Coherent H^∞ Control for Linear Quantum Systems With Uncertainties in the Interaction Hamiltonian

Chengdi Xiang, Shan Ma, Sen Kuang, and Daoyi Dong

***Abstract*—This work conducts robust H^∞ analysis for a class of quantum systems subject to perturbations in the interaction Hamiltonian. A necessary and sufficient condition for the robustly strict bounded real property of this type of uncertain quantum system is proposed. This paper focuses on the study of coherent robust H^∞ controller design for quantum systems with uncertainties in the interaction Hamiltonian. The desired controller is connected with the uncertain quantum system through direct and indirect couplings. A necessary and sufficient condition is provided to build a connection between the robust H^∞ control problem and the scaled H^∞ control problem. A numerical procedure is provided to obtain coefficients of a coherent controller. An example is presented to illustrate the controller design method.**

***Index Terms*—Coherent feedback control, robust control, uncertain quantum system.**

I. Introduction

CONTROLLING quantum phenomena plays an important role in developing quantum technologies, and the exploration of quantum system control theory has drawn wide interests from scientists and engineers in different fields, such as physical chemistry, quantum optics, and quantum information [1]–[8]. In particular, robustness has been recognized as a key issue in the development of quantum control theory and practical quantum technology since the existence of various types of uncertainties and disturbances is unavoidable for most practical quantum systems. These uncertainties may come from decoherence, systematic errors, environmental noises or Hamiltonian identification inaccuracies [9]–[13]. Hence, to deal with different kinds of uncertainties in quantum systems, it is important to develop robust control methods [14]–[31]. For example, a sampling-based learning control (SLC) method has been applied to controller design of quantum systems with uncertainties, even when the uncertainty parameters have large fluctuations [19], [20]. In [21], sliding mode control has been proposed for two-level systems with bounded uncertainties in the Hamiltonian and can guarantee the desired robustness in the presence of uncertainties. A two-step strategy by combining the concepts of feedback and open-loop control methods is presented in [22] to achieve stabilizing control of quantum systems with uncertainties. A sequential convex programming method is proposed in [23] to design robust quantum gates. A noise filtering approach is applied in [24] to enhance robustness in quantum control.

As an effective robust control method, H^∞ control [32], [33] has been used in the quantum domain [25]–[29]. James *et al.* [25] have formulated and solved an H^∞ controller synthesis problem for a class of linear stochastic systems to bound the undesirable effects of disturbance on performance. In [26], the coherent H^∞ control method is applied to a quantum passive system, which is modeled purely in terms of the annihilation operator but not the creation operator. While studies [25], [26] only consider the disturbance input, [27] further investigates coherent robust H^∞ control including disturbance input and parameter uncertainties. A sufficient condition was presented to build the relationship between the robust H^∞ control problem and the scaled H^∞ problem. The work [28] extends the result in [27] to a class of quantum passive systems. It should be noticed that most of the studies [25]–[28] consider the coherent H^∞ control with indirect coupling. The direct and indirect couplings in coherent H^∞ feedback control of linear quantum systems are studied in [29].

In this paper, we present coherent H^∞ control for quantum systems with direct and indirect couplings by taking interaction Hamiltonian uncertainties into consideration. The contributions of this paper are summarized as follows. This paper considers the uncertainties in the interaction Hamiltonian, while most existing papers have only considered the uncertainties in the system Hamiltonian or in the coupling operators [21], [27], [30]. To build a connection between the robust H^∞ control problem and the scaled H^∞ control problem, a necessary and sufficient condition is provided rather than only a sufficient condition given in [27], [28]. This paper also considers the effect of direct and indirect couplings between a quantum system and a coherent controller, instead of only the indirect coupling [25], [26]. Also, a numerical procedure for robust coherent controller design is provided

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based on the change of variable technique, linear matrix inequality (LMI) method [34] and multi-step optimization method.

This paper is organized as follows. In Section II, a class of linear quantum systems via direct and indirect couplings with an exosystem is introduced using quantum stochastic differential equations (QSDEs) and parameters (S,L,H). Then, uncertainties in the interaction Hamiltonian are described. In Section III, for this class of uncertain quantum systems, a robustly strict bounded real condition without uncertainties is presented through a necessary and sufficient condition. In Section IV, a coherent robust H^{∞} controller design method is proposed using robustly strict bounded real lemma. In Section V, a numerical procedure to obtain the corresponding coefficients of desired controller parameters is proposed. Also, an example is presented to illustrate the method in Section V-C, followed by the conclusion in Section VI.

Notations:

In this paper, I_m describes an $m\times m$ identity matrix and $0_{m\times n}$ describes an $m\times n$ zero matrix, where the subscript is omitted when m and n can be determined from the context. Define

$$J_m = \begin{bmatrix} 0 & I_{\frac{m}{2}} \\ -I_{\frac{m}{2}} & 0 \end{bmatrix}$$

$$\Xi_m = \begin{bmatrix} 0 & 2iI_{\frac{m}{2}} \\ -2iI_{\frac{m}{2}} & 0 \end{bmatrix}$$

and

$$\Sigma_{n_a n_b} = \begin{bmatrix} \Lambda_{\frac{n_a}{2}} & 0_{\frac{n_a}{2}\times\frac{n_b}{2}} \\ 0_{\frac{n_a}{2}\times\frac{n_b}{2}} & \Lambda_{\frac{n_a}{2}} \end{bmatrix}$$

where $i=\sqrt{-1}$, n_a and n_b are two positive even integers, and $\Lambda_{\frac{n_a}{2}} = \begin{bmatrix} I_{\frac{n_a}{2}} & 0_{\frac{n_a}{2}\times\frac{(n_b-n_a)}{2}} \end{bmatrix}$.

