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Complex Analysis

Complex Number:-

A complex number is defined as the ordered pair (x, y) of real no., $z = (x, y)$ satisfying rules for addition and multiplication-

$$z_1 + z_2 = (x_1 + x_2,\ y_1 + y_2)$$
$$z_1 z_2 = (x_1 x_2 - y_1 y_2,\ x_1 y_2 + y_1 x_2)$$

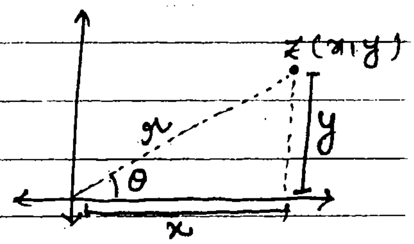
Every complex no. can be written as-

$$z = x + iy \quad , \quad x, y \in \mathbb{R}$$

where $x = r\cos\theta$ and $y = r\sin\theta$

ie. $x + iy = r(\cos\theta + i\sin\theta)$

or, $z = x + iy = re^{i\theta}$



which is polar form of complex no.

And $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

where, θ is argument of z ie the angle with the line joining the point with origin makes the +ve direction of x-axis.

NOTE:-

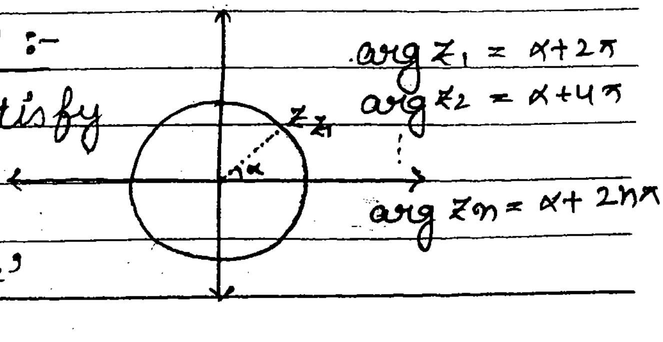
(i) $|z - z_0|$ = Distance of z from z_0

(ii) $|z|$ = Distance of z from origin

(iii) argument of z is not defined at origin

argument of z (arg z):-

Any value of θ which satisfy $x = r\cos\theta$ and $y = r\sin\theta$ is called argument of z And denoted as 'arg z'



we know, $z = e^{i\phi}$

$$\phi = \theta + 2n\pi$$

$$z = e^{i(\theta+2n\pi)}$$

$$z = e^{i\theta} e^{i2n\pi}$$

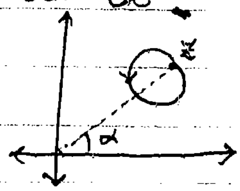
$$\boxed{z = e^{i\theta}} \qquad [\because e^{2n\pi i} = \cos 2n\pi + i \sin 2n\pi]$$

$$\Rightarrow \arg z = \{\cancel{\text{Arg}}+2n\pi \ ; \ n \in \mathbb{Z}\}$$

NOTE :-

If origin is not enclosed after each revolution then argument of z will be unchanged.

ie. $\arg z = \alpha$ for all revolutions



Argument of z (Arg z) :-

For any $z \neq 0$, principal value of $\arg z$ is defined to be unique value that satisfies

$$\boxed{-\pi < \arg z \leq \pi}$$

And it is denoted by $\text{Arg} z$

$$\arg z = \{\text{Arg} z + 2n\pi \ ; \ n \in \mathbb{Z}\}$$

where $-\pi < \text{Arg} z \leq \pi$

NOTE :-

1. 2 complex no are equal iff their principle argument (ie $\text{Arg} z$) and modulus are same.

2. At origin $(0,0)$ [ie $x=0$ & $y=0$] $\text{Arg} z$ is not defined.

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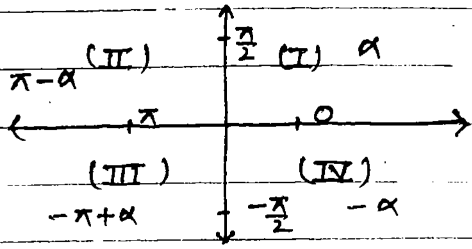
Principal argument of z (ie. $\operatorname{Arg} z$)

Let $\arg z = \tan^{-1}\left(\frac{y}{x}\right) + 2n\pi$



then,

$$\operatorname{Arg} z = \begin{cases} 0, & x>0,\ y=0 \\ \alpha, & x>0,\ y>0 \\ \pi/2, & x=0,\ y>0 \\ \pi-\alpha, & x<0,\ y>0 \\ \pi, & x<0,\ y=0 \\ \alpha-\pi, & x<0,\ y<0 \\ -\pi/2, & x=0,\ y<0 \\ -\alpha, & x>0,\ y<0 \end{cases}$$

where, $\alpha = \tan^{-1}\left|\frac{y}{x}\right|$

Let $z = x+iy$, $x>0,\ y>0$

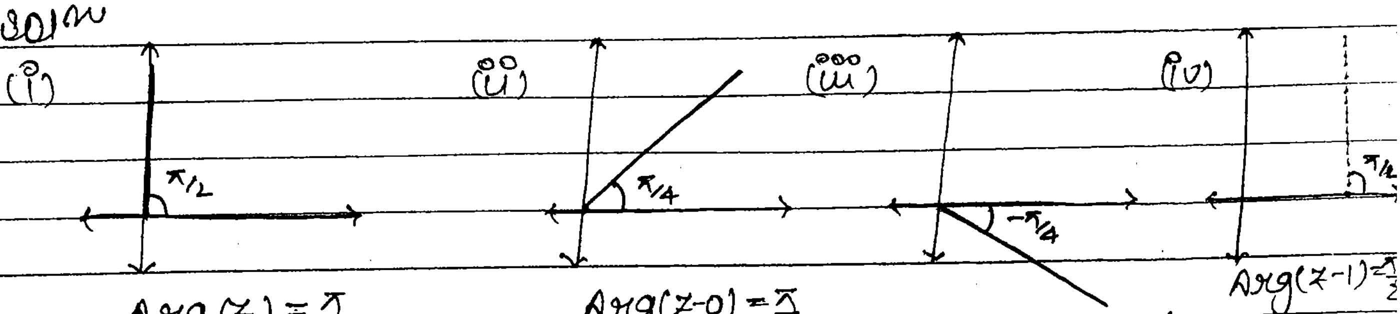
$\operatorname{Arg} z = \tan^{-1}\left(\frac{y}{x}\right) \Rightarrow \cancel{\operatorname{Arg} z = \alpha \text{ and}}$ $\alpha\ \operatorname{Arg} z < \pi/2$

Q. Find the line of —

(i) $\operatorname{Arg} z = \pi/2$ (ii) $\operatorname{Arg} z = \pi/4$

(iii) $\operatorname{Arg} z = -\pi/4$ (iv) $\operatorname{Arg}(z-1) = \pi/2$

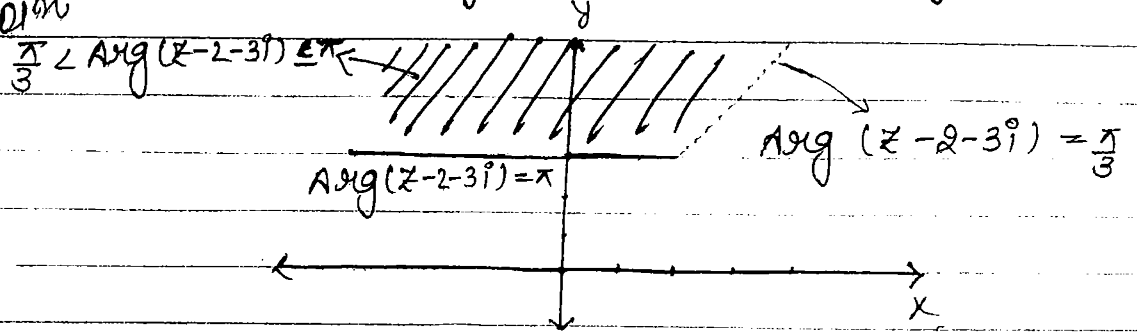
sol^n



$\operatorname{Arg}(z) = \frac{\pi}{2}$, $\operatorname{Arg}(z-0) = \frac{\pi}{4}$ $\operatorname{Arg}(z-0) = -\frac{\pi}{4}$ $\operatorname{Arg}(z-1) = \frac{\pi}{2}$

Q. Find the region of $\pi/3 < \text{Arg}(z-2-3i) \le \pi$

Sol^n



Q. Find $\arg z$, where

(i) $z = 1+i$

(ii) $z = i$

Sol^n (i) $z = 1+i$

$\text{Arg}\, z = \alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}(1) = \frac{\pi}{4}$

$\therefore \arg z = \{\pi/4 + 2n\pi ;\ n \in \mathbb{Z}\}$

(ii) $z = i$

$\text{Arg}\, z = \alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}(\infty) = \pi/2$

$\therefore \arg z = \{\pi/2 + 2n\pi ;\ n \in \mathbb{Z}\}$

Result :-

If z_1 and z_2 are two non-zero complex no. then

1. $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

Pf :- $z_1 = r_1 e^{i\theta_1} \Rightarrow \arg(z_1) = \theta_1 + 2n\pi$

$z_2 = r_2 e^{i\theta_2} \Rightarrow \arg z_2 = \theta_2 + 2n\pi$

$z_1 z_2 = r_1 r_2\, e^{i(\theta_1+\theta_2)}$

$\Rightarrow \arg(z_1 z_2) = \{\theta_1 + \theta_2 + 2n\pi ;\ n \in \mathbb{Z}\}$

$= \{\theta_1 + 2n_1\pi ;\ n_1 \in \mathbb{Z}\} + \{\theta_2 + 2(n-n_1)\pi ;\ n \in \mathbb{Z}\}$

$= \arg z_1 + \arg z_2$

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2. $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

3. $\arg z^m \neq m \arg z$, but $\arg z^m = \underbrace{\arg z + \dots + \arg z}_{(m\text{-times})}$

4. $Arg(z_1 \cdot z_2) \neq Arg z_1 + Arg z_2$

<u>Ex</u> :- $z_1 = 2i$ and $z_2 = -5$

$Arg z_1 = \pi/2$, $Arg z_2 = \pi$

$z_1 \cdot z_2 = -10i$

$\Rightarrow Arg(z_1 \cdot z_2) = -\pi/2 \neq \pi + \pi/2$

5. $Arg(z_1 \cdot z_2) = Arg z_1 + Arg(z_2)$
iff $-\pi < Arg(z_1 + z_2) \leq \pi$

6. $Arg(z_1/z_2) \neq Arg z_1 - Arg z_2$

7. $Arg(z_1/z_2) = Arg z_1 - Arg z_2$, iff $-\pi < Arg(z_1 - z_2) \leq \pi$

<u>Positional Equal Number</u> :-

Complex no. z_1, z_2 are said to be positional equal no. if

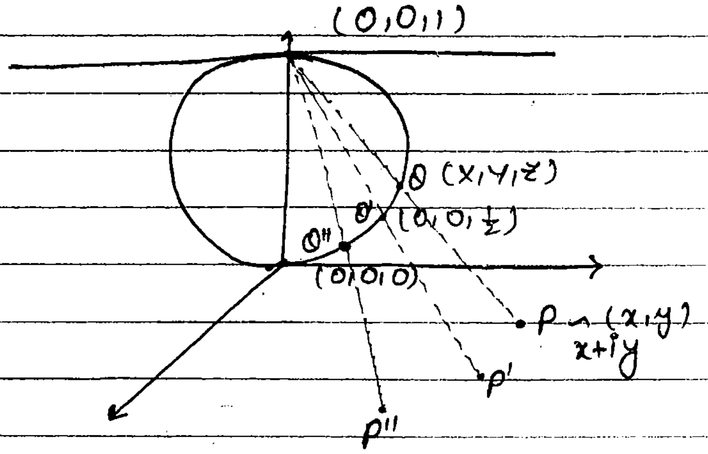
(i) $|z_1| = |z_2|$ and

(ii) $\arg z_1 - \arg z_2 = 2n\pi$, $n \in Z$

Stereographic Projection :-

we consider the extended complex plane $\mathbb{C} \cup \{\infty\}$ as a closed surface having a single point at infinity. we shall then introduce a new metric to describe the behavior of a complex function at infinity and to map the points in $\mathbb{C}$ into the surface of a sphere. This process will be reffered as stereographic projection.

Riemann sphere



A one-to-one correspondence b/w the sphere $(X-0)^2+(Y-0)^2+(Z-1/2)^2 = 1/4$ – ①
and the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ is called stereographic projection

Equation of sphere and eqn of line passes through $(0,0,1)$ and $(x,y,0)$

$$\frac{X-0}{x-0} = \frac{Y-0}{y-0} = \frac{Z-1}{0-1} = r \quad -②$$

$$X = xr \quad , \quad Y = yr \quad , \quad Z = -r+1$$

from ②, $(xr)^2 + (yr)^2 + (-r+1-1/2)^2 = 1/4$

$$r^2(x^2+y^2+1) = r$$

$$r = \frac{1}{x^2+y^2+1}$$

* Vector Space *

D. 16/03/18

Defi^n - Let 'V' be a Nonempty set and $(F, +, \cdot)$ be a field.
Now, define a external composition, $f : F \times V \rightarrow V$
s.t. $f(\alpha, x) = \alpha . x$ then, structure $(V, +, \cdot)$
is said to be a vector space and written as, $V(F)$
under the following conditions,

1> $(V, +)$ is abelian group.

