: Co-ordinate System :

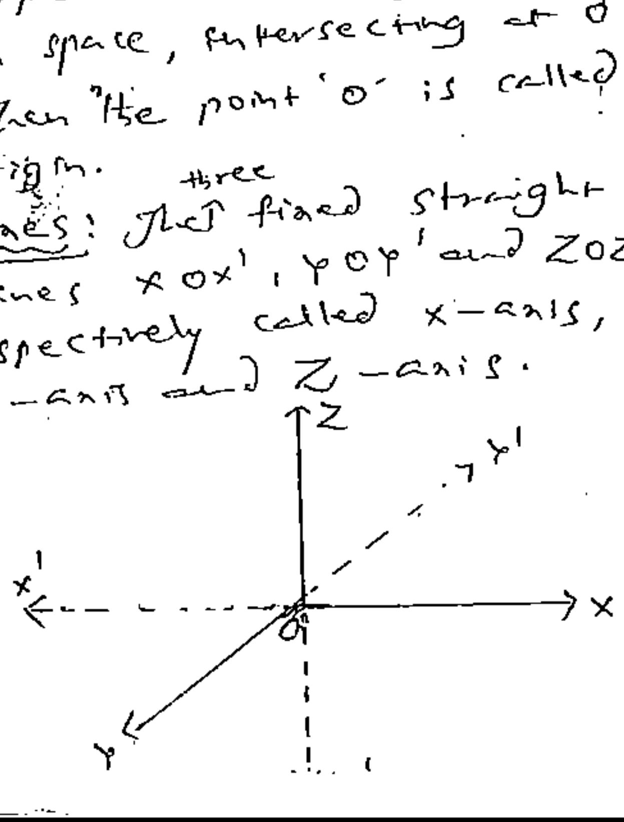
Introduction:

In analytical geometry of two dimensions the position of a point is determined with respect to two axes of reference. But in the space it is not sufficient to determine the point w.r.t two axes.

Thus to locate the position of a point in space, another (third) axis is required in addition to the two axes. That is why, the co-ordinate system in space is called a three dimensional system.

Origin: Let XOX', YOY' and ZOZ' be three mutually perpendicular straight lines in space, intersecting at 'O'. Then the point 'O' is called the origin.

Axes: The three fixed straight lines XOX', YOY' and ZOZ' called X-axis, Y-axis and Z-axis respectively.



The three lines taken together are called rectangular co-ordinate axes.

* Co-ordinate planes:

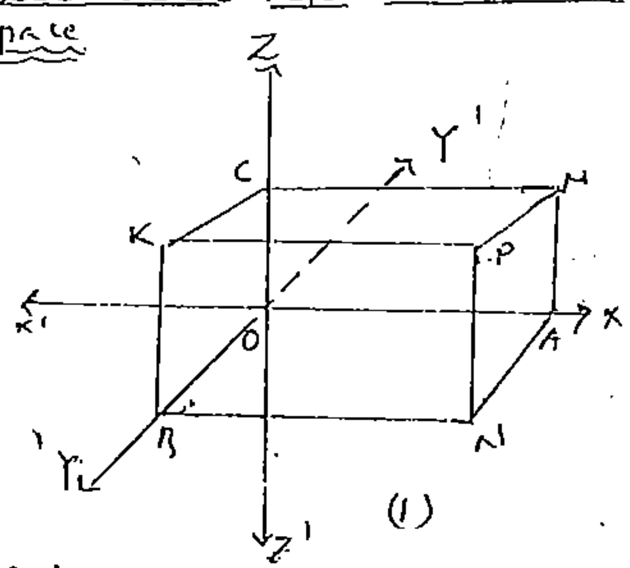
— The plane containing the axes of Y and Z is called the YZ-plane.
Thus YOZ is the YZ-plane.

— The plane containing the axes of Z and X is called the ZX-plane.
Thus ZOX is the ZX-plane.

— The plane containing the axes of X and Y is called the XY-plane.
Thus XOY is the XY-plane.

The above three planes are together called the rectangular co-ordinate planes or simply co-ordinate planes.

* Co-ordinates of a point in space



(1)

Let P be any point in space.

Draw through 'p' three planes parallel to the three co-ordinate planes and cutting x, y and z axes in A, B and C respectively as in the figure.

These planes, together with the co-ordinate planes form a rectangular parallelopiped.

The position of P relative to the co-ordinate system is given by its perpendicular distances from the co-ordinate planes and these distances are given by lengths OA, OB and OC.

Let OA = a, OB = b and OC = c.

Then a, b, c are called x-co-ordinate, y-co-ordinate respectively and z-co-ordinate of the point p. The point P is referred as (a, b, c) or p(a, b, c).

Any one of these a, b, c will be +ve or −ve according as it is measured from 'O' along the corresponding axis in the positive or negative direction.

Let p(x, y, z) be a point in the space.

Then (i) p lies in the xy-plane
$\Rightarrow z = 0$

(ii) p lies in the zx-plane
$\Rightarrow y = 0$

(iii) p lies in the yz-plane
$\Rightarrow x = 0$

(iv) p lies in the x-axis
$\Rightarrow y = 0, z = 0$

(v) p lies in the y-axis
$\Rightarrow x = 0, z = 0$

(vi) p lies in the z-axis
$\Rightarrow x = 0, y = 0$.

(vii) $p = 0 \Rightarrow x = 0, y = 0, z = 0$

* Octants:

The three co-ordinate planes divide the whole space into 8 parts and these parts are called octants.

The sign of a point determine the octant in which it lies.

The signs for the eight octants are given in the tabular form below:

octants	$OXYZ$	$OX'YZ$	$OX'Y'Z$	$OXY'Z$	$OXY'Z'$	$OX'YZ'$	$OX'Y'Z'$	$OXY'Z'$
x	+	−	−	+	+	−	−	+
y	+	+	−	−	−	+	−	−
z	+	+	+	+	−	−	−	−

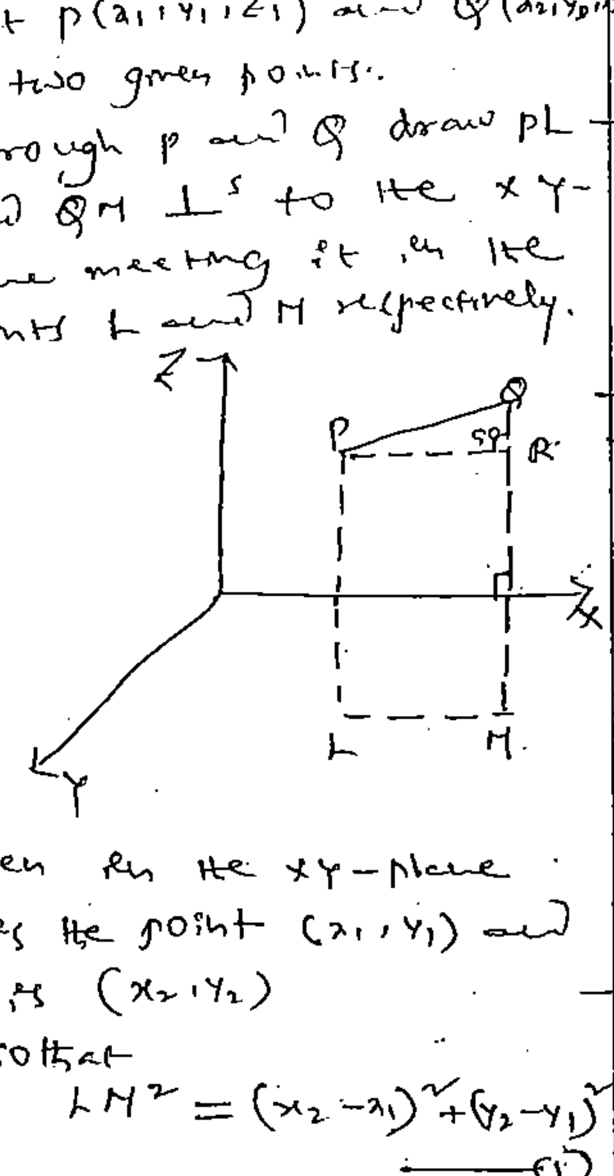
Note:- The co-ordinates of the origin 'O' are (0, 0, 0) and those of A, B, C, N, K and M in fig (1) are (a, 0, 0); (0, b, 0); (0, 0, c); (a, b, 0); (0, b, c) and (a, 0, c) respectively.

* Distance b/w two points:-

→ To find the distance b/w two points (x_1, y_1, z_1) and (x_2, y_2, z_2).

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two given points.

Through P and Q draw PL and QM ⊥s to the XY-plane meeting it in the points L and M respectively.



Then in the XY-plane L is the point (x_1, y_1) and M is (x_2, y_2)

so that

$$LM^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \quad \text{--- (1)}$$

Now through P, draw PR and ⊥ lar to QM.

Then clearly $PR = LM$

and $QR = QM - RM$

$= QM - PL$

$= z_2 - z_1$

∴ In the rt. angled triangle PQR,

$$PQ^2 = PR^2 + QR^2$$

(by Pythagoras theorem)

$$= LM^2 + QR^2 \quad \text{(2)}$$

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

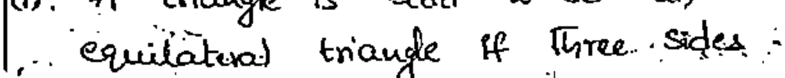
$$\therefore \boxed{PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

→ The distance of the point $P(x, y, z)$ from the origin $(0, 0, 0)$ is $\boxed{OP = \sqrt{x^2 + y^2 + z^2}}$.

→ Method to prove by distance that the three points A, B, C are collinear.

(1) find the three distances AB, BC and CA

(2) Then if the sum of any two distances is equal to the third, the three given points are collinear.



$AB + BC = AC \qquad AC + CB = AB$.

→ Triangles

If Three Points are not collinear, then they form a triangle.

(1). A triangle is said to be an equilateral triangle if three sides of the triangle are equal.

(2). A triangle is said to be an isosceles triangle if any two sides of the triangle are equal.

(3). A triangle is said to be a right angled triangle if one angle of the triangle is a right angle.

(4). A triangle is said to be obtuse angled triangle if one angle of the triangle is an obtuse angle.

, . A triangle is said to be acute angled triangle if the three angles are acute.

Quadrilaterals :-

, A quadrilateral is said to be a parallelogram if opposite sides are parallel and equal. If the opposite sides are equal, Then clearly they are parallel.

, . A quadrilateral is said to be a rectangle if opposite sides are equal and diagonals are equal.

, . A quadrilateral is said to be a rhombus if the four sides are equal and the two diagonals are not equal

, . A quadrilateral is said to be a square if the four sides are equal and the two diagonals are equal.

Note :- In a parallelogram (or) rectangle (or) rhombus (or) square, the diagonals bisect each other.

Note :- In a rhombus (or) a square, the diagonals are perpendicular to each other.

Problems :-

, . Find the distance between the points $(-1,0,6)$ and $(5,3,0)$

, Show that the points $(3,-2,4)$, $(1,1,1)$, $(-1,4,-2)$ are collinear.

→ If $A = (-1,3,5)$ and $B(4,-12,-20)$ find whether O, A, B are collinear.

→ show that the following points are collinear

(i) $(-1,0,7)$, $(3,2,1)$, $(5,3,-2)$ $\begin{vmatrix} - & - & - \\ - & - & - \\ - & - & - \end{vmatrix} = 0$

(ii) $(1,2,3)$, $(7,0,1)$, $(-2,3,4)$

→ show that the three points $(1,1,0)$, $(1,0,1)$, $(0,1,1)$ form an equilateral triangle.

→ $(1,1,1)$, $(-2,4,1)$, $(-1,5,5)$ form a right angles isosceles triangle.

→ show that the points $(-1,-2,-1)$ $(2,3,2)$, $(4,7,6)$ and $(1,2,3)$ form a parallelogram.

→ show that the points $(1,3,4)$, $(-1,6,10)$, $(-7,4,7)$, $(-5,1,1)$ are the vertices of a rhombus.

→ Prove that the four points A, B, C, D whose coordinates are $(1,1,1)$, $(-2,4,1)$, $(-1,5,5)$ and $(2,2,5)$ are the vertices of a square.

* Section formulae for external division:

→ The co-ordinates of a point R which divides the line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ externally in the ratio $m:n$ are

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right)$$

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* **Midpoint formula:**

The co-ordinates of the mid-point R of the line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

→ If $P(x, y, z)$ lies on the line joining $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ then $\frac{x_1 - x}{x - x_2} = \frac{y_1 - y}{y - y_2} = \frac{z_1 - z}{z - z_2}$ and P divides $\overline{AB}$ in the ratio $x_1 - x : x - x_2$ or $y_1 - y : y - y_2$ or $z_1 - z : z - z_2$.

