Unit-I :-

1- Complex Number System

2- Functions of complex Variables ((w = f(z))

3. Limit, Continuity, Differentiability (L, C, D) of w = f(z).

4. Analyticity and its properties

5. Singularities

Unit - II :-

Complex Integration :-

1- Fundamental of complex integration (curve & Defn of complex integration

2- Theorems on complex integration

3- C.I.F., C.I.F.D., C.I.F.H.D. (Cauchy Integral Formula), (Cauchy Integral formula for Derivative), (Cauchy Integral Formula for Higher Derivative)

4- ~~Liovill~~ Liouville's Theorem and its application *

Unit - III :-

Series and Expansions :-

1- Power series *

2- Taylor series

3- Laurent's expansion or, Laurent's series

4 - Application of Laurent's and Taylor's expansion

(a) Zeros of analytic $funct^n$

(b) Extension of Liouville's theorem

(c) Singularities re-visited

(d) Residue at $z = a$: $Res\,[f(z), a]$

Unit - IV

Special Types of $Funct^{ns}$

1. Meromorphic $funct^n$ / Rational $funct^n$

2.

2. Argument Theorem / Argument $Funct^n$

3. Rouche's Theorem *

Unit - V

Conformal mapping

1. Fundamental of Conformality

2. Bilinear / Mobious Transformation & Linear Fractional Transformation

3 - Maximum / Minimum Modulus Principle

4 - Schwarz's lemma and its application.

Unit-I
ch-I

Complex Number System

Date 06/11/2019

#
$x+1=0$ $\quad -1$

$2x-1=0$ $\quad \frac{1}{2}$

$x^2-2=0$ $\quad \sqrt{2}$

x^2+1 $\quad i=\sqrt{-1}$

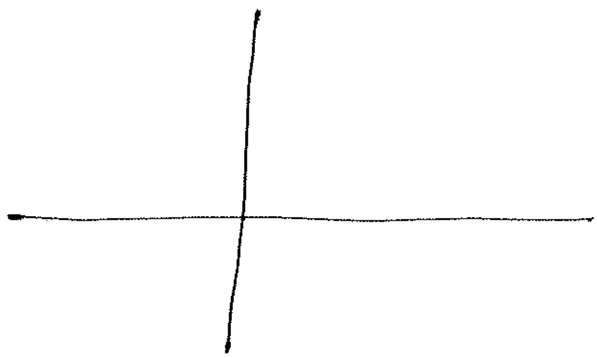
then, we have,

$$\mathbb{C} = \{x+iy : x, y \in \mathbb{R}\}$$

called the complex number system

For every complex no. $z=x+iy$ there is unique $(x,y) \in \mathbb{R}^2$, denoting the position of z.

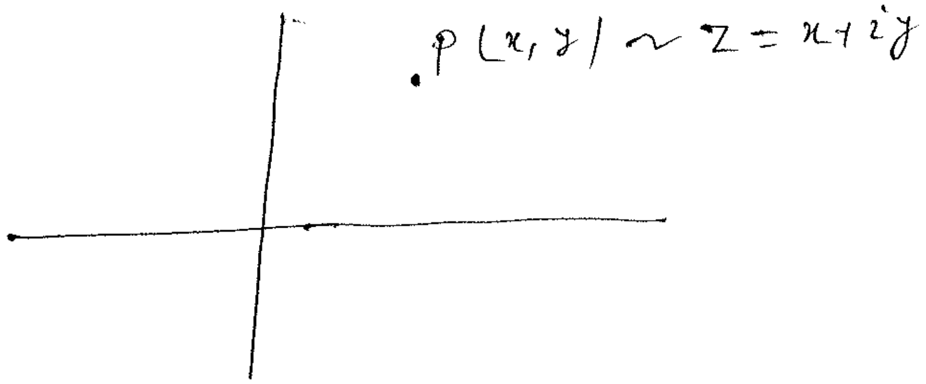
i.e, at every pt. of the cartesian plane there is a complex number and conversely



~~# When in the Cartesian plane~~

When at any point in the Cartesian plane we obtain a complex number, the plane is called Complex-plane or Argand-plane or,

oo, Z-plane.



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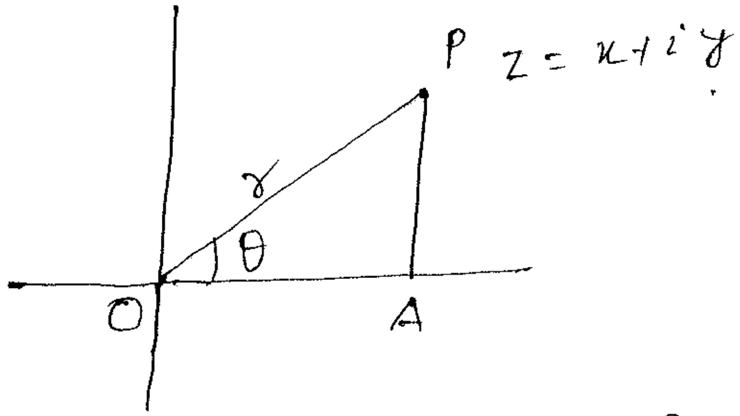


$x = r\cos\theta$, $y = r\sin\theta$

then $z = x+iy$ has equivalent form (r, θ)

called polar form of $z = x+iy$.

$(r, \theta) \in \mathbb{R}^2$

Note:- $z = 0 + i0$ has no polar form. $(\because r = 0)$

v.v.I

Principal Argument of z (Arg z) :-

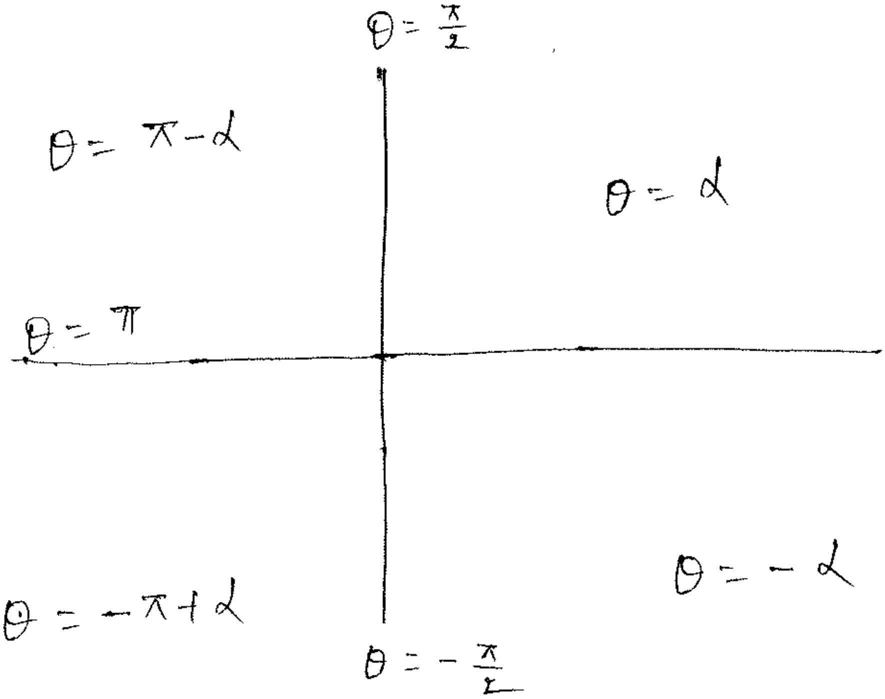
Arg z = Principal Argument

$= \theta \in (-\pi, \pi]$

Let $z = x+iy \neq 0$; $x \neq 0$

& $\alpha = \tan^{-1}\left|\frac{y}{x}\right|$

$\Rightarrow 0 \leq \alpha < \frac{\pi}{2}$



e.g. $Z = -1 + 4i$

$\rightarrow \tan\alpha = \left|\frac{4}{-1}\right| \Rightarrow \alpha = \tan^{-1}(4)$

