

Dynamical robust H_∞ filtering for nonlinear uncertain systems: An LMI approach

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Abstract

In this paper, a new approach to robust H_∞ filtering for a class of nonlinear systems with time-varying uncertainties is proposed in the LMI framework based on a general dynamical observer structure. The nonlinearities under consideration are assumed to satisfy local Lipschitz conditions and appear in both state and measured output equations. The admissible Lipschitz constants of the nonlinear functions are maximized through LMI optimization. The resulting H_∞ observer guarantees asymptotic stability of the estimation error dynamics with prespecified disturbance attenuation level and is robust against time-varying parametric uncertainties as well as Lipschitz nonlinear additive uncertainty.

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1. Introduction

Due to its importance, the problem of state observer design for nonlinear systems has been a subject of extensive research over the past three decades, resulting in numerous algebraic and geometric design methods that achieve asymptotic stability of the observer error dynamics.

Most references on the subject assume the existence of a precisely known mathematical model. In practice, plant disturbances and modeling errors will invariably lead to deviations from the true state unless due precautions are taken during the observer design. These observations motivate the study of *robust* nonlinear observers, understood as

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observers that can perform in the presence of (i) model uncertainties and (ii) persistent disturbances.

To deal with the nonlinear state observation problem in the presence of unknown exogenous disturbance and uncertainties, the robust H_∞ filtering was proposed by de Souza et al. [1,2]. In these references, the authors consider a class of continuous-time nonlinear systems satisfying a Lipschitz continuity condition where time-varying parametric uncertainties exist in the model and obtain Riccati-based sufficient conditions for the stability of the proposed observer with guaranteed disturbance attenuation level. They assume that the Lipschitz constant of the system is known and fixed. In an H_∞ observer, the \mathcal{L}_2 -induced gain from the norm-bounded exogenous disturbance signals to the observer error is guaranteed to be below a prescribed level. The authors also derive matrix inequalities helpful in solving this type of problems. Since then, various methods have been reported in the literature to design robust observers for nonlinear systems [3–12]. In [13,14], an observer design method is proposed using a dynamic observer structure. Using a *Riccati-based* approach, in [14] the nominal stability problem of the observer error dynamics is converted into an H_∞ norm minimization problem satisfying the H_∞ regularity assumptions while no disturbance attenuation level or robustness against uncertainty is considered. On the other hand, the restrictive regularity assumptions in the Riccati approach can be relaxed using linear matrix inequalities (LMIs). An LMI solution to robust H_∞ observer design has been proposed for a class of uncertain Lipschitz nonlinear systems considering static-gain and dynamic observer structures [15–17]. In Refs. [15,16] a Luenberger static-gain observer is considered while the observer contains the nonlinear part of the model. In Ref. [17], the authors propose a *linear strictly proper* dynamical observer. The resulting observers are robust against time-varying parametric uncertainties with guaranteed disturbance attenuation level. Furthermore, in all these works, the Lipschitz constant of the system is assumed to be known and fixed. Recently, the authors of this paper have developed an LMI approach to nonlinear H_∞ filtering in which the Lipschitz constant is one of the LMI variables in order to achieve a maximized admissible Lipschitz constant through convex optimization for systems with uncertainties in linear and nonlinear parts of the state space realization [18–21]. This maximization adds an important extra feature to the observer, making it robust against *nonlinear uncertainties*.

Following the same approach, in this paper we propose a new method of observer design for nonlinear uncertain systems considering a general H_∞ observer design framework. This is a generalization of our previous work of [20,21] in continuous-time domain. The generalization is in twofold: (i) we propose a *generalized observer structure* that can capture both static-gain and dynamical observer structures; (ii) the nonlinearities under consideration appear in *both* the state and the measured output equations and are assumed to satisfy local Lipschitz conditions. A *nonlinear dynamical* H_∞ filter is considered with no assumption on the strict properness of the linear part. In addition, we consider a prespecified decay rate guaranteeing the exponential convergence of the estimations. It will be shown that the proposed dynamical structure has additional degrees of freedom when compared to the conventional static-gain observers and is capable of robustly stabilizing the observer error dynamics for some of those systems for which a static-gain observer cannot be found. Another important conclusion of our work is that, for cases where both static-gain and dynamic observers exist, the maximum admissible Lipschitz constant obtained using the proposed dynamical observer structure can be much larger than that of the static-gain observer. The result is an H_∞ observer with a prespecified disturbance

attenuation level which guarantees asymptotic stability of the estimation error dynamics with guaranteed speed of convergence and is robust against Lipschitz nonlinear uncertainties as well as time-varying parametric uncertainties, simultaneously. Explicit bounds on the nonlinear uncertainty are derived through norm-wise analysis.

Thanks to the linearity of our proposed LMIs in both the admissible Lipschitz constant and the disturbance attenuation level, it is possible to consider a combined objective function. Then, the admissible Lipschitz constant and the disturbance attenuation level are both optimized using a multiobjective optimization technique. The LMI optimization problem can be efficiently solved using the available software with no tuning parameters [22]. The rest of the paper is organized as follows. In Section 2, the problem statement and some preliminaries are mentioned. In Section 3, we propose a new method for robust H_∞ observer design for nonlinear uncertain systems. Section 4 is devoted to robustness analysis in which explicit bounds on the tolerable nonlinear uncertainty are derived. In Section 5, we show the usefulness of our method through an illustrative example.