II. Quantum System Model

A. A Class of Quantum Stochastic Systems

Consider an open quantum system interacting with an exosystem via indirect and direct couplings, which are affected through field couplings and interaction Hamiltonian. This class of quantum system models can be described in the quadrature form by the following non-commutative quantum stochastic differential equations (QSDEs) [29]:

$$\begin{aligned} d\breve{x}(t) &= \mathcal{A}\breve{x}(t)dt + \mathcal{B}_d d\breve{v}(t) + \mathcal{B}_f d\breve{w}(t) \\ d\breve{y}(t) &= \mathcal{C}\breve{x}(t)dt + \mathcal{D}_d d\breve{v}(t) + \mathcal{D}_f d\breve{w}(t) \end{aligned} \tag{1}$$

where $\mathcal{A}\in\mathbb{R}^{n\times n}$, $\mathcal{B}_d\in\mathbb{R}^{n\times n_v}$, $\mathcal{B}_f\in\mathbb{R}^{n\times n_w}$, $C\in\mathbb{R}^{n_y\times n}$, $\mathcal{D}_d\in\mathbb{R}^{n_y\times n_v}$, $\mathcal{D}_f\in\mathbb{R}^{n_y\times n_w}$, and n, n_w, n_v, n_y are positive and even numbers. $\breve{x}(t) = \left[q_1(t)\ q_2(t)...q_{\frac{n}{2}}(t)\ p_1(t)\ p_2(t)...p_{\frac{n}{2}}(t)\right]^T$ is a vector of self-adjoint possibly non-commutative system variables. $q_i(t)$ $(i=1,2,...,\frac{n}{2})$ are the position operators and are known as real quadratures, $p_i(t)$ $(i=1,2,...,\frac{n}{2})$ are momentum operators and are known as phase or imaginary quadratures. The canonical commutation relation for vector $\breve{x}(t)$ is required to satisfy the following relation:

$$[\breve{x}(t),\breve{x}(t)^T] = \breve{x}(t)\breve{x}(t)^T - (\breve{x}(t)\breve{x}(t)^T)^T = \Xi_n \tag{2}$$

where $\Xi_n = 2iJ_n$. The $\breve{w}(t)$ and $\breve{y}(t)$ represent the input and output fields for the nominal system, respectively. The term $\breve{v}(t)$ is an exogenous quantity associated with another system with which the nominal system is directly coupled through the interaction Hamiltonian. With the assumption of independence, $\breve{w}(t)$ and $\breve{v}(t)$ commute with the mode operator $\breve{x}(t)$.

For this class of quantum systems, if it is physically realizable, it can also be described by parameters (S,L,H). Here, the Hamiltonian operator H describes the internal energy of the nominal system. The coupling operator L and the scattering matrix S describe the interface between the system and the fields, and affect indirect coupling. The quadratic Hamiltonian is of the form $H=\breve{x}^T M\breve{x}/2$, where $M\in\mathbb{R}^{n\times n}$ is a real symmetric Hamiltonian matrix. The coupling operator L is denoted as: $L=N\breve{x}$, where $N\in\mathbb{C}^{\frac{n_w}{2}\times n}$ is a complex-valued coupling matrix. L can also be rewritten as

$$\begin{bmatrix} L \\ L^\# \end{bmatrix} = \tilde{N}\breve{x} = \begin{bmatrix} N \\ N^\# \end{bmatrix}\breve{x}. \tag{3}$$

The scattering matrix S is a unitary matrix [35]. Since no interaction between different fields is concerned in most cases, an assumption can be made with $S=I$ [25].

The interaction Hamiltonian H_{int} describes energy exchange between the nominal system and exosystem, affecting the direct coupling, and is in the following form:

$$H_{\text{int}} = \frac{1}{2}(\breve{x}^T R^T\breve{v} + \breve{v}^T R\breve{x}) \tag{4}$$

where $R=i\Xi_{n_v}K/2$ and $K\in\mathbb{R}^{n_v\times n}$ is a direct coupling parameter.

The coefficients of QSDEs (1) can not be arbitrarily chosen. To achieve a physically realizable quantum system, the physical realizability conditions for (1) can be written as follows [25], [29]:

$$\begin{aligned} &\mathcal{A}\Xi_n + \Xi_n\mathcal{A}^T + \mathcal{B}_f\Xi_{n_w}\mathcal{B}_f^T = 0 \\ &\mathcal{B}_f\mathcal{D}_f^T = J_n C^T J_{n_y} \\ &\mathcal{B}_d = -\frac{1}{2}\Xi_n K^T\Xi_{n_v} \\ &\mathcal{D}_f = \Sigma_{n_y n_w}. \end{aligned} \tag{5}$$

For the above systems, these matrices $\mathcal{A},\mathcal{B}_d,\mathcal{B}_f,C,\mathcal{D}_d,\mathcal{D}_f$ can be given by [25], [29], [36]

$$\begin{aligned} \mathcal{A} &= -i\Xi_n M - \frac{1}{2}\Xi_n\tilde{N}^\dagger J_{n_w}\tilde{N} \\ \mathcal{B}_d &= -\frac{1}{2}\Xi_n K^T\Xi_{n_v} \\ \mathcal{B}_f &= -\frac{1}{2}\Xi_n\tilde{N}^\dagger\begin{bmatrix} I & iI \\ -I & iI \end{bmatrix} \\ \mathcal{D}_d &= \Sigma_{n_y n_v} \\ C &= \Sigma_{n_y n_w}\begin{bmatrix} N+N^\# \\ -iN+iN^\# \end{bmatrix} \\ \mathcal{D}_f &= \Sigma_{n_y n_w}. \end{aligned} \tag{6}$$

B. Uncertain Perturbations in the Interaction Hamiltonian

This paper considers the class of uncertain quantum systems subject to uncertain perturbation in the interaction

