\lambda} \frac{(\lambda t)^x}{x!}$$
$$= e^{-\lambda} \sum_{x} \frac{(\lambda t)^x}{x!}$$
$$= e^{\lambda t} \cdot e^{-\lambda}$$
$$= e^{-\lambda(1-t)} = e^{\lambda(t-1)}.$$

3. Negative Binomial Distribution : $\sim$ (with parameter r, p)

$$f(x) = \binom{x+r-1}{x} p^r q^x \; ; x = 0,1,\ldots \text{ ad inf}$$

$$P(t) = E(t^X) = \sum_{x=0}^{\infty} t^x \binom{x+r-1}{x} p^r q^x$$

$$= p^r \sum_{x=0}^{\infty} \binom{x+r-1}{x} (qt)^x$$

$$= p^r (1-qt)^{-r}, \text{ defined for } |qt| < 1$$

(if the series is absolutely convergent) $|t| < \frac{1}{q}$.

USES: — 1. As the name suggests a p.g.f. gives the probabilities corresponding to the mass points of the relevant discrete distribution, i.e. the p.m.f. of a discrete distn. The following theorem is useful in this respect:

Theorem: —
If X, Y are non-negative integer valued r.v.s with p.g.f. $P_X(t)$ and $P_Y(t)$ then, provided X and Y are independent, the p.g.f. of $(X+Y)$ is $P_X(t)P_Y(t)$.

Proof: — The p.g.f of X, $P_X(t)$ be defined for $|t| < t_1$, say and the p.g.f of Y, $P_Y(t)$ be defined for $|t| < t_2$.
Let us take, $t_0 = \min\{t_1, t_2\}$
Then, we have for $|t| < t_0$,
$P_{X+Y}(t) = E(t^{X+Y})$ is defined

$$E(t^{X+Y}) = E(t^X . t^Y)$$

$$= E(t^X).E(t^Y) \quad [\because X \text{ and } Y \text{ are independent, } t^X \,\&\, t^Y \text{ '' } \text{ '' }]$$

$$= P_X(t).P_Y(t). \quad [\text{By product law of expectation}]$$

Example: — Let ~~X has the~~ ~~Binom~~ $X \sim Bin(n_1, p)$ & $Y \sim Bin(n_2, p)$ & they are independent. Then find the probability distn. of $(X+Y)$.

ANS:

$$P_X(t) = (q+pt)^{n_1}$$

$$\& \; P_Y(t) = (q+pt)^{n_2}, \text{ defined for all } t.$$

Since X and Y are independent,

$$P_{X+Y}(t) = P_X(t).P_Y(t)$$

$$= (q+pt)^{n_1+n_2}, \text{ defined for all } t.$$

which is itself the p.g.f. of another Binomial distrn with parameter (n_1+n_2, p). $\therefore X+Y \sim Bin(n_1+n_2, p)$

2. Suppose the moments of all order upto r of a non-negative integer valued random variable X exists. Then the rth factorial moment of X can be obtained from $P_X(t)$ by differentiation.

$$P_X(t) = E(t^X) = \sum_{x=0}^{\infty} t^x p_x$$

$$\frac{d^r}{dt^r} P_X(t) = \sum_{x=0}^{\infty} x(x-1)(x-2)\cdots(x-r+1) \cdot t^{x-r} \cdot p_x$$

$$\left.\frac{d^r}{dt^r} P_X(t)\right|_{t=1} = \sum_{x=0}^{\infty} x_{[r]} p_x$$

$$= E[X_{[r]}]$$

Particular case :-

$P_X'(1) = E(X) \quad \forall t = 1$

$P_X''(1) = E[X(X-1)] = E(X^2) - E(X)$

$\therefore P_X''(1) + P_X'(1) = E(X^2)$

$\therefore P_X''(1) + P_X'(1) - \{P_X'(1)\}^2 = Var(X)$

☆ 8.1. Find the p.g.f. of the no. of points to be obtained in throwing a fair die once. Also find the p.g.f. of the no. of points to be obtained in throwing a fair die n times. Hence s.t. the Prob. of obtaining s points in n times is the prob. of obtaining $(7n-s)$ points.

ANS :- i) $X : 1, 2, 3, 4, 5, 6$.

X = Point obtained in one throw.

$$P_X(t) = E(t^X) = \sum_{x=1}^{6} t^x \cdot P(X=x)$$

$$= \frac{1}{6}(t + t^2 + \cdots + t^6)$$

ii) S = Total points obtained in n throws

$S = X_1 + X_2 + \cdots + X_n$; Note that X_i's are indep. r.v.'s, with the same distribution

$P(X_i = x) = \frac{1}{6} \; \forall x = 1(1)6$, $i = 1(1)n$.

Hence, each of them has the same p.g.f.