$\therefore \arg Z = \pi - \tan^{-1}(4)$

$= 180° - 76°$

$= 104°$

$\arg z = \{ \operatorname{Arg} z + 2n\pi : n \in \mathbb{Z} \}$

$|z|$ = Absolute value of z

$= \sqrt{x^2 + y^2} = r.$

$$\theta = \begin{cases} 0 & x > 0, y = 0 \\ \tan^{-1}\left|\frac{y}{x}\right| & x > 0, y > 0 \\ \frac{\pi}{2} & x = 0, y > 0 \\ \pi - \tan^{-1}\left|\frac{y}{x}\right| & x < 0, y > 0 \end{cases}$$

$$
\begin{cases}
\pi & x<0,\ y=0 \\
-\pi+\tan^{-1}\left|\frac{y}{x}\right| & x<0,\ y<0 \\
-\frac{\pi}{2} & x=0,\ y<0 \\
-\tan^{-1}\left|\frac{y}{x}\right| & x>0,\ y<0
\end{cases}
$$

if $Arg\, z = \theta$, $|z| = r$

then $z = re^{i\theta}$

$= r\cos\theta + ir\sin\theta$

$= x + iy$

$$\left(\because e^{i\theta} = 1 + \frac{i\theta}{\lfloor 1} + \frac{(i\theta)^2}{\lfloor 2} + \cdots \right.$$

$$\left. = \cos\theta + i\sin\theta \right)$$

Log z :-

$\log z = \log(re^{i\theta})$

$= \log r + \log e^{i\theta}$

$= \log r + i\theta$

$= \log r + i(\theta + 2n\pi)$

$\Rightarrow \log z$ has infinite values

$\boxed{\log z = \log r + i\, Arg\, z}$.

Principal value of $\log z$.

Note:- $\log(z_1 z_2)$ may not be $\log z_1 + \log z_2$.

\# if $a, b \in \mathbb{C}$

a^b may have more than one values.

$$\boxed{\text{Principal value of } a^b = e^{b \cdot \log a}}$$

$$= e^{b \cdot (\text{P.V. of } \log a)}$$

e.g. Find principal value of $(1+i)^i$

$\longrightarrow$

$$\text{P.V. of } (1+i)^i = e^{i \log(1+i)}$$

$$= e^{i\left(\log\sqrt{2} + i\frac{\pi}{4}\right)}$$

$$= e^{-\frac{\pi}{4} + i\log\sqrt{2}}$$

$$= e^{-\frac{\pi}{4}} \cdot e^{i\log\sqrt{2}}$$

$$= e^{-\frac{\pi}{4}} \cos(\log\sqrt{2}) + i e^{-\frac{\pi}{4}} \sin\log\sqrt{2}$$

$\therefore \text{Re}\left(\text{P.V. of } (1+i)^i\right) = e^{-\frac{\pi}{4}} \cdot \cos\log\sqrt{2}$.

e.g.

$$\text{P.V. of } i^i = e^{i\log i}$$

$$= e^{i\left(i\frac{\pi}{2}\right)} \qquad \left(\because \log i = \log 1 + i\frac{\pi}{2}\right)$$

$$= e^{-\frac{\pi}{2}}$$

$$= \frac{1}{e^{\pi/2}}$$

Note:-

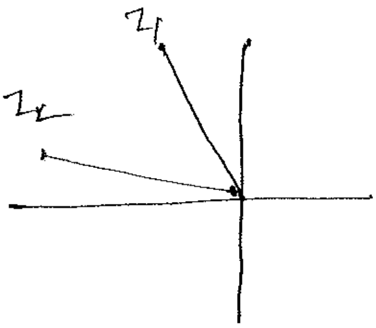
(i) $e^{ix} = \cos x + i\sin x \; ; x \in \mathbb{R}$

but $e^{iz} \neq \cos z + i \sin z \; ; \; z \in \mathbb{C}$.

(ii) $|e^{ix}| = 1 \; ; \; x \in \mathbb{R}$

but $|e^{iz}| \neq 1 \; ; \; z \in \mathbb{C}$ (in general).

e.g.:



Let $\operatorname{Arg} z_1 = \pi - \frac{\pi}{3}$

$\operatorname{Arg} z_2 = \pi - \frac{\pi}{6}$

$\log z_1 + \log z_2 = \log r_1 r_2 + i\underbrace{\left(2\pi - \frac{\pi}{2}\right)}_{\theta'' > \pi}$

but $\log z_1 z_2 = \log r + i\underbrace{\theta}_{\leq \pi}$.

Note:-

$e^z = e^{x+iy}$

$= e^x . e^{iy}$

$= e^x \cos y + i e^x \sin y$

$\operatorname{Re} e^z = e^x \cos y$

$\operatorname{Im} e^z = e^x \sin y$

Index:-

Vector Spaces and Sub-Spaces:-

External Composition:

Let $f: A \times B \longrightarrow C$

adopt '$*$' for $f(a,b) = a*b \quad \forall (a,b) \in A \times B$

i.e, if $f(a,b) = c$

We write

$$a*b = c$$

'$*$' is called an external composition. (b/w A & B in this order)

if $A = B = C \Rightarrow *$ is Binary Operation

or, internal composition.

e.g. $B = R[x] - \{\bar{0}\}$

$A = \mathbb{Q}$

$C = \mathbb{N} \cup \{0\}$

define $f: A \times B \longrightarrow C$

$$f(\alpha, p(x)) = \deg(\alpha \cdot p(x))$$

e.g. $f: \mathbb{Z} \times \mathbb{Q}^* \longrightarrow \mathbb{Q}^*$

$$f(\alpha, a) = a^\alpha$$

$$\alpha * a = a^\alpha$$

Vector Space:

Let $(V, +)$ be an abelian group [+ is a notation for the binary operation on V]

e.g. $V = \{ I, (12), (34), (12)(34) \}$

$(V, +)$

$(12) + (12) = I$

& $(F, +, \cdot)$ be a field.

define $F \times V \longrightarrow V$

an external composition,

$\alpha \in F, X \in V$

$\alpha \cdot X = f(\alpha, X) \in V$

then V forms vector space over F.

if (i) $\forall \alpha, \beta \in F, X \in V$

$(\alpha + \beta) \cdot X = \alpha \cdot X + \beta \cdot X$

(ii) $\forall \alpha \in F, X, Y \in V$

$\alpha \cdot (X + Y) = \alpha \cdot X + \alpha \cdot Y$

(iii) $\forall \alpha, \beta \in F, X \in V$

$(\alpha \cdot \beta) \cdot X = \alpha \cdot (\beta \cdot X)$

(iv) $1 \in F, X \in V$, 1 is unity of F

$1 \cdot X = X$.

The elements of V are called vectors and that of F are scalars, we may also think in ~~this~~ way, the elements of V are objects and that of F are multiples.

The identity of $(V, +)$ will be referred as zero vector and denoted by $\bar{0}$.

And $0 \in F$ is called '0' scalar.

Scalars are always ~~just~~ kept on the left of vectors.

~~# (i) $\bar{0} \cdot X = \bar{0} \quad \forall X \in V$,~~

(i) $0 \cdot X = \bar{0} \quad \forall X \in V, \; 0 \in F$

(ii) $c \cdot \bar{0} = \bar{0} \quad \forall c \in F$

(iii) $(-1) \cdot X = -X$, $-1 \in F$; -1 is the additive inverse of unity.