Hamiltonian. The corresponding uncertain interaction Hamiltonian H_{int} can be denoted as follows:

$$H_{\text{int}} = \frac{1}{2}(\breve{x}^T(R+\Delta R)^T\breve{v} + \breve{v}^T(R+\Delta R)\breve{x}) \tag{7}$$

where $\Delta R = i\Xi_{n_v}\Delta K/2$, $\Delta K = E_1\mathbf{\Delta}E_2$, $E_1 \in \mathbb{R}^{n_v\times p}$ and $E_2 \in \mathbb{R}^{q\times n}$ [27]. Also, we assume that $\mathbf{\Delta} \in \mathbb{R}^{p\times q}$ is an uncertain real matrix satisfying the norm bounded condition

$$\mathbf{\Delta}^T\mathbf{\Delta} \leq I. \tag{8}$$

Accordingly, the dynamics of this class of uncertain quantum systems can be described by the following QSDEs with uncertainties:

$$\begin{aligned} d\breve{x}(t) &= \mathcal{A}\breve{x}(t)dt + (\mathcal{B}_d + \Delta\mathcal{B}_d)d\breve{v}(t) + \mathcal{B}_f d\breve{w}(t)\\ d\breve{y}(t) &= C\breve{x}(t)dt + \mathcal{D}_d d\breve{v}(t) + \mathcal{D}_f d\breve{w}(t). \end{aligned} \tag{9}$$

According to (6) and (7), the uncertain parameter in (9) can be described as

$$\Delta\mathcal{B}_d = -\frac{1}{2}\Xi_n\Delta K^T\Xi_{n_v} = 2J_n\Delta K^T J_{n_v}. \tag{10}$$

III. Strict Bounded Real Properties

This section focuses on the robustly strict bounded real property for this class of quantum systems with uncertainties in the interaction Hamiltonian. We are mainly concerned with the influence of disturbance inputs on the controlled outputs. The uncertain quantum system under consideration is described in the following form:

$$\begin{aligned} d\breve{x}(t) =& \mathcal{A}\breve{x}(t)dt + (\mathcal{B}_d + \Delta\mathcal{B}_d)d\breve{v}(t)\\ &+ \mathcal{B}_f d\breve{w}(t) + \mathcal{B}_h d\breve{v}_{\text{in}}(t)\\ d\breve{z}(t) =& C_p\breve{x}(t)dt + \mathcal{D}_h d\breve{v}_{\text{in}}(t) \end{aligned} \tag{11}$$

where $\mathcal{A} \in \mathbb{R}^{n\times n}$, $\mathcal{B}_d \in \mathbb{R}^{n\times n_v}$, $\mathcal{B}_f \in \mathbb{R}^{n\times n_w}$, $\mathcal{B}_h \in \mathbb{R}^{n\times n_{v_{\text{in}}}}$, $C_p \in \mathbb{R}^{n_z\times n}$, $\mathcal{D}_h \in \mathbb{R}^{n_z\times n_{v_{\text{in}}}}$. Here, $\breve{v}_{\text{in}}(t)$ represents additional quantum noise sources which can be a vacuum noise or a coherent input with finite L_2 energy and $\breve{z}(t)$ is the performance output. The uncertain parameter $\Delta\mathcal{B}_d$ is described in (10). Here, we rewrite (10) in the following way:

$$\Delta\mathcal{B}_d = 2J_n(E_1\mathbf{\Delta}E_2)^T J_{n_v} = \tilde{E}_2\mathbf{\Delta}^T\tilde{E}_1 \tag{12}$$

where $\tilde{E}_1 = E_1^T J_{n_v}$, $\tilde{E}_2 = 2J_nE_2^T$ and $\mathbf{\Delta}$ satisfies (8).

We consider the effect of the variables $\breve{v}(t)$, $\breve{w}(t)$ on the performance variable $\breve{z}(t)$. To analyze this strict bounded real property for (11), we can also rewrite (11) in the following way:

$$\begin{aligned} d\breve{x}(t) &= \mathcal{A}\breve{x}(t)dt + \mathcal{B}d\overline{u}(t) + \mathcal{B}_h d\breve{v}_{\text{in}}(t)\\ d\breve{z}(t) &= C_p\breve{x}(t)dt + \mathcal{D}_h d\breve{v}_{\text{in}}(t) \end{aligned} \tag{13}$$

where $d\overline{u}(t) = \mathcal{P}^T_{n_v+n_w}[d\breve{v}^T(t)\ d\breve{w}^T\ (t)]^T$ and $\mathcal{B} = \begin{bmatrix}\mathcal{B}_d + \Delta\mathcal{B}_d & \mathcal{B}_f\end{bmatrix}\mathcal{P}_{n_v+n_w}$, with

$$\mathcal{P}_{n_v+n_w} = \begin{bmatrix} I_{\frac{n_v}{2}} & 0 & 0 & 0\\ 0 & 0 & I_{\frac{n_v}{2}} & 0\\ 0 & I_{\frac{n_w}{2}} & 0 & 0\\ 0 & 0 & 0 & I_{\frac{n_w}{2}} \end{bmatrix}.$$

$\mathcal{P}_{n_v+n_w}\mathcal{P}^T_{n_v+n_w} = I_{n_v+n_w}$. The strict bounded real property for a class of quantum systems excluding uncertainties is introduced in [29]. It follows from Theorem 5 in [29] that we have the robustly strict bounded real lemma for the uncertain quantum system (13) as follows.

Lemma 1: The quantum stochastic system (13) is robustly strict bounded real with disturbance attenuation $g > 0$ if and only if there exists a positive definite symmetric matrix $P > 0$ such that

$$\mathcal{A}^TP + P\mathcal{A} + C_p^TC_p + g^{-2}P\mathcal{B}\mathcal{B}^TP < 0 \tag{14}$$

or, equivalently,

$$\begin{aligned} &\mathcal{A}^TP + P\mathcal{A} + C_p^TC_p + g^{-2}P\mathcal{B}_f\mathcal{B}_f^TP\\ &\quad + g^{-2}P(\mathcal{B}_d + \Delta\mathcal{B}_d)(\mathcal{B}_d + \Delta\mathcal{B}_d)^TP < 0. \end{aligned} \tag{15}$$

Now we build a relationship between robustly strict bounded real property with uncertainties and strict bounded real property without uncertainties in the following theorem.