2. Preliminaries and problem statement

Consider the following class of continuous-time uncertain nonlinear systems:

$$(\Sigma_s) : \dot{x}(t) = (A + \Delta A(t))x(t) + \Phi(x, u) + Bw(t), \quad (1)$$

$$y(t) = (C + \Delta C(t))x(t) + \Psi(x, u) + Dw(t), \quad (2)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and $\Phi(x, u)$ and $\Psi(x, u)$ contain nonlinearities of second order or higher. We assume that the system (1)–(2) is locally Lipschitz with respect to x in a region \mathcal{D} containing the origin, uniformly in u , i.e.:

$$\Phi(0, u^*) = \Psi(0, u^*) = 0, \quad (3)$$

$$\|\Phi(x_1, u^*) - \Phi(x_2, u^*)\| \leq \gamma_1 \|x_1 - x_2\| \quad \forall x_1, x_2 \in \mathcal{D}, \quad (4)$$

$$\|\Psi(x_1, u^*) - \Psi(x_2, u^*)\| \leq \gamma_2 \|x_1 - x_2\| \quad \forall x_1, x_2 \in \mathcal{D}, \quad (5)$$

where $\|\cdot\|$ is the vector 2-norm, u^* is any admissible control signal and $\gamma_1, \gamma_2 > 0$ are the Lipschitz constants of $\Phi(x, u)$ and $\Psi(x, u)$, respectively. If the nonlinear functions satisfy the Lipschitz continuity condition globally in \mathbb{R}^n , then the results will be valid globally. $w(t) \in \mathfrak{L}_2[0, \infty)$ is an unknown exogenous disturbance, and $\Delta A(t)$ and $\Delta C(t)$ are unknown matrices representing time-varying parameter uncertainties, and are assumed to be of the form

$$\begin{bmatrix} \Delta A(t) \\ \Delta C(t) \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} F(t)N, \quad (6)$$

where M_1 , M_2 and N are known real constant matrices and $F(t)$ is an unknown real-valued time-varying matrix satisfying

$$F^T(t)F(t) \leq I \quad \forall t \in [0, \infty). \quad (7)$$

It is worth noting that the structure of parameter uncertainties in Eq. (6) has been widely used in the problems of robust control and robust filtering for both continuous-time and

discrete-time systems and can capture the uncertainty in a number of practical situations [1,23,15].

2.1. Observer structure

We propose the general observer framework of the following form:

$$\begin{aligned} (\Sigma_o) : \dot{x}_F(t) &= A_F x_F(t) + B_F y(t) + \Phi(x_F, u) + E_1 \Psi(x_F, u), \\ z_F(t) &= C_F x_F(t) + D_F y(t) + E_2 \Psi(x_F, u). \end{aligned} \quad (8)$$

The proposed framework can capture both dynamic and static-gain observer structures by proper selection of E_1 and E_2 . Choosing $E_1 = 0$ and $E_2 = I$ leads to the following dynamic observer structure:

$$\dot{x}_F(t) = A_F x_F(t) + B_F y(t) + \Phi(x_F, u), \quad z_F(t) = C_F x_F(t) + D_F y(t) + \Psi(x_F, u). \quad (9)$$

For the static-gain observer structure we have

$$\dot{x}_F(t) = A x_F(t) + \Phi(x_F, u) + L[y(t) - C x_F(t) - \Psi(x_F, u)], \quad z_F(t) = x_F(t). \quad (10)$$

Hence, with

$$A_F = A - LC, \quad B_F = L, \quad C_F = I, \quad D_F = 0, \quad E_1 = -L, \quad E_2 = 0, \quad (11)$$

the general observer captures the static-gain observer structure as a special case. We prove our result for the general observer of class (Σ_o) .

Suppose that

$$z(t) = Hx(t), \quad (12)$$

stands for the controlled output for states to be estimated where H is a known matrix. The estimation error is defined as

$$e(t) \triangleq z(t) - z_F(t) = -C_F x_F(t) + (H - D_F C - D_F \Delta C)x - D_F \Psi(x, u) - E_2 \Psi(x_F, u) - D_F D w. \quad (13)$$

The observer error dynamics is given by

$$(\Sigma_e) : \dot{\xi}(t) = (\tilde{A} + \Delta \tilde{A})\xi(t) + S_1 \Omega(\xi, u) + \tilde{B} w(t), \quad (14)$$

$$e(t) = (\tilde{C} + \Delta \tilde{C})\xi(t) + S_2 \Omega(\xi, u) + \tilde{D} w(t), \quad (15)$$

where

$$\begin{aligned} \xi &\triangleq \begin{bmatrix} x_F \\ x \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} A_F & B_F C \\ 0 & A \end{bmatrix}, \quad \Delta \tilde{A} = \begin{bmatrix} 0 & B_F \Delta C \\ 0 & \Delta A \end{bmatrix}, \quad \Omega(\xi, u) = \begin{bmatrix} \Phi(x, u) \\ \Psi(x, u) \\ \Phi(x_F, u) \\ \Psi(x_F, u) \end{bmatrix}, \\ \tilde{B} &= \begin{bmatrix} B_F D \\ B \end{bmatrix}, \quad S_1 = \begin{bmatrix} 0 & B_F & I & E_1 \\ I & 0 & 0 & 0 \end{bmatrix}, \quad S_2 = [0 \quad -D_F \quad 0 \quad -E_2], \quad \tilde{D} = -D_F D, \\ \tilde{C} &= [-C_F \quad H - D_F C], \quad \Delta \tilde{C} = [0 \quad -D_F \Delta C]. \end{aligned}$$

For the nonlinear function Ω , it is easy to show that

$$\begin{aligned} \Gamma &\triangleq \begin{bmatrix} 0 & \gamma_1 \\ 0 & \gamma_2 \\ \gamma_1 & 0 \\ \gamma_2 & 0 \end{bmatrix}, \|\Omega(\xi_1, u) - \Omega(\xi_2, u)\| \leq \|\Gamma(\xi_1 - \xi_2)\| \leq \|\Gamma\| \|\xi_1 - \xi_2\| \\ &= \sqrt{\gamma_1^2 + \gamma_2^2} \|\xi_1 - \xi_2\| \triangleq \gamma \|\xi_1 - \xi_2\|. \end{aligned} \quad (16)$$

Thus, the observer error system is Lipschitz with Lipschitz constant γ .