(i) $a + b \in V \quad \forall \; a, b \in V$ — (closure property)

(ii) $a + (b + c) = (a + b) + c \;, \; \forall \; a, b, c \in V$ — (Associative property)

(iii) $\forall \; a \in V \; \exists \; -a \in V$ s.t. $a + (-a) = (-a) + a = 0$ — (Inverse)

(iv) '0' $\in V$ s.t. $a + 0 = a = 0 + a \quad \forall \; a \in V$ — (identity)

(v) $a + b = b + a \;, \; \forall \; a, b \in V$ — (commutative property)

2> $\forall \; \alpha \in F$ & $\forall \; x, y \in V$.
s.t. $\alpha . (x + y) = \alpha . x + \alpha . y$

3> $\forall \; \alpha, \beta \in F$ & $\forall \; x \in V$.
s.t. $(\alpha + \beta) x = \alpha . x + \beta . x$

4) $\forall \; \alpha, \beta \in F$ & $\forall \; x \in V$
s.t. $(\alpha . \beta) . x = \alpha . (\beta . x)$

5) If 1 - unity of field
$1 . x = x \;, \; \forall \; x \in V$.

Note- ① member of $(V,+,\cdot)$ or 'V' is called vector.

② vector addition and scalar multiplication are only allowed.

③ vectors multiplication is not allowed.

ex. ① $V(F) = \mathbb{R}(\mathbb{R})$

if, $x = 2 \in V = \mathbb{R}$

$y = 3 \in V = \mathbb{R}$

but, $x \cdot y$ is not allowed.

$\Rightarrow$ '·' is not a Binary operation over domain (function) from $V \times V \rightarrow V$.

④ $(V,+,\cdot)$ is not an algebraic structure.

ex. ①. $V = \varphi$, $F = \mathbb{R}$

$\Rightarrow V = (\varphi,+,\cdot)$

$F = (\mathbb{R},+,\cdot)$

$\nexists$ any external composition, $f: F \times V \rightarrow V$

i.e, $f: \mathbb{R} \times \varphi \rightarrow \varphi$

$f(\alpha, x) = \alpha \cdot x$

as $f(\sqrt{3}, 2) = \sqrt{3} \cdot 2 = 2\sqrt{3} \notin \varphi$

$\Rightarrow (V,+,\cdot)$ is not vector space over field $(\mathbb{R},+,\cdot)$

ex ②. $V = \{(x,y),\ x \cdot y \geq 0\}$, $F = \mathbb{R}$.

$\Rightarrow \alpha \in F = \mathbb{R}$

$x = (x,y) \in V$

(i) $\alpha \cdot x = (\alpha x, \alpha y) \in V$, when, $(\alpha \cdot x)(\alpha \cdot y) = \alpha^2 \cdot xy \geq 0$

(ii) $(V,+)$ is abelian group.

$x = (-3,-5) \Rightarrow (-3)\cdot(-5) = 15 \in V \quad (\because 15 \geq 0)$

$\Rightarrow x = (-3,-5) \in V$.

$y = (2,6) \Rightarrow 2\cdot 6 = 12 \in V \Rightarrow y \in V$.

$x+y = (-1,1) = (-1)\cdot 1 = -1 \notin V \Rightarrow x+y \notin V$.

$\Rightarrow$ closure not satisfied $\Rightarrow$ Not abelian

$\Rightarrow (V,+,\cdot)$ is not vector space.

ex. (3) $V = (Z_{10}, +_{10}, \times_{10}) \longrightarrow$ is ring not I.D.

$F = \{0, 2, 4, 6, 8, +_{10}, \times_{10}\}$

F is commutative with unity with multiplication inverse.

$\Rightarrow$ 'F' is field

$$\left.\begin{array}{l} 0 \times_{10} 6 = 0 \\ 2 \times_{10} 6 = 2 \\ 4 \times_{10} 6 = 4 \\ 6 \times_{10} 6 = 6 \\ 8 \times_{10} 6 = 8 \end{array}\right\} \Rightarrow \text{'6' is unity of 'F'.}$$

(i) $(V,+) = (Z_{10}, +_{10})$ is abelian group.

$\alpha \in F,\ x \in Z_{10}$

$\alpha \cdot \times_{10}\ x \in Z_{10}$

$\alpha \cdot (x+y) = \alpha \cdot x + \alpha \cdot y$

$(\alpha + \beta)\, x = \alpha \cdot x + \beta \cdot x$

$(\alpha \cdot \beta)\, x = \alpha \cdot (\beta\, x)$

$\cdot \rightarrow \times_{10}$

$+ \rightarrow +_{10}$

Here, unity $= 6$, i.e. $1 = 6$

$6 \times_{10} 5 = 0 \neq 5 \quad (5 \in Z_{10})$

i.e. $1 \cdot x \neq x$ for some $x \in Z_{10}$

$\Rightarrow (V,+,\cdot)$ is not vector space over field F.

ex. ④. $V = \{(a_1, a_2^2, a_3),\ a_1, a_2, a_3 \in \mathbb{R}\}$. if any $(x, y, z) \in \mathbb{R}^3$ want to be member of V, it should be written as,

(x, t^2, z), where, $t^2 = y$, for some $t \in \mathbb{R}$.

let, ex. $(0,-1,0) \in \mathbb{R}^3$ $\nexists$ t in $\mathbb{R}$ s.t. $(0,-1,0) = (0, t^2, 0)$

$\Rightarrow (0,-1,0) \notin V$.

$X = (1,1,1) \in V$ as $(1,1,1) = (1, (-1)^2, 1)$

$-X = (-1,-1,-1) \notin V$ $\Leftarrow$ (additive inverse element not exist)

$\Rightarrow$ 'V' is not vector space.

ex. ⑤. $V = \{(a_1, a_2^3, a_3),\ a_i \in \mathbb{R}\}$, $F = (\mathbb{R}, +, \cdot)$

vector addition - $X = (x_1, y_1, z_1) \in V$

$Y = (x_2, y_2, z_2) \in V$

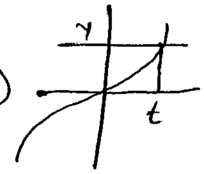
b'coz, $X = (x_1, t_1^3, z_1)$ for some $t_1 \in \mathbb{R}$

$Y = (x_2, t_2^3, z_2)$, for some $t_2 \in \mathbb{R}$

$$X + Y = (\underbrace{x_1 + x_2}_{x_1}, \underbrace{t_1^3 + t_2^3}_{y_1}, \underbrace{z_1 + z_2}_{z_1})$$

$x_1 \in \mathbb{R}$
$y_1 \in \mathbb{R}$
$z_1 \in \mathbb{R}$

$t_1^3 + t_2^3 = y_1 \in \mathbb{R}$ ($\because$ Range of $x^3 = \mathbb{R}$)



$\exists$ t in $\mathbb{R}$. s.t. $y_1 = t^3$

then, $X + Y = (x_1, t_1^3, z_1) \in V$

$\Rightarrow$ 'V' also can be written as,

$V = \{(t_1, t_2, t_3),\ t_1, t_2, t_3 \in \mathbb{R}\}$

$\Rightarrow$ i.e. $V = \mathbb{R}^3$ $\Rightarrow$ V is vector space over $(\mathbb{R}, +, \cdot)$

ex. 6. $V = \{ A = (a_{ij})_{2\times 2},\ a_{ij} \in \mathbb{R} \}$.

$F = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix},\ a \in \mathbb{R} \right\} \Rightarrow$ $\underbrace{\text{subring of matrices of ord with real entries}}_{\text{not c.R.U. (not commute with uni}}$

$$\Rightarrow \begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} e & e \\ e & e \end{bmatrix} = \begin{bmatrix} 2ae & 2ae \\ 2ae & 2ae \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \text{ (unity)}$$

Here, $\alpha . x = \begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in V$.

But, $1 \cdot x = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$= \begin{bmatrix} \frac{a_{11}+a_{21}}{2} & \frac{a_{12}+a_{22}}{2} \\ \frac{a_{11}+a_{21}}{2} & \frac{a_{12}+a_{22}}{2} \end{bmatrix}$$

$$\neq \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \notin V.$$

$\Rightarrow$ 'V' is not vector space over field F

* Examples of the vector space's -

① n- tupple spaces -

(A) $V = \mathbb{R}^n = \{(a_1, a_2, a_3, \dots a_n), a_i \in \mathbb{R}\}, F = (\mathbb{R}, +, \cdot)$ under usual addition and scalar multiplication.

$x+y = (a_1, a_2, \dots a_n) + (b_1, b_2, \dots b_n)$

$= (a_1+b_1, a_2+b_2, \dots + a_n+b_n)$

$\alpha . x = \alpha(a_1, a_2, \dots a_n) = (\alpha a_1, \alpha a_2, \dots \alpha a_n)$

$V(F) = \mathbb{R}^n(\mathbb{R})$ is always a vector space over field $\mathbb{R}$ $\forall n \geq 1$.

(B) $V = \mathbb{C}^n = \{(c_1, c_2, \dots c_n), c_i \in \mathbb{C}\}, F = (\mathbb{C}, +, \cdot)$

$\mathbb{C}^n(\mathbb{C})$ is always a vector space over field $\mathbb{C}$ and over same operation's that defined in above.

$x+y = (c_1, c_2, \dots c_n) + (d_1, d_2 \dots d_n) = (c_1+d_1, c_2+d_2, \dots c_n+d_n)$

$\alpha . x = \alpha . (c_1, c_2 \dots c_n) = (\alpha . c_1, \alpha . c_2, \dots \alpha . c_n)$

If field is $\mathbb{R}$ - (general form of valong with elements of field)

$\alpha . (c_1, c_2, \dots c_n) = \alpha(a_1+ib_1, a_2+ib_2, a_3+ib_3, \dots, a_n+ib_n)$

$= (\alpha a_1 + \alpha i b_1, \alpha a_2 + \alpha i b_2, \dots \alpha a_n + \alpha i b_n)$

* $\mathbb{C}^2 = \{(c_1, c_2)\}, F = \mathbb{R}$ i.e $\mathbb{C}^2(\mathbb{R})$

$\mathbb{C}^2 = \{(a_1+ib_1, a_2+ib_2)\}$

$(1,0)$ $(i,0)$

$(0,1), (0,i)$

(c) If F is field, then, $F^n(F)$ is always a vector space.

$F^n(F) \to \mathbb{R}^n(\mathbb{R})$, $F = \mathbb{R}$; $\mathbb{C}^n(\mathbb{C})$, $F = \mathbb{C}$; $\mathbb{Q}^n(\mathbb{Q})$, $F = \mathbb{Q}$; $[\mathbb{Q}(\sqrt{2})]^n(\mathbb{Q}\sqrt{2})$, $F = \mathbb{Q}(\sqrt{2})$; $[\mathbb{Q}(\sqrt{2},\sqrt{3})]^n \mathbb{Q}(\sqrt{2},\sqrt{3})$, $F = \mathbb{Q}(\sqrt{2},\sqrt{3})$

(2) vector space of Matrices-

(A) $V = \{ (a_{ij})_{m\times n},\ a_{ij} \in \mathbb{R} \}$, $F = (\mathbb{R}, +, \cdot)$

V(F) is a vector space of Matrices of order m by n with real entries over field $\mathbb{R}$ and also denoted as, $M_{mn}(\mathbb{R})$ under the following operation's, usual Addition of matrices and scalar multiplication.

$X + Y = (a_{ij})_{m\times n} + (b_{ij})_{m\times n} = (a_{ij} + b_{ij})_{m\times n} = (c_{ij})_{m\times n}$

$\alpha \cdot X = \alpha \cdot (a_{ij})_{m\times n} = (\alpha \cdot a_{ij})_{m\times n} = (b_{ij})_{m\times n}$.

$\Rightarrow$ V(F) is vector space.

(B) $V = \{ A = (a_{ij})_{m\times n},\ a_{ij} \in \mathbb{C} \}$, $F = (\mathbb{R}, +, \cdot)$

$X + Y = (a_{ij})_{m\times n} + (b_{ij})_{m\times n} = (c_{ij})_{m\times n}$

$\alpha \cdot X = \alpha \cdot (a_{ij})_{m\times n} = \alpha (r_{ij} + i s_{ij})$, r_{ij} & $s_{ij} \in \mathbb{R}$.

$= (\alpha \cdot r_{ij} + i \alpha s_{ij})$

$\Rightarrow$ V(F) is a vector space over field $\mathbb{R}$.

③ Vector space of polynomials-

$V = \{p,\ p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n,\ a_i \in \mathbb{R}\},\ F = (\mathbb{R}$

or

$V = \{p,\ p(x)$ is polynomial with degree $p(x) \le n\}$ (n

= space of polynomials of degree atmost (n). (exc zero poly

under the following operations,

$x + y = p(x) + q(x) = (a_0 + a_1x + \dots + a_nx^n) + (b_0 + b_1x + \dots + b_nx^n)$

$= (a_0 + b_0) + (a_1 + b_1)x + \dots + (a_n + b_n)x^n.$

$\alpha . x = \alpha . p(x) = \alpha . (a_0 + a_1x + a_2x^2 + \dots + a_nx^n)$

$= \alpha . a_0 + \alpha a_1 x + \alpha . a_2 x^2 + \dots + \alpha . a_n x^n.$

④ vector space of functions-

Ⓐ $V = \{f;\ f : \mathbb{R} \to \mathbb{R}\},\ F = (\mathbb{R}, +, \cdot)$

= set of real valued functions defined on $\mathbb{R}$.