$\therefore$ P.g.f. of S is

$P_S(t) = E(t^S)$

$= E(t^{X_1 + X_2 + \cdots + X_n})$

[$\because X_i$'s are indep.] $= E(t^{X_1}) E(t^{X_2}) \cdots E(X^n) = \{P_X(t)\}^n = \frac{1}{6^n}(t + t^2 + \cdots + t^6)^n$

Some Continuous Theoritical Distributions discussed here:

i) Uniform Distribution, ('Rectangular Distribution).
ii) Gamma Distribution.
iii) Beta Distribution.
iv) Exponential Distribution.
v) Normal Distribution.
vi) Double exponential or Laplace Distribution.
xii) Truncated normal distribution.
viii) Log-normal distribution.
ix) Pareto Distribution.
x) Cauchy Distribution.
xi) Logistic Distribution.

SOME CONTINUOUS DISTRIBUTIONS

Rectangular Distribution or UNIFORM DISTRIBUTION :—

An absolutely continuous random variable X defined over $[a,b]$, $-\infty < a < b < \infty$ is said to follow uniform distribution with parameters a, b; if its pdf is given by,

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b, \\ 0, & \text{ow} \end{cases}$$

We will write $X \sim U[a,b]$ if X has a uniform distribution on $[a,b]$

This distribution is also called a rectangular distribution since the area under 'f' in between a and b is rectangular. It is also called Rectangular Distribution.

$$X \sim U[a,b]$$
or,
$$X \sim R[a,b]$$

The end point a or b or both may be excluded. Clearly,

$$\int_{-\infty}^{\infty} f(x)dx = 1.$$

Distribution Function :— The DF of X is given by,—

$$F(x) = \begin{cases} 0 & \text{if } x \le a \\ \frac{x-a}{b-a} & \text{if } a < x < b \\ 1 & \text{if } x \geq b \end{cases};$$

Expectation & Variance :—

$$E(X^k) = \int_a^b x^k f(x)dx = \frac{1}{b-a}\int_a^b x^k dx \quad , k > 0 \text{ is an integer.}$$

$$= \frac{1}{b-a}\left[x^{k+1}\right]_a^b / k+1$$

$$= \frac{b^{k+1} - a^{k+1}}{(k+1)(b-a)}.$$

Putting $k=1$, $E(X) = \frac{b+a}{2}$,

Putting $k=2$, $E(X^2) = \frac{(b^2+ab+a^2)}{3}$

$$\therefore Var(X) = E(X^2) - E^2(X) = \frac{(b-a)^2}{12}.$$

☒ Moment Generating Function: — [C.U. 2003]

$$M_X(t) = E(e^{tX}) = \int_a^b e^{tx}\,dx \cdot \frac{1}{b-a}$$

$$= \frac{e^{tb} - e^{ta}}{t(b-a)}$$

$$= \frac{1}{t(b-a)}\left[\sum_{j=0}^{\infty} \frac{(tb)^j}{j!} - \sum_{j=0}^{\infty} \frac{(ta)^j}{j!}\right]$$

$$= \frac{1}{t(b-a)} \sum_{j=0}^{\infty} \frac{(tb)^j - (ta)^j}{j!}$$

$$= \frac{1}{t(b-a)} \sum_{j=0}^{\infty} \frac{t^j}{j!}(b^j - a^j)$$

$$\therefore \mu_1' = E(X) = \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2}.$$

$$\therefore \mu_2' = E(X^2) = \frac{b^3 - a^3}{3(b-a)} = \frac{b^2 + ab + a^2}{3}$$

$$\therefore V(X) = \frac{b^2 + ab + a^2}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12}.$$

$$\text{S.d.} = \frac{|b-a|}{\sqrt{12}} = \frac{b-a}{\sqrt{12}}$$

NOTE: → The distribution is trivially symmetric about $\frac{a+b}{2}$.

⇨ **Theorem: 1. (Probability Integral Transformation)**

Let X be a continuous R.V. having D.F. F. Then $F(X)$ has the uniform distribution on $[0, 1]$.

Proof:– Let, $Y = F(X)$

Then the DF of Y is given by —

$$F_Y(y) = P[Y \le y]$$

$$= P[F(X) \le y]$$

$$= P[X \le F^{-1}(y)]$$

$$= F[F^{-1}(y)] \quad [\text{as } F(x) \text{ is a monotone non-decreas. function}]$$

$$= \begin{cases} 0 & \text{if } y \le 0 \\ 1 & \text{if } 0 < y < 1 \\ 1 & \text{if } y \ge 1 \end{cases}$$

$$\text{Pdf. of } Y = \frac{d}{dy}[G(y)] = 1$$

$$= \frac{d}{dy}\int_0^y g(y)\,dy$$

$$\therefore Y = F(X) \sim U(0,1).$$

Note that, $E[Y] = \frac{1}{2}$, $V[Y] = \frac{1}{12}$.

NOTE:- The fact can be used to draw random observations from the theoritical distribution of X.