2.2. Disturbance attenuation level and guaranteed decay rate

Our purpose is to design the observer matrices A_F , B_F , C_F and D_F such that the observer error dynamics is asymptotically stable with maximum admissible Lipschitz constant and the following specified H_∞ norm upper bound is simultaneously guaranteed:

$$\|e\| \leq \mu \|w\|. \quad (17)$$

Furthermore we want the observer to have a guaranteed decay rate. Consider the nominal system (Σ_s) with ΔA , $\Delta C = 0$ and $w(t) = 0$. Then, the decay rate of the estimation error (13) is defined to be the largest $\beta > 0$ such that

$$\lim_{t \rightarrow \infty} \exp(\beta t) \|e(t)\| = 0, \quad (18)$$

holds for all trajectories e . We can use the quadratic Lyapunov function $V(e) = e^T P e$ to establish a lower bound on the decay rate of Eq. (13). If $dV(e(t))/dt \leq -2\beta V(e(t))$ for all trajectories, then $V(e(t)) \leq \exp(-2\beta t) V(e(0))$, so that $\|e(t)\| \leq \kappa(P)^{1/2} \exp(-\beta t) \|e(0)\|$ for all trajectories, where $\kappa(P)$ is the condition number (the ratio of the maximum singular value to the minimum singular value) of P and therefore the decay rate of Eq. (13) is at least β [22]. In fact, decay rate is a measure of observer speed of convergence.

We now state two lemmas that will be used later in the proof of our results.

Lemma 1 (Wang et al. [2]). For any $x, y \in \mathbb{R}^n$ and any positive definite matrix $P \in \mathbb{R}^{n \times n}$, we have

$$2x^T y \leq x^T P x + y^T P^{-1} y.$$

Lemma 2 (Wang et al. [2]). Let A , D , E , F and P be real matrices of appropriate dimensions with $P > 0$ and F satisfying $F^T F \leq I$. Then for any scalar $\varepsilon > 0$ satisfying $P^{-1} - \varepsilon^{-1} D D^T > 0$, we have

$$(A + DFE)^T P (A + DFE) \leq A^T (P^{-1} - \varepsilon^{-1} D D^T)^{-1} A + \varepsilon E^T E.$$

3. H_∞ observer synthesis

In this section, an H_∞ observer with guaranteed decay rate β and disturbance attenuation level μ is proposed. The admissible Lipschitz constant is maximized through

LMI optimization. Theorem 1 introduces a design method for such an observer. It worths mentioning that unlike the Riccati approach of [1], in the LMI approach no regularity assumption is needed.

Theorem 1. Consider the Lipschitz nonlinear system (Σ_s) along with the general observer (Σ_o) . The observer error dynamics is (globally) asymptotically stable with guaranteed decay rate β , maximum admissible Lipschitz constant, γ^* , and guaranteed $\mathfrak{L}_2(w \rightarrow e)$ gain, μ , if there exist fixed scalars $\beta > 0$ and $\mu > 0$, scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\alpha_1 > 0$ and $\alpha_2 > 0$ and matrices $P_1 > 0$, $P_2 > 0$, G_1 and G_2 , such that the following LMI optimization problem has a solution:

$$\min(2\alpha_1 + \alpha_2)$$

s.t.

$$\Xi_1 = \begin{bmatrix} A_1 & G_2 C & I & 0 & G_2 M_2 & -C_F^T & 0 & 0 & 0 & G_1 & P_1 & P_1 E_1 & G_2 D & 0 \\ \star & A_2 & 0 & 0 & P_2 M_1 & A_3 & 0 & 0 & P_2 & 0 & 0 & 0 & P_2 B & 0 \\ \star & \star & -\alpha_2 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & -\varepsilon_1 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -\varepsilon_1 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -\frac{1}{3}I & 0 & -D_F M_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -\frac{1}{3}\varepsilon_2 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\frac{1}{3}\varepsilon_2 I & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & -I & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & -I & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -I & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -I & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -\mu^2 I & -D^T D_F^T \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -\frac{1}{3}I \end{bmatrix} < 0, \quad (19)$$

$$\Xi_2 = \begin{bmatrix} \varepsilon_2 I & 0 & -D_F M_2 \\ \star & I & 0 \\ \star & \star & I \end{bmatrix} > 0, \quad (20)$$

$$\Xi_3 = \begin{bmatrix} \alpha_1 I & D_F^T \\ \star & \alpha_1 I \end{bmatrix} > 0, \quad (21)$$

where $A_1 = G_1^T + G_1 + 2\beta P_1$, $A_2 = A^T P_2 + P_2 A + 2\beta P_2 + (\varepsilon_1 + \varepsilon_2)N^T N$ and $A_3 = H^T - C^T D_F^T$. Once the problem is solved:

$$A_F = P_1^{-1} G_1, \quad (22)$$

$$B_F = P_1^{-1} G_2, \quad (23)$$

C_F and D_F are directly obtained

$$\alpha_1^* \triangleq \min(\alpha_1), \quad (24)$$

$$\alpha_2^* \triangleq \min(\alpha_2), \quad (25)$$

$$\gamma^* \triangleq \max(\gamma) = \frac{1}{\sqrt{\alpha_2^*(1 + 3\|E_2\|^2 + 3\alpha_1^{*2})}}. \quad (26)$$