$V(F)$ is a vector space of real valued functions defined on $\mathbb{R}$.

$f(x) + g(x) = (f+g)(x)$

$\alpha . f(x) = (\alpha . f)(x)$

Ⓑ $V = \{f;\ f : S \subseteq \mathbb{R} \longrightarrow \mathbb{R}\},\ F = (\mathbb{R}, +, \cdot)$ (S is any interval)

$\Rightarrow V(F)$ is always a vector space.

Que.
* set of all transfinite no. over field $\mathbb{R}$, is vector space

$\Rightarrow \aleph_0 + c = c.$ ($\aleph_0$ identity)

⑤ <u>vector space of sequences</u> - (Non-fvns)

$V = \{ \langle a_n \rangle ,\ a_n \in \mathbb{R},\ n \geq 1 \}$, $F = (\mathbb{R}, +, \cdot)$

$V(F)$ is a vector space of sequences of real numbers over field $\mathbb{R}$ under the following operations,

$$\langle a_n \rangle + \langle b_n \rangle = \langle a_1, a_2, \ldots a_n \ldots \rangle + \langle b_1, b_2, \ldots b_n \ldots \rangle$$
$$= \langle a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n \ldots \rangle$$

$$\alpha \cdot \langle a_n \rangle = \alpha \langle a_1, a_2, \ldots a_n \ldots \rangle$$
$$= \langle \alpha a_1, \alpha a_2, \ldots \alpha a_n \ldots \rangle$$
$$= \langle \alpha \cdot a_n \rangle$$

* <u>Non-Funs</u>-

⑥ $V = \{ p,\ p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n,\ n \in \mathbb{N},\ a_i \in \mathbb{R} \}$.
$F = (\mathbb{R}, +, \cdot)$

$V(F)$ = space of polynomial over field $\mathbb{R}$.

$V(F)$ = space of polynomial of all degree over $\mathbb{R}$.

or space of polynomial over field $\mathbb{R}$.

<u>sheet ①</u>

(19) $W = \{ X \in \mathbb{R}^n,\ s.t.\ AX = B \}$.

- if $B \neq 0$ (Non-Homogenous)
- if $B = 0$ (Homogenous)
  'W' is sol^n space of $AX = 0$
  'W' is subspace of $\mathbb{R}^n$
  $\Rightarrow$ W is convex
  $\Rightarrow$ W is path connected
  $\Rightarrow$ W is connected.

(19)

$W = \{ X \in \mathbb{R}^n, \text{ s.t. } AX = B \}$.

$B \neq 0$ (NonHomogenous)

$X = 0 \in \mathbb{R}^n$ (n-tupple)

but, $0 \notin W$.

$A \cdot 0 \neq B$

$\Rightarrow$ $\boxed{\text{'W' is not subspace}}$

No solution

~~No solution~~

$W = \phi$

can't discuss as a subspace,

- 'W' is convex. (vacuosly) (void situation)
- W is path connected
- W is connected
- W is bounded.

Unique sol^n

$W = \{ x \}$

$x \neq 0$.

- W is not subspace
- W is convex
- W is path connected
- W is connected
- W is bounded

infinite solution

W is an infinite

$W = \{ x \in \mathbb{R}^n, AX = B \}$

$x, y \in W$, $AX = B$, $AY = B$.

$A \cdot [\lambda \cdot x + (1-\lambda) \cdot y]$

$= A \lambda x + A(1-\lambda) y$

$= \lambda A x + (1-\lambda) A y$

$= B\lambda + (1-\lambda) B$

$= B\lambda + B - B\lambda$

$= B$

- W is not subspace.
- W is convex
- W is connected
- W is path connected.
- $\boxed{\text{W is unbdd}}$

$B = 0$ (Homogenous)

W is solution space of $AX = 0$.

- 'W' is subspace of $\mathbb{R}^n$
- 'W' is convex.
- 'W' is path connected
- 'W' is connected

* Discussion for 'W' is unbdd in infinite sol^n:

Let, W be an infinite set of sol^n of Homogenous system of Linear equation $AX = 0$.

$W = \{ X \in \mathbb{R}^n, \text{ s.t. } AX = 0 \}$.

$x, y \in W$, $\alpha, \beta \in F = \mathbb{R}$.

$\alpha \cdot x + \beta \cdot y = z \in \mathbb{R}^n$

$Az = A[\alpha \cdot x + \beta \cdot y] = \alpha \cdot A \cdot x + \beta \cdot A y$

$= \alpha \cdot 0 + \beta \cdot 0$

'W' is subspace of $\mathbb{R}^n$.

Given, W is an infinite set.

$\Rightarrow \exists\, w = (w_1, w_2, \ldots, w_n) \in W$.

s.t $w \neq 0$ $\Rightarrow$ atleast one $w_i \neq 0$.

say, w_j $(1 \leq j \leq n)$

Now, $\|w\| = \|w\| = \sqrt{w_1^2 + w_2^2 + \ldots + w_n^2}$

$\because$ W is a subspace.

$\alpha, w \in W$

$\|\alpha \cdot w\| = |\alpha| \cdot \|w\|$

As we increase α in $\mathbb{R}$, $\|\alpha \cdot w\|$ increases.

if α larger $(\alpha \to \infty)$

$\|\alpha \cdot w\| \to \infty$, $\nexists$ M s.t. $\|x\| \leq M\ \forall\ x \in W$.

$\Rightarrow$ 'W' is unbdd.

* Subspace-

Let, 'W' be a Non-empty subset of a vector space 'V' over field 'F' then this 'W' is said to be a subspace of vector space 'V', if W is itself a vector space over same field 'F'. denoted as, $\boxed{W \leq V(F)}$

ex. ① $W = \{(a,b) \in \mathbb{R}^2, a, b \geq 0\} \subseteq \mathbb{R}^2$, $W \neq \phi$, $\mathbb{R}^2(\mathbb{R})$ is a vector space.

$W \neq \phi \subseteq \mathbb{R}^2$

as $x = (1,2) \in W$

but, $-x = (-1,-2) \notin W$.

for, $x \in W$ $\nexists$ $-x$

s.t. $x + (-x) = 0$

$(W, +, \cdot)$ is not a group.

$\Rightarrow$ $(W, +, \cdot)$ is not a vector space. (By defin)

* **Test-** Let, $W \neq \phi$ subset of $V(F)$,
'W' is said to be a ~~vector~~ sub space,
iff,

(I) $x-y \in W \quad \forall x, y \in W$.
(II) $\alpha . x \in W, \forall \alpha \in F, \forall x \in W$

$\Rightarrow$ proof- 'W' is a subspace of V

we have,
$x-y \in W, \forall x, y \in W$ — (1)
& $\alpha . x \in W. \forall \alpha \in f. x \in W$ — (2)

by (1), $x = y$.
$\Rightarrow x-y \in W$.
i.e. $x-x = \boxed{0 \in W}$ — identity Axiom

Let, $y \in W$ (arbitrary)
put, $x = 0$
$x-y = 0-y = \boxed{-y \in W}$ — inverse Axiom.

We know, $\forall y \in W, -y \in W$
put, $x = x$.
$y = -y$
$x-y = x-(-y) = \boxed{x+y \in W}$ — closure Axiom.

Associative Axiom (depends only on operation and here operation '+' not changed $\Rightarrow$ satisfy.)

commutative Axiom (also depends only on operation $\Rightarrow$ it also satisfy

$\Rightarrow (W, +)$ is abelian.

(given,)
II) $\alpha . x \in W, \forall \alpha \in f, x \in W$. (external composition done)
$\because \alpha . (x+y) = \alpha . x + \alpha . y \quad \forall \alpha, \beta \in F, \quad \forall x, y \in V$.
$\Rightarrow \underbrace{\alpha . (x+y) = \alpha . x + \alpha . y}_{\text{depends only on operations}}, \quad \forall \alpha, \beta \in F, \forall x, y \in W$.

III) $\underbrace{(\alpha+\beta)x = \alpha\cdot x+\beta\cdot x}$, $\forall\ \alpha,\beta\in F$, $\forall\ x,y\in W$.

depends only on operations.

IV) $\underbrace{(\alpha\cdot\beta)\cdot x = \alpha\cdot(\beta\cdot x)}$, $\forall\ \alpha,\beta\in F$, $\forall\ x,y\in W$.

depends only on operations.

V) $1\cdot x = x\ \ \forall\ x\in W$

Now, $(W,+,\cdot)$ is a vectorspace.

Combining these two conditions,

i.e. (i) $x-y\in W$, $\forall\ x,y\in W$
(ii) $\alpha\cdot x\in W$, $\alpha\in F$ $\forall\ x\in W$.

we have a one step test,

$W\neq\phi\subseteq V(F)$ is said to be a subspace of $V(F)$

iff, $\boxed{\alpha\cdot x+\beta\cdot y\in W\ \ \forall\ x,y\in W,\ \forall\ \alpha,\beta\in F.}$

Let A be a square matrix with trace$(A)=2$ → Not imp.
Now, which of the following are subspace of space of matrices over field $\mathbb{R}$.

(a) $W_1=\{X\in M_n(\mathbb{R})=M_{nn}(\mathbb{R})$. s.t. $A\cdot X=X\cdot A\}$

(b) $W_2=\{X\in M_n(\mathbb{R})$ s.t. $X+A=A+X\}$

(c) $W_3=\{X\in M_n(\mathbb{R})$ s.t. $Tr(AX)=0\}$

(d) $W_4=\{X\in M_n(\mathbb{R})$ s.t. $|AX|=0\}$
↳ det is not satisfying linear property.

Soln- Here A is fixed Matrix.
$x,y\in W_1$ (arbitrary)
$A\cdot X=X\cdot A$.
$A\cdot Y=Y\cdot A$

$\alpha \cdot x + \beta \cdot y = z$ (say)

$$A \cdot z = A \cdot (\alpha \cdot x + \beta \cdot y)$$
$$= A \cdot \alpha x + A \beta y$$
$$= \alpha A x + \beta A y$$
$$= (\alpha x + \beta y) \cdot A$$
$$= z A.$$

$\Rightarrow Az = zA.$

$\Rightarrow \alpha x + \beta y \in W_1$

$\Rightarrow W_1$ is subspace

Simillarly W_2 is also subspace.

* space of n-tupple-

$V = \mathbb{R}^n$, $F = \mathbb{R}$

✓ (1) $W_1 = \{(a_1, a_2, a_3, \dots a_n) \in \mathbb{R}^n$ s.t. $\sum_{i=1}^{n} a_i = 0\}$

✓ (2) $W_2 = \{(a_1, a_2, a_3, \dots a_n) \in \mathbb{R}^n$ s.t. $\sum_{i=1}^{n} (a_i)^2 = 0\} \overset{or}{=} \{0\} \overset{or}{=} \{(0, 0, \dots 0)\}$ = subspace

✓ (3) $W_3 = \{(a_1, a_2, a_3, \dots a_n) \in \mathbb{R}^{100}$, $a_1 = a_2 = \dots = a_{50}$ &
$a_{51} + a_{52} + \dots + a_{100} = 0\}$ satisfy linear condition = subspace

✓ (4) $W_4 = \{(a_1, a_2, a_3, \dots a_{100}) \in \mathbb{R}^{100}$, s.t. $a_1 = 2a_2\}$.

× (5) $W_5 = \{(a_1, a_2, a_3, \dots a_{10}) \in \mathbb{R}^{10}$ s.t. $\frac{a_1}{a_2} = 2\}$.

× (6) $W_6 = \{(a_1, a_2, a_3) \in \mathbb{R}^3$ s.t. $a_1 = a_2^2\}$.

✓ (7) $W_7 = \{(a_1, a_2, a_3 \dots a_n) \in \mathbb{R}^n$, $a_i = a_{[\frac{n}{2}]}$, $i = [\frac{n-1}{2}]\}$

✓ (8) $W_8 = \{(a_1^3, a_2^3, a_3^3, \dots a_{10}) \in \mathbb{R}^{10}$, $a_4 + a_5 + \dots + a_{10} = 0\}$.

✓ (9) $W_9 = \{(a_1, a_2, \dots a_n) \in \mathbb{R}^n$; $\sum_{i=1}^{n} i a_i = 0\}$ = dim = n-1

✓ (10) $W_{10} = \{(a_1, a_2, \dots a_n) \in \mathbb{R}^n$; $a_i = a_j$ when, $i+j$ = even$\}$.

Sol^n - (10) - $X = (a_1, a_2, \ldots, a_n) \in W$, $a_i = a_j$, $i+j = \text{even}$,
$Y = (b_1 b_2, \ldots b_n) \in W$, $b_i = b_j$, $i+j = \text{even}$, } — (1).

$\alpha.X + \beta.Y = (\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \ldots \alpha a_n + \beta b_n)$

$= (c_1, c_2, \ldots c_n)$

$\underbrace{\alpha a_i + \beta b_i}_{c_i} = \underbrace{\alpha a_j + \beta b_j}_{c_j} \quad \forall\, i+j = \text{even}$,

$c_i = c_j$, $\forall\, i+j = \text{even}$

$\Rightarrow (c_1 c_2 \ldots c_n) \in W$.

