Student Information

Name : Ömer Kılınç

ID : 2448603

Answer 1

$$f_{X,Y}(x,y) = \begin{cases} x + ky^3 & 0 \leq x \leq 1 \text{ , } 0 \leq y \leq 1 \\ 0 & 0 \text{ otherwise} \end{cases} \tag{1}$$

a)

By the definition of probability density function:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = 1$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x + ky^3 dxdy = 1$$

$$= \int_0^1\int_0^1 x + ky^3 dxdy$$

$$= \int_0^1 (\frac{x^2}{2} + kxy^3\Big|_0^1)dy$$

$$= \int_0^1 \frac{1}{2} + ky^3 dy$$

$$= \frac{x^2}{2}y + k\frac{y^4}{4}\Big|_0^1$$

$$= \frac{1}{2} + \frac{k}{4} = 1$$

$$k = 2$$

b)

$$P(X = x) = \int_x^x f_X(\tau) d\tau = 0$$

X has continuous distribution.

Probability of X having a particular value is zero for any X = x

$$P(X = x) = 0$$

c)

Computing the probability of Joint Distribution:

$$P((X,Y) \in A) = \int \int_{(x,y) \in A} f_{X,Y}(x,y) dx dy$$

$$P(0 \leq x \leq 1, 0 \leq y \leq 1) = \int_0^{1/2} \int_0^{1/2} x + 2y^3 dx dy$$

$$= \int_0^{1/2} (\frac{x^2}{2} + 2xy^3 \Big|_0^{1/2}) dy$$

$$= \int_0^{1/2} \frac{1}{8} + y^3 dy$$

$$= (\frac{1}{8} y + \frac{y^4}{4}) \Big|_0^{1/2}$$

$$= 1/16 + 1/64 = \frac{5}{64}$$

$$P(0 \leq x \leq 1, 0 \leq y \leq 1) = \frac{5}{64}$$

Answer 2

$$f_{X,Y}(x,y) = \frac{e^{-y-\frac{x}{y}}}{y} \text{ for } 0 < x, y < \infty$$

a)

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$
$$= \int_0^{\infty} \frac{e^{-y-\frac{x}{y}}}{y}dx$$
$$= \frac{e^{-y}}{y}\int_0^{\infty} e^{-\frac{x}{y}}dx$$

Solving the integral part:

$$u = -\frac{x}{y}$$
$$du = -\frac{1}{y}dx$$
$$-ydu = dx$$

$$\int_0^{\infty} e^{-\frac{x}{y}}dx = -y\int_0^{-\infty} e^u du = -y(e^u\big|_0^{-\infty}) = y$$

Inserting the result to equation:

$$f_Y(y) = \frac{e^{-y}}{y}y = e^{-y}$$

$$f_Y(y) = e^{-y} \textbf{ for } x > 0$$

Y has exponential distribution with $\lambda = 1$

b)

$$f_Y(y) = e^{-y} \textbf{ for } x > 0$$

$$E(Y) = \int_{-\infty}^{\infty} yf_Y(y)dy = \int_0^{\infty} ye^{-y}dy$$

$$u = y \Longrightarrow du = dy$$
$$dv = e^{-y}dy \Longrightarrow v = -e^{-y}$$

$$\int_0^{\infty} ye^{-y}dy = -e^{-y}y\big|_0^{\infty} - \int_0^{\infty} -e^{-y}dy$$

$$= 0 + \int_0^\infty e^{-y} dy$$

$$-e^{-y}\Big|_0^\infty = 0 - (-1) = 1$$

$$E(Y) = 1$$

Answer 3

p : probability that a soldier is naval.

$$p = 0.1$$

X : Number of navals in a randomly selected group of n soldiers.
X has binomial distribution since a soldier is either naval or not with p = 0.1 and n

a)

$$n = 1000 \text{ and } p = 0.1$$
$$P(X \geq 1000(0.09)) = P(X \geq 90) = 1 - P(X < 90)$$

Using the Normal approximation to Binomial Distribution
in order to calculate $P(X < 90)$

$$Binomial(n, p) \approx Normal(\mu, \sigma)$$

$$P(X \geq 90) = 1 - P(X < 90))$$

$$\mu = np = 100 \textbf{ and } \sigma = \sqrt{np(1-p)} = \sqrt{90} = 0.487$$

$$P(X < 90) = P(X < 89.5)$$
Since X is strictly less than 90, equality holds

$$P(X < 89.5) = P(\frac{X - 100}{0.487} < \frac{89.5 - 100}{9.487})$$
$$= \Phi(-1.10) = 0.1357$$

$$1 - P(X < 90) = 1 - 0.1357$$
$$= 0.8643$$

b)