Here at first we choose 3 digited random numbers and put a decimal point before the first digit. Let us denote such a quantity by p, clearly p is a realization from $R(0,1)$ distn., now to obtain x we equate $F(x)=p$ and solve for x.

Theorem: 2. Let F be any DF, and let X be a $U[0,1]$ RV. Then there exists a function h such that $h(X)$ has DF F, i.e.,

$$P\{h(X)\le x\}=F(x) \quad \text{for all } x \in (-\infty,\infty).$$

Proof:- If F is the DF of a discrete RV Y, let

$$P[Y=y_k]=p_k, \quad k=1,2,\ldots$$

Define h as follows:-

$$h(x)=\begin{cases} y_1 & \text{if } 0\le x<p_1, \\ y_2 & \text{if } p_1\le x<p_1+p_2, \\ \vdots & \vdots \end{cases}$$

Then

$$P\{h(X)=y_1\}=P\{0\le X<p_1\}=p_1,$$
$$P\{h(X)=y_2\}=P\{p_1\le X<p_1+p_2\}=p_2,$$

and in general,

$$P\{h(X)=y_k\}=p_k, \quad k=1,2,\ldots$$

Thus $h(X)$ is a discrete RV with DF F.

If F is continuous and strictly increasing, F^{-1} is well defined, and if we take $h(X)=F^{-1}(X)$. We have

$$\begin{aligned} P\{h(X)\le x\} &= P\{F^{-1}(X)\le x\} \\ &= P\{X\le F(x)\} \\ &= F(x), \end{aligned}$$

as asserted,

In general, define

$$F^{-1}(y)=\inf\{x: F(x)\ge y\},$$

and let $h(X)=F^{-1}(X)$. Then we have

$$\{F^{-1}(y)\le x\}=\{y\le F(x)\}.$$

$F^{-1}(y)\le x \Rightarrow \forall\ \epsilon>0,\ y\le F(x+\epsilon)$, since $\epsilon>0$ is arbitrary and F is continuous on the right, we let $\epsilon\to 0$ and conclude that $y\le F(x)$.

Since $y\le F(x) \Rightarrow F^{-1}(y)\le x$. Thus,

$$P[F^{-1}(X)\le x]=P[X\le F(x)]=F(x).$$

NOTE:- It is quite useful theorem in generating samples with the help of the uniform distribution.

SOME PROBABILITY INEQUALITIES

The inequalities which contain probability in either left side or right side or in the both side, are called "Probability Inequalities".

MARKOV'S INEQUALITY :—

Statement :— Let X be a r.v. having finite expectation, i.e. $E(X)$ converges. Then for any non-zero quantity 'a', we have the inequality :

$$P(X \geq a) \leq \frac{E(X)}{a}.$$

Proof :— Let us define a r.v. Y such that

$$Y = \begin{cases} a & \text{if } X \geq a \\ 0 & \text{ow} \end{cases}$$

$$X \geq Y \Rightarrow E(X) \geq E(Y)$$

Now, $E(Y) = a \cdot P(X \geq a) \leq E(X)$

$$\Rightarrow P(X \geq a) \leq \frac{E(X)}{a}.$$

NOTE : Markov inequality holds for any function of r.v. X, i.e. for any real valued function $g(x)$, the markov's inequality is given by

$$P[g(x) \geq a] \leq \frac{E(g(x))}{a}, \quad a \neq 0$$

Proof :— Let us define a function of r.v. Y, $g(Y)$

$$g(Y) = \begin{cases} a & \text{if } g(x) \geq a \\ 0 & \text{ow} \end{cases}$$

$$g(x) \geq g(Y)$$

$$\therefore E(g(x)) \geq E(g(Y))$$

$$\therefore E(g(Y)) = a \cdot P[g(x) \geq a] \leq E(g(x))$$

$$\Rightarrow P[g(x) \geq a] \leq \frac{E(g(x))}{a}, \quad a \neq 0$$

Problem1. If X be any r.v. such that $M(t)=E(e^{tX})$ exists for all t. Show that for any $t>0$,

$$P(tX > \delta^2 + \ln M(t)) < e^{-\delta^2}$$

ANS:-
We know that an exponential function is monotonically increasing.