$\alpha.X + \beta.Y \in W$, $\forall \alpha, \beta \in F$, $\forall X, Y \in W$.

$\Rightarrow$ 'W_{10}' is subspace.

sol^n (9) - $X = (a_1, a_2, \ldots a_n) \in W$, $\sum_{i=1}^{n} i a_i = 0$
$Y = (b_1, b_2, \ldots b_n) \in W$, $\sum_{i=1}^{n} i b_i = 0$.

$\alpha.X + \beta.Y = Z$
$= (c_1 c_2 \ldots c_n)$

then, $c_i = \alpha a_i + \beta b_i$. $\forall\, i = 1$ to n

$\sum_{i=1}^{n} i c_i = \sum_{i=1}^{n} i(\alpha a_i + \beta b_i)$

$= \alpha \sum_{i=1}^{n} i a_i + \beta \sum_{i=1}^{n} i b_i$

$= \alpha.0 + \beta.0$

$= 0$

$\Rightarrow \alpha X + \beta Y \in W$.

$\Rightarrow$ 'W_9' is subspace

IMP

* $W_{11} = (X \in \mathbb{R}^n$, s.t. $AX = 0]$ for fixed matrix A.

$X \in W$, $AX = 0$
$Y \in W$, $AY = 0$.

$\alpha.X + \beta.Y \in Z$.

$Z = A(\alpha X + \beta Y)$

$= \alpha AX + \beta AY$

$= \alpha 0 + \beta.0$

$= 0$

$\Rightarrow \alpha.X + \beta.Y \in W$

$\Rightarrow W$ is subspace.

Sol^n ⑫ -

$W_{12} = \{ (x+y, x-y), x, y \in \mathbb{R} \}$.

$\Rightarrow \alpha \cdot (x_1+y_1, x_1-y_1) + \beta(x_2+y_2, x_2-y_2)$

$= (\alpha x_1 + \alpha y_1 + \beta x_2 + \beta y_2, \alpha x_1 - \alpha y_1 + \beta x_2 - \beta y_2)$

$= (A_1 + A_2, A_1 - A_2, A_1, A_2 \in \mathbb{R} \}$.

$\Rightarrow$ W_{12} is a subspace.

Sol^n ⑧ - $W_8 = \{ (a_1^3, a_2^3, a_3^3, \ldots a_{10}^3) \in \mathbb{R}^{10}, a_4 + a_5 + \ldots + a_{10} = 0 \}$.

$\Rightarrow \quad x = (a_1^3, a_2^3, \ldots a_{10}^3) \in \mathbb{R}^{10}, \sum_{i=4}^{10} a_i = 0$

$y = (b_1^3, b_2^3, \ldots b_{10}^3) \in \mathbb{R}^{10}, \sum_{i=4}^{10} b_i = 0$.

$\alpha \cdot x + \beta \cdot y = (\underbrace{\alpha a_1^3 + \beta b_1^3}_{B_1}, \underbrace{\alpha a_2^3 + \beta b_2^3}_{B_2}, \underbrace{\alpha a_3^3 + \beta b_3^3}_{B_3}, \ldots \alpha a_{10}^3 + \beta b_{10}^3)$

$= (B_1, B_2, B_3, A_4 \ldots A_{10})$

$= (B_1, B_2, B_3, A_4, \ldots A_{10})$

$\sum_{i=4}^{10} A_i = \sum_{i=4}^{10} (\alpha a_i + \beta b_i)$

$= \alpha \sum_{i=4}^{10} a_i + \beta \sum_{i=4}^{10} b_i$

$= 0 + 0$

$= 0$.

$\Rightarrow \boxed{\sum_{i=4}^{10} A_i = 0}$

there are B_1, B_2, B_3 are real no.

There are always A_1, A_2, A_3 s.t. $B_1 = A_1^3$, $B_2 = A_2^3$, $B_3 = A_3^3$

$\Rightarrow B_i = A_i^3 \quad \forall i = 1, 2, 3$.

Here $\alpha x + \beta y = (A_1^3, A_2^3, A_3^3, A_4, A_5, \ldots A_{10})$

$\Rightarrow$ 'W_8' is subspace.

Soln (7) $W_7 = \{(a_1, a_2, \ldots a_n),\ a_i = a_{[n/2]},\ i = [\frac{n-1}{2}]\}$

$\Rightarrow$ If $n=4$

$W = \{(a_1, a_2, a_3, a_4) \in \mathbb{R}^4,\ a_i = a_2,\ \forall i = 1\}$.

or

$W = \{(a_1, a_2, a_3, a_4) \in \mathbb{R}^4,\ a_1 = a_2\}$ is subspace.

If $n=5$.

$W = \{(a_1, a_2, a_3, a_4, a_5) \in \mathbb{R}^5,\ a_i = a_2,\ i=2\}$

or

$W = \{(a_1, a_2, a_3, a_4, a_5) \in \mathbb{R}^5,\ a_2 = a_2\} = \mathbb{R}^5$.

$\Rightarrow$ 'W_7' is always form a subspace.

$+ \ W_{13} = \{(a_1, a_2, a_3, \ldots a_n),\ a_i = a_{[n/2]}\ \ \forall i = 1, 2, 3, \ldots [\frac{n-1}{2}].\}$ — subspace

Soln (6) $W_6 = \{(a_1, a_2, a_3) \in \mathbb{R}^3,\ a_1 = a_2^2\}$

$\downarrow$

condition is not linear.

$\Rightarrow$ Not subspace.

Soln (5) - $W_5 = \{(a_1, a_2, a_3, \ldots a_{10}) \in \mathbb{R}^{10}$ s.t. $\frac{a_1}{a_2} = 2\}$

by default whatever condition given we need to check over that.

& also,

$\Rightarrow (0, 0, 0 \ldots 0) \notin W_5$. as $\frac{0}{0}$. (not defined)

Soln (5) $(0, 0, 0 \ldots 0) \in W_4 \Rightarrow$ subspace

(14) $W_{14} = \{ (a_1, a_2, a_3, \ldots a_n \in \mathbb{R}^n,$ s.t. $a_1 + 2a_2 + a_3 = 0$ and $a_2 + a_4 + a_5 = 0 \}$ is subspace

W_{14} can be written as,

$W_{14} = \{ X \in \mathbb{R}^5,\ AX = 0,$ where $A = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}_{2\times 5}$ ($n > m$) (infinite sol^n)

* Space of Matrices -

$V = M_{mn}(\mathbb{R})$ - Space of Matrices over field $\mathbb{R}$ of order $m \times n$.

$V = M_n(\mathbb{R})$ - space of Matrices of order 'n' over field $\mathbb{R}$

(1) $W_{15} = \{ A \in M_n(\mathbb{R}),\ A \cdot A^T = 0 \} = W = \{0\}$.

2) $W_{16} = \{ A = (a_{ij})_{m\times n}$ s.t. $\underbrace{\sum_{j=1}^{n} a_{ij} = 0}_{\text{sum of each row, is zero}},\ \forall i = 1$ to $m \} = m \times (n-1)$

(3) $W_{17} = \{ A = (a_{ij})_{m\times n}$ s.t. $\underbrace{\sum_{i=1}^{m} a_{ij} = 0}_{\text{sum of each coloumn, is zero}},\ \forall j = 1$ to $n \} \Rightarrow (m-1) \times n$

(4) $W_{18} = \{ A = (a_{ij})_{m\times n}$ s.t. $\sum_{j=1}^{n} a_{ij} = 0,\ \forall i = 1$ to m &

$\sum_{i=1}^{m} a_{ij} = 0,\ \forall j = 1$ to $n \} \Rightarrow (m-1) \times (n-1)$

(5) $W_{19} = \{ A \in M_n(\mathbb{R})$ s.t. $\text{trace}(A \cdot A^T) = 0 \} \Rightarrow W_{19} = \{0\}$ (contains only zero matrix)

(6) $W_{20} = \{ A \in M_n(\mathbb{R})$ s.t. $A^T = A \} = \Sigma n$

(7) $W_{21} = \{ A \in M_n(\mathbb{R})$ s.t. $A^T = -A \} = \frac{n(n-1)}{2} = \Sigma n-1$

(8) $W_{22} = \{ A \in M_n(\mathbb{R})$ s.t. $\text{Trace}(A) = 0 \} = n^2 - 1$

(9) $W_{23} = \{ A = (a_{ij})_{n\times n},\ \underbrace{\sum_{i=1}^{n} a_{ii} = 0}_{\text{trace}(A)=0},\ \&\ a_{11} = 0 \} = n^2 - 2$

(10) $W_{24} = \{ A = (a_{ij})_{n\times n},\ \underbrace{\sum_{i=1}^{n} a_{ii} = 0}_{\text{trace}(A)=0},\ a_{11} = 0 + \underbrace{a_{ij} = a_{ji}}_{\text{Symmetric}},\ \forall i \neq j \} = (n-2) + \frac{(n^2-n)}{2}$

IMP

(11) $W_{25} = \{ A = (a_{ij})_{m \times n},\ a_{ij} = 0$, where, $i+j =$ even $\}$ = dim = $\begin{cases} \frac{n^2}{2}, & n \text{ is even} \\ \frac{n^2-1}{2}, & n \text{ is odd} \end{cases}$

(12) $W_{26} = \{ A = \begin{bmatrix} a_1 & a_2^3 \\ a_3^3 & a_4 \end{bmatrix},\ a_i \in \mathbb{R},\ \forall\ i = 1 \text{ to } 4 \}$.

(13) $W_{27} = \{ A = (a_{ij})_{n \times n}$, s.t. $a_{ij} = 0$, whenever, $i+j = n+1 \}$ = $\dim W_{27} = n^2 - n$

(14) $W_{28} = \{ A \in M_n(\mathbb{R})$ s.t. $|A| = 0 \}$. , $\boxed{W = \{ A \in M_n(\mathbb{R}),\ \text{s.t. } |A| \neq 0 \}}$

(15) $\{$ Let, $x \neq 0$ vector in $\mathbb{R}^n$, Now define,

$W_{29} = \{ A \in M_n(\mathbb{R})$, s.t. $AX = X \}$.

IMP

(16) Let, 'B' be a fixed matrix of order 'n' with real entries,

Now, define a set,

$W_{30} = \{ A \in M_n(\mathbb{R})$ s.t. $AB = BA \}$. = (null space)

(17) Let, 'B' be a orthogonal matrix, Now define a set,

$W_{31} = \{ BAB^T ;\ A \in M_n(\mathbb{R}) \}$

* with field $\mathbb{C}$ -

$W_{32} = \{ A \in M_n(\mathbb{C}),\ A^\theta = A \}$. (Here, $a_{ij} \in \mathbb{C}$ & $F = \mathbb{C}$) not subspace

$W_{33} = \{ A \in M_n(\mathbb{C}),\ A^\theta = -A \}$. (— " —)

$W_{34} = \{ A = (a_{ij})_{n \times n},\ a_{ij} \in \mathbb{C},\ A^\theta = A \}$, $F = \mathbb{R}$. ⇒ $\dim W_{34} = 2\frac{n^2-n}{2} + n = n^2 - n + n = n^2$

$W_{35} = \{ A = (a_{ij})_{n \times n},\ a_{ij} \in \mathbb{C},\ A^\theta = -A \}$, $F = \mathbb{R}$.

$\boxed{\dim W_{35} = \frac{n^2-n}{2}}$

$W_{36} = \{$

*Note-
operation based matrices Not form a subspace, entries based matrices form a subspace,

* space of polynomial -

*** for vector space**

1) if $\exists$ $\alpha \in F$ or $x \in V$ s.t. $\alpha . x \notin V$.

2) if $\exists$ x & y in V s.t. $x + y \notin V$.

3) if $0 \notin V$.

4) for some x, $\nexists$ $-x$ in V.

5) $\exists$ x in V s.t. $1 . x \neq x$

If any one of the above is satisfied then 'V' is not a vector space.

*** for subspace**

1) $0 \notin W$.

2) If $\exists$ $x, y \in W$ s.t. $x + y \notin W$.

3) If $\exists$ $\alpha \in F$ or $x \in W$. s.t. $\alpha . x \notin W$.

If any one of the above is satisfy then, 'W is not a subspace.

Sol^n - $W_{15} = \{A \in M_n(\mathbb{R}) : A . A^T = 0\}$

$A = O_{n \times n} \in W_{15}$, $0 . 0^T = 0$

$W_{15} \neq \phi$

$A, B \in W_{15}$ $\Rightarrow$ $A . A^T = 0$
$B . B^T = 0$

$\alpha A + \beta B = C$

$$
\begin{aligned}
C . C^T &= (\alpha A + \beta B) . (\alpha A + \beta B)^T \\
&= (\alpha A + \beta B) . (\alpha A^T + \beta B^T) \\
&= \alpha^2 A A^T + \alpha \beta A B^T + \beta B \alpha A^T + \beta^2 B . B^T . \\
&= 0 + \alpha \beta A B^T + \beta B \alpha A^T + 0 \\
&= \alpha \beta (A B^T + B A^T)
\end{aligned}
$$

$A B^T = - B A^T$.

generalised over 2x2 matrix,

If $A_{n\times n}$ s.t. $A \cdot A^T = 0$, then diagonal entry in ith row

$$= \sum_{j=1}^{n} (a_{ij})^2 = 0$$

$\Rightarrow a_{ij} = 0, \forall i \& j.$

$\Rightarrow A = 0$. (zero matrix)

Here $W_{15} = \{0\}$.