Proof. Consider the following Lyapunov function candidate:

$$V = \xi^T P \xi. \quad (27)$$

We have

$$\dot{V} = \dot{\xi}^T P \xi + \xi^T P \dot{\xi} = 2\xi^T (\tilde{A} + \Delta\tilde{A})^T P \xi + 2\xi^T P S_1 \Omega + 2\xi^T P \tilde{B} w. \quad (28)$$

Now, we define

$$J \triangleq \int_0^\infty (e^T e - \mu^2 w^T w) dt. \quad (29)$$

Therefore, we have

$$J < \int_0^\infty (e^T e - \zeta w^T w + \dot{V}) dt. \quad (30)$$

It follows that a sufficient condition for $J \leq 0$ is that

$$\forall t \in [0, \infty), \quad e^T e - \mu^2 w^T w + \dot{V} \leq 0. \quad (31)$$

We have

$$\begin{aligned} e^T e &= \xi^T (\tilde{C} + \Delta\tilde{C})^T (\tilde{C} + \Delta\tilde{C}) \xi + 2\xi^T (\tilde{C} + \Delta\tilde{C})^T S_2 \Omega + 2\xi^T (\tilde{C} + \Delta\tilde{C})^T \tilde{D} w \\ &\quad + 2w^T \tilde{D}^T S_2 \Omega + \Omega^T S_2^T S_2 \Omega + w^T \tilde{D}^T \tilde{D} w. \end{aligned} \quad (32)$$

Thus, using Lemma 1 we have

$$\begin{aligned} \dot{V} + e^T e - \mu^2 w^T w &\leq 2\xi^T (\tilde{A} + \Delta\tilde{A})^T P \xi + 2\xi^T P S_1 \Omega + 2\xi^T P \tilde{B} w - \mu^2 w^T w \\ &\quad + 2\xi^T (\tilde{C} + \Delta\tilde{C})^T (\tilde{C} + \Delta\tilde{C}) \xi + 2\xi^T (\tilde{C} + \Delta\tilde{C})^T \tilde{D} w \\ &\quad + 3\Omega^T S_2^T S_2 \Omega + 2w^T \tilde{D}^T \tilde{D} w \\ &\leq \xi^T [2(\tilde{A} + \Delta\tilde{A})^T P + P S_1^T S_1 P + 2(\tilde{C} + \Delta\tilde{C})^T (\tilde{C} + \Delta\tilde{C})] \xi \\ &\quad + \xi^T [2P \tilde{B} + 2(\tilde{C} + \Delta\tilde{C})^T \tilde{D}] w + \Omega^T \Omega + 3\Omega^T S_2^T S_2 \Omega \\ &\quad + w^T (2\tilde{D}^T \tilde{D} - \mu^2 I) w \\ &\leq \xi^T [2(\tilde{A} + \Delta\tilde{A})^T P + P S_1^T S_1 P + 3(\tilde{C} + \Delta\tilde{C})^T (\tilde{C} + \Delta\tilde{C})] \xi \\ &\quad + 2\xi^T P \tilde{B} w + \Omega^T \Omega + 3\Omega^T S_2^T S_2 \Omega + w^T (3\tilde{D}^T \tilde{D} - \mu^2 I) w. \end{aligned} \quad (33)$$

Without loss of generality, we assume that $\|D_F\| < \alpha_1$ where, $\alpha_1 > 0$ is an unknown variable. Thus,

$$\begin{aligned} \Omega^T \Omega + 3\Omega^T S_2^T S_2 \Omega &\leq (1 + 3\|S_2^T S_2\|) \Omega^T \Omega \\ &= (1 + 3\|S_2 S_2^T\|) \Omega^T \Omega = [1 + 3\|E_2 E_2^T + D_F D_F^T\|] \Omega^T \Omega \\ &\leq [1 + 3(\|E_2 E_2^T\| + \|D_F D_F^T\|)] \Omega^T \Omega = (1 + 3\|E_2\|^2 + 3\|D_F\|^2) \Omega^T \Omega \\ &< (1 + 3\|E_2\|^2 + 3\alpha_1^2) \Omega^T \Omega \leq (1 + 3\|E_2\|^2 + 3\alpha_1^2) \xi^T \Gamma^T \Gamma \xi \end{aligned}$$

$$\leq (1 + 3\|E_2\|^2 + 3\alpha_1^2)\gamma^2 \xi^T \xi. \quad (34)$$

Note that $\Omega(0, u) = 0$. Now, defining the change of variables

$$\alpha_2 \triangleq \frac{1}{(1 + 3\|E_2\|^2 + 3\alpha_1^2)\gamma^2} \Rightarrow \gamma = \frac{1}{\sqrt{\alpha_2(1 + 3\|E_2\|^2 + 3\alpha_1^2)}}, \quad (35)$$

we have

$$\Omega^T \Omega + 3\Omega^T S_2^T S_2 \Omega < \alpha_2^{-1} \xi^T \xi. \quad (36)$$

It is worth mentioning that the change of variables in Eq. (35) plays a vital role here. The alternative changes of variables such as $\alpha_2 = \alpha_1 \gamma$, which may seem more straightforward, would make γ appear in Ξ_1 and then due to the existence of α_1 in the LMI (21), the variables α_1 and γ would be over-determined. On the other hand, we have

$$\Delta \tilde{A} = \begin{bmatrix} 0 & B_F \Delta C \\ 0 & \Delta A \end{bmatrix} = \begin{bmatrix} 0 & B_F M_2 F N \\ 0 & M_1 F N \end{bmatrix} = \begin{bmatrix} 0 & B_F M_2 \\ 0 & M_1 \end{bmatrix} F \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} \triangleq \tilde{M}_1 F \tilde{N}, \quad (37)$$

$$\begin{aligned} \Delta \tilde{C} &= [0 \quad -D_F \Delta C] = [0 \quad -D_F M_2 F N] = [0 \quad -D_F M_2] F \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} \\ &\triangleq \tilde{M}_2 F \tilde{N}. \end{aligned} \quad (38)$$