So, $P(tX > \delta^2 + \ln M(t))$

$= P\left(e^{tX} > e^{\delta^2 + \ln M(t)}\right)$

$= P\left[e^{tX} > e^{\delta^2} \cdot e^{\ln M(t)}\right]$

Let $g(X) = e^{tX}$ then by Markov's inequality, we have

$$P\left(e^{tX} > e^{\delta^2} \cdot e^{\ln M(t)}\right) < \frac{E(e^{tX})}{e^{\delta^2} \cdot e^{\ln M(t)}} = \frac{M(t)}{M(t) \cdot e^{\delta^2}}$$

$= e^{-\delta^2}$ (Proved)

Problem2. For any random variable X, show that,

$$P[|X|>t] \le \frac{1+t^2}{t^2} E\left(\frac{X^2}{1+X^2}\right) \text{ for any } t>0.$$

[2002]

ANS:- Here, $P[|X|>t]$

$= P[X^2 > t^2]$

$= P[1+X^2 > 1+t^2]$

$= P\left[\frac{X^2}{1+X^2} > \frac{t^2}{1+t^2}\right]$

Now by Markov's inequality,

$$P\left[\frac{X^2}{1+X^2} > \frac{t^2}{1+t^2}\right] \le E\left(\frac{X^2}{1+X^2}\right) \cdot \frac{1+t^2}{t^2}$$ [Proved]

C.U. S.T. $P[X>t] < E(e^{aX})/e^{at}$.

ANS:- $P[aX > at] = P[X > t]$

$= P(e^{aX} > e^{at})$ [$\because$ e is monotonically increasing]

By Markov's inequality,

$< \dfrac{E(e^{aX})}{e^{at}}$, where $E(e^{aX})$ exists where $a>0$.

Problem 3. A fair die is rolled n times. Find a lower bound to n such that, the probability of at least one six in rolling is $\geq \frac{1}{2}$.

ANS:- Let us define a random variable X representing the number of six by throwing a die n times.

$$\therefore X \sim bin(n, \tfrac{1}{6})$$

By Markov's inequality,

$$P[X \geq 1] \leq \frac{E(X)}{1}$$

$$\Rightarrow P[X \geq 1] \leq \frac{n}{6} \text{ ——— } \langle i \rangle$$

Again it is given that $P[X \geq 1] \geq \frac{1}{2}$

$\therefore$ From $\langle i \rangle$,

$$\frac{n}{6} \geq \frac{1}{2}$$

$$\Rightarrow n \geq 3$$

$\therefore$ the die should be at least thrown 3 times.

Problem 4. $X_1, X_2, \dots, X_K$ are independent r.v.'s having zero mean and unit variance. Find an upper bound to,

$$P\left[\sum_{i=1}^{K} X_i^2 \geq \lambda K\right], \quad \lambda > 0$$

ANS:- $X_1, X_2, \dots, X_K$ are independent r.v.'s with mean 0 and variance 1.

i.e. $E(X_i) = 0 \quad \forall \ i = 1(1)K$

$$V(X_i) = E(X_i^2) - E^2(X_i)$$

$$\Rightarrow E(X_i^2) = 1 \qquad [\because E(X_i) = 0]$$

$$\Rightarrow \sum_{i=1}^{K} E(X_i^2) = K$$

$$\Rightarrow E\left(\sum_{i=1}^{K} X_i^2\right) = K \qquad [\because X_i\text{'s are independent}]$$

Now by Markov's inequality,

$$P\left[\sum_{i=1}^{K} X_i^2 \geq \lambda K\right] \leq \frac{E\left(\sum_{i=1}^{K} X_i^2\right)}{\lambda K} = \frac{K}{\lambda K} = \frac{1}{\lambda}$$

$\therefore$ Required upper bound $= \frac{1}{\lambda}$.

Chebyshev's Inequality :—

Statement :— For a random variable X having finite mean and variance σ^2, then for any $t > 0$, the chebyshev's inequality is given as follows:

$$P(|X-\mu| \geq t\sigma) \leq \frac{1}{t^2}$$

or

$$P(|X-\mu| \leq t\sigma) \geq 1-\frac{1}{t^2}$$

Proof :— In order to prove chebyshev's inequality, we will first prove Markov's inequality,

Let us define a random variable Z,

$$Z = \begin{cases} a & , Y \geq a \\ 0 & , \text{ow} \end{cases}$$

where Y is another RV.

From the definition of Z, it is such that,

$$Y \geq Z$$

$$\Rightarrow E(Y) \geq E(Z)$$

$$\Rightarrow E(Y) \geq a.P[Y \geq a]$$

$$\Rightarrow P[Y \geq a] \leq \frac{E(Y)}{a}$$

This is the required Markov's inequality.

Now for the RV X,

$E(X) = \mu < \infty$, $V(X) = \sigma^2 = E(X-\mu)^2 > 0$

Now, $P[|X-\mu| \geq t\sigma] = P[(X-\mu)^2 \geq t^2\sigma^2]$

Now, let us choose $Y = (X-\mu)^2$ and $a = t^2\sigma^2$, then by Markov's inequality, we have,

$$P[(X-\mu)^2 \geq t^2\sigma^2] \leq \frac{E(X-\mu)^2}{t^2\sigma^2} = \frac{\sigma^2}{t^2\sigma^2}$$

$$= \frac{1}{t^2}$$

$$\therefore P[|X-\mu| \geq t\sigma] \leq \frac{1}{t^2} \quad \text{———} \langle i \rangle$$

Hence proved.