Similarly, n=2000 and p = 0.1

$$Binomial(n, p) \approx Normal(\mu, \sigma)$$

$$\mu = np = 200 \textbf{ and } \sigma = \sqrt{np(1-p)} = \sqrt{180} = 13.416$$

$$P(X \geq 200) = P(X \geq 180) = 1 - P(X < 180)$$

$$P(X < 180) = P(X < 179.5)$$

$$= P(\frac{X - 200}{13.416} < \frac{179.5 - 200}{13.416})$$

$$= \Phi(-1.52) = 0.0643$$

$$P(X \geq 180) = 1 - 0.0643 = 0.9357$$

Increasing the sample size makes the distribution get closer to Normal Distribution (Cental Limit Theorem)

Answer 4

T : Lifespan of an elephant in years.

T has normal distribution with:

$\mu = 65$ years and $\sigma = 6$ years

a)

$$P(60 < T < 75)$$

Converting from Normal to Uniform Normal:

$$P(\frac{60 - 65}{6} < \frac{T - 65}{6} < \frac{75 - 65}{6})$$

$$= P(-0.83 < Z < 1.66)$$

$$= \Phi(1.66) - \Phi(-0.83)$$
$$= 1 - \Phi(-1.66) - \Phi(-0.83)$$
$$= 1 - 0.0485 - 0.2061$$
$$= 0.7454$$

b)

```
u = 65;
q = 6;

N = [20, 100, 1000];

for i = 1:3

    sample = normrnd(u, q, N(i), 1);

    figure;

    hist(sample, 25);

    xlabel('Lifespan (years)');
    ylabel('Frequency');

end
```