Therefore, based on Eqs. (34) and (35) and using Lemma 2 we can write

$$\begin{aligned} \dot{V} + e^T e - \mu^2 w^T w &< \xi^T [\tilde{A}^T P + P \tilde{A} + \varepsilon_1 \tilde{N}^T N + \varepsilon_1^{-1} P \tilde{M}_1 \tilde{M}_1 P \\ &+ 3\tilde{C}^T (I - \varepsilon_2^{-1} \tilde{M}_2 \tilde{M}_2^T)^{-1} \tilde{C} + \varepsilon_2 \tilde{N}^T N + P S_1 S_1^T P + \alpha_2^{-1}] \xi + 2\xi^T P \tilde{B} w \\ &+ w^T (3\tilde{D}^T \tilde{D} - \mu^2 I) w. \end{aligned} \quad (39)$$

In order to guarantee the decay rate $\beta > 0$, we need $\dot{V} \leq -2\beta V$ instead of $\dot{V} < 0$. This adds the term $2\beta \xi^T P \xi$ to the right hand side of Eq. (39). It guarantees the decay rate β for $\xi(t)$, i.e. $\|\xi(t)\| \leq \kappa^{1/2}(P) \exp(-\beta t) \|\xi(0)\|$. In order to show that $e(t)$ has the same decay rate, based on Eq. (15) with $\Delta A, \Delta C = 0$ and $w(t) = 0$, we can write

$$\begin{aligned} \|e(t)\| &\leq \|\tilde{C}\| \|\xi(t)\| + \|S_2\| \|\Omega\| \leq (\|\tilde{C}\| + \|S_2\| \|\Gamma\|) \|\xi(t)\| \\ &\leq (\|\tilde{C}\| + \|S_2\| \|\Gamma\|) \kappa^{1/2}(P) \exp(-\beta t) \|\xi(0)\| \\ &= (\|\tilde{C}\| + \|S_2\| \|\Gamma\|) \kappa^{1/2}(P) \exp(-\beta t) \|\xi(0)\|^2 \|\xi(0)\|^{-1}. \end{aligned} \quad (40)$$

On the other hand, since $\Omega(0) = 0$,

$$\|e(0)\| \leq \|\tilde{C}\| \|\xi(0)\| \Rightarrow \|\xi(0)\|^{-1} \leq \frac{\|\tilde{C}\|}{\|e(0)\|}. \quad (41)$$

Substituting Eqs. (41) and (40) leads to

$$\begin{aligned} \|e(t)\| &\leq \|\xi(0)\|^2 (\|\tilde{C}\| + \|S_2\| \|\Gamma\|) \kappa^{1/2}(P) \exp(-\beta t) \|\tilde{C}\| \cdot \frac{1}{\|e(0)\|} \\ &= \frac{\|\xi(0)\|^2 \|\tilde{C}\| (\|\tilde{C}\| + \|S_2\| \|\Gamma\|)}{\|e(0)\|^2} \cdot \kappa^{1/2}(P) \exp(-\beta t) \|e(0)\| \\ &\triangleq \eta \cdot \kappa^{1/2}(P) \exp(-\beta t) \|e(0)\|, \end{aligned} \quad (42)$$

which means that the estimation error $e(t)$ has guaranteed decay rate β . Now, a sufficient condition for Eq. (31) is that the right hand side of Eq. (39) be negative definite. Using Schur complements, this is equivalent to the following LMI. Note that having $w=0$, Eq. (28) is already included in Eq. (33) and consequently in Eq. (39):

$$\begin{bmatrix} Y & I & P\tilde{M}_1 & \tilde{C}^T & 0 & PS_1 & P\tilde{B} & 0 \\ \star & -\alpha_2 I & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & -\varepsilon_1 I & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & -\frac{1}{3}I & \tilde{M}_2 & 0 & 0 & 0 \\ \star & \star & \star & \star & -\frac{1}{3}\varepsilon_2 I & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -I & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -\mu^2 I & \tilde{D}^T \\ \star & \star & \star & \star & \star & \star & \star & -\frac{1}{3}I \end{bmatrix} < 0, \quad (43)$$

$$Y = \tilde{A}^T P + P\tilde{A} + (\varepsilon_1 + \varepsilon_2)\tilde{N}^T \tilde{N}. \quad (44)$$

Substituting from Eqs. (37) and (38), defining change of variables $G_1 \triangleq P_1 A_F$ and $G_2 \triangleq P_1 B_F$ with $P = \text{diag}(P_1, P_2)$ and using Schur complements successively, the LMI (19) is obtained. The LMI (20) is equivalent to the condition $I - \varepsilon_2^{-1} \tilde{M}_2 \tilde{M}_2^T > 0$ needed in Lemma 2. Condition $\|D_F\| < \alpha_1$ is equivalent to the LMI (21). Maximization of γ is equivalent to the simultaneous minimization of α_1 and α_2 . Combining the two objective functions, we will minimize the scalarized linear objective function $c_1 \alpha_1 + c_2 \alpha_2$. To determine the weights c_1 and c_2 in the objective function, we compute the sensitivity of γ to the changes of α_1 and α_2 . We have

$$\mathcal{S}_{\alpha_1}^\gamma = \frac{\partial \gamma}{\partial \alpha_1} \cdot \frac{\alpha_1}{\gamma} = \frac{-3\alpha_1^2}{1 + 3\|E_2\|^2 + 3\alpha_1^2} > -1, \quad (45)$$

$$\mathcal{S}_{\alpha_2}^\gamma = \frac{\partial \gamma}{\partial \alpha_2} \cdot \frac{\alpha_2}{\gamma} = -\frac{1}{2}. \quad (46)$$

Hence, γ is up to twice more sensitive to the changes of α_1 than those of α_2 . Note that the absolute value of the sensitivity function determines the sensitivity of γ with respect to the changes in α_i , while its sign determines the direction of change. Therefore, a reasonable choice can be $c_1 = 2$ and $c_2 = 1$. \square

Remark 1. Maximization of γ guarantees the robust asymptotic stability of the error observer dynamics for any Lipschitz nonlinear function $\Omega(\xi, u)$ with Lipschitz constant less than or equal γ^* . It is clear that if an observer for a system with a given fixed Lipschitz constant is to be designed, the proposed LMI optimization problem will reduce to an LMI feasibility problem and therefore, in this special case, there is no need for the change of variable (35).