→ xy-plane divides the line segment joining (x_1, y_1, z_1) (x_2, y_2, z_2) in the ratio $-z_1 : z_2$ -

→ If $D(\alpha_1, \beta_1, \gamma_1)$, $E(\alpha_2, \beta_2, \gamma_2)$, $F(\alpha_3, \beta_3, \gamma_3)$ are midpoints of sides $\overline{BC}, \overline{CA}, \overline{AB}$ of the triangle ABC then

$A(\alpha_2 + \alpha_3 - \alpha_1, \beta_2 + \beta_3 - \beta_1, \gamma_2 + \gamma_3 - \gamma_1)$

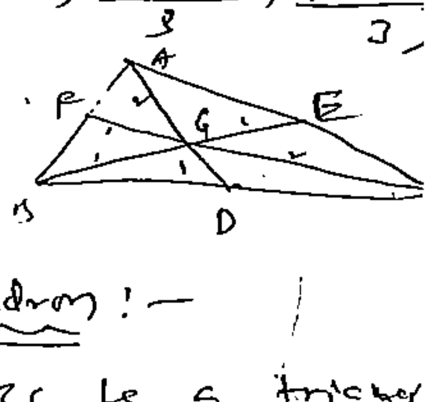
$B(\alpha_1 + \alpha_3 - \alpha_2, \beta_1 + \beta_3 - \beta_2, \gamma_1 + \gamma_3 - \gamma_2)$

and

$C(\alpha_1 + \alpha_2 - \alpha_3, \beta_1 + \beta_2 - \beta_3, \gamma_1 + \gamma_2 - \gamma_3)$.

→ **Centroid of a triangle**

The centroid of a triangle with vertices $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$



* **Tetrahedron:-**

Let ABC be a triangle and D is a point in the space which is not in the plane of the triangle ABC. Then ABCD is called a tetrahedron.

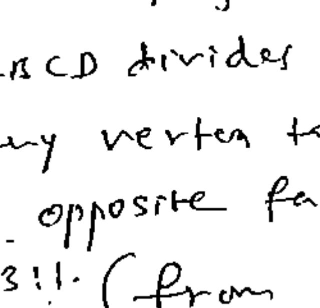
→ The tetrahedron ABCD has four faces namely △ABC, △ACD, △ABD & △BCD.

→ It has four vertices, namely A, B, C, D and it has six edges, namely, AB, AC, AD, BC, BD, CD

→ The centroid G of the tetrahedron ABCD divides the line joining any vertex to centroid of its opposite face in the ratio 3:1. (from statics)

Thus if G_1 is the centroid of △BCD, then G the centroid of tetrahedron ABCD divides AG_1 in the ratio 3:1

i.e $\frac{AG}{GG_1} = \frac{3}{1}$.



→ If all the edges are of equal length, then it is called a regular tetrahedron

→ Centroid of tetrahedron:

Let ABCD be a tetrahedron with vertices $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$. Then the co-ordinates of its centroid are

$$\left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right)$$

→ find the points dividing the line segment joining (1,-1,2) and (2,3,7) in the ratio.

i) 2:3 ii) -2:3.

→ Find the middle point of the line segment with end points (1,2,-3) and (-1,6,7).

→ find the ratio in which the line joining the points (x_1, y_1, z_1), (x_2, y_2, z_2) is divided by xy-plane.

Sol: Let the ratio be $\lambda:1$ and let R be the point of intersection of plane and line segment

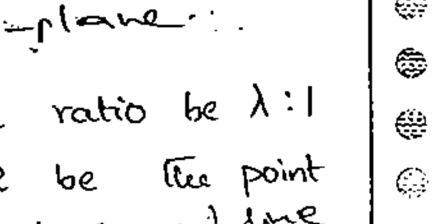
∴ The coordinates of R are

$$\left[\frac{\lambda x_2 + x_1}{\lambda+1}, \frac{\lambda y_2 + y_1}{\lambda+1}, \frac{\lambda z_2 + z_1}{\lambda+1}\right]$$

Since the point R lies on XY-plane.

∴ z - coordinate must be zero.

$$\therefore \frac{\lambda z_2 + z_1}{\lambda+1} = 0$$



$$\Rightarrow \lambda z_2 + z_1 = 0$$

$$\Rightarrow \lambda z_2 = -z_1$$

$$\Rightarrow \lambda = \frac{-z_1}{z_2}$$

$$\Rightarrow \frac{\lambda}{1} = \frac{-z_1}{z_2}$$

$$\Rightarrow \boxed{\lambda : 1 = -z_1 : z_2}$$

Note : → on XY-plane is $-z_1 : z_2$

Set - I

* Complex Analysis *

→ Introduction :-

In the field of real numbers, the equation $x^2+1=0$ has no solution. To permit the solution of this and similar equations (i.e $x^2-2x+3=0$ etc), the real number system was extended to the set of complex numbers. Euler introduced the symbol i with the property that $i^2=-1$. He also called i as the imaginary unit.

A number of the form $a+ib$ where a, b are real numbers, was called complex number.

If we write $z=x+iy$ then z is called a complex variable.

Also x is called real part of z and is denoted by $R(z)$ i.e $R(z)=x$ and y is called imaginary part of z and is denoted by $I(z)$ i.e. $I(z)=y$.

- Some times we express z as $z=(x,y)$
- If $x=0$ i.e. $z=iy$ then z is called pure imaginary number.
- the conjugate of $z=x+iy$ is $\bar{z}=x-iy$.
- $Re(z) = x = \frac{z+\bar{z}}{2}$
- $I(z) = y = \frac{z-\bar{z}}{2i}$

* Fundamental operations with Complex Numbers :-

Addition : $(a+ib)+(c+id) = (a+c)+i(b+d)$

Subtraction : $(a+ib)-(c+id) = (a-c)+i(b-d)$.

Multiplication : $(a+ib)(c+id) = (ac-bd)+i(bc+ad)$

Division : $\frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)}$

$= \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$

$= \frac{ac+bd}{c^2+d^2} + i\left(\frac{bc-ad}{c^2+d^2}\right)$ if $c^2+d^2 \neq 0$.

* Absolute value :

The absolute value (or) modulus of a complex number $z=a+ib$ is denoted by $|z|$ and is defined as

$|z| = |a+ib|$

$= \sqrt{a^2+b^2}$

Evidently $|z|^2 = a^2+b^2$

$= (a+ib)(a-ib)$

$= z\bar{z}$

$\therefore |z|^2 = z\bar{z}$

Also $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

* Geometrical Representation of Complex Numbers :-

Consider the complex number $z=x+iy$. A complex number can be regarded as an ordered pair of reals. i.e $z=(x,y)$.

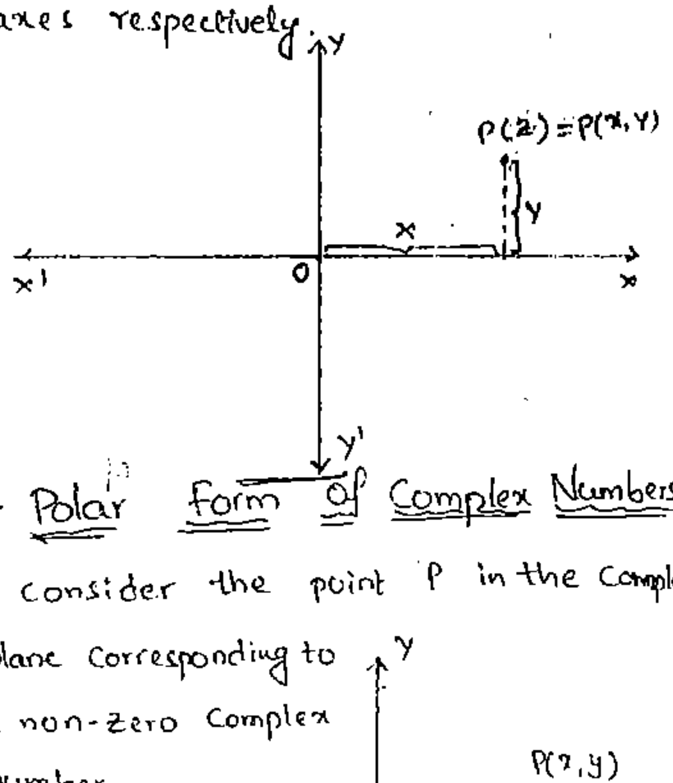
This form of z suggests that z can be represented by a point P whose coordinates are x & y relative to the rectangular axes x & y

To each complex number there corresponds one and only one point in the xy-plane and conversely to each point in the plane there exists one and only one complex number. Due to this fact, the complex number z is referred to the point z in this plane.

This plane is called complex plane or Gaussian plane or Argand plane.

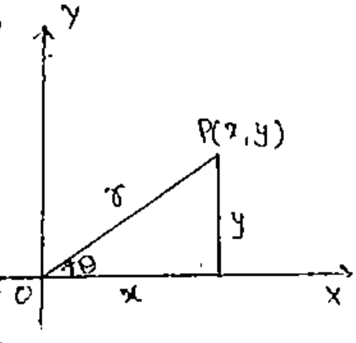
The representation of complex numbers is called Argand diagram.

The complex number $x+iy$ is called complex coordinate and x, y axes are called real and imaginary axes respectively.



* Polar Form of Complex Numbers:

Consider the point P in the complex plane corresponding to a non-zero complex number.



From the figure,

$\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$

$\Rightarrow x = r\cos\theta,\ y = r\sin\theta$

$\therefore r = \sqrt{x^2+y^2}$

$= |x+iy|$

$= |z|$

$\boxed{\therefore r = |z|}$

and $\tan\theta = \frac{y}{x} \Rightarrow \boxed{\theta = \tan^{-1}\left(\frac{y}{x}\right)}$

It follows that,

$z = x+iy = r(\cos\theta + i\sin\theta)$

$= re^{i\theta}$ —— (1)

It is called polar form of the complex numbers z.

r and θ are called polar coordinates of z.

→ r is called modulus (or) absolute value of z.

→ The angle θ which the line OP makes with the +ve x-axis, is called argument (or) amplitude of z and is denoted by $\theta = \arg(z)$ (or) $\theta = \mathrm{amp}(z)$

→ The argument of z is not unique, since the equation (1) does not alter, if we replace θ by $2\pi+\theta$. So θ can have infinite number of values which differ from each other by 2π.

→ If a value of θ satisfies (1) and lies b/w $-\pi$ & π.

i.e $-\pi < \theta \leq \pi$ then that value of θ is called principal value of the argument.

Note:- It is evident from the definition of difference and modulus that $|z_1 - z_2|$

is the distance b/w two points z_1 & z_2

i.e $z_1 = x_1 + iy_1$ & $z_2 = x_2 + iy_2$

$\therefore |z_1 - z_2| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

It follows that for fixed Complex number z_0 and a real numbers 'r'. The equation $|z-z_0| = r$ represents a circle with centre z_0 and radius r.

* **Point Set** :- Any collection of points in the Complex (two dimensional) plane is called a point set and each point is called a member(or) element of the point set.

— The set of Complex numbers is denoted by 'C'.

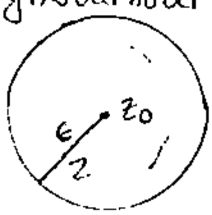
* **ϵ neighbourhood of a Complex number z_0** :-

The set of all points $z \in C$ satisfying the condition $|z-z_0| < \epsilon$ is defined as ϵ-neighbourhood of the z_0.

— A deleted neighbourhood of z_0 is neighbourhood of z_0 in which the point z_0 is omitted

i.e. $0 < |z - z_0| < \epsilon$

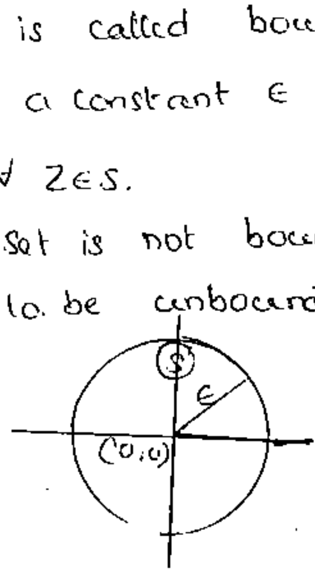
— In general ϵ-neighbourhood of z_0 is denoted by $N(z_0, \epsilon)$.



* **Bounded Set** :- A set 's' is said to be bounded if it is contained in some neighbourhood of the origin. (or)

A set 's' is called bounded if we can find a constant ϵ such that $|z| < \epsilon \ \forall \ z \in S$.

— If a set is not bounded then it is said to be unbounded.



* **Interior Point** :-

A Point z_1 of a set 's' is said to be an interior point of the set 's' if there exist a neighbourhood of z_1 which is contained completely in the set 's'.

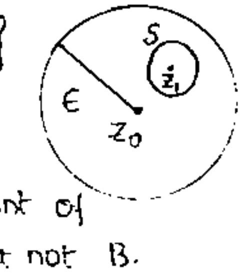
— If every neighbourhood of z_1 contains some points of 's' and some points that doesnot belong to 's' is called a boundary point.

— A point z_0 which is neither interior nor boundary point is called exterior point.