Theorem 1: For the uncertain quantum system (13) and a given disturbance attenuation constant $g > 0$, there exists a positive symmetric matrix $P > 0$ such that (15) holds for $\Delta\mathcal{B}_d$ described in (12) and uncertainty $\mathbf{\Delta}$ satisfying (8), if and only if there exists a constant $\epsilon > 0$ such that

$$\begin{bmatrix} \mathcal{A}^TP + P\mathcal{A} + C_p^TC_p & P\mathcal{B}_f & P\tilde{E}_2 & P\mathcal{B}_d\\ \mathcal{B}_f^TP & -g^2I & 0 & 0\\ \tilde{E}_2^TP & 0 & -\frac{1}{\epsilon}I & 0\\ \mathcal{B}_d^TP & 0 & 0 & -g^2I + \frac{1}{\epsilon}\tilde{E}_1^T\tilde{E}_1 \end{bmatrix} < 0. \tag{16}$$

In order to prove the theorem, we require a number of lemmas.

Fact 1 (see [27] for proof): For any real matrices X and Y with appropriate dimensions and a free parameter $\epsilon > 0$, the following inequality is valid:

$$X^TY + Y^TX \leq \epsilon X^TX + \frac{1}{\epsilon}Y^TY. \tag{17}$$

Lemma 2: For $C \in \mathbb{R}^{m\times n}$, $C^TC \leq rI_n$ if and only if $CC^T \leq rI_m$.

Lemma 2 can be proven using the result in Section 3.8 of [37].

Lemma 3 (see [38] for proof): Given any $x \in \mathbb{R}^n$,

$$\max\{(x^TPDFEx)^2 : F^TF \leq I\} = x^TPDD^TPxx^TE^TEx. \tag{18}$$

Lemma 4 (see [39] for proof): Let X, Y, and Z be given $n\times n$ symmetric matrices such that $X \geq 0$, $Y < 0$, and $Z \geq 0$. Furthermore, assume that

$$(\xi^TY\xi)^2 - 4(\xi^TX\xi\xi^TZ\xi) > 0$$

for all non-zero $\xi \in \mathbb{R}^n$. Then, there exists a constant $\lambda > 0$ such that the matrix $\lambda^2X + \lambda Y + Z$ is negative-definite.

Proof of Theorem 1: Based on the Schur complement [40], (15) is equivalent to

$$\begin{bmatrix} G_1 + g^{-2}P\mathcal{B}_f\mathcal{B}_f^TP & P(\mathcal{B}_d + \Delta\mathcal{B}_d)\\ (\mathcal{B}_d + \Delta\mathcal{B}_d)^TP & -g^2I \end{bmatrix} < 0 \tag{19}$$

where $G_1 = \mathcal{A}^TP + P\mathcal{A} + C_p^TC_p$. Define

$$\mathcal{W}_1 = \begin{bmatrix} \mathcal{A}^T P + P\mathcal{A} + C_p^T C_p + g^{-2} P\mathcal{B}_f \mathcal{B}_f^T P & P\mathcal{B}_d \\ \mathcal{B}_d^T P & -g^2 I \end{bmatrix}$$
$$\mathcal{W}_2 = \begin{bmatrix} 0 & P\Delta\mathcal{B}_d \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} P\tilde{E}_2 \\ 0 \end{bmatrix} \mathbf{\Delta}^T \begin{bmatrix} 0 & \tilde{E}_1 \end{bmatrix}. \quad (20)$$

Then (19) can be rewritten as

$$\mathcal{W}_1 + \mathcal{W}_2^T + \mathcal{W}_2 < 0. \quad (21)$$

First, we prove the sufficiency part of the theorem. Applying Schur complement, (16) is equivalent to

$$\begin{bmatrix} G_1 + g^{-2} P\mathcal{B}_f \mathcal{B}_f^T P + \epsilon P\tilde{E}_2 \tilde{E}_2^T P & P\mathcal{B}_d \\ \mathcal{B}_d^T P & -g^2 I + \frac{1}{\epsilon}\tilde{E}_1^T \tilde{E}_1 \end{bmatrix} < 0. \quad (22)$$

Based on the norm bounded condition for $\mathbf{\Delta}$ in (8) and Lemma 2, we have

$$\mathbf{\Delta}\mathbf{\Delta}^T \le I. \quad (23)$$

Using (17) and the condition (23), we obtain the following form:

$$\begin{aligned} &\mathcal{W}_2^T + \mathcal{W}_2 \\ &= \left(\begin{bmatrix} P\tilde{E}_2 \\ 0 \end{bmatrix} \mathbf{\Delta}^T \begin{bmatrix} 0 & \tilde{E}_1 \end{bmatrix} \right)^T + \begin{bmatrix} P\tilde{E}_2 \\ 0 \end{bmatrix} \mathbf{\Delta}^T \begin{bmatrix} 0 & \tilde{E}_1 \end{bmatrix} \\ &\le \epsilon \begin{bmatrix} P\tilde{E}_2 \\ 0 \end{bmatrix} \begin{bmatrix} P\tilde{E}_2 \\ 0 \end{bmatrix}^T + \frac{1}{\epsilon} \begin{bmatrix} 0 & \tilde{E}_1 \end{bmatrix}^T \mathbf{\Delta}\mathbf{\Delta}^T \begin{bmatrix} 0 & \tilde{E}_1 \end{bmatrix} \\ &\le \epsilon \begin{bmatrix} P\tilde{E}_2 \tilde{E}_2^T P & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{\epsilon} \begin{bmatrix} 0 & 0 \\ 0 & \tilde{E}_1^T \tilde{E}_1 \end{bmatrix}. \end{aligned} \quad (24)$$

Hence, we have

$$\begin{aligned} &\begin{bmatrix} G_1 + g^{-2} P\mathcal{B}_f \mathcal{B}_f^T P & P(\mathcal{B}_d + \Delta\mathcal{B}_d) \\ (\mathcal{B}_d + \Delta\mathcal{B}_d)^T P & -g^2 I \end{bmatrix} \\ &\le \begin{bmatrix} G_1 + g^{-2} P\mathcal{B}_f \mathcal{B}_f^T P + \epsilon P\tilde{E}_2 \tilde{E}_2^T P & P\mathcal{B}_d \\ \mathcal{B}_d^T P & -g^2 I + \frac{1}{\epsilon}\tilde{E}_1^T \tilde{E}_1 \end{bmatrix} \\ &< 0. \end{aligned} \quad (25)$$