Ex:-

$$(V, +) = (\mathbb{R}^+, \cdot)$$

$$(F, +, \cdot) = (\mathbb{R}, +, \cdot)$$

$$\alpha \in F, \; X \in V = \mathbb{R}^+$$

$$\alpha \cdot X = X^\alpha$$

V is a vector space over $\mathbb{R}$.

Date
24/09/2019

Let $(V,+)$ be an abelian. ($+$ is the notation for the B.O. on V) & $(F,+,\cdot)$ be a field.

if $f : F \times V \longrightarrow V$ s.t.

$$f(\alpha, X) = \alpha \cdot X$$

then V forms V.S. over F.

if (i) $(\alpha+\beta)\cdot X = \alpha \cdot X + \beta \cdot X$

(ii) $\alpha \cdot (X+Y) = \alpha \cdot X + \alpha \cdot Y$

(iii) $(\alpha \cdot \beta)\cdot X = \alpha \cdot (\beta \cdot X)$

(iv) $1 \cdot X = X$

$\forall\, \alpha, \beta \in F$
& $X, Y \in V$

e.g.-1. $(V,+) = (\mathbb{R}^+, \cdot)$

$F = \mathbb{R}$

$f : F \times V \longrightarrow V$

$f(\alpha, X) = X^{\alpha}$

$\rightarrow$ Here, $(\mathbb{R}^+, \cdot)$ is an abelian gp ✓

$\mathbb{R}$ is a field ✓

(i) $(\alpha+\beta)\cdot X = X^{\alpha+\beta}$

$\alpha \cdot X + \beta \cdot X = X^{\alpha} \cdot X^{\beta} = X^{\alpha+\beta}$

$\therefore (\alpha+\beta)\cdot X = \alpha \cdot X + \beta \cdot X$

(ii) $\alpha \cdot (X+Y) = (X+Y)^{\alpha}$

$= (X \cdot Y)^{\alpha} = X^{\alpha} \cdot Y^{\alpha}$

$= [illegible] \cdot (X^{\alpha} \cdot Y^{\alpha})$

(iii) $(\alpha \cdot \beta) X = X^{\alpha\beta}$

$\therefore \alpha \cdot (\beta \cdot X) = \alpha \cdot X^{\beta}$

$= (X^{\beta})^{\alpha}$

$= X^{\beta\alpha} = X^{\alpha\beta}$

(iv). $1 \cdot X = X^{1} = X$

Ex 2:-

Let $(R, +, \cdot)$ be a ring

then if $V = R \Rightarrow (V, +)$ is an abelian group.

Let $F < R$ s.t. F is a field.

define $f : F \times V \longrightarrow V$

as $f(\alpha, X) = \alpha \cdot X$ wher '$\cdot$' is that of $(R, +, \cdot)$

f is well defined as $\alpha, X \in R$.

Now, $\forall\, \alpha, \beta \in F,\ X, Y \in V \Rightarrow \alpha, \beta, X, Y \in R$

as

and $(R, +, \cdot)$ is ring $\Rightarrow$ (i) ✓ $(\alpha + \beta) \cdot X = \alpha \cdot X + \beta \cdot X$

(ii) ✓ $\alpha \cdot (X + Y) = \alpha \cdot X + \alpha \cdot Y$

(iii) ✓ $(\alpha \cdot \beta) X = \alpha \cdot (\beta \cdot X)$

if the unity of F & R are same, then

same $1 \cdot X = X \quad \forall X \in R$.

$\Rightarrow$ "the additive gp of any ring R forms V.S. over its subring (which is field) if unity of R & the subring are same and ~~the~~ external composition is the multiplication of $(R, +, \cdot)$".

e.g. 3

Let $(R, +, \cdot) = (\mathbb{Z}_{10}, +_{10}, \times_{10})$

$(V, +) = (\mathbb{Z}_{10}, +_{10})$

$\Rightarrow$ V is abelian gp.

$F = \{0, 2, 4, 6, 8\}$

$\Rightarrow (F, +_{10}, \times_{10})$ is field

but V doesn't form V.S. over F as unity of R is 1 and that of F is 6.

Ex 4:

$R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$

$(R, +, \cdot)$ is a ring

$\Rightarrow V = R$, then $(V, +)$ is an abelian gp.

$F = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} : a \in \mathbb{R} \right\}$

$(F, +, \cdot)$ is a field and $F \subset R$

if '.' of $(R, +, \cdot)$ is taken the external compo., V doesn't form V.S. over F.

as $1 = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \in F$, $X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in V$

$1 \cdot X = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \neq X.$

Jai Maa Saraswati

Set :— A collection of well defined distinct objects is defined as a set.

By well defined we mean, there is no confusion or ambiguity regarding the inclusion or the exclusion of the object.

* If cardinality of A is n i.e, $|A| = n$ then $|P(A)| = 2^n$.

Proof :— Let $|A| = n$, then

No. of subsets of A having no element $= {}^nC_0$

" " 1 " $= {}^nC_1$

" " "

No. of subsets of A having n element $= {}^nC_n$

We have,

$$ {}^nC_0 + {}^nC_1 + \cdots + {}^nC_n = |P(A)| \quad \text{— (I)} $$

By Binomial theorem,

$$ (1+x)^n = {}^nC_0 + {}^nC_1 x + \cdots + {}^nC_n x^n \quad \text{— (II)} $$

On comparing (I) & (II),

we get $x = 1$

$$ \therefore |P(A)| = (1+1)^n = 2^n. $$

IInd method:-

Let $A = \{x_1, x_2, \ldots, x_n\}$

inclusion exclusion

2 ways

$$\therefore \; |P(A)| = \underbrace{2 \times 2 \times \cdots \times 2}_{n \text{ times}}$$

$$= 2^n.$$

Q. Let A be the set ~~having~~ containing $(2n+1)$ elements, then the number of subsets of A having more than n elements is

(a) 2^{n-1} (b) 2^n

(c) 2^{n+1} (d) 2^{2n}

Soln:-

$${}^{2n+1}C_0 + {}^{2n+1}C_1 + \cdots + {}^{2n+1}C_n + \underbrace{{}^{2n+1}C_{n+1} + \cdots + {}^{2n+1}C_{2n+1}}_{\alpha}$$

$$= 2^{2n+1}$$

$$\Rightarrow \alpha + \alpha = 2^{2n+1} \qquad \left\{ \because \; {}^{2n+1}C_n = {}^{2n+1}C_{n+1}, \quad {}^{2n+1}C_0 = {}^{2n+1}C_{2n+1} \right.$$

$$\Rightarrow 2\alpha = 2^{2n+1}$$

$$\alpha = 2^{2n}$$

Ans - (d)

Cartesian Product:-

Let A & B be any two non-empty sets

define $A \times B = \{(a, b) : a \in A, b \in B\}$

Then, $A \times B$ is defined as Cartesian product of A & B.

If either A or B is empty, then

$$A \times B = \phi .$$

Properties:

i) If $|A| = m$, $|B| = n$
then $|A \times B| = m \times n = mn$

ii) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

iii) $(A \cap B) \times C = (A \times C) \cap (B \times C)$.

iv) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Q. If A and B have 99 elements in common then the no. of elements common to $A \times B$ and $B \times A$ is.

(i) 100 (ii) 2^{99} (iii) 99^2 (iv) 101.

Sol^n:- $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$

$$= 99 \times 99$$

$$= 99^2$$

Relation:-

Let A and B be any two sets any subset of $A \times B$ is defined as relation from A to B.

Two relations are said to be ~~different~~ distinct if and only if they correspond to different subsets.

if $|A| = m$ and $|B| = n$

Then, No. of relations from A to B $= 2^{mn}$

$$= 2^{|A| \times |B|}.$$

i.e, ~~Relation~~.