Example :-

Let $A = \{z \in C / |z - z_0| < \epsilon\}$

$B = \{z \in C / |z - z_0| \leq \epsilon\}$



In this example every point of A is an interior point but not B.

* **Open Set** :— A set 's' is called an open set if every point in 's' is an [illegible]

interior Point.

i, in the empty set

ii, the set of all complex numbers.

iii, $\{z : |z| > r\}$, $r \geq 0$

iv, $\{z : r_1 < |z| < r_2\}$, $0 \leq r_1 < r_2$

* **Limit Point** :- A point z_0 is said to be a limit point of 's' if every deleted neighbourhood of z_0 contains a point of 's'.

— Limit point is also known as Cluster point (or) point of accumulation.

— The limit Point of the set may (or) may not belong to the set.

Ex:- ① the limit points of open set $|z| < 1$ are $|z| \leq 1$.

i.e. all the points of the set and all the points on the boundary $|z| = 1$

②. The set $\{1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n}, \ldots\}$ has '0' as a limit point.

③. the set $\left\{\frac{3+2ni}{1+n} \,\middle|\, n = 1, 2, 3 \ldots\right\}$

$$= \left\{\frac{3+2i}{2}, \frac{3+4i}{3}, \frac{3+6i}{4}, \ldots\right\}$$

has $2i$ as a limit point.

* **Closed Set** : A set is said to be closed if it contains all its limit point.

Ex:- ① the empty set

(2). the set of all complex numbers

(3). $\{z : |z| > r\}$, $r \geq 0$.

(4). $\{z : r_1 \leq |z| \leq r_2\}$, $0 \leq r_1 < r_2$

(5) the union of any two closed sets.

* **Closure of a set** :- the union of a set and its limit points is called closure.

* **Domain (Region)** :-

— A set 's' of points in the complex plane is said to be connected set if any two of its points can be joined by a continuous curve, all of whose belong to 's'.

— An open connected is called an open domain (or) open region

— If the boundary point of 's' are also added to an open domain, then it is called closed domain.

* **Complex Variable** :-

If a symbol 'z' takes any one of the values of a set of complex numbers, then z is called a complex variable. (or)

Let D be an arbitrary non-empty point set of xoy-plane. If z is allowed to denote any point of D, then z is called a complex variable. and D is the domain of definition of z (or) simply domain.

* **Functions of a Complex Variable**

we say that 'w' is a function of the complex variable z with domain D and Range R ; if D and R are two non-empty point sets of Complex plane, if to each z in D there corresponds at least one ω in R and to each ω of R, there is at least one z of D to which ω corresponds.

Then we symbolically write

$\omega = f(z)$.

— The variable z is sometimes called independent variable and 'ω' is called dependent variable.

the value of a function at $z=a$ is written as $f(a)$.

Thus if $f(z) = z^2$, $f(2i) = (2i)^2 = -4$

— If we have only one value 'ω' of R to each value of z in D, then we say that 'ω' is a single valued function of z (or) $f(z)$ is single valued.

— If more than one value of ω corresponds to each value of z, we say that 'ω' is a multi valued (or) multiple valued function.

Ex:- ① Let $\omega = z^2$. Then corresponding to each value of z, we get only one value to 'ω'. So ω is a single valued function.

This is because:

the function $\omega = z^2$ may be expressed as $\omega = f(z)$

$= f(x+iy)$

$= f(x,y)$

$= (x+iy)^2$

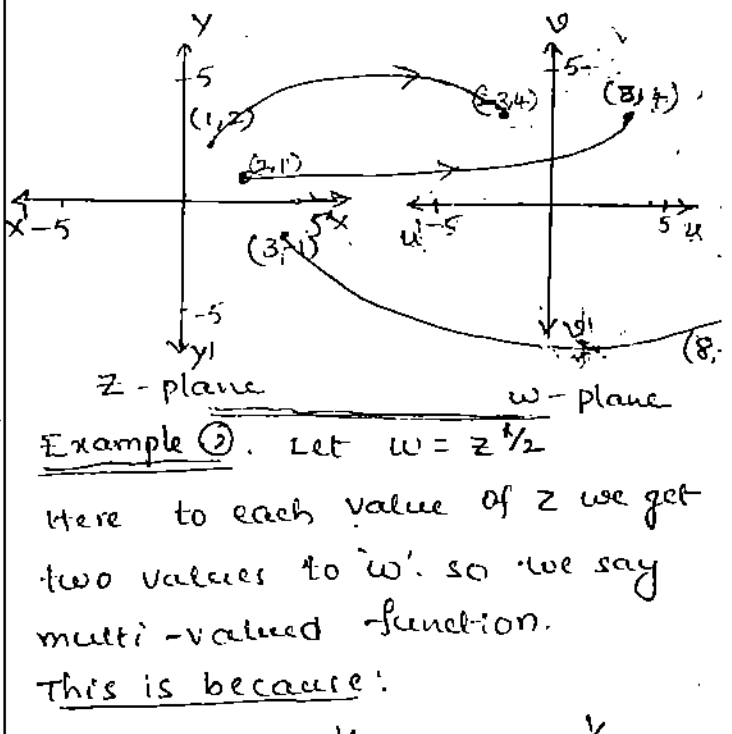
$= x^2 - y^2 + i(2xy)$

where $Re(\omega) = x^2 - y^2$

$= u(x,y)$ say

and $Im(\omega) = 2xy$

$\boxed{\because \omega = u+iv}$ $= v(x,y)$ say



Example ②. Let $\omega = z^{1/2}$

Here to each value of z we get two values to 'ω'. so we say multi-valued function.

This is because:

$$\omega = z^{1/2} = (x+iy)^{1/2}$$

$$= \sqrt{r}\, e^{i\theta/2} \quad \text{where } x = r\cos\theta,\ y = r\sin\theta$$

Let $\theta = \theta_1$ then $\omega = \sqrt{r}\, e^{i\theta_1/2}$;

$\theta = \theta_1 + 2\pi$ then $\omega = \sqrt{r}\, e^{i(\theta_1+2\pi)/2}$

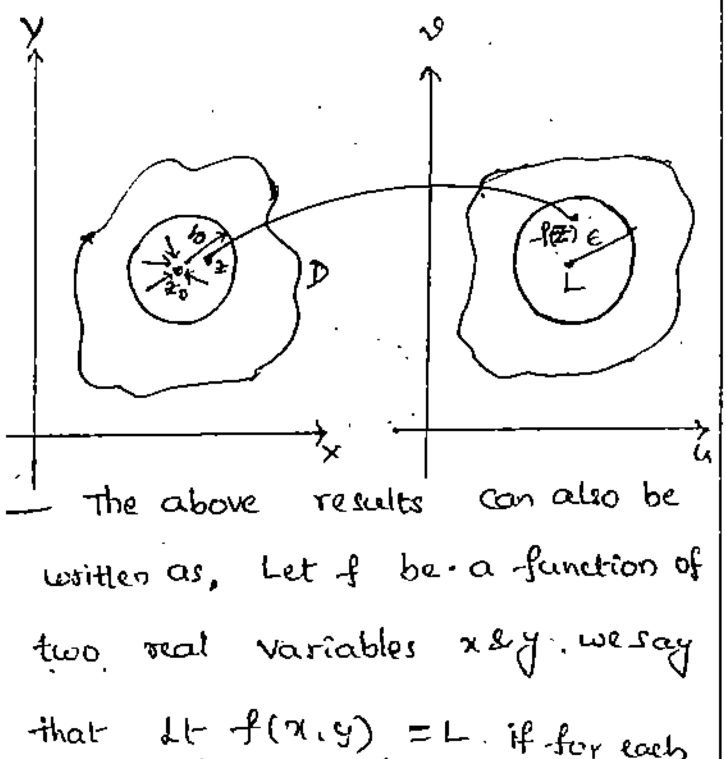
$= \sqrt{r}\left[\cos(180 + \theta_1/2) + i\sin(180+\theta_1/2)\right]$

$= \sqrt{r}\left[-\cos\theta_1/2 - i\sin\theta_1/2\right]$.

We can verify that 'w' gets the same values for θ_1 and $\theta_1+4\pi$.

* Limit of a Function:-

Let $f(z)$ be a function of a complex variable z. Then we say that $\underset{z\to z_0}{Lt} f(z) = L$, if for any given $\epsilon>0$ (however small), $\exists$ a $\delta>0$ (depending on ϵ) such that $|f(z)-L|<\epsilon$ whenever $0<|z-z_0|<\delta$.



The above results can also be written as, Let f be a function of two real variables x & y. we say that $\underset{(x,y)\to(x_0,y_0)}{Lt} f(x,y) = L$ if for each $\epsilon>0$, $\exists$ a $\delta>0$ such that $|f(x,y)-L|<\epsilon$ for every $0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta$.

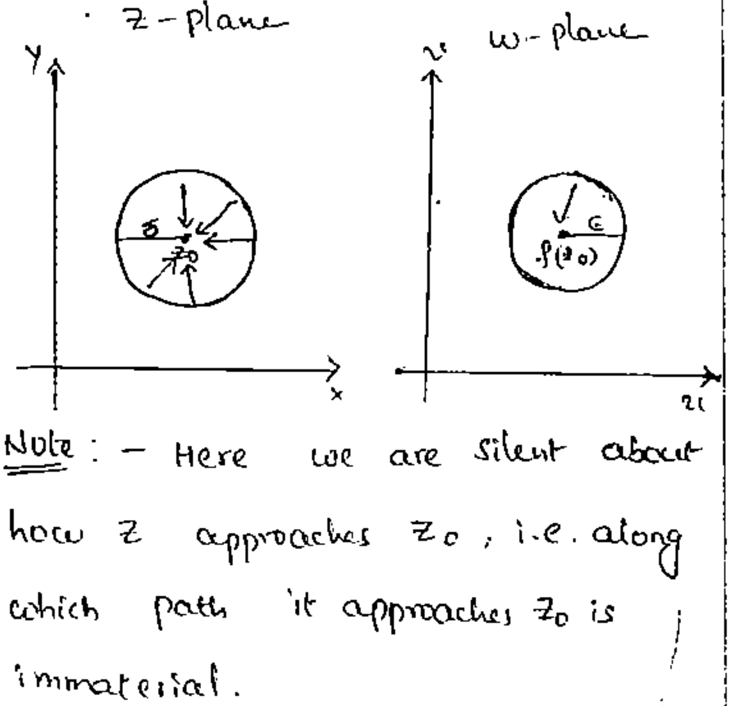
* Continuity of a Function:-

If $f(x_0,y_0)=L$, then we say that $f(x,y)$ is continuous at (x_0,y_0) (or) $f(z_0)=L$.

i.e. the value of the function at $z=z_0$ is equal to 'L', then we say that $f(z)$ is continuous at $z=z_0$.

(or)

$f(z)$ is continuous at $z=z_0$ if $\underset{z\to z_0}{Lt} f(z) = f(z_0)$ i.e. if given $\epsilon>0$ (however small), $\exists$ a $\delta>0$ depending on ϵ such that $|f(z)-f(z_0)|<\epsilon$ whenever $|z-z_0|<\delta$.



Note:- Here we are silent about how z approaches z_0, i.e. along which path it approaches z_0 is immaterial.

Note:- Let us consider $f(z)=z^2+3z+5$

Let $z=x+iy$ then $z^2=x^2-y^2+i2xy$

$\therefore f(z) = z^2+3z+5$

$=(x^2-y^2+3x+5)+i(2xy+3y)$

$=f_1(x,y)+if_2(x,y)$

Set VI

* The Riemann Integral *

Introduction :-

In elementary treatments. the Process of integration is generally introduced as the inverse of differentiation.

If $F'(x) = f(x)$ for all x belonging to the domain of the function f, F is called an integral of the given function f.

Historically, however the Subject of Integral arose in connection with the problem of finding areas of plane regions in which the area of a plane region is calculated as the limit of a Sum. This notion of Integral as Summation is based on geometrical Concepts.

A German mathematician. G.F.B. Riemann gave the first rigorous arithmetic treatment of definite integral free from geometrical Concepts.

Riemann's definition covered only bounded functions.

It was Cauchy who extended this definition to unbounded functions.

In the present Chapter we shall study the Riemann integral of real valued, bounded functions defined on some closed interval.

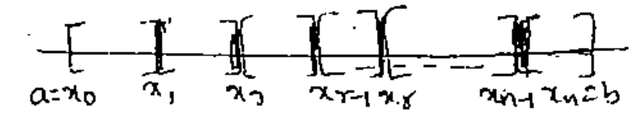
* Partition of a Closed Interval

Let $I = [a,b]$ be a Closed bounded interval. If $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ then the finite ordered set

$$P = \{x_0, x_1, x_2, \dots, x_{r-1}, x_r, \dots x_n\}$$

where $r = 1, 2, \dots n$, is called a partition of I.

The $(n+1)$ Points $x_0, x_1, \dots, x_{n-1}, x_n$ are called Partition points of P.