$\Rightarrow W_{15}$ is a subspace.

Sol^n —

* $W_{16} = \{ A = (a_{ij})_{m\times n}, \sum_{j=1}^{n} a_{ij} = 0, \forall i = 1 \text{ to } n \}$ $\Rightarrow \boxed{\dim(W_{16}) = n\times(m-1)}$

$A = (0)_{m\times n} \in W_{16}$, Hence $W_{16} \neq \phi$.

$A = (a_{ij})_{m\times n} \in W_{16}$ as $\sum_{j=1}^{n} a_{ij} = 0 \quad \forall i = 1 \text{ to } m$

$B = (b_{ij})_{m\times n} \in W_{16}$ as $\sum_{j=1}^{n} b_{ij} = 0, \quad \forall i = 1 \text{ to } m.$

$\alpha . A + \beta . B = C \in W_{16}$

as, $\alpha (a_{ij})_{m\times n} + \beta (b_{ij})_{m\times n} = (c_{ij})_{m\times n}$

$c_{ij} = \alpha a_{ij} + \beta b_{ij}$.

$$\sum_{j=1}^{n} c_{ij} = \sum_{j=1}^{n} \alpha a_{ij} + \sum_{j=1}^{n} \beta b_{ij}$$

$$= \alpha \sum_{j=1}^{n} a_{ij} + \beta \sum_{j=1}^{n} b_{ij}$$

$$= 0 + 0$$

$$= 0$$

$\Rightarrow W_{16}$ is a subspace.

$W_{16} = \{ A = (a_{ij})_{m\times n}, \sum_{j=1}^{n} a_{ij} = 0 \quad \forall i = 1 \text{ to } n \}$

~~for~~ any 3×3 matrix.

$$\begin{bmatrix} a_{11} & a_{12} & -a_{11}-a_{12} \\ a_{21} & a_{22} & -a_{21}-a_{22} \\ a_{31} & a_{32} & -a_{31}-a_{32} \end{bmatrix}_{3\times 3}$$

$\Rightarrow$ dim of W_{16} for $n=3$, $m=3$ $= 6$

$\Rightarrow \dim(W_{16}) = m\times(n-1)$

Similarly, W_{17} & W_{18} are also subspace.

$\boxed{\dim W_{17} = (n-1)\times m}$ & $\dim W_{18} =$ ......

* $W_{19} = \{ A \in M_n(\mathbb{R}),\ Tr(A \cdot A^T = 0 \}$.

$a_{11}^2 + a_{22}^2 + a_3^2 + \dots + a_n^2 = 0$

$\Rightarrow W_{19} = \{0\}$

$\Rightarrow W_{19} = \{0\}$

$\Rightarrow W_{19}$ is subspace.

* $W_{20} = \{ A \in M_n(\mathbb{R}),\ A^T = A \}$. $\Rightarrow$ $W_{20} \Rightarrow$ space of symmetric matrix.

$\cdot\ 0 \in W_{20}$, as $0 = 0^T$, $W_{20} \neq \phi$. $\boxed{\dim(W_{20}) = \frac{n(n+1)}{2} = \sum n}$

$\Rightarrow$ let, $A \in W_{20}$ as $A^T = A$

$B \in W_{20}$ as $B^T = B$.

$(\alpha A + \beta B)^T = (\alpha A)^T + (\beta B)^T$

$= \alpha A^T + \beta B^T$

$= \alpha A + \beta B$

$\Rightarrow \alpha A + \beta B \in W_{20}$

$\Rightarrow$ $\underline{W_{20}\text{ is subspace}}$

Simillarly, $\underline{W_{21}\text{ is subspace}}$. $\rightarrow \boxed{\dim(W_{21}) = \frac{n(n-1)}{2}}$

* $W_{22} = \{ A \in M_n(\mathbb{R}),\ Trace(A) = 0 \}$ $\Rightarrow \boxed{\dim(W_{22}) = n^2 - 1}$

$0 \in W_{22}$, $trace(0) = 0$. $\circ\ W_{22} \neq \phi$.

$\Rightarrow$ let, $A \in W_{22}$, as $trace(A) = 0$

$B \in W_{22}$, as $trace(B) = 0$.

$(\alpha A + \beta B) = C$.

$Trace(C) = Trace(\alpha A + \beta B)$

$= \alpha\, Trace(A) + \beta\, Trace(B)$

$= \underline{\underline{0}}$

$\Rightarrow W_{22}$ $\underline{\text{is subspace}}$.

$W_{22} = \{ A \in M_n(\mathbb{R}),\ trace(A) = 0 \}$

$\Rightarrow$ $\underline{\text{or}}$ $n = 3$

$$W_{22} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & (-a_{11} - a_{22}) \end{bmatrix}$$

$\Rightarrow$ dim of W for $n = 3 = \underline{8}$

$\Rightarrow \boxed{\dim(W_{23}) = n^2 - 1}$

* $W_{23} = \{ A \in M_n(\mathbb{R}),\ \underbrace{\sum_{i=1}^{n} a_{ii} = 0}_{\text{trace}(A)=0},\ a_{11} = 0 \}$. $\Rightarrow \boxed{\dim W_{23} = n^2 - 2}$

$\Rightarrow 0 \in W_{23} \neq \phi$.

$A \in W_{23} \Rightarrow \sum_{i=1}^{n} a_{ii} = 0,\ a_{11} = 0$

$B \in W_{23} \Rightarrow \sum_{i=1}^{n} b_{ii} = 0,\ b_{11} = 0$.

$\alpha . A + \beta B = C$

$\alpha (a_{ij}) + \beta (b_{ij}) = c_{ij}$

$\sum_{i=1}^{n} c_{ii} = \alpha \sum_{i=1}^{n} a_{ii} + \beta \sum_{i=1}^{n} b_{ii}$

$= \alpha . 0 + \beta . 0$

$= \underline{0}$.

$c_{11} = \alpha a_{11} + \beta b_{11}$

$= \alpha . 0 + \beta . 0$

$c_{11} = \underline{0}$

$W_{23} = \{ A \in M_n(\mathbb{R}),\ \text{trace}(A)=0,\ a_{11} = 0 \}$

$\underline{n=3} \Rightarrow W = \{ A \in M_3(\mathbb{R}),\ \text{trace}(A)=0,\ a_{11}=0 \}$

$\left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},\ \text{trace}(A)=0,\ a_{11}=0 \right\}$

$\Rightarrow = \begin{bmatrix} 0 & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & -a_{22} \end{bmatrix}$

$\Rightarrow \dim(W) = \underline{7}$

$\Rightarrow \boxed{\dim(W_{23}) = n^2 - 2}$

$\Rightarrow$ $\underline{W_{23} \text{ is subspace.}}$

* $W_{25} = \{ A \in M_n(\mathbb{R}),\ A = (a_{ij})_{n \times n},$ with $a_{ij} = 0$, where $i+j =$ even $\}$

Here if we take $\{ A = M_2(\mathbb{R}),\ A = (a_{ij})_{2\times 2}$. with $a_{ij} = 0$, where, $i+j =$ even $\}$

$\underline{\text{or}}$

$\left\{ A = \begin{bmatrix} 0 & a_{12} \\ a_{21} & 0 \end{bmatrix},\ \forall a_{ij} \in \mathbb{R} \right\}$

Basis, $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$

$\boxed{\dim = 2}$

$\Rightarrow$ W_{25} is a subspace of

$\dim W_{25} = \begin{cases} \dfrac{n^2}{2}, & n \text{ is even} \\ \dfrac{n^2-1}{2}, & n \text{ is odd.} \end{cases}$

Note -

* $L_1 = \{(x,1), x \in \mathbb{R}\}$, $L_2 = \{(x,2), x \in \mathbb{R}\}$ — two parallel line about x-axis, may or may not be from a subspace.

$L_1 \cup L_2$ is not subspace,

---

* space of polynomial — $V = P_n(x)$ — (of degree atmost 'n').

1) $W_{36} = \{p \in V$ s.t. $p(0) = 0\}$ = {p; p is a polynomial passing through origin}

2) $W_{37} = \{p \in V$ s.t. $p(\alpha) = 0\}$

3) $W_{38} = \{p \in V$ s.t. $p(x) = p(1-x)\}$ $\Rightarrow \dim W_{38} = [\frac{n}{2}] + 1$

4) $W_{39} = \{p \in V$ s.t. $p(x) = p(x^2)\}$.

5) $W_{40} = \{p \in V$ s.t. $p(x) = [p(x)]^2\}$.

6) $W_{41} = \{p \in V$ s.t. graph of $p(x)$ intersect x-axis atleast two point

7) $W_{42} = \{p \in V$ s.t. graph of $p(x)$ intersect x-axis atleast two given points

8) $W_{43} = \{p = a_0 + a_1 x + \dots + a_{10} x^{10},\ a_i \in \mathbb{R}$ with $a_1 + a_2 = 0,\ a_1 + a_3 = 0,\ a_1 - a_3 + a_4 = 0\}$.

9) $W_{44} = \{p \in V$ s.t. $p(\alpha) = p(\beta)\}$.

10) $W_{45} = \{p \in V$ s.t. 'p' has local maxima$\}$.

11) $W_{46} = \{p = a_0 + a_1 x + a_2 x^2;\ a_i \in \mathbb{R}$, 'p' has extreema at $x = 1\}$.

Soln - $W_{36} = \{p \in V;$ s.t $p(0) = 0\}$

$p, q \in W_{36}$ as $p(0) = 0,\ q(0) = 0$.

$\alpha p(x) + \beta q(x) = h(x)$

$h(0) = \alpha p(0) + \beta q(0)$

$= 0$ $\Rightarrow \alpha p(x) + \beta q(x) \in W$.

$\Rightarrow W_{36}$ is subspace.

* $V = \{p;\ p(x) = a_0 + a_1 x + a_2 x^2 + \dots a_n x^n,\ a_i \in \mathbb{R}\}$

$B = \{1, x, x^2, \dots x^n\}$ is a Basis of V.

* $W_{36} = \{p \in V,\ p(0) = 0\}$.

or

$W_{36} = \{p \in V,\ x(b_0 + b_1 x + \dots + b_n x^{n-1})\}$
$b_i \in \mathbb{R}$.

$B = \{x, x^2, x^3 \dots x^n\}$.

* $W_{37} = \{p \in V,\ p(\alpha) = 0\}$.

or

$W_{37} = \{p \in V,\ (x-\alpha)(b_0 + b_1 x + \dots + b_n x^{n-1},\ b_i \in \mathbb{R}\}$.

$B = \{(x-\alpha),\ x(x-\alpha), \dots\ x(x-\alpha)^{n-1}\}$.

* $W_{47} = \{p \in V,\ p(1) = p(-1) = 0\}$.

or

$W_{47} = \{p \in V,\ (x-1)(x+1)(b_0 + b_1 x + \dots + b_n x^{n-1},\ b_i \in \mathbb{R}\}$.

* $W_{38} = \{p \in V,\ \text{s.t } p(x) = p(1-x)\}$.

$\Rightarrow\ p, q \in W_{38},\ p(x) = p(1-x)$
$q(x) = q(1-x)$

$\alpha p(x) + \beta \cdot q(x) = h(x)$

replace x by $1-x$.

$\alpha p(1-x) + \beta \cdot q(1-x) = h(1-x)$

$\alpha p(x) + \beta \cdot q(x) = h(1-x)$

$h(x) = h(1-x)$

$\Rightarrow W_{38}$ is subspace

* $W_{38} = \{ p \in V \text{ s.t. } p(x) = p(1-x) \}$

$p(x) = ax + b$.

$p(1-x) = a(1-x) + b = ax + b$.

$= a - ax + \cancel{b} = ax + \cancel{b}$

$2ax = a$

$a(2x-1) = 0$

$a = 0$

$p(x) = b$ (constant polynomial)

$a \neq 0$.

$x = \frac{1}{2}$.

$p(x) = p(1-x)$ only for $x = \frac{1}{2}$ for degree 1. polynomial.

$\nexists$ any polynomial of degree 1, s.t. $p(x) = p(1-x)$, $\forall x$.

$p(x) = ax^2 + bx + c$.

$\cancel{ax^2} + bx + \cancel{c} = a(1-x)^2 + b(1-x) + c$

$= a + \cancel{ax^2} - 2ax + b - bx + \cancel{c}$

$2bx = a - 2ax + b$

$2bx - b = a - 2ax$

$b(2x-1) = a(1-2x)$

$b(2x-1) + a(2x-1) = 0$

$(2x-1)(a+b) = 0$

$a = -b$

$x = \frac{1}{2}$.

a general polynomial of degree '2' we have, $p(x) = p(1-x)$

$p(x) = ax^2 - ax + c$

$= ax(x-1) + c$.

$B = \{ 1, x(x-1) \}$

$W_{38}' = \{ p,\ \text{degree } p(x) \leq 2, \ p(x) = p(1-x) \}$ $= [\frac{n}{2}] + 1$

$B' = \{ 1, x(x-1) \}$

$W_{38}'' = \{ p,\ \text{degree } p(x) \leq 3,\ p(x) = p(1-x) \}$

$B'' = \{ 1, x(x-1) \}$

$W_{38}''' = \{ p,\ \text{degree } p(x) \leq 4,\ p(x) = p(1-x) \}$

$B''' = \{ 1, x(x-1), x^2(x-1)^2 \}$

* $W_{39} = \{ p \in V,\ \text{s.t. } p(x) = p(x^2) \}$

$\Rightarrow \quad \alpha \cdot p(x^2) + \beta \cdot q(x^2) = h(x^2)$

$\alpha \cdot p(x) + \beta \cdot q(x) = h(x^2)$

$h(x) = h(x^2)$

$\alpha \cdot p(x) + \beta \cdot q(x) \in W$.