1 — ⟨i⟩ gives

$$P[|X-\mu| \leq t\sigma] \geq 1-\frac{1}{t^2} \quad \text{———} \langle ii \rangle$$

Hence proved.

REAL/MATHEMATICAL ANALYSIS

• SEQUENCES OF REAL NUMBERS :

The word "sequence" is used to convey the idea that the things are arranged in order.

Definition :→ A 'sequence' of real numbers is a function defined on the set N of natural numbers whose range is a subset of the set $\mathbb{R}$ of real numbers ; i.e. if for every $n \in N$, $\exists$ a real number a_n, then the order set

$$a_1, a_2, \ldots\ldots, a_n, \ldots\ldots$$

is said to define a sequence of real nos.

Remark :→ 1) $f : A \to B$ is a mapping or function if for every $x \in A$, $\exists$ a unique value of $y \in B$.

Here, we write $y = f(x)$ where $x \in A$, $y \in B$.

2) $y = \pm x$ is not a function, it's a relation.

Notation :→ 1) If a_n is the n^{th} term of a sequence, then we write $a_1, a_2, \ldots, a_n, \ldots\ldots$, to describe the sequence.

2) $f : N \to \mathbb{R}$ is a sequence.

3) $\{a_n\}, \{b_n\}, \{x_n\}, \{y_n\}$.

The main question we are concerned with here is to decide whether or not the term a_n tends to a finite quantity when n increases indefinitely.

Definition :→ A sequence $\{a_n\}$ is said to have a limit $l \in \mathbb{R}$ if, for every $\epsilon > 0$, $\exists$ a natural number $N(\epsilon)$, $\ni |a_n - l| < \epsilon$, for all $n \geq N(\epsilon)$

Example. (1) Prove that $\lim_{n\to\infty} \left(\frac{1}{n}\right) = 0$.

Soln.→ Let $\epsilon > 0$ be an arbitrary number, then

$$|a_n - l| < \epsilon$$

$$\Rightarrow \left|\frac{1}{n} - 0\right| < \epsilon$$

$$\Rightarrow \frac{1}{n} < \epsilon$$

$$\Rightarrow n > \frac{1}{\epsilon}$$

In particular if $\epsilon = 0.1$, then

$$\left|\frac{1}{n} - 0\right| < \epsilon = 0.1$$

$$\Rightarrow n > \frac{1}{\epsilon} = 10$$

$$\therefore \left|\frac{1}{n} - 0\right| < \epsilon = 0.1$$

whenever $n \geq 11 = N(\epsilon = 0.1)$

If $\epsilon = 0.01$, then

$$\left|\frac{1}{n} - 0\right| < \epsilon = 0.01$$

$$\Rightarrow n > \frac{1}{\epsilon} = 100$$

$$\Rightarrow n > 101$$

Hence take $N(\epsilon) = \left[\frac{1}{\epsilon}\right] + 1$

- Or, choose a natural no. $N(\epsilon)$ which is $> \frac{1}{\epsilon}$.

Then $\exists$ a natural no. $N(\epsilon) \ni \left|\frac{1}{n} - 0\right| < \epsilon \quad \forall\ n \geq N(\epsilon)$

Hence, by definition, $\lim_{n\to\infty} \left(\frac{1}{n}\right) = 0$

$$|a_n - l| < \epsilon \text{ for } n \geq N(\epsilon)$$

$$\Rightarrow l - \epsilon < a_n < l + \epsilon \text{ for } n \geq N(\epsilon)$$

For $0 < \epsilon' < \epsilon$, then in general,

$$N(\epsilon') > N(\epsilon)$$

If ϵ is small, then $N(\epsilon)$ will be sufficiently large to ensure $|a_n - l| < \epsilon$ for $n \geq N(\epsilon)$. And all the members a_n, $n \geq N(\epsilon)$ are in the small interval, $(l-\epsilon, l+\epsilon)$ i.e. then a_n is very close to l.

If a sequence $\{a_n\}$ has a finite limit 'l', then we say that the sequence $\{a_n\}$ converges to l or the sequence is convergent. If a sequence does not converge to a finite limit, then it is said to be divergent. If $\{a_n\}$ converges to 'l'; we write

$$\lim_{n\to\infty}(a_n) = l, \text{ or, } \lim(a_n) = l.$$

Example. (2). Prove that $\lim_{n\to\infty}\left(\frac{1}{n+1}\right) = 0$.

Soln. → Let $\epsilon > 0$ be an arbitrary number,

Then $|a_n - l| < \epsilon$

$$\Rightarrow \left|\frac{1}{n+1} - 0\right| < \epsilon$$

$$\Rightarrow \frac{1}{n+1} < \epsilon$$

$$\Rightarrow n < \frac{1}{\epsilon} - 1$$

Choose a natural number $N(\epsilon)$ which is $> \frac{1}{\epsilon} - 1$.