Types of relⁿs on the set A:

1) Identity relation:

Let A be any set and $S \subset A \times A$ is defined as identity relation if every element of A is related to itself only.

i.e, $I = \{(a, a) : a \in A\}$

e.g. $I_1 = \{(1,2), (1,1), (2,2), (3,3)\}$ ✗

$I_2 = \{(1,1), (2,2)\}$ ✗

$I_3 = \{(1,1), (2,2), (3,3)\}$ ✓

Note:- Identity relation is always unique.

2) <u>Reflexive reln:</u>

Let A be any set and $S \subseteq A \times A$ then S is said to be reflexive reln if

$I \subseteq S$.

e.g. $S = \{(1,1), (2,2), (1,2)\}$ ✗

$S = \{(1,1), (2,2), (3,3), (1,2)\}$ ✓

$S = \{(1,1), (2,2), (3,3)\}$ ✓

If $|A| = n$, then

Total no. of reflexive reln on A $= 2^{n^2-n}$

Proof:-

No. of choices for elements of the type (a_i, a_i) is 1

" " " $(a_i, a_j) = 2$ where $i \neq j$

$\therefore$ No. of reflexive reln $= \underbrace{1 \times 1 \times \cdots \times 1}_{n \text{ times}} \times \underbrace{2 \times 2 \times \cdots \times 2}_{n^2-n \text{ times}}$

$= 2^{n^2-n}$.

3) <u>Irreflexive reln:-</u>

Let A be any set and $S \subseteq A \times A$ then S is said to be irreflexive reln if

$I \cap S = \phi$.

e.g. $A = \{1, 2, 3\}$

$S = \{(1,1), (1,2), (2,3)\}$ ✗

$S = \{(1,2), (3,2)\}$ ✓

If $|A| = n$, then

Total no. of irreflexive rel^{n}s $= 2^{n^2-n}$

Proof:-

No. of choices for the elements of the type $(a_i, a_i) = 1$

" " $(a_i, a_j) = 2$

$\therefore$ No. of irreflexive rel^{n}s

$$= \underbrace{1 \times 1 \times \cdots \times 1}_{n \text{ times}} \times \underbrace{2 \times 2 \times \cdots \times 2}_{n^2-n \text{ times}}$$

$$= 2^{n^2-n}.$$

4) <u>Symmetric rel^n:-</u>

Let A be a rel^n and $S \subseteq A \times A$ then S is said to be symmetric rel^n if

$(a, b) \in S \Rightarrow (b, a) \in S$

i.e, if $a \sim b$ then $b \sim a$.

Eg:- $A = \{1, 2, 3\}$

$S = \{(1,1), (2,2), (2,3)\}$ ✗

$S = \{(1,1), (2,3), (3,2)\}$

If $|A| = n$, then

Total no. of symmetric relⁿs $= 2^{\frac{n(n+1)}{2}}$

Proof :-

No. of choices for the elements of type $(a_i, a_i) = 2$

" " of the pair (a_i, a_j) & $(a_j, a_i) = 2$ where $i \neq j$

$\therefore$ No. of symmetric relⁿs

$$= \underbrace{2 \times 2 \times \cdots \times 2}_{n \text{ times}} \times \underbrace{2 \times 2 \times \cdots \times 2}_{\frac{n^2-n}{2} \text{ times}}$$

$$= 2^n \times 2^{\frac{n^2-n}{2}}$$

$$= 2^{n + \frac{n^2-n}{2}}$$

$$= 2^{\frac{n^2+n}{2}}$$

$$= 2^{\frac{n(n+1)}{2}}$$

$$= 2^{\Sigma n}$$

$(a_i, a_j) \notin (a_j, a_i) \in S$

Or, Neither

$(a_i, a_j) \notin (a_j, a_i) \in S$

16/07/2019

5) Asymmetric relations:-

Let A be any set and $S \subseteq A \times A$, then S is said to be asymmetric relation if $(a,b) \in S \Rightarrow (b,a) \notin S$

Note:- If rel^n is asymmetric then it is ~~not~~ irreflexive.

i.e, Asymmetric $\Rightarrow$ Irreflexive Rel^n

Eg:- $A = \{1, 2, 3\}$

$S = \{(1,1), (2,3)\}$ ✗

$S = \{(2,3), (3,2)\}$ ✗

$S = \{(2,3)\}$ ✓

Syllabus

5–6

18 – 23

1. Introduction
2. Order & Degree of the D. E.
3. Formation of O.D.E.
4. First Order and First degree D.E.
   (i) Separation of Variable
   (ii) Reducible to separation of variable
   (iii) Homogeneous D. E.
   (iv) Reducible to Hom^n. D. E.
   (v) Exact + I. F.
   (vi) Reducible to Exact D. E.
   (vii) Linear D. E.
   (viii) Reducible L. D. E.
   (ix) Bernoulli Eq^n
5. First order ~~and~~ but not first degree (Singular Sol^n)
6. Linear D. E. with constant Coefficient
7. Linear D. E. with variable coefficient
8. Wronskian + Zeros — 1
9. Uniqueness & Existence — 1
10. Boundary Value Problem — 1
11. System of O.D.E. — 1
12. Green funct^n.

Function

- Algebraic
  - Poly^n d
  - Rational
  - irrational
  - Piecewise
    - Modulus
    - Signum
    - G.I.F.
    - L.I.F.
    - F.P.F.
- Transcedental
  - Trigonometric
  - Inverse fn
  - Exp^n
  - log^n
  - Geometric

Dependent Variable & Independent Variable:

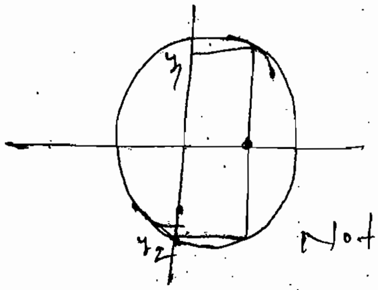
The variable whose value is assigned is called independent variable and the variable whose value is obtained corresponding to assigned value is called dependent variable.

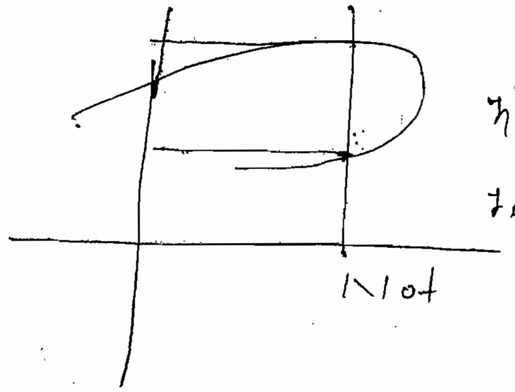
Function:

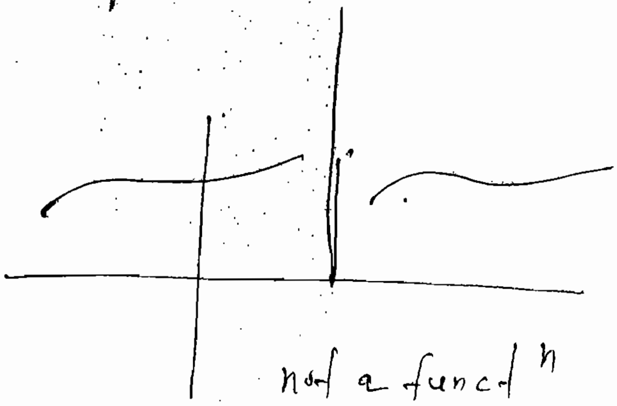
(i) Every element in domain having a unique image in codomain.