$\Rightarrow$ W_{39} is subspace.

$\Rightarrow$ $\nexists$ any polynomial of degree ≥ 1 which have the property $p(x) = p(x^2)$

Hence, $W_{39} = \cancel{\phi}$ is subspace of constant polynomial.

$B = \{ 1 \}$.

* $W_{40} = \{$

* $W_{41} = \{ p \in V,$ s.t graph of $p(x)$ intersect x-axis atleast two point $\}$

$p(x) = \hat{0}(x)$ (zero polynomial) $\in W_{41}$

Let, $p(x) \in W_{41}$, $p(x)$ intersect x-axis atleast two point

$h(x) = \alpha \cdot p(x) + \beta \cdot q(x) \in W_{41}$,

$\alpha(p(x))$ also intersect x-axis as atleast two points.

$p(x) = x^2 - 1$

$q(x) = x^2 - 2$

$p(x) - q(x) = 1.$

$\Rightarrow W_{41}$ is Not subspace.

* $W_{43} = \{ p = a_0 + a_1 x + a_2 x^2 + \cdots + a_{10} x^{10},\ a_i \in \mathbb{R},\ a_1 + 2a_2 = 0,$
$a_2 + a_3 = 0,$
$a_1 - a_3 + a_4 = 0 \}$.

$\Rightarrow \dim W_{43} = \dim W + 7 \Leftarrow \underbrace{(a_0, a_5, a_6, a_7, a_8, a_9, a_{10})}_{7 \text{ choices}}$

$W = \{ p ;\ p(x) = \underline{a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4} \}$

$, a_1 + 2a_2 = 0$

$, a_2 + a_3 = 0$

$a_1 - a_3 + a_4 = 0$

$\dim W =$ dim of solution space of system $AX = 0$,

where $A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}_{3 \times 4}$, $X = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$

$\dim W = n - r$

$= 4 - \operatorname{rank}(A)$

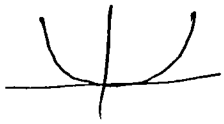
$= 4 - 3$

$\boxed{\dim W = 1}$

$\Rightarrow \boxed{\dim W_{43} = 8}$

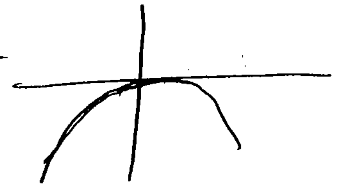
* $W_{45} = \{ p \in V,$ s.t. 'p' has local maxima $\}$ ⇒ Not subspace.

$p(x) = x^2$



$x=0$, local minima.

$\alpha \cdot p(x) = -1 \cdot p(x) = -x^2$



have no local minima

* <u>space of sequences</u> - over field $\mathbb{R}$.

① $W_{48} = \{ \langle a_n \rangle \in V$ s.t $\langle a_n \rangle$ is convergent $\}$.

② $W_{49} = \{ \langle a_n \rangle \in V,$ s.t. $\langle a_n \rangle$ is divergent $\}$. ex. $\langle a_n \rangle = n$, $\langle b_n \rangle = -n$; $\langle a_n + b_n \rangle = 0$ cgt $\notin W_{49}$

③ $W_{50} = \{ \langle a_n \rangle \in V,$ s.t. $\langle a_n \rangle$ is monotonic $\}$.

④ $W_{51} = \{ \langle a_n \rangle \in V$ s.t. $\langle a_n \rangle$ has atleast one limit point $\}$.

⑤ $W_{52} = \{ \langle a_n \rangle \in V$ s.t. $\langle a_n \rangle$ is bounded $\}$ ⇒ sum of bdd is bdd.

$W_{53} = \{ \langle a_n \rangle \in V$ s.t. $\langle a_n \rangle$ is unbdd $\}$, $\langle a_n \rangle = n$, unbdd; $\langle b_n \rangle = -n$, unbdd

⑥ $W_{54} = \{ \langle a_n \rangle \in V$ s.t. $\langle a_n \rangle$ has infinite no. of limit point $\}$

⑦ $W_{55} = \{ \langle a_n \rangle \in V$ s.t. $\langle a_n \rangle$ is cauchy $\}$ ⇒ sum of two cauchy is cauchy.

⑧ $W_{56} = \{ \langle a_n \rangle \in V$ s.t. $\langle a_n \rangle$ has unique limit point $\}$.

⑨ $W_{57} = \{ \langle a_n \rangle \in V,$ s.t. $\langle a_n \rangle$ is oscillatory $\}$

⑩ $W_{58} = \{ \langle a_n \rangle \in V,$ s.t. $\langle a_n \rangle$ has uncble limitpoints $\}$, $0 \notin W_{58}$

⑪ $W_{59} = \{ \langle a_n \rangle \in V,$ s.t. $\langle a_n \rangle$ has countably infinite no. of limit points $\}$, $0 \notin W_{59}$

⑫ $W_{60} = \{ \langle a_n \rangle \in V,$ s.t. $\langle a_n \rangle$ is eventually monotonic $\}$

⑬

<u>ex W_{50}</u> ⇒ $\langle a_n \rangle = \{1, 1, 1, 1, 2, 2, 2, 2 \ldots\}$

ex W_{60} = $\langle b_n \rangle$ $\{-1, -1, -2, -2, \ldots\}$

ex. W_{51} and W_{56} = $\langle a_n \rangle = \langle 1, 0, 2, 0, 3, 0 \ldots \rangle$
$\langle b_n \rangle = \langle 0, 1, 0, 2, \ldots \rangle$ } one limit point.

$\langle a_n + b_n \rangle = \langle 1, 1, 2, 2, \ldots \rangle \Rightarrow$ no limit point

ex. W_{53} = $\langle a_n \rangle = n$ unbdd
$\langle b_n \rangle = -n$ unbdd

$\langle a_n + b_n \rangle = \langle 0 \rangle \notin W_{53} \Rightarrow$ not subspace

ex. W_{54} = $\langle 0 \rangle \notin W_{54} \Rightarrow$ not subspace

ex. W_{57} = $\langle a_n \rangle$ =

$0 \notin W_{57} \ldots \rightarrow$ not subspace

ex. $W_{60} \Rightarrow \langle a_n \rangle = \langle 0, 1, 2, 1, 1, 1, 1, 1, 2, 2 \rangle$
$\langle b_n \rangle = \langle 0, 1, 2, -1, -1, -2, -2, -2, -2 \rangle$ } eventually monotonic

$\langle a_n + b_n \rangle = \langle 0, 2, 4, 0, 0,$

$\Rightarrow 0 \in W_{60}$
$= \alpha . x \in W_{60}$

W_{60}: $\langle a_n \rangle = \langle 1, 5, 8, 10, 15, 20, \ldots \rangle$
$\langle b_n \rangle = \langle 1, -6, -9, -9, -16, -19 \ldots \rangle$ } $\Rightarrow$ eventually $\rightarrow$ monotonic

$\langle a_n + b_n \rangle = \langle 2, -1, 1, -1, 1, \ldots \rangle \Rightarrow$ not eventually monotonic

$\Rightarrow$ W_{60} is not subspace.

* Space of functions ..

$V =$ space of functions defined on I:

✓ $W_{62} = \{ f \in V$ s.t. 'f' is bdd funⁿ on $I \}$

✗ $W_{63} = \{ f \in V$, s.t. 'f' is monotonic on $I \}$ $\{x\} = x - [x]$

✗ $W_{64} = \{ f \in V$, s.t. 'f' is periodic on $I \}$. = Linear combination P. F may or may not be periodic.

✓ $W_{65} = \{ f \in V$, s.t. 'f' is cts on $I \}$

✗ $W_{66} = \{ f \in V$. s.t. f is discts on $I \}$, zero funⁿ ∉ W_{66}

✓ $W_{67} = \{ f \in V$ s.t. 'f' is Lipschitz on $I \}$ – sum of Lipschitz is Lipsch

✓ $W_{68} = \{ f \in V$ s.t. 'f' is Riemann integrable on $I = [a,b] \}$

✓ $W_{69} = \{ f \in V$ s.t. f is of Bounded variation on $I = [a,b] \}$

✓ $W_{70} = \{ f \in V$ s.t. f is U.C. on $I \}$.

✓ $W_{71} = \{ f \in V$, s.t. f is diff. on $I \}$

✗ $W_{72} = \{ f \in V$, s.t. f is convex on $I \}$

✗ $W_{73} = \{ f \in V$. s.t. 'f' is concave on $I \}$

✓ $W_{74} = \{ f \in V$. s.t. f satisfy the Rolles thᵐ on $[a,b] \}$

✓ $W_{75} = \{ f \in V$ s.t. 'f' satisfy the Lagranges mean value thᵐ on $[a,b] \}$.

✗ $W_{76} = \{ f \in V$ s.t. $\int_a^b f(x)\,dx = 0$ on $[a,b] \}$.

✓ $W_{77} = \{ f \in V$, s.t. f is cts s.t. $\int_0^1 [f(x)]^2 dx = 0 \}$ This is only = zero space

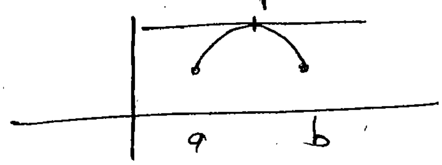
$\Rightarrow W_{74} =$

Rolle's th^m = Let, $f : [a,b] \xrightarrow{cts} \mathbb{R}$ $\exists$ no break in the graph

$f : (a,b) \xrightarrow{diff} \mathbb{R}$ $\exists$ at $c \in (a,b)$ have unique

s.t. $f(a) = f(b)$.

$\exists\, c \in (a,b)$ s.t. $f'(c) = 0$.



$0 \in W_{74}$

$\Rightarrow W_{74} \neq \phi$

$f(x), g(x) \in W_{74}$

f, g cts on $[a,b]$ $\Rightarrow$ $f+g$ is cts on $[a,b]$

$f \cdot g$ diff on (a,b) $\Rightarrow$ $f+g$ diff. on (a,b)

$f(a) = f(b)$, $g(a) = g(b)$

$f(a) + g(a) = f(b) + g(b)$

$(f+g)(a) = (f+g)(b)$

$\exists\, c \in (a,b)$ s.t. $f'(c) = 0$.

$\Rightarrow W_{74}$ is subspace.

$\Rightarrow \underline{W_{76}} =$ $f, g \in W_{76}$

$\int_a^b f(x)\,dx = 0$, $\int_a^b g(x)\,dx = 0$.

$\alpha \cdot f + \beta \cdot g = h(x)$.

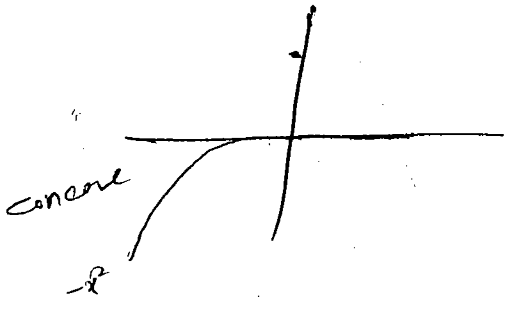
$\int_a^b h(x)\,dx = \int_a^b [\alpha f(x) + \beta g(x)]\,dx$

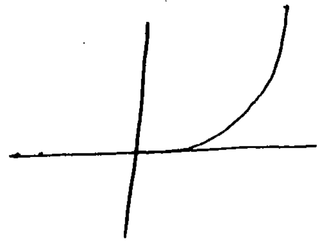
$= 0$.

$\Rightarrow$ W_{76} is subspace

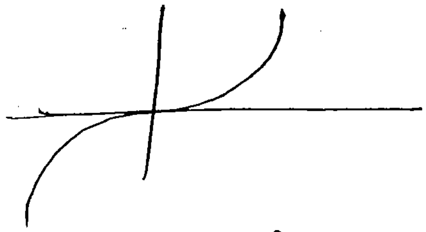
78. ✗ $W_{78} = \{ f \in V$ s.t. f is either convex or concave on $I \}$

$$g(x) = \begin{cases} -x^2, & x \leq 0, \\ 0, & x \geq 0 \end{cases} \qquad f(x) = \begin{cases} 0, & x \leq 0 \\ x^2, & x \geq 0, \end{cases}$$





$$f(x) + g(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x \leq 0 \end{cases}$$



Neither, convex Nor concave.

$\Rightarrow W_{78}$ is not subspace

$W_{79} = \{ f$ is complex valued function s.t. f is analytic fun$^n \}$

$W_{80} = \{ f$ is complex valued function s.t. f is satisfy C.R eq$^n \}$

* Linear combination -

Let, 'V' is a vector space over field 'F', and let, $x_1, x_2, \ldots, x_n$ are n-vectors of vector space 'V' and let, a vector $x \in V$ and the scalars $\alpha_1, \alpha_2, \alpha_3 \ldots, \alpha_n \in F$ s.t.

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n$$

Then, 'x' is called a Linear combination of the vectors x_i's by using the scalars α_i's. & they are always unique.

* Note ① - Linear combination can be discussed over FDVS or over Non-FDVS.