Then, $\exists$ a natural no. $N(\epsilon) \ni \left|\frac{1}{n+1} - 0\right| < \epsilon$ for all $n \geq N(\epsilon)$

By defn. $\lim_{n\to\infty}\left(\frac{1}{n+1}\right) = 0$

AH. → $\left|\frac{1}{n+1} - 0\right| = \frac{1}{n+1} < \frac{1}{n} < \epsilon$

$$\Rightarrow n > \frac{1}{\epsilon}$$

Take $N(\epsilon) = \left[\frac{1}{\epsilon}\right] + 1$

Then $\exists$ a natural no. $N(\epsilon)$, $\ni \left|\frac{1}{n+1} - 0\right| < \epsilon \quad \forall\ n \geq N(\epsilon)$

$\therefore$ By defn. $\lim_{n\to\infty}\left(\frac{1}{n+1}\right) = 0$.

Example (3) Prove that $\lim_{n\to\infty}\left(\frac{1}{n^2+1}\right)=0$

Soln. $\rightarrow$ Let $\epsilon>0$ be an arbitrary number.

$$|a_n-l|<\epsilon$$

$$\Rightarrow \left|\frac{1}{n^2+1}-0\right| = \frac{1}{n^2+1} < \frac{1}{n^2} < \epsilon$$

$$\Rightarrow n > \frac{1}{\sqrt{\epsilon}}$$

Take $N(\epsilon)=\left[\frac{1}{\sqrt{\epsilon}}\right]+1$

Then $\exists$ a natural no. $N(\epsilon)$, $\ni$ $\left|\frac{1}{n^2+1}-0\right|<\epsilon \quad \forall\, n\geq N(\epsilon)$.

$\therefore$ By defn. $\lim_{n\to\infty}\left(\frac{1}{n^2+1}\right)=0$.

Example (4). Prove that $\lim_{n\to\infty}\left(\frac{2n^2+1}{n^2+n}\right)=2$.

Soln. $\rightarrow$ Let $\epsilon>0$ be an arbitrary number.

$$\therefore |a_n-l|<\epsilon$$

$$\Rightarrow \left|\frac{2n^2+1}{n^2+n}-2\right|<\epsilon$$

$$\Rightarrow \frac{\cancel{2n^2}+1-\cancel{2n^2}-2n}{n^2+n}<\epsilon$$

$$\Rightarrow \frac{1-2n}{n^2+n}<\epsilon$$

$$\Rightarrow \frac{2n-1}{n^2+n}<\frac{2n}{n^2}=\frac{2}{n}<\epsilon$$

$$\Rightarrow n>\frac{2}{\epsilon}.$$

Take $N(\epsilon)=\left[\frac{2}{\epsilon}\right]+1$

Then $\exists$ a natural no. $N(\epsilon)$, $\ni$ $\left|\frac{2n^2+1}{n^2+n}-2\right|<\epsilon \;\forall\, n\geq N(\epsilon)$

$\therefore$ By defn. $\lim_{n\to\infty}\left(\frac{2n^2+1}{n^2+n}\right)=2$.

Example (5). Prove that $\lim_{n\to\infty}\left(\frac{1}{n^p}\right)=0$, $p>0$.

Soln. $\rightarrow$ Let $\epsilon>0$ be an arbitrary number.

$$|a_n-l|<\epsilon$$

$$\Rightarrow \left|\frac{1}{n^p}-0\right|<\epsilon$$

$$\Rightarrow \frac{1}{n^p}<\epsilon$$

$$\Rightarrow n^p>\frac{1}{\epsilon}$$

$$\Rightarrow n>\left(\frac{1}{\epsilon}\right)^{1/p}$$

since $p>0$, Take $N(\epsilon)=\left[\frac{1}{\sqrt[p]{\epsilon}}\right]+1$.

Then $\exists$ a natural no. $N(\epsilon)$, $\ni$ $\left|\frac{1}{n^p}-0\right|<\epsilon \;\forall\, n\geq N(\epsilon)$

Example (7). ~~Proof~~ Prove that $\lim_{n\to\infty}(r^n)=0$ if $|r|<1$.

Soln.→ Let $\epsilon>0$ be an arbitrary number.

Then If $|r^n-0|<\epsilon$

$\Rightarrow |r^n|<\epsilon$

$\Rightarrow n\ln|r| < \ln\epsilon$

$\Rightarrow n > \dfrac{\ln\epsilon}{\ln|r|}$ [since $|r|<1$, $\ln|r|<0$]

Choose a natural no. $N(\epsilon)$ which is $\geq \dfrac{\ln\epsilon}{\ln|r|}$

Then $\exists$ a natural no. $N(\epsilon)$, $\ni |r^n-0|<\epsilon \;\forall\; n\geq N(\epsilon)$

$\therefore$ By defn. $\lim_{n\to\infty}(r^n)=0$ if $|r|<1$.