The n closed Subintervals $I_1 = [x_0, x_1]$, $I_2 = [x_1, x_2]$, $\dots$ $I_r = [x_{r-1}, x_r]$ $\dots$ $I_n = [x_{n-1}, x_n]$ determined by P are called segments of the partition P.

clearly $\bigcup_{r=1}^{n} I_r = \bigcup_{r=1}^{n} [x_{r-1}, x_r] = [a,b] = I$

(or)

$$P = \{[x_{r-1}, x_r]\}_{r=1}^{n}$$

-- The length of the r^{th} Subinterval $I_r = [x_{r-1}, x_r]$ is denoted by δ_r

i.e. $\delta_r = x_r - x_{r-1}$; $r = 1, 2, \dots n$

Note: (1) By changing the partition points, the partition can be changed and hence, there can be an infinite number of partitions of the interval I.

we shall denote it by $P[a,b]$, the set (or family) of all partitions of $[a,b]$.

). Partition is also known as dissection (or) net.

Norm of a partition :-

The maximum of the lengths of the subintervals of a partition P is called norm (or) mesh of the partition P and is denoted by $\|P\|$ (or) $\mu(P)$.

i.e. $\|P\| = \text{Max}\{\delta_r / r = 1,2 --- n\}$

$= \text{Max}\{x_r - x_{r-1} / r = 1,2 --- n\}$

$= \text{Max}\{x_1 - x_0, x_2 - x_1, --- x_n - x_{n-1}\}$

Note (1): If $P = \{x_0, x_1, ---- x_n\}$ is a partition of $[a,b]$ then

$$\sum_{r=1}^{n} \delta_r = \delta_1 + \delta_2 + --- + \delta_n$$

$$= (x_1 - x_0) + (x_2 - x_1) + --- + (x_n - x_{n-1})$$

$$= x_n - x_0$$

$$= b - a$$

* Refinement of a partition:-

If P, P' be two partitions of $[a,b]$ and $P \subset P'$ then the partition P' is called a refinement of partition P on $[a,b]$. we also say P' is finer than P.

i.e. If P' is finer than P, then every point of P is used in the construction of P'. and P' has atleast one additional point.

→ If P_1, P_2 are two partitions of $[a,b]$ then $P_1 \subset P_1 \cup P_2$ and $P_2 \subset P_1 \cup P_2$. Therefore $P_1 \cup P_2$ is called a common refinement of P_1 & P_2.

Note: If $P_1, P_2 \in P[a,b]$ and $P_1 \subset P_2$ then $\|P_2\| \leq \|P_1\|$.

* Upper and Lower Darboux Sums:

Let $f:[a,b] \rightarrow \mathbb{R}$ be a bounded function and

$P = \{a = x_0, x_1, ----- x_n = b\}$ be a partition of $[a,b]$.

Since f is bounded on $[a,b]$, f is also bounded on each of the subintervals. (i.e. $I_r = [x_{r-1}, x_r], r = 1,2 -- n$)

Let M, m be the supremum and infimum of f in $[a,b]$ and M_r, m_r be the supremum and infimum of f. in the rth subintervals.

$I_r = [x_{r-1}, x_r]$; $r = 1, 2, --- n$.

The sum $M_1\delta_1 + M_2\delta_2 + --- + M_r\delta_r + --- + M_n\delta_n = \sum_{r=1}^{n} M_r\delta_r$.

is called the Upper Darboux Sum of f corresponding to the partition P and is denoted by $U(P,f)$ or $U(f,P)$

→ The Sum $m_1\delta_1 + m_2\delta_2 + \cdots + m_r\delta_r + \cdots + m_n\delta_n = \sum_{r=1}^{n} m_r\delta_r$

is called the lower Darboux sum of f corresponding to the partition P and is denoted by $L(P,f)$ or $L(f,P)$.

i.e. $U(P,f) = \sum_{r=1}^{n} M_r\delta_r$

$L(P,f) = \sum_{r=1}^{n} m_r\delta_r$

* Oscillatory Sum:

Let $f:[a,b] \to \mathbb{R}$ be a bounded function and $P = \{a = x_0, x_1, \cdots x_{n-1}, x_n = b\}$ be a partition of $[a,b]$.

Let m_r and M_r be the infimum and Supremum of f on $I_r = [x_{r-1}, x_r]$ $r = 1, 2 \cdots n$. Then

$$U(P,f) - L(P,f) = \sum_{r=1}^{n} M_r\delta_r - \sum_{r=1}^{n} m_r\delta_r$$
$$= \sum_{r=1}^{n} (M_r - m_r)\delta_r$$
$$= \sum_{r=1}^{n} O_r\delta_r .$$

where $O_r = M_r - m_r$ denotes the oscillation of f on I_r.

$U(P,f) - L(P,f) = \sum_{r=1}^{n} O_r\delta_r$ is Called the oscillatory Sum of f Corresponding to the partition P and is denoted by $\omega(P,f)$.

i.e. $\omega(P,f) = \sum_{r=1}^{n} O_r\delta_r$

→ If $f:[a,b] \to \mathbb{R}$ is bounded function and $P \in \mathcal{P}[a,b]$ then

$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$

where m, M are the infimum and Supremum of f on $[a,b]$

Proof: Let $P = \{a = x_0, x_1, \cdots x_n = b\}$ be partition of $[a,b]$.

Since f is bounded on $[a,b]$

$\Rightarrow f$ is bounded on each subinterval of $[a,b]$.

i.e. f is bounded on $I_r = [x_{r-1}, x_r]$, $r = 1, 2 \cdots n$.

Let m_r and M_r be the infimum & Supremum of f on $I_r = [x_{r-1}, x_r]$

$\therefore m \leq m_r \leq M_r \leq M$.

$\Rightarrow m\delta_r \leq m_r\delta_r \leq M_r\delta_r \leq M\delta_r$

$\Rightarrow \sum_{r=1}^{n} m\delta_r \leq \sum_{r=1}^{n} m_r\delta_r \leq \sum_{r=1}^{n} M_r\delta_r \leq \sum_{r=1}^{n} M\delta_r$

$\Rightarrow m\sum_{r=1}^{n}\delta_r \leq L(P,f) \leq U(P,f) \leq M\sum_{r=1}^{n}\delta_r$

$\Rightarrow m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$

$\left[\because \sum_{r=1}^{n}\delta_r = b-a\right]$.

Note: The above theorem implies that $L(P,f)$ & $U(P,f)$ are bounded if f is bounded.

* Upper and Lower Riemann Integrals:—

Let $f:[a,b] \to \mathbb{R}$ be a bounded function. and $P \in \mathcal{P}[a,b]$ then we have

$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$

where

m, M are infimum and supremum of 'f' on $[a,b]$.

for every $P \in P[a,b]$,

we have

$L(P,f) \leq M(b-a)$ and

$U(P,f) \geq m(b-a)$

$\Rightarrow$ the set $\{L(P,f)\}_{P \in P[a,b]}$ of lower sums is bounded above by $M(b-a)$.

$\therefore$ It has the least upper bound. (lub)

the set $\{U(P,f)\}_{P \in P[a,b]}$ of the upper sums is bounded below by $m(b-a)$.

$\therefore$ It has the greatest lower bound. (glb)

Now the $\sup\{L(P,f)\}_{P \in P[a,b]}$ is called lower Riemann Integral of 'f' on $[a,b]$ and is denoted by

$$\underline{\int_a^b} f(x)\,dx.$$

i.e. $$\underline{\int_a^b} f(x)\,dx = \text{Lub}\{L(P,f)\}_{P \in P[a,b]}$$

and the $\text{glb}\{U(P,f)\}_{P \in P[a,b]}$ is called Upper Riemann Integral of f on $[a,b]$ and is denoted by

$$\overline{\int_a^b} f(x)\,dx$$

i.e. $$\overline{\int_a^b} f(x)\,dx = \underline{\underline{\text{glb}}}\{U(P,f)\}_{P \in P[a,b]}$$

* Riemann Integral:

A bounded f is said to be Riemann integrable (or R-integrable) on $[a,b]$ if its lower and upper Riemann integrals are equal.

i.e. if $$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx.$$

— The common value of these integrals is called the Riemann integral of f on $[a,b]$ and is denoted by $\int_a^b f(x)\,dx$.

i.e., $$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx.$$

Note: (1) the interval $[a,b]$ is called the range of the integration. The numbers a and b are called the lower and upper limits of integration respectively.

(2) the family of all bounded functions which are R-integrable on $[a,b]$ is denoted by $R[a,b]$.

If f is R-integrable on $[a,b]$ then $f \in R[a,b]$.

(3) f is R-integrable on $[a,b]$

$\Rightarrow$ (i) f is bounded on $[a,b]$

(ii) $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx$

(4) A bounded function f on $[a,b]$ is such that

$$\underline{\int_a^b} f(x)\,dx \neq \overline{\int_a^b} f(x)\,dx$$

then f is not R-integrable on $[a,b]$

Problems:

→ Let $f(x) = x \;\forall\; x \in [0,1]$ and let $P = \{0, 1/3, 2/3, 1\}$ be a partition of $[0,1]$. Compute $U(P,f)$ and $L(P,f)$

Sol^n: Partition set P divides the interval $[0,1]$ into subintervals.

$I_1 = [0, 1/3]$, $I_2 = [1/3, 2/3]$, $I_3 = [2/3, 1]$

Now $\delta_1 = 1/3 - 0 = 1/3$

$\delta_2 = 2/3 - 1/3 = 1/3$

$\delta_3 = 1 - 2/3 = 1/3$

Since $f(x) = x$ is an increasing function on $[0,1]$.

$\therefore$ $M_1 = 1/3$, $m_1 = 0$

$M_2 = 2/3$, $m_2 = 1/3$

$M_3 = 1$, $m_3 = 2/3$

$\therefore$ $U(P,f) = \sum_{r=1}^{3} M_r \delta_r$

$= M_1\delta_1 + M_2\delta_2 + M_3\delta_3$

$= \frac{1}{3}\cdot\frac{1}{3} + \frac{2}{3}\cdot\frac{1}{3} + 1\cdot\frac{1}{3}$

$= \frac{1}{3}\left(\frac{1}{3} + \frac{2}{3} + 1\right) = \frac{2}{3}$

Now $L(P,f) = \sum_{r=1}^{3} m_r \delta_r$

$= 1/3$

→ Compute $L(P,f)$ and $U(P,f)$ for the function f defined by $f(x) = x^2$ on $[0,1]$ and $P = \{0, 1/4, 2/4, 3/4, 1\}$

Sol^n: The partition set P divides $[0,1]$ into subintervals $I_1 = [0, 1/4]$, $I_2 = [1/4, 2/4]$, $I_3 = [2/4, 3/4]$ and $I_4 = [3/4, 1]$

$\therefore$ $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 1/4$.

Since $f(x) = x^2$ is an increasing on $[0,1]$.

$\therefore$ $m_1 = 0$; $M_1 = 1/16$

$m_2 = 1/16$; $M_2 = 4/16$

$m_3 = 4/16$; $M_3 = 9/16$

$m_4 = 9/16$; $M_4 = 1$

$L(P,f) = \sum_{r=1}^{4} m_r \delta_r$

$= 7/32$

and $U(P,f) = \sum_{r=1}^{n} M_r \delta_r = 15/32$

→ If f is defined on $[a,b]$ by $f(x) = K \;\forall\; x \in [a,b]$ where K is constant then $f \in R[a,b]$ and $\int_a^b f(x)\,dx = K(b-a)$. (or)

A constant function is R-integrable.

Sol^n:- Let $P=\{a=x_0, x_1, \cdots x_n=b\}$ be any partition of $[a,b]$.

Let $I_r=[x_{r-1}, x_r]$; $r=1,2\cdots n$ be the r^{th} subinterval of $[a,b]$.

Since $f(x)=K$ (Constant).

$\therefore M_r=m_r=K$.

$$U(P,f)=\sum_{r=1}^{n} M_r \delta_r$$

$$=\sum_{r=1}^{n} K(x_r-x_{r-1})$$

$$=K\sum_{r=1}^{n}(x_r-x_{r-1})$$

$$=K(b-a)$$

and $L(P,f)=\sum_{r=1}^{n} m_r\delta_r$

$$=K(b-a)$$

Now $\underline{\int_a^b} f(x)dx = \text{lub}_{P\in P[a,b]}\{L(P,f)\}$

$$=K(b-a).$$

and $\overline{\int_a^b} f(x)dx = \text{glb}_{P\in P[a,b]}\{U(P,f)\}$

$$=K(b-a).$$

$\therefore \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = K(b-a)$

$\therefore f\in R[a,b]$

and $\int_a^b f(x)dx = K(b-a)$.

2009 show that the function f defined by $f(x)=\begin{cases}0 & \text{when } x \text{ is rational}\\ 1 & \text{when } x \text{ is irrational}\end{cases}$ is not Riemann integrable on any interval.

(or)

show by an example that every bounded function need not be R-integrable.