② - set of vectors x_i's should be finite.

* Linear Span / (smallest) Generating set / Spanning set -

Let, $S = \{x_1, x_2, x_3, \ldots x_n\}$ be the set of vectors of V(F) then the linear span of 'S' is denoted by L(S) and defined as,

L(S) is the intersection of all the subspaces containing 'S'.

(Hence, $L\{\phi\} = \{0\}$, as $L(\phi) = \cap$ of all the subspaces containing ϕ.

$$= \{0\} \cap W_1 \cap W_2 \cap \ldots$$

$$= \{0\}$$

$$\Rightarrow \boxed{L\{\phi\} = \{0\}}\ )$$

also, $\boxed{L\{0\} = \{0\}}$)

Hence, L(S) is the smallest subspace of 'V' which contains 'S'.

If 'S' is Nonempty, then L(S) is the collection of all possible linear combinations.

$S = \{x_1, x_2, x_3 \dots x_n\} \subseteq V.$

$|F| = p^r$

Let, $x \in L(S)$ (arbitrary vector of $L(S)$)

$\Rightarrow \exists \alpha_1, \alpha_2, \dots \alpha_n \in F$ s.t.

$$x = \underbrace{\alpha_1}_{p^r \text{ choices}} x_1 + \underbrace{\alpha_2}_{p^r \text{ choices}} x_2 + \dots + \underbrace{\alpha_n}_{p^r \text{ choices}} x_n$$

total choices for $x = p^r \cdot p^r \dots p^r$ (n-times)

$= p^{n \cdot r}$

Hence, $|L(S)| \leq p^{n \cdot r}$

$$\boxed{|L(S)| \leq |F|^{|S|}}$$

Let $S = \{x_1, x_2, x_3 \dots x_n\}$ is L.D.

WLG, x_q $(1 \leq q \leq n)$ is L.C of preceeding vectors.

$x_q \in L\{x_1, x_2, \dots x_{q-1}, x_{q+1}, \dots x_n\}$

Here, $L(S) = L(x_1, x_2, \dots x_{q-1}, x_{q+1}, x_n)$

$|L(S)| \leq p^{q(n-1)} < p^{n \cdot q}$

Hence, If 'S' is L.D., $|S| = n$, with $|F| = p^r$.

then, $\boxed{|L(S)| < p^{nr}}$

MODERN * ALGEBRA

'Monu Kummar'

Page No.
Date : / /

Set : Collection of well-defined distinct objects is called set.

Cartesian Product :- Let A, B be two sets.

$$A \times B = \{(a,b) : a \in A, b \in B\}$$

is called cartesian product of A & B.

e.g. Let $A = \{1,2,3\}$, $B = \{1,2\}$

$\Rightarrow A \times B = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,2)\}$

Note : If $|A| = m$, $|B| = n$ then $|A \times B| = m \cdot n$.

Relation :- A relation from A to B is a subset of $A \times B$. Infact, every subset of $A \times B$ is a relation from A to B.

$\Rightarrow$ No. of relations from A to B = No. of subsets of $A \times B = 2^{|A \times B|}$.

Note : A relation from A to A is a subset of $A \times A$

$\Rightarrow$ No. of relations on A = No. of subsets of $A \times A$
$= 2^{|A \times A|}$.

Type of Relation :-

(1) **Identity Relation** - Let $I \subseteq A$, such that if $a \in A$ then $(a,a) \in I$ (only).

i.e. if $a \in A \Rightarrow a R a$ only.

Ques: Let $A = \{1,2,3\}$

$\Rightarrow A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$ Then -

x i) $S = \{(1,1), (2,2)\}$ is identity relation ?

✓ ii) $S = \{(1,1), (2,2), (3,3)\}$ is identity relation

✗ iii) $S = \{(1,1), (2,2), (3,3), (1,3)\}$ is identity relation

✗ iv) $S = \phi$ is identity relation

Note: If $A = \phi \Rightarrow A \times A = \phi \times \phi$ & $S = \phi \subseteq \phi \times \phi$ then S is an identity relation.

[2] Reflexive Relation - Let $S \subseteq A \times A$ s.t. if $a \in A$ then $(a,a) \in S$.

i.e. if $a \in A$ then aRa then S is called reflexive relation.

Ques - Let $A = \{1,2,3\}$

$\Rightarrow A \times A = \{(1,1), (1,2), \text{------} (3,3)\}$ Then -

✗ i) $S = \{(1,1), (2,2)\}$ is reflexive

✓ ii) $S = \{(1,1), (2,2), (3,3)\}$ is reflexive

✓ iii) $S = \{(1,1), (2,2), (3,3), (1,3)\}$ is reflexive

✗ iv) $S = \phi$. is reflexive

Notes:- If S is a reflexive relation on a set A & I is identity relation then $I \subseteq S$.

[3] Irreflexive relation - Let $S \subseteq A \times A$ s.t. if $a \in A$ then $(a,a) \notin S$.

i.e. if $a \in A$ then $a \not R a$ then S is called Irreflexive relation.

Ques: Let $A = \{1,2,3\}$

$\Rightarrow A \times A = \{(1,1), (1,2), \text{.....} (3,3)\}$ then

✗ i) $S = \{(1,1), (1,2)\}$ is irreflexive relation

Monu Kummar

✓ ii) $S = \{(1,2),(2,1)\}$ is irreflexive relation
✓ iii) $S = \{(1,2),(1,3),(2,3)\}$ is irreflexive relation.
✗ iv) $S = \{(1,1),(1,3)\}$ is irreflexive relation.
✓ v) ϕ is irreflexive relation

Note - If S is Irreflexive relation then - $S \cap I = \phi$.

4] **Symmetric Relation** - Let $S \subseteq A \times A$ s.t. if $(a,b) \in S$ then $(b,a) \in S$.

i.e. if aRb then bRa

then S is called symmetric relation.

Ques - Let $A = \{1,2,3\}$

$\Rightarrow A \times A = \{(1,1),(1,2),(1,3),(2,1), \ldots\ldots (3,3)\}$ Then -

✓ i) $S = \{(1,1)\}$ is symmetric relation.
✓ ii) $S = \{(1,1),(2,2),(3,3)\}$ is symmetric relation
✓ iii) $S = \{(1,2),(2,1)\}$ is symmetric relation.
✗ iv) $S = \{(1,3)\}$ is symmetric relation
✓ v) $S = \phi$ is symmetric relation

5] **Asymmetric Relation** - Let $S \subseteq A \times A$ if $(a,b) \in S$ $\Rightarrow$ $(b,a) \notin S$ then S is called asymmetric relation.

if $a = b \Rightarrow (a,a) \notin S$.

Ques - Let $A = \{1,2,3\}$

$\Rightarrow A \times A = \{(1,1),(1,2), \ldots\ldots (3,3)\}$. Then which of the following is ~~ias~~

✗ i) $S = \{(1,1)\}$
✓ ii) $S = \{(2,3)\}$
✗ iii) $S = \{(1,2),(2,1)\}$
✓ iv) $S = \{(1,3),(1,2),(2,3)\}$
✓ v) $S = \phi$

[6] **Anti-symmetric Relation** :- Let $S \subseteq A \times A$ such that if $(a,b), (b,a) \in S \Rightarrow a = b$

then S is called anti-symmetric relation.

i.e. if aRb & $bRa \Rightarrow a = b$

if $a = b \Rightarrow (a,a) \in S$.

Ques : Let $A = \{1,2,3\}$

$\Rightarrow A \times A = \{(1,1),(1,2), \ldots (3,3)\}$ Then -

✓ i) $S = \{(1,1),(2,2)\}$ is antisymmetric relation

✓ ii) $S = \{(1,1),(2,2),(3,3),(1,3)\}$ is antisymmetric relation

✓ iii) $S = \{(1,1),(1,2)\}$ is antisymmetric relation

✓ iv) $S = \phi$ is antisymmetric relation

✗ v) $S = \{(1,1),(2,2),(3,3),(1,2),(2, \ )\}$ is antisymmetric relat'

[7] **Transitive Relation** :- Let $S \subseteq A \times A$ such that if (a,b) & $(b,c) \in S \Rightarrow (a,c) \in S$.

i.e. if aRb & $bRc \Rightarrow aRc$

then S is called transitive relation.

Ques : Let $A = \{1,2,3\}$

$\Rightarrow A \times A = \{(1,1),(1,2), \ldots (3,3)\}$ Then -

✓ i. $S = \{(1,2),(2,3),(1,3)\}$ is transitive relation

✓ ii. $S = \{(1,1),(2,2),(3,3)\}$ is transitive relation

✓ iii. $S = \{(1,2)\}$ is transitive relation

✓ iv. $S = \phi$. is transitive relation

Equivalence Relation :- Let $S \subseteq A \times A$ such that -

S is - i) Reflexive ii) Symmetric &

iii) Transitive

then S is called equivalence relation.

Page No.
Date : / /

Monu Kummar

Ques :- Let $A = \{1, 2, 3\}$

$\Rightarrow$ $A \times A = \{(1,1), (1,2), \ldots\ldots (3,3)\}$ Then -

✓ i) $S = \{(1,1), (2,2), (3,3)\}$ is equivalence relation

x ii) $S = \{(1,1), (2,2)\}$ is equivalence relation

x iii) $S = \phi$ is equivalence relation

✓ iv) $S = \{(1,1), (2,2), (3,3), (1,3), (3,1)\}$ is equi. relation

Ques- Let A = collection of all straight line in plane. define a relation R on A. such that - if $L_1, L_2 \in A$ then $L_1 R L_2$ iff $L_1 \parallel L_2$.
Then R is an equi. relation on A.?

Ques :- Let A = collection of all straight line in plane. define a relation R on A s.t. if $L_1, L_2 \in A$ then $L_1 R L_2$ if $L_1 \perp L_2$.
Then R is not an equi. relation on A.?

Ques :- Define a relation R on A (collectⁿ of all human bein s.t. $a, b \in A$, $a R b$ iff a is brother b
Then - R is not an equivalence relation. ?

Ques :- A = collectⁿ of all mens.
define a relation R on A s.t. $a, b \in A$, $a R b$ iff a is brother b
Then - R is an equivalence relation. ?

Counting of Relation :⇒

Let $A = \{a_1, a_2, \ldots\ldots a_n\}$

$\Rightarrow$ $A \times A = \{(a_1, a_1), (a_1, a_2), \ldots\ldots (a_n, a_n)\}$

i) Identity relation - if $a \in A \Rightarrow (a, a) \in S$ only.

Page No.
Date : / /

Now :- (a_1, a_1) — 1 (choice) (a_1, a_2) - 1 (choice)

(a_2, a_2) — 1 ⋮

⋮

(a_n, a_n) — 1 (choice) (a_i, a_j) - 1 if $i \neq j$

$\Rightarrow$ No. of identity relations $= \underbrace{1 \cdot 1 \cdot 1 \cdots\cdots}_{n^2 \text{ times}} = 1$

[2] **Reflexive** - $S \subseteq A \times A$, if $a \in A \Rightarrow (a, a) \in S$ (Reflexive)

(a_1, a_1) - 1 (choice) (a_1, a_2) - 2 (Choices)

(a_2, a_2) - 1 (choice) (a_2, a_1) - 2 (Choices)

⋮ ⋮

(a_n, a_n) - 1 (choice) (a_i, a_j) - 2 if $i \neq j$

$\Rightarrow$ No. of reflexive relations $= (\underbrace{1 \cdot 1 \cdot 1 \cdots 1}_{n\text{-times}})(\underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{n^2-n \text{ times}})$

$= 2^{n^2-n}$

[3] **Irreflexive** - $S \subseteq A \times A$, if $a \in A \Rightarrow (a, a) \notin S$. (Irreflexive)

(a_1, a_1) - 1 (choice) (a_1, a_2) - 2 (choices)

(a_2, a_2) - 1 (choice) (a_2, a_1) - 2 (Choices)

⋮ ⋮

(a_n, a_n) - 1 (choice) (a_i, a_j) - 2 if $i \neq j$

$\Rightarrow$ No. of Irreflexive relations $= (1 \cdot 1 \cdots 1)(\underbrace{2 \cdot 2 \cdots 2}_{n^2-n \text{ times}})$

$= \underline{2^{n^2-n}}$.

[4] **Symmetric** - if $(a, b) \in S \Rightarrow (b, a) \in S$ (Symmetric)

if $a = b$, $(a, a) \in S$.

Vinit Raj (1)

15/5/17

Partial Differential Equation

(4-5 question)
(15 to 20 marks)

Syllabus:-

also imp [1) Origin of pde.
2) classification of linear and non-linear pde.

100% imp [3) First order and first degree pde and Solution by lagrange's method. (1) ques
(Extension of lagrange's method).

100% imp [4) Non-linear Partial diff. eqn [Ist order but not of first degree]. (1) ques
5) classification of second order pde → Parabola, hyperbola, ellipse. and their canonical form.

also imp [(6) Product method (seperation method).
7) Heat equation (1-2) ques
8) Wave equation
9) Laplace equation.
10) Dirichlet eqn. } Boundary on interior of circles.
11) Neumann eqn. }

12) characteristic of non-linear P.d.e.

* Partial diffⁿ eqⁿ :- P.D.E is partial diffⁿ eqⁿ is a relation between dependent variable and some of its derivatives w.r to more than one independent variables.