Example (8). Prove that $\lim_{n\to\infty}\left(2-\dfrac{1}{2^n}\right)=2$

Soln.→ Let $\epsilon>0$ be an arbitrary no.

$\therefore \left|2-\dfrac{1}{2^n}-2\right|<\epsilon$

$\Rightarrow \left|-\dfrac{1}{2^n}\right|<\epsilon$

$\Rightarrow \dfrac{1}{2^n}<\dfrac{1}{n}<\epsilon$

$\Rightarrow n>\dfrac{1}{\epsilon}$

Take $N(\epsilon)=\left[\dfrac{1}{\epsilon}\right]+1$

Then $\exists$ a natural no. $N(\epsilon)$, $\ni \left|2-\dfrac{1}{2^n}-2\right|<\epsilon \;\forall\; n\geq N(\epsilon)$

$\therefore$ By defn. $\lim_{n\to\infty}\left(2-\dfrac{1}{2^n}\right)=2$.

Example (9). Prove that $\lim_{n\to\infty}(2^{1/n})=1$.

Soln.→ Let $\epsilon>0$ be an arbitrary no.

$\therefore |2^{1/n}-1|<\epsilon$

$\Rightarrow 2^{1/n}<\epsilon+1$

$\Rightarrow \dfrac{1}{n}\ln 2<\ln(\epsilon+1)$

$\Rightarrow \dfrac{1}{n}<\dfrac{\ln(\epsilon+1)}{\ln 2}$

$\Rightarrow n>\dfrac{\ln 2}{\ln(\epsilon+1)}$

Take $N(\epsilon)=\left[\dfrac{\ln 2}{\ln(\epsilon+1)}\right]$

$\therefore n>N(\epsilon)$. [P]

RANDOM SAMPLING AND SAMPLING DISTRIBUTION

Definition of Some Terms :

1. **Parameter :** — A constant which changes its value from one situation to another. Specially, it is denoted by θ. A parameter labels a distribution uniquely.

2. **Parameter Space :** — Set of all admissible values of the parameter, denoted by Θ

Example : — i) $X \sim N(\mu, 1)$
μ = Parameter,
$\mathbb{R}$ = Parameter space.

ii) $X \sim N(\mu, \sigma^2)$
(μ, σ) = Parameter (vector valued)
Parameter Space $= \{(\mu, \sigma) : \mu \in \mathbb{R}, \sigma \in \mathbb{R}^+\}$

iii) $X \sim Bin(n, p)$
(n, p) = Parameter
Parameter space $= \{(n, p) : n \in \mathbb{N}, 0 < p < 1\}$

3. **Labelling Parameter :** — Suppose X is normally distributed with mean μ and s.d. unity. Then the parameter μ labels the distribution uniquely and hence termed as labelling parameter.

On the other hand, the parameter $\xi_{1/2}$, the median of a distribution though reflects a feature (regarding location) of the distribution, but it fails to label the distribution. But in case of one parameter Cauchy distribution with median θ, which labels the distribution.

Thus if a random variable X has distribution Function F, where the distribution is labelled or, indexed by the parameter θ. We denote the distn. by $F_\theta(\cdot)$.

Family of Distribution : —

Let X be a random variable having distribution function F_θ, $\theta \in \Theta$, then $\{F_\theta(\cdot) : \theta \in \Theta\}$ is said to be a family of distribution function, similarly, one may define a family of PDF or PMF's namely $\{f_\theta : \theta \in \Theta\}$, where $f_\theta(\cdot)$ is the PDF or PMF of X.

Example : — $\{\Phi(x-\mu) : \mu \in \mathbb{R}\}$

is a family of normal distribution with mean μ and s.d. unity.

4. Random Sample : If $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variable each having distribution function F then $(X_1, X_2, \ldots, X_n)$ constitutes a random sample drawn from F.

5. Sample Space : Let $(X_1, X_2, \ldots, X_n)$ be a random sample drawn from a distribution having distribution function F. Suppose $(x_1, x_2, \ldots, x_n)$ is the realization on $(X_1, X_2, \ldots, X_n)$ then $(x_1, x_2, \ldots, x_n)$ is said to be a sample point. Clearly, these sample points may vary from one sampling location to another. The totality of all such sample points constitutes the sample space, commonly denoted by $\mathcal{X}$.

Example: Suppose, we have a random sample of size 2 from $N(\mu, 1)$ distribution. Then the sample space will be $\mathbb{R}^2$.

6. Statistic : Let $(X_1, X_2, \ldots, X_n)$ be a random sample drawn from a population having distribution function $F_\theta(\cdot)$. suppose $T(X_1, X_2, \ldots, X_n)$ is a measurable function $\ni$ $T: \mathbb{R}^n \to \mathbb{R}^k$ if $k=1$, T is said to be a real valued statistic and for $k>1$, T will be a vector valued statistic. In simple words, statistic is a function of sample observation which is independent of any unknown parameters, i.e. here T does not depend on the labelling parameter θ.