Remark 2. The proposed LMIs are linear in α_1 , α_2 and $\zeta (= \mu^2)$. Thus, either can be a fixed constant or an optimization variable. So, either the admissible Lipschitz constant or the

disturbance attenuation level can be considered as an optimization variable in Theorem 1. Given this, it may be more realistic to have a combined performance index. This leads to a multiobjective convex optimization problem optimizing both γ and μ , simultaneously. See [21] for details of the approach.

3.1. Static-gain observer

It is worth mentioning that in the case of static-gain observer, some simplification can be made. First of all, since in this structure $D_F=0$, the LMIs (20) and (21) are eliminated. Besides, since $E_2=0$ and $\alpha_1=0$, the inequality (34) reduces to $\Omega^T\Omega + 3\Omega^TS_2^TS_2\Omega \leq \gamma^2\xi^T\xi$ and there is no need to the change of variables (35). Consequently, the cost function simplifies to $\max(\gamma)$. In addition, for this structure we have

$$A_F = A-LC \Rightarrow P_1A_F = P_1A - P_1LC, \quad (47)$$

$$B_F = L, E_1 = -L \Rightarrow P_1B_F = -P_1E_1 = P_1L. \quad (48)$$

Therefore, instead of variables G_1 and G_2 , a change of variables $G = P_1L$ is enough. Obviously, the dynamic observer structure has additional degrees freedom and can provide a robust observer in some of the cases for which a static-gain observer does not exist.

In the next section we discuss an important feature of the proposed observer, namely; robustness against nonlinear uncertainty.

4. Robustness against nonlinear uncertainty

As mentioned earlier, the maximization of Lipschitz constant makes the proposed observer robust against some Lipschitz nonlinear uncertainty. In this section this robustness feature is studied and a norm-wise bound on the nonlinear uncertainty are derived. The norm-wise analysis provides an upper bound on the Lipschitz constant of the nonlinear uncertainty and the norm of the Jacobian matrix of the corresponding nonlinear function.

Assume nonlinear uncertainty of the form

$$\Phi_\Delta(x, u) = \Phi(x, u) + \Delta\Phi(x, u), \quad (49)$$

$$\Psi_\Delta(x, u) = \Psi(x, u) + \Delta\Psi(x, u), \quad (50)$$

$$\dot{x}(t) = (A + \Delta A)x(t) + \Phi_\Delta(x, u) + Bw(t), \quad (51)$$

$$y(t) = (C + \Delta C)x(t) + \Psi_\Delta(x, u) + Dw(t), \quad (52)$$

where Φ_Δ and Ψ_Δ are uncertain nonlinear functions and $\Delta\Phi$ and $\Delta\Psi$ are unknown nonlinear uncertainties. Suppose that

$$\|\Delta\Phi(x_1, u) - \Delta\Phi(x_2, u)\| \leq \Delta\gamma_1 \|x_1 - x_2\|,$$

$$\|\Delta\Psi(x_1, u) - \Delta\Psi(x_2, u)\| \leq \Delta\gamma_2 \|x_1 - x_2\|.$$

Proposition 1. Suppose that the actual Lipschitz constant of the nonlinear functions Φ and Ψ are γ_1 and γ_2 , respectively, and the maximum admissible Lipschitz constant achieved by

Theorem 1, is γ^* . Then, the observer designed based on Theorem 1 can tolerate any additive Lipschitz nonlinear uncertainties over Φ and Ψ with Lipschitz constants $\Delta\gamma_1$ and $\Delta\gamma_2$ such that $\sqrt{(\gamma_1 + \Delta\gamma_1)^2 + (\gamma_2 + \Delta\gamma_2)^2} \leq \gamma^*$.

Proof. We have

$$\Omega_A(\zeta, u) = \Omega(\zeta, u) + \Delta\Omega(\zeta, u) = \begin{bmatrix} \Phi_A(x, u) \\ \Psi_A(x, u) \\ \Phi_A(x_F, u) \\ \Psi_A(x_F, u) \end{bmatrix} = \begin{bmatrix} \Phi(x, u) \\ \Psi(x, u) \\ \Phi(x_F, u) \\ \Psi(x_F, u) \end{bmatrix} + \begin{bmatrix} \Delta\Phi(x, u) \\ \Delta\Psi(x, u) \\ \Delta\Phi(x_F, u) \\ \Delta\Psi(x_F, u) \end{bmatrix}.$$

Based on Schwartz inequality,

$$\begin{aligned} \|\Phi_A(x_1, u) - \Phi_A(x_2, u)\| &\leq \|\Phi(x_1, u) - \Phi(x_2, u)\| + \|\Delta\Phi(x_1, u) - \Delta\Phi(x_2, u)\| \\ &\leq \gamma_1 \|x_1 - x_2\| + \Delta\gamma_1 \|x_1 - x_2\| = (\gamma_1 + \Delta\gamma_1) \|x_1 - x_2\|. \end{aligned}$$