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} + z = 0$$

dependent

independent

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} = \sin(x+y)$$

not in discussion at

$$\frac{\partial^2 z}{\partial x_1^2} + \frac{\partial z}{\partial x_1} + \frac{\partial^2 z}{\partial x_2^2} + \frac{\partial z}{\partial x_2} + \frac{\partial^2 z}{\partial x_3^2} + \frac{\partial z}{\partial x_3} = x_1 + x_2 + x_3$$

z - dependent

$x_1, x_2, x_3 \longrightarrow$ independent

* Order of P.D.E :-

The order of highest derivative occurring in p.d.e is called the order of PDE.

eg :- i) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} = \sin(x+y)$

highest derivatives is two

$\Rightarrow$ order is 2.

ii) $\frac{\partial z}{\partial y} + \frac{\partial z}{\partial x} + z = \sin z$

$\Rightarrow$ order is 1 but degree not define

iii) $\frac{d^2 y}{dx^2} + \frac{dy}{dx} + y = \sin y \longrightarrow$ independent vari

$= \{1 - \frac{y^3}{3!} + \text{---}$

$\Rightarrow$ order is 2

but degree is not defined.

* **Degree of P.D.E :-**

The power of highest derivative in p.d.e after made it free from radicals and fractions, so far as derivative are concerned.

eg :- $\frac{dy}{dx} = \sqrt{\sin x}$ → power is radical

$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \sin x$ → order is 1 → degree is 2 (free from radicals)

Example :- $\left(\frac{dy}{dx}\right) = \left(\frac{2(dy/dx)^2 + y}{dy/dx}\right)$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 = 2\left(\frac{dy}{dx}\right)^2 + y$ (free from fraction)

$\Rightarrow \left(\frac{dy}{dx}\right)^2 + y = 0$ → order is 1 → degree is 2.

eg $\sin\left(\frac{dy}{dx}\right) + y = 2$ → can't be free from radicals.

⇒ order is 1 but degree not defined.

> Note :- If there is any radicals or fractions in pde the order is not change.

> Note :- If dependent variable or its derivative containing in pde in such a way that they can't free from the radicals. (eg :- $\sin\left(\frac{dy}{dx}\right)$, $e^{\left(\frac{dy}{dx}\right)}$) degree is not defined.

> Note :- $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$; $\frac{\partial^2 z}{\partial x^2} = r$, $\frac{\partial^2 z}{\partial x \partial y} = s$, $\frac{\partial^2 z}{\partial y^2} = t$.
>
> z - dependent, x, y → independent.

* Linear and non-linear pde :-

A pde is said to be linear if the dependent var. and its derivative occurs only in the first degree and are not multiplied.

And if it is not linear then is called non linear.

Example :- 1) $\frac{\partial z}{\partial y} + \frac{\partial z}{\partial x} + y = x$ → linear.

2) $zy \cdot \frac{\partial z}{\partial x} + x \cdot \frac{\partial z}{\partial y} = 0$ → not linear

two dependent z & $\frac{\partial z}{\partial x}$ is multiply

3) $\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial z}{\partial y} = 2$ → not linear
→ first order but not first degree.
⇒ non-linear.

derivative multiply

* Classification of first order p.de :-

1) Linear p.d.e :- A pde $F(x, y, z, p, q) = 0$ is said to be linear pde of order 1 if it is linear in p, q and z.

And equation can be written in the form

$$\boxed{P(x,y) \cdot p + Q(x,y) \cdot q + R(x,y)\, z = F(x,y)}$$

where $P(x,y)$, $Q(x,y)$, $R(x,y)$ are f^n of x and y only.

Example :- 1) $x^2 p + y^2 q = 0$
2) $e^{x+y} \cdot p + \sin(y-x)\, q = x^2 + y^2$ } linear

3) $x^2 p - y^2 q = z^2 + xy$ → not linear

not linear in z, R is not in form (x,y)

2) Semi linear :-

A pde eqⁿ $F(x,y,z,p,q)=0$ is said to be Semi linear if it is linear in p and q and can be written in the form

$$P(x,y)\cdot p + Q(x,y)\cdot q + R(x,y,z) = F(x,y)$$

(or)

$$\boxed{P(x,y)\, p + Q(x,y)\, q = R(x,y,z)}$$

Example :- $x^2 p + y^2 q + \underset{R(x,y,z)}{z^2} = \underset{F(x,y)}{(x+y)}$

(or)

$$x^2 p + y^2 q = \underbrace{(x-y) - z^2}_{R(x,y,z)}$$

Example :- $p+q+z=0 \rightarrow$ (linear, semi linear)

$p+q+z^2=0 \rightarrow$ (semi linear, not linear)

z is not linear

Note :- Linear pde $\Rightarrow$ semi linear pde
but Semi linear $\not\Rightarrow$ linear pde

3) Quassi linear pde :-

A pde $F(x,y,z,p,q)=0$ is said to be quassi linear pde if it can be written as

$$\boxed{P(x,y,z)\cdot p + Q(x,y,z)\cdot q = R(x,y,z)}$$

Example :- $x^2 z p + y^2 q = z^2(x+y)$

Note :- Linear $\Rightarrow$ Semi linear $\Rightarrow$ Quassi linear.

16/5/17.

Q1.) The diff^n eq^n is $y \cdot \frac{\partial u}{\partial x} + u \cdot \frac{\partial u}{\partial y} = 5u \cdot x$, $(x \& y > 0)$

i) linear homogeneous.
ii) linear non homogeneous.
~~iii)~~ Quasi linear.
iv) Semi linear.

Q2) The given p.d.e is $(x^2+y^2) z_x + x^3 \cdot z_y = xyz^2$

$\left(\because z_x = \frac{\partial z}{\partial x} \right.$
$z_y = \frac{\partial z}{\partial y}$

i) Linear.
ii) Semi but not quasi.
~~iii)~~ Semi but not linear.
iv) non-linear.

Sol^n 1) $y \cdot u_x + u \cdot u_y = 5u \cdot x$

$y \cdot p + u \cdot q = 5u \cdot x$

here is multiplication of $u \cdot \frac{\partial u}{\partial y}$ $\Rightarrow$ not linear
~~de~~variable $\Rightarrow$ not semi lin
~~and~~ dependent variables
& of its derivative

Sol^n 2) $(x^2+y^2) \cdot p + x^3 \cdot q = xy z^2 \rightarrow$ z is not linear

$\Rightarrow$ semi linear
$\Rightarrow$ Quasi linear

-+- SETS AND FUNCTIONS -+-

Sets :- The collection of well-defined objects, is called set. Collection should be of distinct objects.

That is, we can say collection of all such type of elements or objects which satisfy some rule, it is posible to say, whether a particular objects belongs to the collection or not. ①

e.g $S = \{$ collection of elements from $\mathbb{Z}$ which satisfy $2x^3+x^2-2x-1=0\}$

i.e $S = \{x \in \mathbb{Z}$ s.t. $2x^3+x^2-2x-1=0\}$

as we put $x = 1, -1, -\frac{1}{2}$ in $2x^3+x^2-2x-1$ we get 0; but out of these only 1 and -1 belongs to $\mathbb{Z}$. 1 and -1 are only two objects which satisfy thegiven rule.
The given rule is that, x should be from $\mathbb{Z}$ and satisfies $2x^3+x^2-2x-1=0$.

Hence S is nothing only $S = \{1, -1\}$

e.g. $T = \{x \in \mathbb{R}$ s.t. $x \neq x\}$

As we know every element or every real number is always equal to itself. So there is no real number which satisfies the given rule. Hence T is empty

i.e $T = \phi$

Equality of Sets :- Two sets are said to be equal when they consists of exactly the same elements.

Subset :- If S and T are two sets s.t. each member of S is also member of T. Then S is subset of T and denoted as $S \subseteq T$.

i.e. $x \in S \Rightarrow x \in T$

$\Rightarrow \boxed{S \subseteq T}$

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Note :- If $x \in A \Rightarrow x \in B$ and if there is $y \in B$ but $y \notin A$, then A is called proper Subset of B. Written as $\boxed{A \subset B}$

Note :- Every set is subset of itself.

Note :- Empty Set is a Subset of every set.

Note : Empty set is unique.

Russel paradox :- There is no set of all sets

Let $A = \{ X \text{ is a set}, \; X \notin X \}$

Let if posible A is a set

$\Rightarrow A \in A$; but if any element belongs to A say X satisfies $X \notin X$, so here if $A \in A$ if $A \notin A$ [contradiction],

contradiction comming by taking wrong assumption

Hence, There is no set of all sets

$\boxed{\text{Paradox means self contradictory statement}}$

Cardinality of a Set :- The number of elements in any set A is called cardinality of A. It is denoted by Card(A) or $|A|$.

e.g. $S = \{e, i, o, a, u\}$.

$\text{Card}(S) = |S| = 5$

Power Set :- The set of all subsets of any set S is said to be a power set, and It is denoted by $P(S)$ ②.

e.g. $S = \{p, q, r\}$, then

$P(S) = \{\phi, S, \{p\}, \{q\}, \{r\}, \{p,q\}, \{p,r\}, \{q,r\}\}$

here $|P(S)| = \text{card } P(S) = 8 = 2^3$

Note :- If set S contains n elements then its power set will contain 2^n elements i.e cardinality of power set is 2^n. $(|P(S)| = 2^n)$

Proof Card(P(S)) = Number of sets with no elements
+
Number of subsets with 1 elements
+
Number of subsets with 2 elements
+
⋮
Number of subsets with n elements

i.e $|P(S)| = {}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n$

$= 1 + {}^nC_1 + {}^nC_2 + \cdots + 1$

$= (1+1)^n$

$\boxed{|P(S)| = 2^n}$

Cartesian Product of Sets :-

for any two sets A and B, cartesian product of A and B is denoted by $A \times B$ and defined as -

$$A \times B = \{(a,b);\ a \in A,\ b \in B\}$$

For three sets A, B and C;

$$A \times B \times C = \{(a,b,c),\ a \in A,\ b \in B,\ c \in C\}.$$

In General

$$\boxed{\prod_{i=1}^{n} A_i = A_1 \times A_2 \times A_3 \times \cdots \times A_n = \{(a_1 a_2 \cdots a_n);\ a_i \in A_i\ \forall\ i = 1 \text{ to } n\}}$$

here $(a_1, a_2, \ldots, a_n)$ called an ordered n- tupple.

Results :-

(1) If $|A| = m$ & $|B| = n$ then $|B \times A| = |A \times B| = m \times n = m \cdot n$

(2) $A \times B \Leftrightarrow B \times A$ iff $A = B$

(3) If $|A \cap B| = m$ then $|(A \times B) \cap (B \times A)| = m^2$

(4) $A \subset B \Rightarrow A \times C \subset B \times C$

(5) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(6) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(7) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

(8) $(A \times C) - B \times C = (A - B) \times C$

(9) $A \subset B$ and $C \subset D$ then $A \times C \subset B \times D$

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Union and Intersection of sets :

$$A \cup B = \{x;\ x \in A \text{ or } x \in B\}$$

$$A \cap B = \{x;\ x \in A \text{ and } x \in B\}.$$

Properties, for any sets A, B and C

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(1) $A \subset A \cup B$

(2) $B \subset A \cup B$

(3) $A \cup A = A$ (Idempotent law)

(4) $A \cup (B \cup C) = (A \cup B) \cup C$ (Associative law).

(5) $A \cap B \subset A$

(6) $A \cap B \subset B$

3

(7) $A \cap \phi = \phi$

(8) $A \cap A = A$ (Intersection is Idempotent)

(9) $A \cap (B \cap C) = (A \cap B) \cap C$ (Associative)

Disjoint Set : Two sets A and B are said to be disjoint if they have no element in common, means there intersection should be ϕ.

i.e $\boxed{A \text{ and } B \text{ are disjoint if } A \cap B = \phi}$

Note Every disjoint sets are distinct sets, not Conversily.

e.g. $S = \{1, 2, 3, 4\}$, $T = \{2, 5, 7, 8\}$.

here S and T are distinct but not disjoint because $S \cap T = \{2\} \neq \phi$.

Difference of two sets :-

$A - B = \{x;\ x \in A \text{ and } x \notin B\}$.

All elements which are in A but not in B.

Note (1) Difference not Commutative (i.e $A - B \neq B - A$)

Note (2) Difference not associative (i.e. $A - (B - C) \neq (A - B) - C$)

Symmetric Difference :- Let X and Y are two sets then symmetric difference of sets X and Y denoted by $X \Delta Y$ and defined as

$$\boxed{X \Delta Y = (X - Y) \cup (Y - X) = (X \cup Y) - (X \cap Y)}$$

Functions Or Mappings :- Let A and B are two sets; if there is a rule 'f' which assigns to every element of A to a unique element of B. Then such a rule 'f' is called a function from Set A to B.

We write it as $f : X \longrightarrow B$. Here A is called domain and B is called co-domain of function f.

Note : If we take two different domain for same rule f, then can be consider as two distinct function.

i.e. Let $f : A \longrightarrow B$ s.t. $f(x) = \sin x$.

& $f : C \longrightarrow B$ s.t. $f(x) = \sin x$.

here rule for both are same $f(x) = \sin x$ but domain are different, that's why $f : A \longrightarrow B$ and $f : C \longrightarrow B$ can consider as two distinct function.

Note Two functions 'f' and 'g' are equal iff

(I) domain of 'f' = domain of g

(II) $f(x) = g(x) \quad \forall \; x \in$ domain.