Example: Let $(X_1, X_2, \ldots, X_n)$ be a random sample drawn from $N(\mu, 1)$ population.

Here, the sample mean $\bar{X}$ is a statistic, we know, $\bar{X} \sim N(\mu, \frac{1}{n}) \Rightarrow \sqrt{n}(\bar{X}-\mu) \sim N(0,1)$.

It is to be noted that unless μ is specified, $\sqrt{n}(\bar{X}-\mu)$ would not be a statistic.

Once μ is specified as 2, $\sqrt{n}(\bar{X}-2)$ becomes a statistic.

Some real valued statistic are sample mean $\bar{X}$, sample Range R, sample s.d. 's'.

$X_{(1)}$, the minimum of the sample observation,

$X_{(n)}$, the maximum of the sample observation,

where as $(\bar{X}, s)$, $(X_{(1)}, X_{(n)})$, $(X_{(1)}, X_{(2)}, \ldots, X_{(n)})$ are vector valued statistic.

7. **Sampling Distribution:** → The probability distribution of any statistic is termed as sampling distribution.

(2006) In a problem of parametric inference the population feature of interest can objectively be written as a function of labelling parameters, say $\gamma(\theta)$, where, an observation $X_1 \sim F_\theta$, which is not known completely except the form of F.

Now in order to guess $\gamma(\theta)$ [Problems of estimation] or, to validate any conjuctured regarding $\gamma(\theta)$ [Problem of hypothesis testing], we proceed with a specific statistic and make use of its sampling distribution. In such inferential problem we always associate a measure of error with the conclusion where this error is nothing but the sampling error. As a measure the s.d. of the sampling distribution of the statistic would serve the purpose and this is termed as the standard error.

- Exhibiting a sampling distribution in case of sampling from a finite identifiable population: —

Here the term identifiability means that the population units can easily be distinguished.

8. **Simple Random Sampling:** → Suppose we have a finite identifiable population of size N $(u_1, u_2, \ldots, u_N)$, where u_α is the αth member of the population. By a sample we mean, a non-empty collection of units from $(u_1, u_2, \ldots, u_N)$ with, or, without repeatations. Here the sampling procedure may be subjective (purposive sampling, deliberate sampling, haphazard sampling) or, objective (Probabilistic, non-probabilistic, mixed).

The probabilistic sampling may be an equal probability sampling where each of the possible sample has the same probability to occur (or, every unit of the population has the same probability to be included in the sample) or, an unequal probability sampling.

- **Definition:** → Simple random sampling (SRS) is an equal probability sampling. An SRS may be drawn with replacement termed as SRSWR or, without replacement termed as SRSWOR.

SRSWR :— Suppose a sample of n units is drawn from the population of size N one by one with replacement. Clearly, number of possible samples is N^n on each has probability $\frac{1}{N^n}$ to occur.

SRSWOR :— Suppose a sample of n units is drawn at random one by one without replacement from the population of size N. If we ignore the order of the units in the sample. The number of possible sample will be $\binom{N}{n}$ and each has probability $\frac{1}{\binom{N}{n}}$ to occur.

On the other hand, if the order is taken into account the number of possible sample is $(N)n$ and each has the probability $\frac{1}{(N)n}$ to occur.

Suppose an SRSWOR of size 3 is drawn from a population of size 5.

$(u_1, u_2, u_3, u_4, u_5)$

Let us ignore the order of the units in the sample.

Further assume that the variate values of the population units are 6, 8, 4, 6, 8, respectively.

Let s be a typical sample and $\bar{x}(s)$ be the sample mean. Then we have the following sampling distribution of the sample mean.

Serial No.	s	$\bar{x}(s)$
1	(u_1, u_2, u_3)	6
2	(u_1, u_2, u_4)	6.67
3	(u_1, u_2, u_5)	7.33
4	(u_1, u_3, u_4)	5.33
5	(u_1, u_3, u_5)	6
6	(u_1, u_4, u_5)	6.67
7	(u_2, u_3, u_4)	6
8	(u_2, u_3, u_5)	6.67
9	(u_2, u_4, u_5)	7.33
10	(u_3, u_4, u_5)	6

$\bar{x}$	
5.33	1/10
6	2/5
6.67	3/10
7.33	1/5

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Policy Details

Servicing LIC Branch Code	**5012**
Instalment Premium	**40168.0**
Mode / Number Of Instalments	**Yearly / 1**
Due from - Due To	**28/05/2020 - 28/05/2020**
Total Premium	**40168.0**
CDA Charges	**0.0**
Late Fee	**0.0**
SGST / UTGST	**451.89**
CGST	**451.89**
Next Due	**28/05/2021**
Grand Total	**41071.78**
Mudrank Number	**CSD/323/2020/802/2020**

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