Then we move to the necessity part. We know that (15) is equivalent to (21) for all $\mathbf{\Delta}$ satisfying (8). It follows from (21) that we have:

$$\mathcal{W}_1 < -\mathcal{W}_2^T - \mathcal{W}_2. \quad (26)$$

Thus, for any given $\xi \in \mathbb{R}^n$, $\xi \ne 0$ and $\mathbf{\Delta}$ satisfying (8), we have

$$\xi^T \mathcal{W}_1 \xi < -2\xi^T \mathcal{W}_2 \xi. \quad (27)$$

It follows that:

$$\xi^T \mathcal{W}_1 \xi < -2\max\{\xi^T \mathcal{W}_2 \xi : \mathbf{\Delta}^T \mathbf{\Delta} \le I\} \le 0. \quad (28)$$

Hence, for any given $\xi \in \mathbb{R}^n$, $\xi \ne 0$, we have

$$(\xi^T \mathcal{W}_1 \xi)^2 > 4\max(\xi^T \mathcal{W}_2 \xi)^2 : \mathbf{\Delta}^T \mathbf{\Delta} \le I\}. \quad (29)$$

Also, Lemma 3 implies

$$(\xi^T \mathcal{W}_1 \xi)^2 > 4\xi^T \begin{bmatrix} P\tilde{E}_2 \tilde{E}_2^T P & 0 \\ 0 & 0 \end{bmatrix} \xi \xi^T \begin{bmatrix} 0 & 0 \\ 0 & \tilde{E}_1^T \tilde{E}_1 \end{bmatrix} \xi \quad (30)$$

for all $\xi \in \mathbb{R}^n$, $\xi \ne 0$. Then, it follows from Lemma 4 that there exists a constant $\epsilon > 0$ such that:

$$\epsilon^2 \xi^T \begin{bmatrix} P\tilde{E}_2 \tilde{E}_2^T P & 0 \\ 0 & 0 \end{bmatrix} \xi + \epsilon \xi^T \mathcal{W}_1 \xi + \xi^T \begin{bmatrix} 0 & 0 \\ 0 & \tilde{E}_1^T \tilde{E}_1 \end{bmatrix} \xi < 0. \quad (31)$$

Therefore, (31) leads to (22) which is equivalent to (16). ■

Using Theorem 1, the connection between the robustly strict bounded real property (15) with uncertainties and the strict bounded real property (16) without uncertain parameters has been built. Therefore, based on Lemma 1 and Theorem 1, the uncertain quantum system (13) is robustly strict bounded real with disturbance attenuation $g > 0$ if and only if (16) is satisfied.

IV. Synthesis of Coherent Robust H^∞ Controller With Direct and Indirect Couplings

In this section, we focus on coherent robust H^∞ control for a class of quantum systems with uncertainties in the interaction Hamiltonian. The controller is connected with the quantum plant via direct and indirect couplings. The closed-loop system consisting of an uncertain quantum system and the coherent controller is introduced in Section IV-A. The relationship between the robust H^∞ control problem and the scaled H^∞ control problem is built through strict bounded real property in Section IV-B.

A. Closed-Loop System and H^∞ Control Objective

1) Closed-Loop Plant-Controller System: The quantum plant under consideration is a class of uncertain quantum systems and the desired controller is a coherent controller. The connection between the system and the controller is through the interaction Hamiltonian and field coupling, generating direct and indirect couplings. Also, the quantum system model is subject to uncertainties in the interaction Hamiltonian. The interaction Hamiltonian between the quantum system and the coherent controller can be described as follows:

$$H_{\text{int}} = \frac{1}{2}(\check{x}^T (R + \Delta R)^T \check{x}_K + \check{x}_K^T (R + \Delta R)\check{x}) \quad (32)$$

where $\check{x}$ is a vector of plant variables, $\check{x}_K$ is a vector of controller variables, $R = i\Xi_{n_K} K/2$ and $K \in \mathbb{R}^{n_K \times n}$ is the direct coupling parameter. The uncertain part of the interaction Hamiltonian is described by $\Delta R = i\Xi_{n_K}\Delta K/2$, where $\Delta K = E_1 \mathbf{\Delta} E_2$ and $\mathbf{\Delta}$ is an uncertain norm bounded real matrix satisfying (8).

We consider the quantum plant described by a non-commutative stochastic model in the following form:

$$\begin{aligned} d\check{x}(t) &= \mathcal{A}\check{x}(t)dt + (\mathcal{B}_{PK} + \Delta\mathcal{B}_{PK})\check{x}_K(t)dt \\ &\quad + \mathcal{B}_f d\check{w}(t) + \mathcal{B}_u d\check{u}(t) + \mathcal{B}_h d\check{v}_{\text{in}}(t) \\ d\check{z}(t) &= C_p \check{x}(t)dt + \mathcal{D}_u d\check{u}(t) \\ d\check{y}(t) &= C\check{x}(t)dt + \mathcal{D}_h d\check{v}_{\text{in}}(t) + \mathcal{D}_f d\check{w}(t) \end{aligned} \quad (33)$$

where $\mathcal{A} \in \mathbb{R}^{n\times n}$, $\mathcal{B}_{PK} \in \mathbb{R}^{n\times n_K}$, $\mathcal{B}_h \in \mathbb{R}^{n\times n_{v_{\text{in}}}}$, $\mathcal{B}_f \in \mathbb{R}^{n\times n_w}$, $\mathcal{B}_u \in \mathbb{R}^{n\times n_u}$, $C_p \in \mathbb{R}^{n_z\times n}$, $\mathcal{D}_u \in \mathbb{R}^{n_z\times n_u}$, $C \in \mathbb{R}^{n_y\times n}$, $\mathcal{D}_h \in \mathbb{R}^{n_y\times n_{v_{\text{in}}}}$, $\mathcal{D}_f \in \mathbb{R}^{n_y\times n_w}$. Here, the quantity $\check{z}(t)$ represents the performance output and the quantity $\check{y}(t)$ represents the measured output. $\check{w}(t)$ is a quantum disturbance vector and satisfies the decomposition $d\check{w}(t) = \beta_{\check{w}}(t)dt + d\hat{w}(t)$, where $\hat{w}(t)$

is the noise part of $\breve{w}(t)$ and $\beta_{\breve{w}}(t)$ is an adapted process [41]. $\breve{v}_{\rm in}(t)$ represents any additional quantum noise. The term $(\mathcal{B}_{PK}+\Delta\mathcal{B}_{PK})\breve{x}_K(t)$ represents direct coupling and uncertain part of direct coupling between the system and the controller.