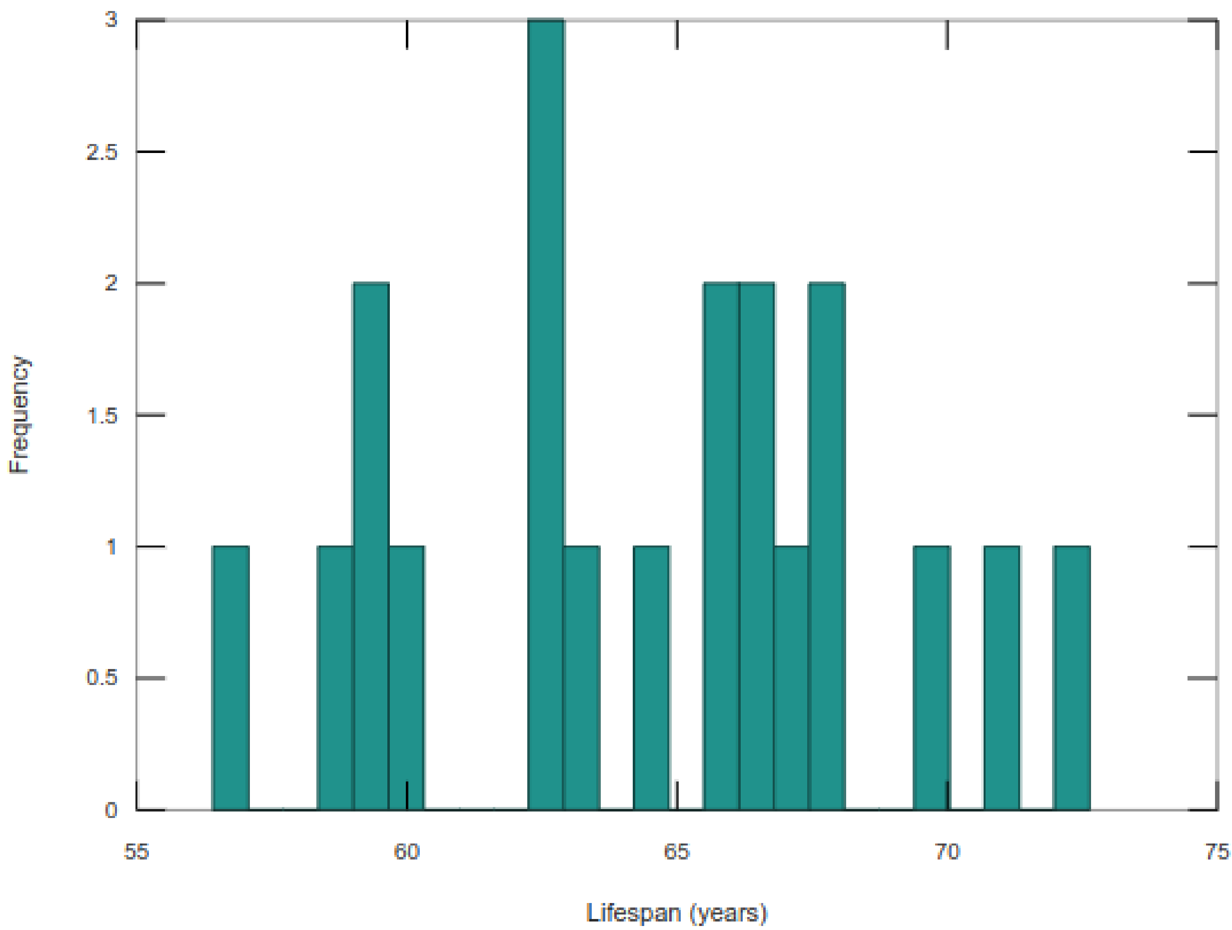

Figure 1: N=20

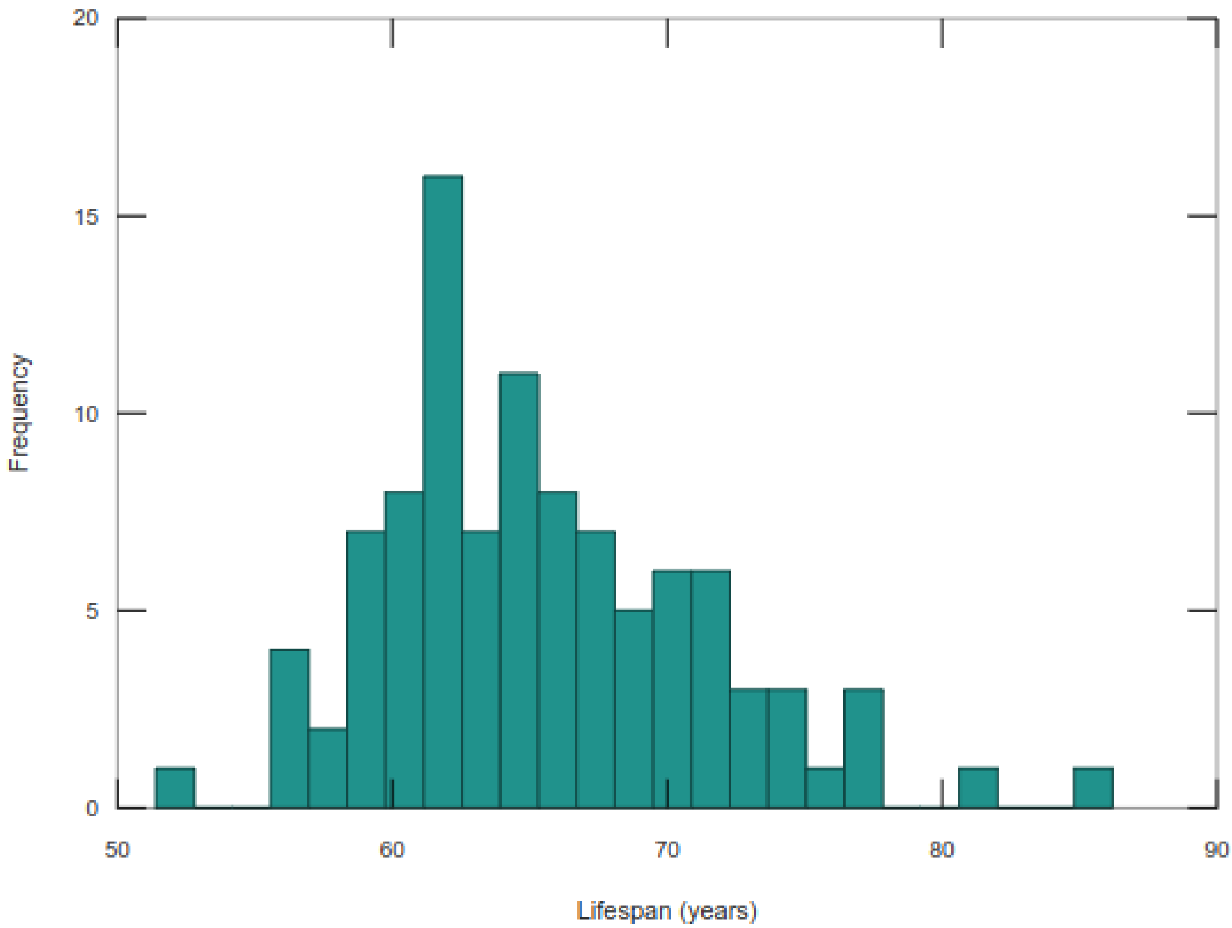