Sol^n:- Let f be denoted on $[a,b]$ by

$$f(x)=\begin{cases}0 & \text{when } x \text{ is rational}\\ 1 & \text{when } x \text{ is irrational}\end{cases}$$

Clearly $f(x)$ is bounded on $[a,b]$ because $0\le f(x)\le 1 \ \forall x\in[a,b]$.

Let $P=\{a=x_0, x_1 \cdots x_n=b\}$ be a partition of $[a,b]$.

Let $I_r=[x_{r-1}, x_r]$; $r=1,2\cdots n$. be r^{th} subinterval of $[a,b]$.

$\therefore M_r=1$; $m_r=0$.

$$U(P,f)=\sum_{r=1}^{n} M_r\delta_r=\sum_{r=1}^{n} 1.\delta_r$$

$$=b-a$$

and $L(P,f)=\sum_{r=1}^{n} m_r\delta_r$

$$=\sum_{r=1}^{n} 0.\delta_r$$

$$=0$$

Now $\underline{\int_a^b} f(x)\,dx = \text{Lub}_{P\in P[a,b]}\{L(P,f)\}$

$$=0$$

and $\overline{\int_a^b} f(x)\,dx = \text{glb}_{P\in P[a,b]}\{U(P,f)\}$

$$=b-a.$$

$\therefore \underline{\int_a^b} f(x)\,dx \neq \overline{\int_a^b} f(x)\,dx$

VECTOR SPACES Set-I

Field: Let F be a non-empty set and $+^n$ and $\times^n$ be binary operations on F. Then algebraic structure $(F, +, \cdot)$ is said to be field if the following properties are satisfied

(I) $(F, +)$ is an abelian group.

i) Closure property : $\forall a, b \in F \Rightarrow a+b \in F$

ii) Asso. prop :- $\forall a, b, c \in F \Rightarrow (a+b)+c = a+(b+c)$.

(iii) Existence of left identity : $\forall a \in F \ \exists 0 \in F$ s.t $0+a=a$

Here '0' is the identity elt.

(iv) Existence of left inverse :

$\forall a \in F, \exists -a \in F$ s.t $(-a)+a=0$ (left identity)

Here $-a$ is the inverse of a in F.

(v) comm. prop : $\forall a, b \in F$; $a+b=b+a$

(II) $(F, \cdot)$ is an abelian group

i) Closure prop : $\forall a, b \in F \Rightarrow a \cdot b \in F$

ii) Asso. prop : $\forall a, b, c \in F \Rightarrow (a \cdot b) \cdot c = a \cdot (b \cdot c)$

iii) Existence of left identity:

$\forall a \in F \ \exists 1 \in F$ s.t $1 \cdot a = a$.

Here 1 is the identity in F

(iv) Existence of left inverse:

$\forall 0 \neq a \in F \ \exists \frac{1}{a} \in F$ s.t $\frac{1}{a} \cdot a = 1$

$\therefore \frac{1}{a}$ is the inverse of a in F.

(v) comm. prop : $\forall a, b \in F$; $ab = ba$

III. $\times^n$ is distributive w.r.t $+^n$

i.e, $\forall a, b, c \in F \Rightarrow a \cdot (b+c) = ab + ac$.

Ex: $(\mathbb{Z}, +, \cdot)$ is not a field. Integers, not fractions ($a \cdot \frac{1}{a}$) not integer

$(\mathbb{Q}, +, \cdot)$, $(\mathbb{R}, +, \cdot)$, $(\mathbb{C}, +, \cdot)$ are fields.

$(\mathbb{Q}^*, +, \cdot)$, $(\mathbb{R}^*, +, \cdot)$, $(\mathbb{C}^*, +, \cdot)$ are not fields.

$a+b=a$ Identity

Defn **Subfield**: Let F be a field and $K \subseteq F$.
If K is a field w.r.t same binary operations in F then K is called subfield of F.

Ex - I is not a subfield of Q

Q is a subfield of R

R is " " C

I is not field because $\odot$ is fail inverse

Defns **Internal Composition**:

Let A be any set. If $a * b \in A$ $\forall a, b \in A$ then $*$ is said to be internal composition on A.

→ **External Composition**:

Let V and F be any two sets. If $a \circ \alpha \in V$ $\forall a \in F, \alpha \in V$ then 'o' is said to be an external composition in V over F

→ **Vector Space or Linear Space**

Let $(F, +, \cdot)$ be a field. The elts of F are called scalars. Let V be a non-empty set whose elts are called vectors. The following compositions are defined.

i) An internal composition in V called vector addition

(ii) An external composition in V over the field F called scalar multiplication.

If these compositions satisfy the following axioms then V is called vector space over the field F.

$\boxed{I}$ $(V, +)$ is an abelian group.

(i) Closure prop: $\forall \alpha, \beta \in V \Rightarrow \alpha + \beta \in V$

(ii) Asso. prop: $\forall \alpha, \beta, \gamma \in V \Rightarrow (\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$.

(iii) Existence of identity:

$\forall \alpha \in V, \exists 0 \in V$ s.t $\alpha + 0 = 0 + \alpha = \alpha$

Here the identity elt $0 \in V$ is called zero vector

(iv) Existence of inverse:

$\forall \alpha \in V, \exists -\alpha \in V$ s.t $\alpha + (-\alpha) = -\alpha + \alpha = 0$

(v) Comm. prop:

$\forall \alpha, \beta \in V \Rightarrow \alpha + \beta = \beta + \alpha$

$\boxed{II}$. The two compositions i.e, Scalar x^n and vector +

$\forall\ a, b \in F;\ \alpha, \beta \in V \Rightarrow$

(i) $a \cdot (\alpha + \beta) = a\alpha + a\beta$

(ii) $(a+b)\alpha = a\alpha + b\alpha$

(iii) $(ab)\alpha = a(b\alpha)$

iv) $1\alpha = \alpha$; 1 is the unity elt of the field F.

Note: $\boxed{1}$. When V is a vector space over field F then we shall denote it by V(F) and we say that V(F) is a vector space

$\boxed{2}$. If F is the field $\mathbb{R}$ of real no.s then V is called real vector space Similarly V(Q), V(C) are called rational, complex vector spaces respectively

Problems: (1) $V = I$, $F = Q$

Is V(F) a vector space ?

$I \subseteq Q$

$V \subseteq F$

$\therefore$ V is not a vec space

Sol^n Internal Composition:

$\forall \alpha, \beta \in I \Rightarrow \alpha + \beta \in I$

$\therefore$ vector $+^n$ is an internal composition on I.

External Composition:

$\forall a \in Q, \alpha \in I \Rightarrow a\alpha$ need not be an integer.

Ex $a = \frac{1}{2} \in Q$, $\alpha = 3 \in I \Rightarrow \frac{1}{2} \cdot 3 = \frac{3}{2} \notin I$.

$\therefore$ scalar x^n is not an external composition in I over Q

$\therefore I(Q)$ is not a vector space

Note: If $V \subseteq F$ then $V(F)$ is not a vector space.
(except $V = \{0\} \subseteq F$)

(2) $V = \mathbb{R}$; $F = \mathbb{Q}$ $\qquad Q \subseteq R$, $F \subseteq V$

Solⁿ $\forall \alpha, \beta \in \mathbb{R} \Rightarrow \alpha + \beta \in \mathbb{R}$.

and $\forall a \in \mathbb{Q} \subseteq \mathbb{R}$, $\alpha \in \mathbb{R} \Rightarrow a\alpha \in \mathbb{R}$

$\therefore$ Internal and external compositions are satisfied

$\boxed{I}$ i) $\forall \alpha, \beta \in \mathbb{R} \Rightarrow \alpha + \beta \in \mathbb{R}$

$\therefore$ Closure prop. is satisfied.

ii) $\forall \alpha, \beta, \gamma \in \mathbb{R}$

$\Rightarrow (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

$\therefore$ Asso. prop. is satisfied.

(iii) $\forall \alpha \in \mathbb{R}$ $\exists 0 \in \mathbb{R}$ s.t $\alpha + 0 = 0 + \alpha = \alpha$

$\therefore$ Identity prop is satisfied.

$\therefore 0$ is identity elt.

(iv) $\forall \alpha \in \mathbb{R}$ $\exists -\alpha \in \mathbb{R}$ s.t $\alpha + (-\alpha) = (-\alpha) + \alpha = 0$ (identity elt in $\mathbb{R}$)

$\therefore$ Inverse of α is $-\alpha$.

$\therefore$ Inverse prop. is satisfied.

(v) $\forall \alpha, \beta \in \mathbb{R} \Rightarrow \alpha + \beta = \beta + \alpha$

$\therefore$ Comm. prop. is satisfied

$\therefore (\mathbb{R}, +)$ is an abelian group.

$\boxed{II}$ $\forall a, b \in \mathbb{Q} \subseteq \mathbb{R}$; $\alpha, \beta \in \mathbb{R}$

(i) $a(\alpha + \beta) = a\alpha + a\beta$ (LDL in $\mathbb{R}$)

(ii) $(a+b)\alpha = a\alpha + b\alpha$ (RDL in $\mathbb{R}$)

(iii) $(ab)\alpha = a(b\alpha)$ (Asso. prop. in $\mathbb{R}$)

(iv) $1 \cdot \alpha = \alpha$ $\forall \alpha \in \mathbb{R}$. (1 is identity w.r.t x^n in $\mathbb{R}$)

$\therefore \mathbb{R}(\mathbb{Q})$ is vector space.

Note: If $F \subseteq V$ then $V(F)$ is a vector space.

Similarly $\mathbb{C}(\mathbb{Q})$, $\mathbb{C}(\mathbb{R})$ are also vector spaces

→ A field k can be regarded as a vector space over any subfield F of k. (3)

Sol^n: Given that k is a field and F is a subfield of k.

$\therefore$ F is also field w.r.t some b-o's defined in k

Let us consider the elts of k as vectors.

$\forall\ \alpha, \beta \in k \Rightarrow \alpha + \beta \in k$.

and let us consider the elts of the subfield F as scalars.

Now $a \in F \subseteq k$, $\alpha \in k \Rightarrow a\alpha \in k$.

$\therefore$ Internal and external Compositions are satisfied.

I. Since k is a field.

$\therefore$ $(k, +)$ is an abelian group

II. $\forall\ a, b \in F \subseteq k$; $\alpha, \beta \in k$

(i) $a(\alpha+\beta) = a\alpha + a\beta$ (LDL in k)

(ii) $(a+b)\alpha = a\alpha + b\alpha$ (RDL in k)

(iii) $(ab)\alpha = a(b\alpha)$ (Asso. prop in k)

(iv) $1\alpha = \alpha$ $\forall \alpha \in k$. and 1 is the identity elt of the subfield F.

($\because$ 1 is also identity elt of the field k).

$\therefore 1\alpha = \alpha$ $\forall \alpha \in k$.

$\therefore$ $K(F)$ is a vector space.

Note: If F is any field, then F itself is a vector space over the field F.

i.e, $F(F)$ is a vector space.

→ V = Set of all vectors and F is a field of real no's.

Sol^n $\forall \bar{\alpha}, \bar{\beta} \in V \Rightarrow \bar{\alpha} + \bar{\beta} \in V$ and

$a \in F, \bar{\alpha} \in V \Rightarrow a\bar{\alpha} \in V$

$\therefore$ Internal and external compositions are satisfied.

$\boxed{I}$ (i) $\forall \bar{\alpha}, \bar{\beta} \in V \Rightarrow \bar{\alpha} + \bar{\beta} \in V$

$\therefore$ Closure prop. is satisfied.

(ii) $\bar{\alpha}, \bar{\beta}, \bar{\gamma} \in V \Rightarrow (\bar{\alpha} + \bar{\beta}) + \bar{\gamma} = \bar{\alpha} + (\bar{\beta} + \bar{\gamma})$

$\therefore$ Asso. prop. is satisfied.

(iii) $\forall \bar{\alpha} \in V \ \exists \ \bar{0} \in V$ s.t $\bar{\alpha} + \bar{0} = \bar{0} + \bar{\alpha} = \bar{\alpha}$

$\therefore \bar{0}$ is the identity vector in V.

iv) $\forall \bar{\alpha} \in V \ \exists -\bar{\alpha} \in V$ s.t $\bar{\alpha} + (-\bar{\alpha}) = (-\bar{\alpha}) + \bar{\alpha} = \bar{0}$ (zero vector)

$\therefore$ inverse of $\bar{\alpha}$ is $-\bar{\alpha}$

(v) $\forall \bar{\alpha}, \bar{\beta} \in V \Rightarrow \bar{\alpha} + \bar{\beta} = \bar{\beta} + \bar{\alpha}$

$\therefore$ comm. prop. is satisfied.

$\boxed{II}$. $\forall a, b \in \mathbb{R}$; $\bar{\alpha}, \bar{\beta} \in V$

(i) $a(\bar{\alpha} + \bar{\beta}) = a\bar{\alpha} + a\bar{\beta}$

(ii) $(a+b)\bar{\alpha} = a\bar{\alpha} + b\bar{\alpha}$

(iii) $(ab)\bar{\alpha} = a(b\bar{\alpha})$

(iv) $1\bar{\alpha} = \bar{\alpha} \quad \forall \bar{\alpha} \in V$

$\therefore$ V(R) is a vector space.