The coherent controller to be designed is assumed to have the form

$$\begin{aligned} d\breve{x}_K(t) =& \ \mathcal{A}_K\breve{x}_K(t)dt+(\mathcal{B}_{KP}+\Delta\mathcal{B}_{KP})\breve{x}(t)dt \\ &+\mathcal{B}_{Kh}d\breve{v}_K(t)+\mathcal{B}_K d\breve{y}(t) \\ d\breve{u}(t) =& \ C_K\breve{x}_K(t)dt+\mathcal{D}_K d\breve{v}_K(t) \end{aligned} \tag{34}$$

where $\mathcal{A}_K\in\mathbb{R}^{n_K\times n_K}$, $\mathcal{B}_{KP}\in\mathbb{R}^{n_K\times n}$, $\mathcal{B}_{Kh}\in\mathbb{R}^{n_K\times n_{v_K}}$, $\mathcal{B}_K\in\mathbb{R}^{n_K\times n_y}$, $C_K\in\mathbb{R}^{n_u\times n_K}$, $\mathcal{D}_K\in\mathbb{R}^{n_u\times n_{v_K}}$. Here, $\breve{x}_K(t)$ is a vector of self-adjoint controller variables and $\breve{v}_K(t)$ is a vector of additional quantum noise. The term $\mathcal{B}_{KP}\breve{x}(t)$ is introduced because of the direct coupling between the system and the controller, and $\Delta\mathcal{B}_{KP}\breve{x}(t)$ represents the uncertainties in the interaction Hamiltonian. Based on the interaction Hamiltonian (32) [29], we have

$$\begin{aligned} \mathcal{B}_{PK} &= 2J_nK^TJ_{n_K} \\ \mathcal{B}_{KP} &= 2K \\ \Delta\mathcal{B}_{PK} &= 2J_nE_2^T\mathbf{\Delta}^TE_1^TJ_{n_K} \\ \Delta\mathcal{B}_{KP} &= 2E_1\mathbf{\Delta}E_2. \end{aligned} \tag{35}$$

Also, $\mathcal{B}_{PK}$ and $\mathcal{B}_{KP}$ satisfy the following relationship:

$$\mathcal{B}_{KP} = J_{n_K}\mathcal{B}_{PK}^TJ_n. \tag{36}$$

Assume that $\breve{x}(0)$ commutes with $\breve{x}_K(0)$ at time $t=0$. We interconnect (33) with (34) through direct and indirect couplings, the closed loop system is in the following way:

$$\begin{aligned} d\eta(t) =& \ (\overline{\mathcal{A}}+\Delta\overline{\mathcal{A}})\eta(t)dt+\overline{\mathcal{B}}_f d\breve{w}(t)+\overline{\mathcal{B}}_h d\zeta(t) \\ d\breve{z}(t) =& \ \overline{C}\eta(t)dt+\overline{\mathcal{D}}d\zeta(t) \end{aligned} \tag{37}$$

where

$$\begin{aligned} \eta(t) &= \begin{bmatrix} \breve{x}(t) \\ \breve{x}_K(t) \end{bmatrix}, \quad \zeta(t) = \begin{bmatrix} \breve{v}_{\rm in}(t) \\ \breve{v}_K(t) \end{bmatrix} \\ \overline{\mathcal{A}} &= \begin{bmatrix} \mathcal{A} & \mathcal{B}_uC_K+\mathcal{B}_{PK} \\ \mathcal{B}_KC+\mathcal{B}_{KP} & \mathcal{A}_K \end{bmatrix} \\ \overline{\mathcal{B}}_f &= \begin{bmatrix} \mathcal{B}_f \\ \mathcal{B}_K\mathcal{D}_f \end{bmatrix} \\ \overline{\mathcal{B}}_h &= \begin{bmatrix} \mathcal{B}_h & \mathcal{B}_u\mathcal{D}_K \\ \mathcal{B}_K\mathcal{D}_h & \mathcal{B}_{Kh} \end{bmatrix} \\ \overline{C} &= \begin{bmatrix} C_P & \mathcal{D}_uC_K \end{bmatrix} \\ \overline{\mathcal{D}} &= \begin{bmatrix} 0 & \mathcal{D}_u\mathcal{D}_K \end{bmatrix} \\ \Delta\overline{\mathcal{A}} &= \begin{bmatrix} 0 & \Delta\mathcal{B}_{PK} \\ \Delta\mathcal{B}_{KP} & 0 \end{bmatrix} = \overline{E}_1\overline{F}\,\overline{E}_2 \\ \overline{E}_1 &= \begin{bmatrix} 0 & 2J_nE_2^T \\ 2E_1 & 0 \end{bmatrix} \\ \overline{F} &= \begin{bmatrix} \mathbf{\Delta} & 0 \\ 0 & \mathbf{\Delta}^T \end{bmatrix}, \ \overline{E}_2 = \begin{bmatrix} E_2 & 0 \\ 0 & E_1^TJ_{n_K} \end{bmatrix}. \end{aligned} \tag{38}$$