Similarly

$$\|\Psi_A(x_1, u) - \Psi_A(x_2, u)\| \leq (\gamma_2 + \Delta\gamma_2) \|x_1 - x_2\|. \quad (53)$$

Based on (16), we can write

$$\Gamma_A \triangleq \begin{bmatrix} 0 & \gamma_1 + \Delta\gamma_1 \\ 0 & \gamma_2 + \Delta\gamma_2 \\ \gamma_1 + \Delta\gamma_1 & 0 \\ \gamma_2 + \Delta\gamma_2 & 0 \end{bmatrix}, \quad (54)$$

$$\|\Omega_A(\xi_1, u) - \Omega_A(\xi_2, u)\| \leq \|\Gamma_A(\xi_1 - \xi_2)\| \leq \sqrt{(\gamma_1 + \Delta\gamma_1)^2 + (\gamma_2 + \Delta\gamma_2)^2} \|\xi_1 - \xi_2\|.$$

On the other hand, according to Theorem 1, $\Omega_A(x, u)$ can be any Lipschitz nonlinear function with Lipschitz constant less than or equal to γ^* ,

$$\|\Omega_A(\xi_1, u) - \Omega_A(\xi_2, u)\| \leq \gamma^* \|\xi_1 - \xi_2\|,$$

so, there must be

$$\sqrt{(\gamma_1 + \Delta\gamma_1)^2 + (\gamma_2 + \Delta\gamma_2)^2} \leq \gamma^*. \quad \square \quad (55)$$

In addition, we know that for continuously differentiable functions $\Delta\Phi$ and $\Delta\Psi$,

$$\begin{aligned} \|\Delta\Phi(x_1, u) - \Delta\Phi(x_2, u)\| &\leq \left\| \frac{\partial \Delta\Phi}{\partial x}(x_1 - x_2) \right\|, \quad \forall x, x_1, x_2 \in \mathcal{D}, \\ \|\Delta\Psi(x_1, u) - \Delta\Psi(x_2, u)\| &\leq \left\| \frac{\partial \Delta\Psi}{\partial x}(x_1 - x_2) \right\|, \quad \forall x, x_1, x_2 \in \mathcal{D}, \end{aligned}$$

where $\partial\Delta\Phi/\partial x$ and $\partial\Delta\Psi/\partial x$ are the Jacobian matrices [24]. So $\Delta\Phi(x, u)$ and $\Delta\Psi(x, u)$ can be any additive uncertainties with $\sqrt{(\gamma_1 + \|\partial\Delta\Phi/\partial x\|)^2 + (\gamma_2 + \|\partial\Delta\Psi/\partial x\|)^2} \leq \gamma^*$.

Remark 3. Alternatively, we can write

$$\|\Omega_A(\xi_1, u) - \Omega_A(\xi_2, u)\| \leq \|\Omega(\xi_1, u) - \Omega(\xi_2, u)\| + \|\Delta\Omega(\xi_1, u) - \Delta\Omega(\xi_2, u)\|$$

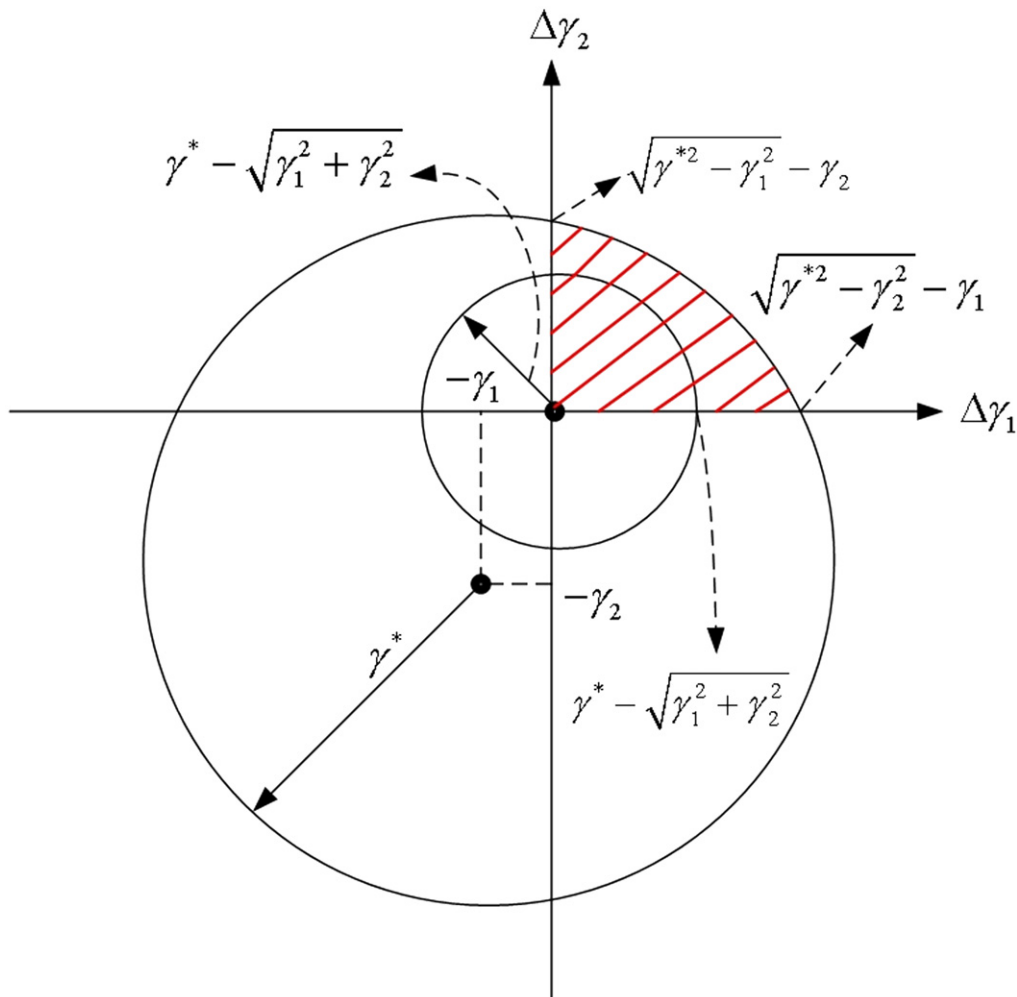


Fig. 1. Geometric representation of uncertainty bounds. The admissible region is hachured.