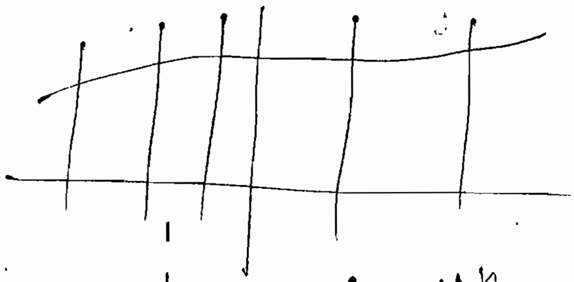
(ii) $f : A \to B$

$\forall x \in A$: $\exists$ unique $y \in B$ such that $y = f(x)$.









(iii) Graphical def^n:

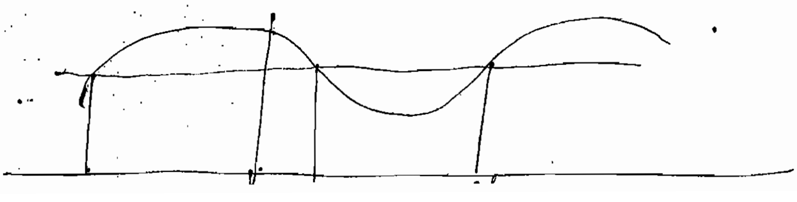
A mapping $f : A \to B$ is called a funct^n if any line passing through domain and $\parallel$ to y-axis should intersect the curve $y = f(x)$ exactly once.

1-1 funct^n:

$f : A \to B : \to$ 1-1

$\Leftrightarrow$ if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

or $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$



Since $f(x_1) = f(x_2) = f(x_3) \not\Rightarrow x_1 = x_2 = x_3$

'A funct^n $f: A \to B$ is called 1-1, iff any line passing through co-domain and $\parallel$ to x-axis should intersect the curve $y = f(x)$ <u>at most</u> once.

Onto funct^n:

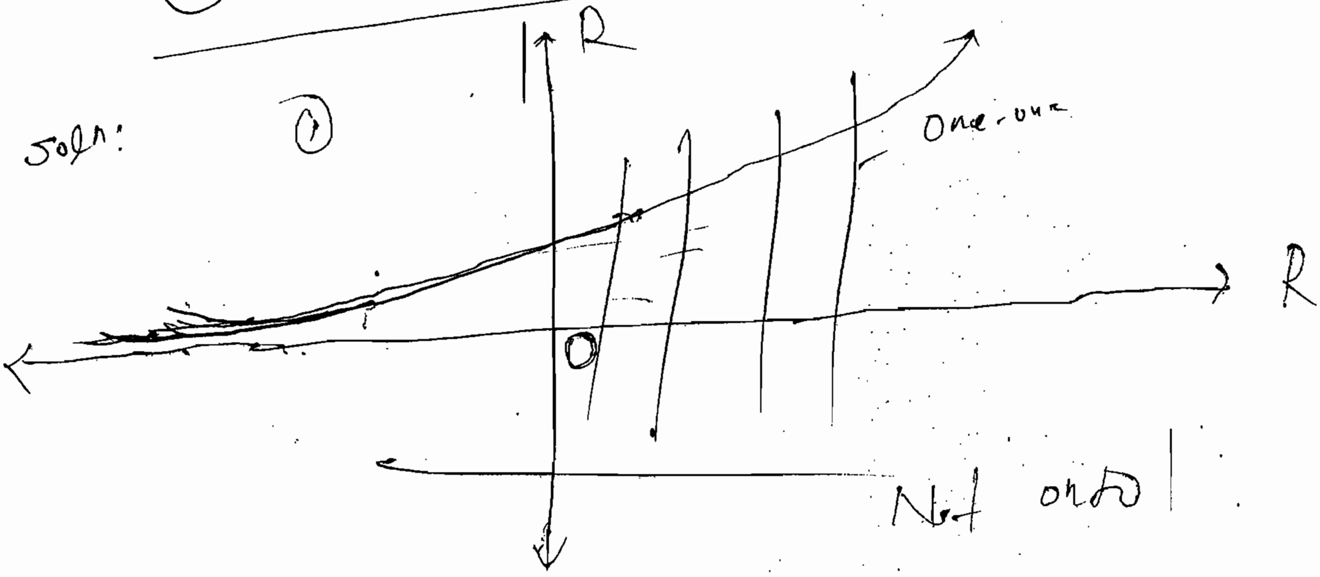
Range = Codomain

A funct^n $f: A \to B$ is called onto iff any line passing through co-domain and $\parallel$ x-axis should ~~should~~ intersect the curve $y = f(x)$ <u>at least</u> once.

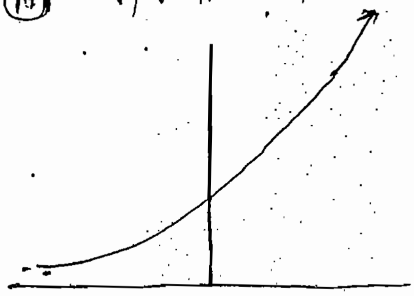
Ex:- $f(x) = e^x$

	functi	1-1	Onto
	✓	✓	✗
(i) $f: R \to R$	✓	✓	✓
(ii) $f: R \to R^+$	✓	✓	✗
(iii) $f: R^+ \to R$	✓	✓	✗
(v) $f: R^+ \to R^+$			

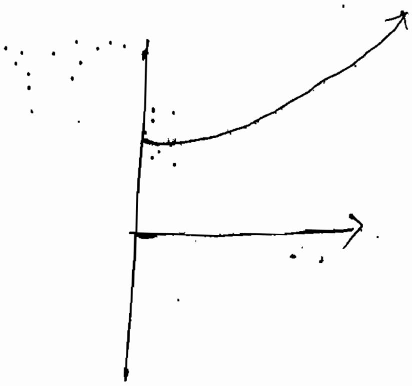
Soln: (i)



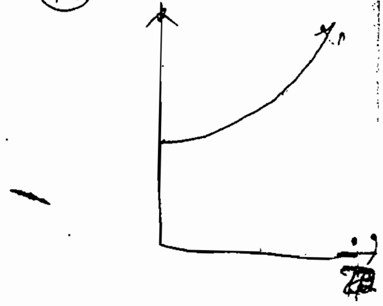
(ii) $f: \mathbb{R} \to \mathbb{R}^+$



(iii) $f: \mathbb{R}^+ \to \mathbb{R}$



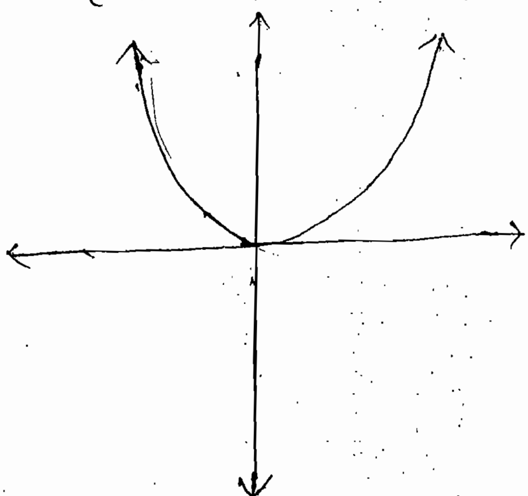
(iv) $f: \mathbb{R} \to \mathbb{R}^-$



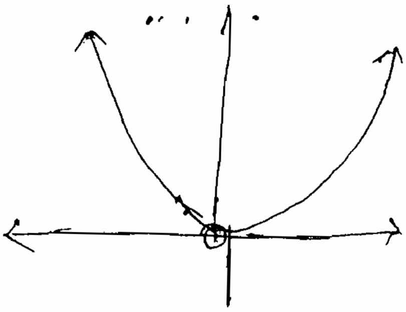
Ex 2:

$f(x) = x^2$

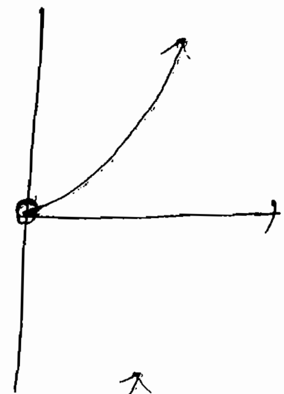
(i) $f: \mathbb{R} \to \mathbb{R}$

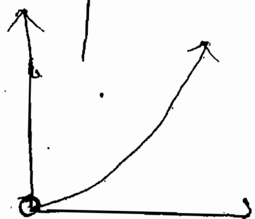


(ii) $f: \mathbb{R} \to \mathbb{R}^+$



(iii) $f: \mathbb{R}^+ \to \mathbb{R}^+$

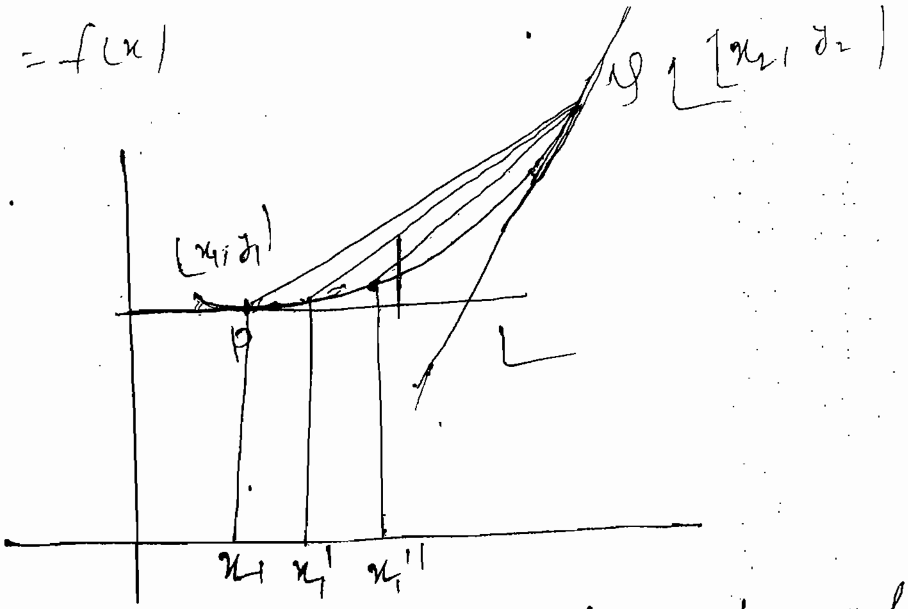




(iv) $f: \mathbb{R}^+ \to \mathbb{R}^+$

$f(x) = x^2$	Funct^n	1 - 1	Onto
(i) $f: \mathbb{R} \to \mathbb{R}$	✓	✗	✗
(ii) $f: \mathbb{R} \to \mathbb{R}^+$	✗	✗	✗
(iii) $f: \mathbb{R}^+ \to \mathbb{R}$	✓	✓	✗
(iv) $f: \mathbb{R}^+ \to \mathbb{R}$	✓	✓	✓

$y = f(x)$



$\Delta y = y_2 - y_1$ = distance b/w y_1 & y_2

$\Delta x = x_2 - x_1$ = " " x_1 & x_2

$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$ = slope of L

~~$\frac{\Delta y}{\Delta x} = y_2 -$~~

$\frac{dy}{dx} = \lim_{x_1 \to x_2} \frac{\Delta y}{\Delta x}$

= slope of L when $P \to Q$

$\Rightarrow \frac{dy}{dx}$ = slope of tangent at point Q

Algebraic term — Graphical term

rate of change $\to \frac{dy}{dx}$

Date 21/08/2019

Differential Equation:

Any eqⁿ between dependent variable & independent variable and ~~contain total~~ derivative of D.V. of w.r.t. independent variable is called D.E.

Eg:-

(i) $y = f(x)$

$$\frac{dy}{dx} + y = \sin x$$

$$\frac{d^2y}{dx^2} + f(x)y = e^x$$

(ii) $y_1 = f(x)$

$y_2 = g(x)$

$$\frac{dy_1}{dx} + y_2 = \sin x$$

$$\frac{d^2y}{dx^2} + \frac{dy_2}{dx} = 0$$

(iii) $z = f(x, y)$

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \sin(x+y)$$

Simple P.D.E.

(IV) $z_1 = f(x, y)$

$z_2 = g(x, y)$

$$\frac{\partial^2 z_1}{\partial x^2} + \frac{\partial^2 z_2}{\partial x \partial y} = 0$$

$$\frac{\partial z_1}{\partial y} + \frac{\partial z_2}{\partial x} = e^{x+y}$$

System of P.D.E.

Ordinary differential eqⁿ:

Any differential eqⁿ in which unique independent variable and total derivative of D.V. w.r.t. I.V. is called O.D.E.

Eg:- $y = f(x)$

$$\frac{dy}{dx} + y = \sin x$$

$$\frac{d^2y}{dx^2} + f(x)\,y = e^x$$

Simple O.D.E

$y_1 = f(x)$

$y_2 = g(x)$

$$\frac{dy_1}{dx} + y_2 = \sin x$$

$$\frac{d^2y_1}{dx^2} + \frac{dy_2}{dx} = 0$$

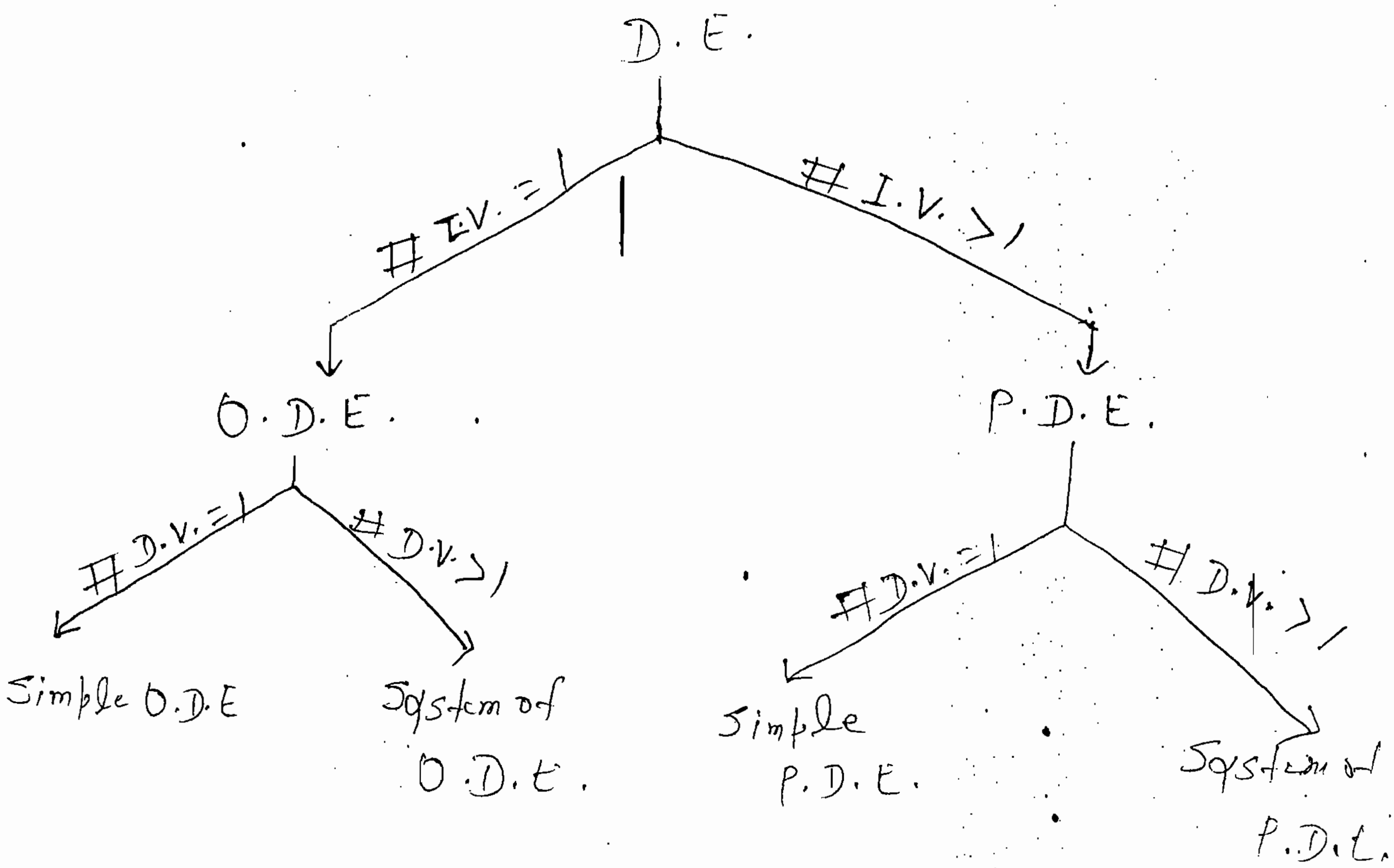
System of O.D.E.