→ V = Set of all $m \times n$ matrices with their elts as real numbers.

and $F = \mathbb{R}$.

Note : If V = the set of all $m \times n$ matrices with their elts as rational numbers and $F = \mathbb{R}$ then V(F) is not a vector space.

Because there is no external composition.

Ex: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \in V$; $\sqrt{7} \in \mathbb{R}$ then $\sqrt{7}A = \begin{bmatrix} \sqrt{7} & 2\sqrt{7} & 3\sqrt{7} \\ 0 & \sqrt{7} & 2\sqrt{7} \end{bmatrix} \notin V$

($\because$ the elts of resulting matrix are not rational numbers)

* Linear programming *

Introduction:

The linear programming originated during world war II (1939–1945), when the British and American Military management called upon a group of scientists to study and plan the war activities, so that maximum damages could be inflicted on the enemy camps at minimum cost and loss. Because of the success in military operations, it quickly spread in all phases of industry and government organisations.

It was first coined in 1940 by Mc Closky and Treften (by using the term Operations Research) in a small town, Bowdsey, of the United Kingdom.

In India, it came into existence in 1949, with opening of an operations research unit at the regional research laboratory at Hyderabad.

Linear programming problems:

In the competitive world of business and Industry, the decision maker wants to utilize his limited resources in a best possible manner. The limited resources may include material, money, time, man power, machine capacity etc. Linear programming can be viewed as a scientific approach that has evolved as an aid to a decision maker in business, industrial, agri-cultural, hospital, government and military organisations.

→ Now, suppose a vendor has a sum of Rs. 350 with which he wishes to purchase two types of tape, say, red and blue. Red tape costs Rs. 2 per metre and blue tape costs Rs. 3 per metre. He doesnot want to buy more than 40 metres of red tape. The question arises, " How many metres of red and blue tapes can he buy ?" Assume that he buys x metres of red tape and y metres of blue tape. The above problem can be stated mathematically as follows:

find x and y such that

$$2x + 3y \leq 350 \quad \text{---(i)}$$
$$x \leq 40 \quad \text{---(ii)}$$
$$x \geq 0,\ y \geq 0 \quad \text{---(iii)}$$

There can be a number of solution pairs (x, y). Now, further suppose that the vendor sells red tape at a profit of Rs. 0.75 per metre while blue tape at a profit of Rs. 1 per metre. Obviously, vendor likes to pick up a pair (x, y) which gives him the maximum profit. Now, the problem arises to find out the pair (x, y) which give maximum profit to the vendor, i.e, which will maximize $0.75x + 1 \cdot y$.

The above kind of problem is called a linear programming problem.

In a linear programming problem, we have constraints expressed in the form of linear inequalities. Therefore, to study linear programming, we must know the system of linear inequalities particularly their graphical solutions. Now we shall confine our discussion to the graphical solutions of inequalities. ②

— Closely linked with the system of linear inequalities is the theory of convex sets. This theory has very important applications not only in linear programming but also in Economics, Game theory etc.

Due to these applications, a great deal of work has been done to develope the theory of convex sets.

Thus, now we discuss the inequalities and convex sets. In addition, we need the notion of extreme points, Hyper-plane and Half spaces. These notions will be defined and explained with the help of some simple examples.

Inequalities and their graphs:

We know that a general equation of a line is

$$ax + by = c,$$

where a, b, c are real constants.

It is also called a linear eqn in two variables x and y.

If we put $y=0$, we get $x = c/a$, provided $a \neq 0$.

$x = c/a$ is the intercept of the line on x-axis.

Similarly on taking $x=0$, we get

$y = c/b$, $b \neq 0$.

as the intercept on y-axis.

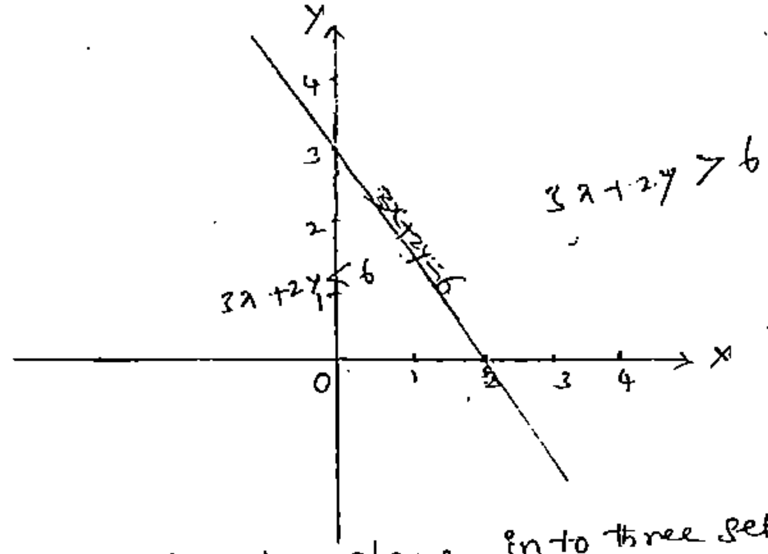
By joining the points $\left(\frac{c}{a}, 0\right)$ & $\left(0, \frac{c}{b}\right)$, $a \neq 0, b \neq 0$

we can trace the line.

<u>for example</u>,

Consider the line $3x+2y=6$.

Draw this line as shown in the figure



This line divides the plane into three sets or regions as shown in the figure.

These Regions may be described as follows:

(i) The set of points (x, y) such that

$3x+2y=6$

i.e, those points which lie on the line.

(ii) The set of points (x, y) such that

$3x+2y<6$.

- The set of points (x, y) for which $3x+2y<6$ is called the half plane bounded by the line $3x+2y=6$

(iii) The set of points (x, y) such that $3x+2y>6$. (3)
the other half plane bounded by the line $3x+2y=6$

- The inequality $3x+2y \leq 6$ represents the set of points (x,y) which either lie on the line $3x+2y=6$ or belong to the half-plane
$3x+2y<6$

Similarly, the inequality $3x+2y \geq 6$ represents the set of points (x,y) which either lie on the line $3x+2y=6$ or belong to the half-plane $3x+2y>6$.

- Most of the inequalities that we study here will be of the form

$ax+by \leq c$ or $ax+by \geq c$.

- In general we can say that a line $ax+by=c$ divides the xy-plane into <u>three regions</u> viz

(i) the set of points (x, y) such that $ax+by=c$, that is the line itself.

(ii) the set of points (x, y) such that $ax+by<c$ i.e, one of the half-planes bounded by the line.

(iii) the set of points (x,y) such that $ax+by>c$ the other half plane bounded by the line.

→ Draw the graph of the inequality $15x+8y \geq 60$

First consider the line.

$$15x+8y=60$$

If we take $y=0$, then $x=4$.

If $x=0$, then $y=15/2$

$\therefore$ we can trace the line by joining the points $(4,0)$ and $(0, 15/2)$.

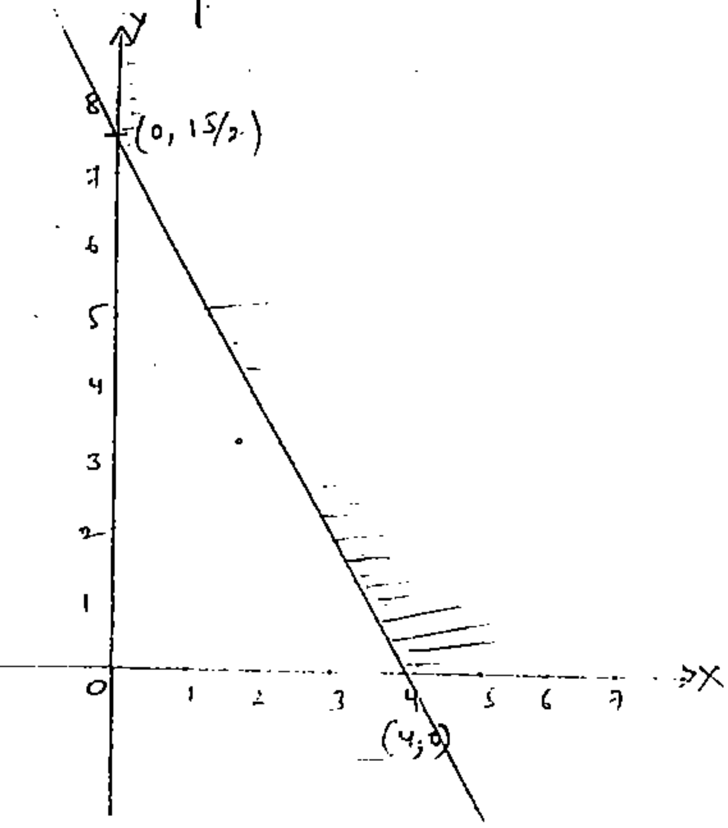
Let us now determine the location of the half plane.

for this, we put

$x=0$ and $y=0$

$15(0)+9(0)=0 \leq 60$

This shows that $15x+8y \geq 60$ is that half plane in which origin doesnot lie. Hence the shaded region as shown in the figure, represents the $15x+8y \geq 60$



GROUPS

Some sets of Numbers:

→ $N = \{1, 2, 3, \dots\}$

→ $W = \{0, 1, 2, 3, \dots\}$

→ $I = \{\dots -3, -2, -1, 0, 1, 2, \dots\}$

→ The set of all rational numbers

$$Q = \left\{\frac{p}{q} \,/\, p, q \in I;\ q \neq 0\right\}$$

→ Q' = The numbers which cannot be expressed in the form of $\frac{p}{q}$, $(q \neq 0)$ are known as irrational numbers.

Ex: $\sqrt{2}, \sqrt{3}, \sqrt{5}, e, 2+\sqrt{3}$ etc.

Note: (i) A rational number can be expressed either as a terminating decimal or a non-terminating recurring decimal.

(ii) An irrational number can be expressed as non-terminating non recurring decimal.

→ $R = Q \cup Q'$.

i.e, the set of all real numbers $\mathbb{R}$ which contains the set of rational and irrational numbers.

→ $C = \{a+ib \,/\, a, b \in \mathbb{R},\ i = \sqrt{-1}\}$

→ $I^+, Q^+, \mathbb{R}^+$ are the sets of +ve members of $I, Q, \mathbb{R}$ respectively.

→ $Z^*, Q^*, \mathbb{R}^*$ and C^* are the sets of non-zero members of I, Q, R and C respectively.

→ I_o and I_e are the sets of odd and even numbers of I

Some definitions

→ Let A and B be two sets. If $a \in A$ and $b \in B$, then (a, b) is called an ordered pair. 'a' is called the first component (co-ordinate) and 'b' is called the second component of the ordered pair (a, b).

→ Let A and B be two sets then $\{(a, b) \,/\, a \in A, b \in B\}$ is called the Cartesian product of A and B and is denoted by $A \times B$.

i.e, $A \times B = \{(a, b) \,/\, a \in A, b \in B\}$.

Ex: If $A = \{1, 2, 3\}$ and $B = \{3, 4\}$ then $A \times B = \{(1,3), (1,4), (2,3), (2,4), (3,3), (3,4)\}$.

Note: (1) If A and B are finite sets, $n(A) = m$ and $n(B) = k$ then $n(A \times B) = n(B \times A) = mk$.

(2) $A \times B \neq B \times A$ unless $A = B$

(3) If one of A and B is empty then $A \times B$ is also empty.
i.e, $A \times \phi = \phi$, $\phi \times B = \phi$.

→ If A and B are non-empty sets, then any subset of $A \times B$ is called a relation from A to B.

→ Let A be a non-empty set then subset of $A \times A$ is called a binary relation on A.

Ex: If $A = \{1, 2, 3\}$, $B = \{4, 5\}$:

$$A \times B = \{(1,4), (1,5), (2,4), (2,5), (3,4), (3,5)\}.$$

then $f = \{(1,4), (2,4)\} \subseteq A \times B$ is a relation from A to B.

and $A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$.

then $g = \{(1,1), (2,1), (3,2), (3,3)\} \subseteq A \times A$ is a binary relation on A.

Function:

Let A and B be two non-empty sets and f be a relation from A to B. If for each elt $a \in A$ $\exists$ a unique $b \in B$ s.t. $(a,b) \in f$ then f is called function (or mapping) from A to B. or A into B. It is denoted by $f: A \to B$.

Binary operation (or) Binary composition

→ Let S be a non-empty set, $S \times S = \{(a,b) / a \in S, b \in S\}$.
If $f: S \times S \to S$ (i.e, for each ord pair (a,b) of elts of S $\exists$ a uniquely defined an elt of S) then f is said to binary operation on S.

→ The image of the ordered pair (a,b) under the function f is denoted by $f((a,b))$ or afb.

Ex:- Let $\mathbb{R}$ be the set of all real numbers.