$$\leq \sqrt{\gamma_1^2 + \gamma_2^2} \|\xi_1 - \xi_2\| + \sqrt{\Delta\gamma_1^2 + \Delta\gamma_2^2} \|\xi_1 - \xi_2\| = \left(\sqrt{\gamma_1^2 + \gamma_2^2} + \sqrt{\Delta\gamma_1^2 + \Delta\gamma_2^2} \right) \|\xi_1 - \xi_2\|.$$

Then, we can conclude that, $\Delta\Phi(x, u)$ and $\Delta\Psi(x, u)$ can be any additive uncertainties with

$$\sqrt{\Delta\gamma_1^2 + \Delta\gamma_2^2} \leq \gamma^* - \sqrt{\gamma_1^2 + \gamma_2^2}. \quad (56)$$

However, it is not hard to show that

$$\sqrt{(\gamma_1 + \Delta\gamma_1)^2 + (\gamma_2 + \Delta\gamma_2)^2} \leq \sqrt{\gamma_1^2 + \gamma_2^2} + \sqrt{\Delta\gamma_1^2 + \Delta\gamma_2^2}, \quad \forall \gamma_1, \gamma_2, \Delta\gamma_1, \Delta\gamma_2 \geq 0. \quad (57)$$

Therefore, the bound in Eq. (55) is less conservative. The geometric representations of the two bounds are shown in Fig. 1. The admissible region is hachured.

5. Numerical example

Consider a system of class Σ_s with

$$A = \begin{bmatrix} 0 & 10 \\ -16 & -15 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 0.1 & 0.15 \\ -0.2 & -0.1 \end{bmatrix}, \quad N = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ C = [1 \ 0], \quad M_2 = [-0.2 \ 0.2], \quad D = 0.2.$$

Using the method given in [21] with parameters $\mu = 0.1$ and $H = 0.25I_2$ and $\beta = 0.15$, the maximum admissible Lipschitz constant and the observer static gain are

$$\gamma^* = 0.3105,$$

$$L = \begin{bmatrix} 4.9949 \\ 4.9976 \end{bmatrix}.$$

Now we use the proposed method in this paper. Solving the LMI optimization problem in Theorem 1, if we design a robust H_∞ static-gain observer for this system with the same parameters, we have

$$\gamma^* = 0.4484,$$

$$L = \begin{bmatrix} 5.0003 \\ 4.9993 \end{bmatrix},$$

in which the observer gain is almost the same as before with a slight increase in the maximum admissible Lipschitz constant meaning that the observer is less conservative than the one designed using the method in [21]. Note that there is also another important advantage in using this observer over the previous one. The class of systems that this observer can handle may have nonlinearities both in the state and output equations with Lipschitz constants γ_1 and γ_2 , respectively, as long as $\sqrt{\gamma_1^2 + \gamma_2^2} \leq \gamma^*$; while the observer designed based on [21] can only handle systems with nonlinearity only in the state equations. There is however, still room for improvement. Using Theorem 1 and the same parameters as before, the dynamic H_∞ observer is obtained as

$$A_F = \begin{bmatrix} -45.8141 & -21.7799 \\ -21.7797 & -76.8907 \end{bmatrix}, \quad B_F = \begin{bmatrix} 1.0472 \\ 1.2378 \end{bmatrix}, \quad C_F = \begin{bmatrix} -0.0062 & 0.0070 \\ -0.0027 & -0.0247 \end{bmatrix}, \\ D_F = 1e-5 \times \begin{bmatrix} 0.9547 \\ -0.3700 \end{bmatrix}, \quad \varepsilon_1 = 0.6671, \quad \varepsilon_2 = 0.9739, \\ \alpha_1 = 4.7283e-4, \quad \alpha_2 = 8.2989e-4, \quad \gamma^* = 17.3564.$$

Therefore, there is a major improvement (in the order of tens) over the maximum admissible Lipschitz constant γ^* in the dynamic observer in comparison with the static-gain observer. Since the maximum admissible Lipschitz constant is directly related to the nature of nonlinearity and size of the operating region (since the Lipschitz constant is usually region based and increases with the enlargement of the operating region), this means that the dynamic H_∞ observer can handle a much larger class of systems for which a

static-gain observer either does not exist or can work only in a much smaller operating region.

6. Conclusion

A new nonlinear H_∞ dynamical observer design method for a class of Lipschitz nonlinear uncertain systems is proposed through LMI optimization. The developed LMIs are linear both in the admissible Lipschitz constant and the disturbance attenuation level allowing both to be an LMI optimization variable. The proposed dynamical structure has more degree of freedom than the conventional static-gain observers and is capable of robustly stabilizing the observer error dynamics for some of those systems for which a static-gain observer cannot be found. In addition, when the static-gain observer also exists, the maximum admissible Lipschitz constant obtained using the proposed dynamical observer structure can be much larger than that of the static-gain observer. The achieved H_∞ observer guarantees asymptotic stability of the error dynamics with a prespecified decay rate (exponential convergence) and is robust against Lipschitz additive nonlinear uncertainty as well as time-varying parametric uncertainty. Explicit bounds on the nonlinear uncertainty are derived through norm-wise and element-wise analysis.

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