Partial Differential eqn.:

Any D.E. contain partial derivative is called partial D.E.

Eg:

Classification of D.E.:



No. of Ques	Marks
6-7	8-22

1. Formation of 1st order P.D.E. → by eliminating of arbitrary constant → by eliminating of arb. funct^n (1)

2. First order P.D.E.

(i) Linear
(ii) Semi-linear
(iii) Quassi-linear → Lagrange's method (1)
(iv) Non-linear → Charpit's Method (1)

3. Integral surface passing through a given curve → Quassi linear → Non-linear

4. Surface orthogonal to surface
5. Classification of 2nd Order P.D.E. → Hyperbolic → parabolic → eliptic (1)
6. P.D.E. with constant coefficient. (1)
7. Separation of variable → Heat eq^n → Wave eq^n → Laplace Eq^n. — 2 or 3

Note:-

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = p,$$

$$\frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \partial y} = s; \quad \frac{\partial^2 z}{\partial y^2} = t.$$

Date
21/10/2019

Def. Partial Differential Equation:

An eqⁿ which contains partial derivative of dependent variable w.r.t. two or more than two independent variable, is called P.D.E.

Note:–

No. of D.V. = 1

No. of I.V. ≥ 2

e.g.

(i) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = x + y$

(ii) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$

Classification of 1st order P.D.E.

1. Linear:–

A 1st order P.D.E. is said to be linear, if it is linear in p, q & z and of the form

$$P(x,y)\,p + Q(x,y)\,q = R(x,y)\,z + S(x,y).$$

e.g.

(i) $xp + yq = xyz + x^3y^3$

(ii) $p + q = xyz$

(iii) $x^2p + y^2q = 1$

Note:–

(i) If $S(x,y) = 0$, then it is called homogⁿ linear P.D.E.

(ii) If $S(x,y) \neq 0$, then it is called non-homogⁿ linear P.D.E.

2. Semi linear :-

A 1st order P.D.E. is said to be semi linear if it is linear in p & q but not necessary in z and of the form

$$P(x,y)\,p + Q(x,y) = R(x,y,z).$$

e.g.

(i) $p+q = xyz$

(ii) $x^2 p + y^2 q = x^2y^2z^2 \longrightarrow$ Semi linear

3. Quassi linear :-

A 1st order P.D.E. is said to be Quassi linear, if it is linear in ~~p & q~~ p & q and of the form

$$P(x,y,z)\,p + Q(x,y,z)\,q = R(x,y,z)$$

e.g.

(i) $p+q = xyz$

(ii) $(y-z)p + (z-x)q = x-y$

(iii) $y^2 p + (x-z)q = z^2$

**

1. $P(x,y)\,p + Q(x,y)\,q = R(x,y)z + S(x,y)$ - linear
2. $P(x,y)\,p + Q(x,y)\,q = R(x,y,z)$ ——— semi-linear
3. $P(x,y,z)\,p + Q(x,y,z)\,q = R(x,y,z)$

Note :-

Linear $\subset$ Semi-linear $\subset$ Quassi linear.

4. Non-linear

A 1st order P.D.E. is said to be non-linear if it doesn't come under any one of the above types.

e.g. (i) $pq = z^2$

(ii) $p^2 q^2 = 1$

Formation of 1st order P.D.E. :—

by eliminating of arb^r constant

by eliminating of arb^r funct^n

1. By eliminating of arb^r constant :—

Let $F(x, y, z; a, b) = 0$ —— (1)

where a & b are arb^r constant & z is dependent variable and x & y are independent variable.

Differentiate partially w.r.t. x & w.r.t. y

$F_1(x, y, z, \frac{\partial z}{\partial x}, a, b) = 0$ —— (2)

$F_2(x, y, z, \frac{\partial z}{\partial y}, a, b) = 0$ —— (3)

We eliminate a & b from eq^n (1), (2) & (3), then we get $\boxed{\psi(x, y, z, p, q) = 0}$

e.g. ~~The P~~

Eg: The P.D.E. representing the set of all sphere of unit radius with centre in the xy-plane is

(i) $1+p^2+q^2=0$ (ii) $yp-x-q=0$

(iii) $z^2(1+p^2+q^2)=1$ (iv) N.O.T.

$\rightarrow$ $$(x-a)^2+(y-b)^2+z^2=1$$

$$\cancel{2(x-a)=0}$$

$$2(x-a)+2zp=0$$

$$\Rightarrow x-a=-zp$$

$$\text{Hly, } y-b=-zq$$

$$\therefore z^2p^2+z^2q^2+z^2=1$$

$$\Rightarrow z^2(p^2+q^2+1)=1$$

GATE
Eg:- The P.D.E. for $z^2(1+a^3)=8(x+ay+b)^3$ is

(i) $z=p^2+q^2$ (ii) $27z=p^2+q^2$

(iii) $27z=p^3+q^3$ (iv) $z=p^3+q^3$

$\rightarrow$ $$z^2(1+a^3)=8(x+ay+b)^3 \quad \text{--- (I)}$$

$$2(1+a^3)zp=24(x+ay+b)^2 \quad \text{--- (II)}$$

$$2(1+a^3)zq=24a(x+ay+b)^2 \quad \text{--- (III)}$$

$$\therefore \cancel{\frac{24(x+ay+b)^2}{p}=\frac{24a(x+ay+b)^2}{q}}$$

$$\Rightarrow \cancel{aq=bp}. \quad \text{eq}^n \text{ (II)} \div \text{eq}^n \text{ (III)}$$

$$\frac{p}{q}=\frac{1}{a} \Rightarrow \boxed{a=\frac{q}{p}}$$

eqⁿ (i) ÷ (iv)

$$\frac{z}{2p} = \frac{1}{3}(x+ay+b)$$

$$x+ay+b = \frac{3z}{p}$$

$$\therefore \left(z^2+\frac{q^3}{p^3}\right) = \cancel{8}\times\frac{27z^3}{\cancel{8}p^3}$$

$$\Rightarrow \frac{p^3+q^3}{p^2} = \frac{27z}{p^3}$$

$$\Rightarrow 27z = p^3+q^3$$

Ex:- $z = ax-y$

Part Diffⁿ w.r.t. x,

$$p = a$$

$$\boxed{z = px-y}$$

Again partially diffⁿ w.r.t. y

$$q = -1$$

$$\boxed{q+1=0}$$

Ex:- $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$ —— (i)