"+", "×", and "−" of any two real numbers is again a real number
i.e, $\forall a,b \in \mathbb{R} \Rightarrow a+b \in \mathbb{R}$, $a \times b \in \mathbb{R}$ and $a-b \in \mathbb{R}$.

Now we define
$+: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\times: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $-: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

are three mappings

$\therefore$ $+((a,b))$ or $a+b \in \mathbb{R}$.
$\times((a,b))$ or $a \times b \in \mathbb{R}$
$-((a,b))$ or $a-b \in \mathbb{R}$.

→ An operation which combines two elements of a set to give another elt of the same set is called binary operation.

Generally the b-o is denoted by 'o' or '*'

i.e, $\forall a,b \in S$ and * is an operation if $a * b \in S$ then * is called b-o on S.

Examples 1) $S = N, W, I, Q, \mathbb{R}, C$.
$\forall a,b \in S \Rightarrow a+b \in S$ and $a.b \in S$.

$\therefore +^n$ and $\times^n$ are b-o operations on S.

Here, $-$ is b-o on I, Q, R & C.

i.e, $a, b \in I, Q, R, C \Rightarrow a-b \in I, Q, R, C$.

but $-$ is not b-o on N and W

i.e, $a, b \in N, W \Rightarrow a-b \notin N, W$

$a, b \in S \Rightarrow a \div b \notin S$.

$\therefore \div$ is not a b-o on S.

but $a, b \in Q, R, C$

$\Rightarrow a \div b \in Q, R, C$ if $b \neq 0$

$\therefore \div$ is a b-o on Q, R, C.

(2) $S = Q^*, R^*, C^*$ (Non zero sets)

$a, b \in S \Rightarrow a \div b \in S$.

$\therefore \div$ is a b-o on S.

(3) Addition and subtraction are not b-os on the set of odd integers.

Types of binary operations

Closure operations:- A binary operation $*$ on a set 'S' is said to be closure if $a*b \in S \; \forall a, b \in S$.

Ex: (1) $S = N, W, I, Q, R, C$.

$\forall a, b \in S \Rightarrow a+b \in S$ &

$a \cdot b \in S$.

$\therefore$ S is closed w.r.t b-o. $+^n$ & $\times^n$

$\rightarrow a, b \in I, Q, R, C \Rightarrow a-b \in I, Q, R, C$

$\therefore$ I, Q, R, C are closed under b-o '$-$'.

but $a, b \in N, W \Rightarrow a-b \notin N, W$.

$\therefore$ N, W are not closed under b-o $\div$

(2) $S = Q, R, C$

$a, b \in S \Rightarrow a \div b \in S$ if $b \neq 0$.

$\therefore$ S is closed w.r.t b-o $\div$

(3) $S = Q^*, R^*, C^*$

$a, b \in S \Rightarrow a \div b \in S$

$\therefore$ S is closed w.r.t b-o $\div$

Commutative operations:

A binary operation $*$ on a set 'S' is Commutative if $a*b = b*a \; \forall a, b \in S$.

Ex: $S = N, W, I, Q, R, C$

$\forall a, b \in S \Rightarrow a+b = b+a$

$a \cdot b = b \cdot a$.

$\therefore$ S is commutative w.r.t b-o + & $\cdot$.

but $a, b \in S \Rightarrow a-b \neq b-a$

$\therefore$ S is not Commutative w.r.t b-o '$-$'

$\rightarrow S = Q^*, R^*, C^*$

$a, b \in S \Rightarrow a \div b \neq b \div a$.

$\therefore$ S is not commutative under $\div$

→ S = The set of all $m\times n$ matrices
$\forall A, B \in S \Rightarrow A+B = B+A$.
$\therefore S$ is commutative under b-o $+^n$.
but $A, B \in S \Rightarrow A-B \neq B-A$.

→ S = The set of all $n\times n$ matrices
$\forall A, B \in S \Rightarrow A+B = B+A$
$A-B \neq B-A$
$A\cdot B \neq B\cdot A$.

→ S = The set of all matrices with real entries.
The usual matrix addition, subtraction, $\times^n$ are not b-os on S.
[$\because A, B \in S \Rightarrow A+B, A-B$ & $A\cdot B$ are not defined]

→ S = The set of all vectors.
$\bar{a}, \bar{b} \in S \Rightarrow \bar{a}+\bar{b} = \bar{b}+\bar{a}$
$\bar{a}-\bar{b} \neq \bar{b}-\bar{a}$
$\bar{a}\times\bar{b} \neq \bar{b}\times\bar{a}$.
but the usual '$\cdot$' is not b-o on S. [$\because \bar{a}\cdot\bar{b}$ is scalar $\notin S$]

Associative operations:-

A binary operation $*$ on S is said to be associative if
$(a*b)*c = a*(b*c)$ $\forall a, b, c \in S$.

Ex:- $S = N, W, I, Q, R, C$.
$\forall a, b, c \in S \Rightarrow (a+b)+c = a+(b+c)$
$(a\cdot b)\cdot c = a\cdot(b\cdot c)$
but $(a-b)-c \neq a-(b-c)$.

→ S = The set of all $m\times n$ matrices
$\forall A, B, C \in S \Rightarrow (A+B)+C = A+(B+C)$
but $(A-B)-C \neq A-(B-C)$.

→ S = The set of all $n\times n$ matrices
$\forall A, B, C \in S \Rightarrow (A+B)+C = A+(B+C)$
$(A\cdot B)\cdot C = A\cdot(B\cdot C)$
but $(A-B)-C \neq A-(B-C)$.

→ S = The set of all vectors.
$\forall \bar{a}, \bar{b}, \bar{c} \in S \Rightarrow (\bar{a}+\bar{b})+\bar{c} = \bar{a}+(\bar{b}+\bar{c})$
$(\bar{a}-\bar{b})-\bar{c} \neq \bar{a}-(\bar{b}-\bar{c})$.

Identity element:

Let S be a non-empty set and $*$ be a b-o on S.
if $\exists$ an elt $b \in S$ s.t
$a*b = b*a = a$ $\forall a \in S$.
then b is called an identity element in S w.r.t b-o $*$.
→ The identity elt can be denoted by e. i.e, $b = e$.

Ex (1) $\exists$ an elt $b = 0 \notin N$
s.t $a+0 = 0+a = a$ $\forall a \in N$.
$\therefore 0$ is not an identity elt in N w.r.t b-o $+^n$
$\exists$ an elt $b = 1 \in N$ s.t
$a\cdot 1 = 1\cdot a = a$ $\forall a \in N$.
$\therefore 1$ is an identity elt in N w.r.t $\times^n$.

(2) $S = I, Q, R, C$.
$\exists$ an elt $b = 0 \in S$ s.t $a+0 = 0+a = a$ $\forall a \in S$.
$\exists b = 1 \in S$ s.t $a\cdot 1 = 1\cdot a = a$ $\forall a \in S$.

Note: In any number system identity elt w.r.t ordinary addition is zero. and w.r.t ordinary multiplication is 1.

(3) S = The set of all $m\times n$ matrices.

$A, B \in S \Rightarrow A+B = B+A = A$.

then $B = O$ (null matrix) is the identity elt in S

4) S = The set of all $n\times n$ matrices.

$A, B \in S \Rightarrow A\cdot B = B\cdot A = A$

then $B = I$ (unit matrix) is the identity ~~matrix~~ elt in S.

Inverse element:-

Let S be a non-empty set and $*$ be a b-o on S.

for each $a \in G$ $\exists$ an elt $b \in S$ s.t

$a * b = b * a = e$

then 'b' is said to be an inverse of 'a' and is denoted by a^{-1} i.e, $b = a^{-1}$.

Ex:-

for each $a \in \mathbb{Z}$ $\exists$ an elt $b = -a \in \mathbb{Z}$

s.t $a + (-a) = 0 = (-a) + a$.

$\therefore -a$ is an inverse of a in I

(2) for each $a \in G$ $\exists$ $b = \frac{1}{a} \notin I$ s.t $a \cdot \frac{1}{a} = 1 = \frac{1}{a} \cdot a$ $(a \neq 0)$

$\therefore$ $1/a$ is an inverse of a.

→ $S = \mathbb{Q}, \mathbb{R}, \mathbb{C}$: for each $a \in S$ $\exists$ $b = -a \in S$ s.t

$a + (-a) = (-a) + a = 0$

for each $a \in S$

$\exists$ $b = \frac{1}{a} \in S$ (if $a \neq 0$) s.t

$a \cdot \frac{1}{a} = 1 = \frac{1}{a} \cdot a$

$\therefore \frac{1}{a}$ is an inverse of a.

→ S = The set of all $m\times n$ matrices.

for each $A \in S$

$\exists$ $B = -A \in S$ s.t

$A + (-A) = O_{m\times n} = (-A) + A$

then $-A$ is the inverse of A

→ S = The set of all $n\times n$ matrices.

$\exists$ $B = A^{-1} = \frac{\operatorname{adj} A}{|A|}$ (if $|A| \neq 0$)

s.t $A \cdot A^{-1} = A^{-1} A = I$.

Note: In any number system the inverse of 'a' w.r.t ordinary addition is '$-a$' and the inverse of 'a' w.r.t ordinary multiplication is $1/a$.

Problems

Determine whether the binary operation $*$ defined is commutative and whether $*$ is associative.

1) $*$ defined on $\mathbb{Z}$ by letting $a * b = a - b$.

2) $*$ defined on Q by letting $a*b = ab+1$

3) $*$ defined on Q by letting $a*b = 2^{ab}$

4) $*$ defined on Z^+ by letting $a*b = a^b$

5) $*$ defined on Z by letting $a*b = a+b-ab$

6) $*$ defined on Q by letting $a*b = \frac{ab}{3}$

Determine whether the b-o $*$ defined is identity

7) $*$ defined on Q by letting $a*b = \frac{ab}{3}$

8) $*$ defined on Z by letting $a*b = a+b+ab$

Answers:

1. Since $a*b = a-b \quad \forall a,b \in Z$

$b*a = b-a$

$\therefore a*b \neq b*a$.

$\therefore *$ is not commutative in Z

Since $a*b = a-b \quad \forall a,b \in Z$

Let $a,b,c \in Z$

$\Rightarrow (a*b)*c = (a-b)*c$
$= a-b-c$

and $a*(b*c) = a*(b-c)$
$= a-(b-c)$
$= a-b+c$

$\therefore (a*b)*c \neq a*(b*c)$

$\therefore *$ is not associative in Z

(2) Not associative.

(3) not associative

both not a

(7) Since $a*b = \frac{ab}{3} \quad \forall a,b \in Q$

Let $a \in Q$, $e \in Q$ then

$a*e = a = e*a$

Now $a*e = a \Rightarrow \frac{ae}{3} = a$

$\Rightarrow \frac{ae}{3} - a = 0$

$\Rightarrow \frac{a}{3}(e-3) = 0$

$\Rightarrow e-3 = 0 \quad (\text{if } \frac{a}{3} \neq 0)$

$\Rightarrow e = 3$.

$\therefore a*e = \frac{ae}{3} = \frac{a}{3} \times 3$
$= a$
$= e*a$.

$\therefore$ 3 is the identity elt in Q

Algebraic structure.

G is a non-empty set and $*$ is a b-o on it, G together with the b-o is called an algebraic structure and is denoted by $(G,*)$.

(or)

A non-empty set equipped with one or more b-os is called an algebraic structure.

Ex: $(N,+)$, $(N,+,\cdot)$, $(I,+,\cdot,-)$ etc are algebraic structures.

but $(N,-)$, $(I,\div)$ etc are not algebraic structures.

Groupoid (or) Quasi group:

An algebraic structure $(G,*)$ is said to be groupoid if it satisfies the closure property.

i.e, $\forall a,b \in G \Rightarrow a*b \in G$

Ex: $(N,+)$, $(I,+)$ etc are groupoids.

ॐ Set-I

IMS
(INSTITUTE OF MATHEMATICAL SCIENCES)
INSTITUTE FOR IAS/IFoS EXAMINATION
NEW DELHI-110009
Mob: 09999197625

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ERRORS

INTRODUCTION:

Most of the numerical methods give answers that are approximations to the desired solutions. In this situation, it is important to measure the accuracy of the approximate solution compared to the actual solution. To find the accuracy we must have an idea of the possible errors that can arise in computational procedures. Now we shall introduce different forms of errors, which are common in numerical computations.

Numbers and their accuracy:

There are two kinds of numbers - exact and approximate numbers.

The numbers 1, 2, 3, ---- $\frac{1}{2}, \frac{3}{2}, \frac{3}{5}$, ---- etc are all exact and π, $\sqrt{2}$, e, ---- etc; written in this manner are also exact.

Approximate numbers are those that represent the numbers to a certain degree of accuracy i.e, an approximate number 'x' is a number that differs but slightly from an exact number X.