Figure 2: N=100

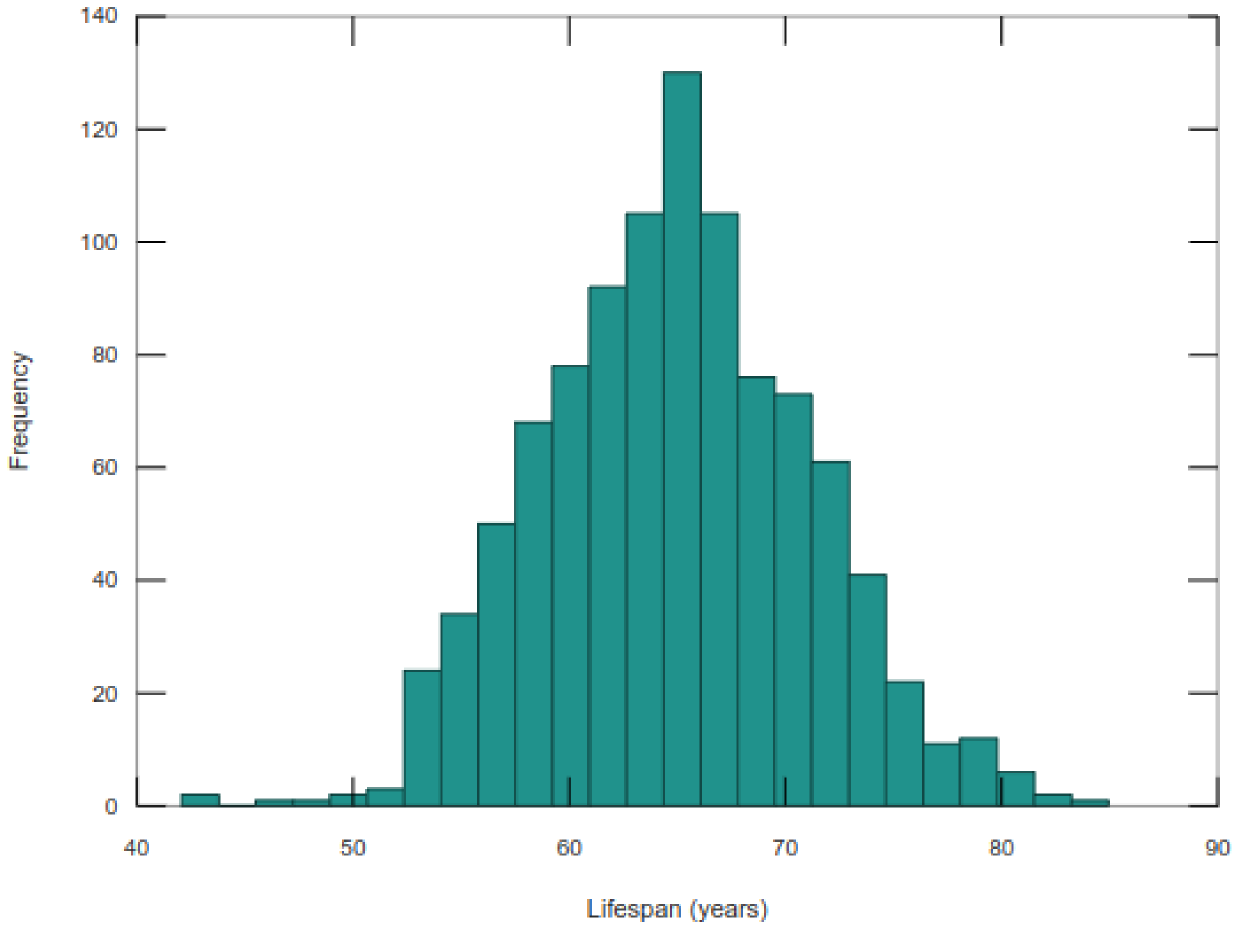

Figure 3: N=1000