2) H^∞ Control Synthesis Objective: The goal of H^∞ control synthesis is to design a coherent controller such that for a given disturbance attenuation parameter $g>0$, the closed-loop system (37) consisting of quantum system (33) and quantum controller (34) satisfies

$$\begin{aligned} &\int_0^t\langle\beta_{\breve{z}}(s)^T\beta_{\breve{z}}(s)+\tilde{\epsilon}\eta(s)^T\eta(s)\rangle ds \\ &\quad\le(g^2-\tilde{\epsilon})\int_0^t\langle\beta_{\breve{w}}(s)^T\beta_{\breve{w}}(s)\rangle ds+\mu_1+\mu_2t, \quad \forall t>0 \end{aligned} \tag{39}$$

where $\tilde{\epsilon}$, μ_1, $\mu_2>0$ are some real constants, $\beta_{\breve{z}}(t)=\overline{C}\eta(t)$ is controlled output operator and the notation $\langle\cdot\rangle$ refers to expectation over all initial variables and noises [25]. Hence, the effect of the disturbance signal $\beta_{\breve{w}}(t)$ on the performance output $\breve{z}(t)$ is within a specified bound $g>0$. Moreover, it should be noticed that the H^∞ control synthesis objective (39) will be satisfied when the closed-loop system (37) is robustly strict bounded real with disturbance attenuation $g>0$.

B. Relationship Between the Robust H^∞ Control Problem and the Scaled H^∞ Control Problem

The designed controller aims to make the closed-loop system (37) robustly strict bounded real disturbance attenuation $g>0$. In this section, we need to develop Theorem 2 to build a connection between the robust H^∞ control problem and the scaled H^∞ control problem.

Theorem 2: For a given constant $g>0$, there exists a symmetric matrix $P>0$ such that

$$(\overline{\mathcal{A}}+\Delta\overline{\mathcal{A}})^TP+P(\overline{\mathcal{A}}+\Delta\overline{\mathcal{A}})+\overline{C}^T\overline{C}+g^{-2}P\overline{\mathcal{B}}_f\overline{\mathcal{B}}_f^TP<0 \tag{40}$$

where $\Delta\overline{A}$ is shown in (38) and $\mathbf{\Delta}$ satisfies (8), if and only if there exists $\epsilon>0$ such that the following inequality holds:

$$\begin{aligned} &\overline{\mathcal{A}}^TP+P\overline{\mathcal{A}}+\overline{C}^T\overline{C}+g^{-2}P\overline{\mathcal{B}}_f\overline{\mathcal{B}}_f^TP \\ &\quad+\epsilon P\overline{E}_1\overline{E}_1^TP+\frac{1}{\epsilon}\overline{E}_2^T\overline{E}_2<0 \end{aligned} \tag{41}$$

where $\overline{E}_1$ and $\overline{E}_2$ are indicated in (38).

Proof of Theorem 2: The sufficiency part of the proof is shown as below. Based on (38), we know that

$$\Delta\overline{\mathcal{A}}=\overline{E}_1\overline{F}\,\overline{E}_2, \quad \overline{F}=\begin{bmatrix} \mathbf{\Delta} & 0 \\ 0 & \mathbf{\Delta}^T \end{bmatrix}. \tag{42}$$

According to (8), we have $\mathbf{\Delta}^T\mathbf{\Delta}\le I$. Also, it follows from Lemma 2 that $\mathbf{\Delta}\mathbf{\Delta}^T\le I$. Hence, it is straightforward to verify that

$$\overline{F}^T\overline{F}\le I. \tag{43}$$

It follows from (17) and (43) that for any $\epsilon>0$ and $\mathbf{\Delta}$ satisfying (8) we have:

$$\begin{aligned} \Delta\overline{\mathcal{A}}^TP+P\Delta\overline{\mathcal{A}} &= \overline{E}_2^T\overline{F}^T\overline{E}_1^TP+P\overline{E}_1\overline{F}\,\overline{E}_2 \\ &\le\epsilon P\overline{E}_1\overline{E}_1^TP+\frac{1}{\epsilon}\overline{E}_2^T\overline{F}^T\overline{F}\,\overline{E}_2 \\ &\le\epsilon P\overline{E}_1\overline{E}_1^TP+\frac{1}{\epsilon}\overline{E}_2^T\overline{E}_2. \end{aligned} \tag{44}$$

Therefore, if (41) holds, (40) is automatically satisfied.

To prove necessity, suppose (40) is satisfied for all $\mathbf{\Delta}$ satisfying (8). Hence, it follows from (40) that:

$$\begin{aligned}\mathcal{W} := {} & \overline{\mathcal{A}}^T P + P\overline{\mathcal{A}} + \overline{C}^T\overline{C} + g^{-2}P\overline{\mathcal{B}}_f\overline{\mathcal{B}}_f^T P \\ & < -\Delta\overline{\mathcal{A}}^T P - P\Delta\overline{\mathcal{A}}. \end{aligned} \tag{45}$$

Thus, for any given $\xi \in \mathbb{R}^n$, $\xi \neq 0$ and F satisfying (43), we have

$$\xi^T\mathcal{W}\xi < -2\xi^T P\overline{E}_1\overline{F}\,\overline{E}_2\xi. \tag{46}$$

By applying a similar technique to that used in Theorem 1, we know that there exists a constant $\epsilon > 0$ such that

$$\epsilon^2\xi^T P\overline{E}_1\overline{E}_1^T P\xi + \epsilon\xi^T\mathcal{W}\xi + \xi^T\overline{E}_2^T\overline{E}_2\xi < 0. \tag{47}$$

Therefore, we obtain the following relation:

$$\begin{aligned}&\overline{\mathcal{A}}^T P + P\overline{\mathcal{A}} + \overline{C}^T\overline{C} + g^{-2}P\overline{\mathcal{B}}_f\overline{\mathcal{B}}_f^T P \\ &+ \epsilon P\overline{E}_1\overline{E}_1^T P + \frac{1}{\epsilon}\overline{E}_2^T\overline{E}_2 < 0. \end{aligned} \tag{48}$$

■

According to Theorem 2, to design the required H^∞ controller for uncertain quantum system (33), we need to solve (41) without uncertainties.