The approximate value of π is 3.1416 and to a better approximation it is 3.14159265 but not exact value.

significant digits (figures)

The digits that are used to express a number are called significant digits or significant figures.
- A significant digit of an approximate number is any non-zero digit in its decimal representation, or any zero lying between significant digits or used as place holder to indicate a retained place.

The digits 1, 2, 3, 4, 5, 6, 7, 8, 9 are significant digits. 0 is also a significant figure except when it is used to fix the decimal point, or to fill the places of unknown or discarded digits.

for eg:, in the number 0.0005010, the first four 0's are not significant digits, since they serve only to fix the position of the decimal point and indicate the place values of the other digits. The other two 0's are significant.

Two notational conventions which make clear how many digits of a given number are significant are given below.

1. The significant figure in a number in positional notation consists of:
- a) All non zero digits
b) zero digits which (i) lie between significant figures

(ii) lie to the right of decimal point, and at the same time to the right of a non-zero digit.

(iii) are specifically indicated to be significant.

[2]. The significant figure in a number written in scientific notation ($M \times 10^n$) consists of all the digits explicitly in M. (when n is negative)

- Significant figures are counted from left to right starting with the left most non zero digit.

Number	significant figure	No. of significant figures
37.89	3, 7, 8, 9	4
0.00082	8, 2	2
0.000620	6, 2, 0	3
3.506×10	3, 5, 0, 6	4
8×10^{-3}	8	1
3.14167	3, 1, 4, 1, 6, 7	6
2.35698	2, 3, 5, 6, 9, 8	6

Rounding-off numbers

Some times, we come across numbers with a large number of digits and in making calculations it might be necessary to cut them to a useable number of figures. This process is known as rounding-off and will be done by the following rule

To round-off a number to a significant digit, we shall discard all digits right of the n^{th} digit. If discarded number is

a) greater than '$\frac{1}{2}$' a unit, in the n^{th} place, the n^{th} digit would be increased by unity.

b) less than '$\frac{1}{2}$' a unit, in the n^{th} place, the n^{th} digit would be left unaltered.

c) exactly half a unit, in the n^{th} place, the n^{th} digit would be increased by unity if odd otherwise left unchanged.

Ex-1

Number	Round-off to Three figures	Round-off to four figures	Round-off to five figures
00.522341	00.522	00.5223	00.52234
93.2155	93.2	93.22	93.216
00.66666	00.667	00.6667	00.66667

Ex-2

Number	Round-off to four significant figures
9.6782	9.678
29.1568	29.16
8.24159	8.242
30.0567	30.06

→ In numerical analysis, the analysis of error is of great importance. Errors may occur at any stage of the process of solving a problem. By the error we mean the difference between the true value and the approximate value.

$\boxed{\therefore \text{Error} = \text{True value} - \text{Approximate value}}$

Ex: The true value of π is 3.14159265....

In some mensuration problems the value $\frac{22}{7}$ is commonly used as an approximation to π. What is the error in this approximation?

Sol: The true value of π is 3.14159265...

Now, we convert $\frac{22}{7}$ to decimal form, so that we can find the difference between the approximate value and true value. Then the approximate value of π is:

$$\frac{22}{7} = 3.1428571 4$$

$\therefore$ Error = True value − approximate value

$$= -0.0012644 9.$$

Note: In this case the error is negative.

Error can be positive or negative. We shall in general be interested in absolute value of the error which is defined as

$$|\text{error}| = |\text{True value} - \text{approximate value}|$$

In the above example, the absolute Error is:

$$|error| = |-0.00126449\ldots|$$
$$= 0.001264\ldots$$

→ Sometimes, when the true value is very large or very small we prefer the error by comparing it with the true value. This is known as Relative error and we define this as

$$|\text{Relative error}| = \left|\frac{\text{True value} - \text{approximate value}}{\text{True value}}\right|$$

i.e
$$|\text{Relative error}| = \left|\frac{\text{error}}{\text{True value}}\right|$$

→ The percentage error is 100 times the relative error. It is denoted by E_P: $E_P = E_R \times 100$

∴ The errors classified into 3 types.

1) Inherent error 2) Round off - error
3) Truncation error

(1) → The Inherent error is that quantity which is already present in the statement of the before its solution.

The inherent error arises either due to the simplified assumptions in the mathematical formulation of the problem or due to the physical measurements of the parameters of the problem.

(2) → Round-off error: when depicting even ~~rational numbers in decimal system or some other positional system~~, there may be

Differential Equations

Differential eqn: An equation involving derivatives of a dependent variable w.r.t one or more independent variables, is called a differential eqn.

Ex: (1) $\frac{dy}{dx} = x\log x$

(2) $\frac{d^2y}{dx^2} + 3x\left(\frac{dy}{dx}\right)^2 - 5y = \log x$

(3) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 12y = 5e^x + \sin x + x^3$

(4) $\left(\frac{d^3y}{dx^3}\right)^{2003} + P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)\,y = S(x)$.

(5) $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x\partial y} + \frac{\partial^2 z}{\partial y^2} = 0$

(6) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = kz$

Note: $\frac{dy}{dx} = y'$ (or) $y^{(1)}$ (or) y_1 ; $\frac{d^2y}{dx^2} = y''$ (or) $y^{(2)}$ (or) y_2

$\frac{d^3y}{dx^3} = y'''$ (or) $y^{(3)}$ (or) y_3 .... $\frac{d^ny}{dx^n} = y^{(n)}$ (or) y_n

Types of Differential equations:

(i) Ordinary Diff. eqn: An eqn involving the derivatives of a dependent variable w.r.t a single independent variable, is called an ordinary diff. eqn.

The above examples (1), (2), (3), & (4) are ordinary diff. eqns.

(ii). Partial Diff. eqn: An equation involving the derivatives of a dependent variable w.r.t more than one independent variable, is called a partial diff. eqn.

The above examples (5) & (6) are partial. diff. eqns.

Order of a Diff. eqn: The order of the highest order derivative involving in a differential eqn is called the order of the diff. eqn.

Ex: (1) $\frac{d^2y}{dx^2} + 4y = e^x$. is of 2^{nd} order.

(2) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 12y = 5e^x + \sin x + x^3$ is of second order.

(3) $\frac{d^2y}{dx^2} = k\left[1 + \left(\frac{dy}{dx}\right)^3\right]^{5/3}$ is of 2^{nd} order.

(4) $\log\left(\frac{dy}{dx}\right) = ax + by$ is of 1^{st} order.

(5) $\sin\left(\frac{dy}{dx}\right) = x^{100}$
order $= 1$

(6) $\cos\left(\frac{dy}{dx}\right) = x^{100}$
order $= 1$

Note [1]. A differential eqn of order one is of the form $f(x, y, \frac{dy}{dx}) = 0$

[2]. A diff. eqn of order two is of the form
$F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}) = 0$

[3]. In general, diff. eqn of order 'n' is of the form $F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots \frac{d^ny}{dx^n}) = 0$

Degree of a diff. eqn: The degree (i.e, power) of the highest order derivative involving in a diff. eqn, when the derivatives are made free from radicals (roots) and fractions, is called the degree of the diff. eqn.

(2)

Ex: (1) $x\left(\frac{d^2y}{dx^2}\right)^3 + y^2\left(\frac{dy}{dx}\right)^4 + xy = 0$ is of order 2 and degree 3.

(2) $\frac{d^2y}{dx^2} = k\left[1+\left(\frac{dy}{dx}\right)^3\right]^{5/3}$ (radical form)

cubing on both sides, we get,

$\left(\frac{d^2y}{dx^2}\right)^3 = k^3\left(1+\left(\frac{dy}{dx}\right)^3\right)^5$ order = 2, Degree = 3.

(3) $y\left(\frac{dy}{dx}\right) = \sqrt{x} + \frac{k}{dy/dx}$ (fractions form)

$\Rightarrow y\left(\frac{dy}{dx}\right)^2 = \sqrt{x}\left(\frac{dy}{dx}\right) + k$

$\therefore$ $\boxed{\text{order} = 1,\ \text{Degree} = 2}$

(4) $y = x\frac{dy}{dx}\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$\Rightarrow y^2 = x^2\left(\frac{dy}{dx}\right)^2\left(1+\left(\frac{dy}{dx}\right)^2\right)$

$\Rightarrow y^2 = x^2\left(\frac{dy}{dx}\right)^2 + x^2\left(\frac{dy}{dx}\right)^4.$

Order = 1
Degree = 4

(5) $\frac{d^3y}{dx^3} = \sqrt{1+\left(\frac{dy}{dx}\right)^5}$ squaring both sides

$\Rightarrow\left(\frac{d^3y}{dx^3}\right)^2 = 1+\left(\frac{dy}{dx}\right)^5$

order = 3
degree = 2

(6) $\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$ fractions form

$$\Rightarrow \quad e\left(\frac{d^2y}{dx^2}\right) = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}$$

$$\Rightarrow \quad e^2\left(\frac{d^2y}{dx^2}\right)^2 = \left[1+\left(\frac{dy}{dx}\right)^2\right]^3$$

Order $=2$

Degree $=2$

(7) $$(y''')^{4/3} + \sin x\left(\frac{dy}{dx}\right) + xy = x$$

$$\Rightarrow (y''')^{4/3} = x - \sin x\left(\frac{dy}{dx}\right) - xy$$

$$\Rightarrow (y''')^4 = \left(x - \sin x\left(\frac{dy}{dx}\right)^3 - xy\right)^3$$

order $=3$

Degree $=4$

(8) $$(y''')^{1/2} - 2(y')^{1/4} + xy = 0$$

$$\Rightarrow (y''')^{1/2} + xy = 2(y')^{1/4}$$

$$\Rightarrow \left((y''')^{1/2} + xy\right)^4 = 2^4\, y'$$

$$\Rightarrow \left[\left((y''')^{1/2} + xy\right)^2\right]^2 = 16y'$$

$$\Rightarrow \left[y''' + x^2y^2 + 2xy(y''')^{1/2}\right]^2 = 16y'$$

$$\Rightarrow (y''')^2 + x^4y^4 + 4x^2y^2y''' + 2x^2y^2y''' + 4x^3y^3(y''')^{1/2}$$
$$+ 4xy(y''')^{3/2} = 16y'$$

$$\Rightarrow \quad 4xy(y''')^{1/2}\left[y''' + x^2y^2\right] = \left[16y' - (y''')^2 - x^4y^4 - 6x^2y^2y'''\right]$$

squaring on both sides

$$16x^2y^2y'''(y''' + x^2y^2)^2 = \left(16y' - (y''')^2 - x^4y^4 - 6x^2y^2y'''\right)^2$$

$$\Rightarrow 16x^2y^2y'''\left[(y''')^2 + x^4y^4 + 2x^2y^2y'''\right] = (16y')^2 + (y''')^4 + \cdots$$

$\therefore$ order $=3$ & Degree $=4$

(9) $(y''')^{4/3} + (y')^{1/5} + y = 0$

Order $= 3$

Degree $= 60$

(10) $(y''')^{3/2} + (y''')^{2/3} = 0$

Order $= 3$

Degree $= 9.5$

Note:

[1] $y = \sin\left(\frac{dy}{dx}\right)$

order $= 1$

Degree = not defined.

Because $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\cdots$

$= \frac{dy}{dx} - \frac{1}{3!}\left(\frac{dy}{dx}\right)^3 + \frac{1}{5!}\left(\frac{dy}{dx}\right)^5 - \frac{1}{7!}\left(\frac{dy}{dx}\right)^7 + \cdots\cdots$

Here $x = \frac{dy}{dx}$

Similarly $\cos\left(\frac{dy}{dx}\right)$; $\tan\left(\frac{dy}{dx}\right)$; $\cot\left(\frac{dy}{dx}\right)$, $\sec\left(\frac{dy}{dx}\right)$ and $\operatorname{cosec}\left(\frac{dy}{dx}\right)$ degrees donot exist (or not defined.

[2]. $y = x\left(\frac{dy}{dx}\right) + \sin\left(\frac{dy}{dx}\right)$

Order $= 1$

Degree = not defined.

(3) $\frac{d^2y}{dx^2} + 2e^{x\frac{dy}{dx}} - 3y = x$

$\Rightarrow 2e^{x\frac{dy}{dx}} = x + 3y - \frac{d^2y}{dx^2}$

$$\Rightarrow x\frac{dy}{dx} = \log\left[\frac{1}{2}\left(x+3y-\frac{d^2y}{dx^2}\right)\right]$$

$\therefore$ order $= 2$

Degree = not defined.

(4) $3x^2 \frac{d^3y}{dx^3} - \sin\frac{d^2y}{dx^2} - \cos(xy) = 0$

(5) $(y''')^{1/3} + xy'' = 2005$

$$\Rightarrow (y''')^{1/3} = -xy'' + 2005$$

$$\Rightarrow y''' = (2005 - xy'')^3$$

Order $= 3$

Degree $= 1$

(6) $\left[y'' - 4(y')^2\right]^{5/2} = ay''$

$$\left[y'' - 4(y')^2\right]^5 = a(y'')^2$$

Order $= 2$

degree $= 5$