$\rightarrow$ ~~$\frac{2x}{a^2}+$~~ partially diffⁿ w.r.t. x,

(case-1)

$$\frac{2x}{a^2}+\frac{2z}{c^2}p = 0 \quad \text{—— (ii)}$$

$$\Rightarrow \frac{x}{a^2}+\frac{zp}{c^2} = 0$$

Again diffⁿ w.r.t. x

$$\frac{1}{a^2}+\frac{1}{c^2}(zr+p^2) = 0$$

$$\Rightarrow \frac{1}{a^2} = \frac{-1}{c^2}(zr+p^2)$$

$$\frac{-x}{c^2}\{zr+p^2\}+\frac{zp}{c^2}=0 \qquad \{\text{fr } (11)\}$$

$$\boxed{xzr+xp^2=zp}$$

Case-II) Partially diffⁿ w.r.t. y;

$$\frac{2y}{b^2}+\frac{2zq}{c^2}=0$$

$$\frac{y}{b^2}+\frac{zq}{c^2}=0$$

Again diffⁿ w.r.t. y

$$\frac{1}{b^2}+\frac{1}{c^2}\{zt+q^2\}=0$$

$$\frac{1}{b^2}=\frac{-1}{c^2}\{zt+q^2\}$$

$$\frac{-y}{c^2}\{zt+q^2\}+\frac{zq}{c^2}=0$$

$$\boxed{yzt+yq^2=zq}$$

Case-III) Partially diffⁿ w.r.t. x

$$\frac{x}{a^2}+\frac{zp}{c^2}=0$$

Again diffⁿ w.r.t. y

$$0+\frac{1}{c^2}\{zs+pq\}=0$$

$$\boxed{zs+pq=0}$$

Note:-

(i) By eliminating arbtr constant, we can get both non-linear as well as Quasi linear.

(ii) No. of arbtr constant = No. of independent variable then, we will get unique partial D.E.

JMS

Unit-I :

Chapter-1 - Sets & Its Fundamentals

Chapter-2 Point Set topology of $\mathbb{R}$

Unit-2

1 - Sequence of Real number

2 - Series of Real Number

Unit-3

1 - Fundamentals of functions

2 - Limits and Continuity

3 - Differentiability

4 - Application of Derivative

Unit-4

1 - Sequence of Function

2 - Series of Function

Unit-5

1 - Riemann Integrability

2 - Function of Bounded Variation

Unit-6

1 - Function of Several Variables

Sets and its fundamentals

Chapter - I

A well defined collection of distinct objects is called sets.

Note:- (I) By well defined, we mean there is no ambiguity or confusion regarding inclusion or exclusion of object.

(II) Empty collection is well defined so it is a set called void set or null set or empty set and denoted by { } or ϕ.

(III) Set itself considered as an object, hence eligible to collection for any set.

(iv) Generally, sets are denoted by capital letters and objects included in the set, elements are denoted by small letters a, b, c, x, y, z etc and if a is an object included in the set X, a belongs to X and write

$a \in X$

* Axiom of Regularity (AOR):

No set can belong to itself i.e, if X is a set then $X \notin X$.

→ Ordinary and Extra Ordinary Sets:—

If X is a set s.t. $X \in X$, X is called extra ordinary set.

A set is called ordinary if it doesn't contain itself as an element.

$X = \{\alpha$ is an object : α is not a tea cup$\}$

X is a set $\Rightarrow X$ is not a tea cup.

$\Rightarrow$ Russel's Paradox:

$X = \{A : A$ is an ordinary set$\}$

$=$ collection of all the ordinary sets

If X is a set

If X is ordinary $\Rightarrow X \in X$

$\Rightarrow X$ is extraordinary

X is extra-ordinary $\Rightarrow X \in X$

$\Rightarrow X$ is ordinary

i.e, X is not a set

$\therefore$ There is no set of all the ordinary sets.

$X = \{A$ is a set : $A \notin A\}$

$=$ The collection of all the sets

If X is a set $\Rightarrow X \notin X$

$\Rightarrow X \in X$.

Subset :-

Let A & B are sets, if $x \in A \Rightarrow x \in B$, then we say A is a subset of B denoted by $A \subset B$.

OR,

$$\nexists x \in A \text{ s.t. } x \notin B$$

Note: (i) Empty set is a subset of every set.
(ii) Every set is a subset of itself.

$\Rightarrow$ (i) $A \cup B = \{x : x \in A \lor x \in B\}$

(ii) $A \cap B = \{x : x \in A \land x \in B\}$

(iii) $A - B = \{x : x \in A,\ x \notin B\}$

(iv) $A \Delta B = (A - B) \cup (B - A)$
$= (A \cup B) - (A \cap B)$.

(v) $A^c = U - A$

(vi) $(A \cup B)^c = A^c \cap B^c$

(vii) $(A \cap B)^c = A^c \cup B^c$

(viii) $A = B \Leftrightarrow A \subset B$ & $B \subset A$.

* Power set :-

Let A be a set

$$P(A) = \{X : X \subset A\}$$
= The set of all subsets of A.

e.g. Let $A = \{a, b, c\}$

$$P(A) = \{ \phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\} \}.$$

For any set A, $P(A) \neq \phi$.

Proof:- Define $P(A) = \{ X : X \subset A \}$

Cartesian product:-

Let A & B are sets.

Define $A \times B = \{ (a,b) : a \in A, b \in B \}$

where (a, b) is called ordered pair.

Note:- (i) $A_1 \times A_2 \times \cdots \times A_n = \prod_{i=1}^{n} A_i = \{ (a_1, a_2, \ldots, a_n) : a_i \in A_i \}$

n-tupples

(ii) $\prod_{i=1}^{n} A_i = \phi \Leftrightarrow A_i = \phi$ for some i.

Functions:-

Let A & $B \neq \phi$. Then a rule by which every element is assigned to some unique element of B is defined as a function from A to B and defined by $f : A \to B$.

if $f: A \to B$ then,

(i) A is called domain

(ii) B is called co-domain

(iii) If $x \in A$ is assigned to $y \in B$, we write

$$y = f(x)$$

And y is called the image of x & x - pre-image of y.

(iv) $f(A) = \{ f(x) : x \in A \} \subset B$ called the range of f.

(v) One-one function (Injection):

$$f(a) = f(b) \Leftrightarrow a = b$$

(vi) Surjection (onto)

$$f(A) = B.$$

(vii) Bijection - One-one and onto.

Note:- $\mathbb{N} = \{1, 2, 3, \ldots, n, n+1, \ldots\}$

$S(n) = \{1, 2, 3, \ldots, n\}$.

Similar sets:-

Two non-empty sets are said to be similar if $\exists$ a bijection between them. The words like equivalent, equinumerous, equipotential are also used in the place of similar.

Note:-

(i) If $A \& B \neq \phi$, we say B has more potential than that of A if onto function from A to B can't be defined.

(ii) If one-one function can't be defined then we say A has more potential than B.

Finite Set:-

A non-empty set is said to be finite if it is similar to S_n.

if $A \sim S(n)$, we say cardinality of A is n, denoted by $card(A) = |A| = n$.

Note:- (i) By extension of def^n of finite set, empty set is also considered as finite set and its cardinality is 0.

$$\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$$

= The set of all the finite cardinals.

(ii) The intersection of two power sets can't be disjoint.