V. Numerical Procedure of Coherent H^∞ Controller

In Section IV, the relationship between the robust H^∞ control problem and the scaled H^∞ control problem has been built. A necessary and sufficient condition for robustly strict bounded real property of the closed loop system (37) is developed. This section will focus on determining the coefficients of the robust coherent controller (34). In Section V-A, (41) is converted into a linear matrix inequality using various techniques. Section V-B determines a method to make the resulted controller physically realizable. An illustrative example of determining the desired coherent robust H^∞ controller is proposed in Section V-C.

A. Numerical Method

From Theorem 2, we know that the closed-loop system (37) is robustly strict bounded real with disturbance attenuation $g > 0$ if and only if (41) is satisfied.

Since (41) is not a linear matrix inequality, we apply the Schur complement [40], (41) is equivalent to

$$\begin{bmatrix} \overline{\mathcal{A}}^T P + P\overline{\mathcal{A}} & \overline{C}^T & \overline{E}_2^T & P\overline{\mathcal{B}}_f & P\overline{E}_1 \\ \overline{C} & -I & 0 & 0 & 0 \\ \overline{E}_2 & 0 & -\epsilon I & 0 & 0 \\ \overline{\mathcal{B}}_f^T P & 0 & 0 & -g^2 I & 0 \\ \overline{E}_1^T P & 0 & 0 & 0 & -\frac{1}{\epsilon}I \end{bmatrix} < 0. \tag{49}$$

Now, we use a change of variables technique [42], [43] to convert (49) into a linear matrix inequality. The first step of this method is to introduce auxiliary variables $\boldsymbol{M}, \boldsymbol{N}, X, Y$, where X, Y are symmetric, $\boldsymbol{N}$, $\boldsymbol{M}$ are invertible, and $\boldsymbol{M}\boldsymbol{N}^T + XY = I$. Assume that

$$P = \begin{bmatrix} Y & \boldsymbol{N} \\ \boldsymbol{N}^T & -\boldsymbol{N}^T X(\boldsymbol{M}^T)^{-1} \end{bmatrix}, \Pi = \begin{bmatrix} X & I \\ \boldsymbol{M}^T & 0 \end{bmatrix}. \tag{50}$$

It can be shown that $-\boldsymbol{N}^T X(\boldsymbol{M}^T)^{-1}$ is symmetric and $P\Pi = \begin{bmatrix} I & Y \\ 0 & \boldsymbol{N}^T \end{bmatrix}$. Hence, the changes of controller variables can be defined as follows:

$$\begin{aligned} \hat{\mathcal{A}} &= \boldsymbol{N}(\mathcal{A}_K\boldsymbol{M}^T + \mathcal{B}_K CX) + Y(\mathcal{B}_u C_K\boldsymbol{M}^T + \mathcal{A}X) \\ \hat{\mathcal{B}} &= \boldsymbol{N}\mathcal{B}_K \\ \hat{C} &= C_K\boldsymbol{M}^T. \end{aligned} \tag{51}$$

Next, we apply a congruence transformation using $\Gamma = \mathrm{diag}(\Pi^T, I, I, I, I)$ and its transpose on both sides of (49). The following inequality is obtained:

$$\begin{bmatrix} G_2 & \Pi^T\overline{C}^T & \Pi^T\overline{E}_2^T & \Pi^T P\overline{\mathcal{B}}_f & \Pi^T P\overline{E}_1 \\ \overline{C}\Pi & -I & 0 & 0 & 0 \\ \overline{E}_2\Pi & 0 & -\epsilon I & 0 & 0 \\ \overline{\mathcal{B}}_f^T P\Pi & 0 & 0 & -g^2 I & 0 \\ \overline{E}_1^T P\Pi & 0 & 0 & 0 & -\frac{1}{\epsilon}I \end{bmatrix} < 0 \tag{52}$$

where $G_2 = \Pi^T\overline{\mathcal{A}}^T P\Pi + \Pi^T P\overline{\mathcal{A}}\Pi$. By replacing parameters of (52) with the corresponding explicit expression, (52) can be obtained in (54). Hence, we have that (41) holds if and only if the following inequalities:

$$-\begin{bmatrix} X & I \\ I & Y \end{bmatrix} < 0 \tag{53}$$

and (54) hold.

$$\begin{aligned}&\begin{bmatrix} \mathcal{A}X + X\mathcal{A}^T + \mathcal{B}_u\hat{C} + (\mathcal{B}_u\hat{C})^T & \hat{\mathcal{A}}^T + \mathcal{A} & XC_p^T + \hat{C}^T\mathcal{D}_u^T & \star & \star & \mathcal{B}_f & 0 & 2J_nE_2^T \\ \hat{\mathcal{A}} + \mathcal{A}^T & \mathcal{A}^T Y + Y\mathcal{A} & C_p^T & \star & \star & Y\mathcal{B}_f + \hat{\mathcal{B}}\mathcal{D}_f & 2NE_1 & 2YJ_nE_2^T \\ \star & \star & -I & 0 & 0 & 0 & 0 & 0 \\ E_2X & E_2 & 0 & -\epsilon I & 0 & 0 & 0 & 0 \\ E_1^T J_{n_K} M^T & 0 & 0 & 0 & -\epsilon I & 0 & 0 & 0 \\ \star & \star & 0 & 0 & 0 & -g^2 I & 0 & 0 \\ \star & \star & 0 & 0 & 0 & 0 & -\frac{1}{\epsilon}I & 0 \\ \star & \star & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\epsilon}I \end{bmatrix} \\ &+ \begin{bmatrix} \mathcal{B}_{PK}\boldsymbol{M}^T + (\mathcal{B}_{PK}\boldsymbol{M}^T)^T & (\boldsymbol{N}\mathcal{B}_{KP}X)^T + (Y\mathcal{B}_{PK}\boldsymbol{M}^T)^T & 0 & 0 & 0 & 0 & 0 & 0 \\ \boldsymbol{N}\mathcal{B}_{KP}X + Y\mathcal{B}_{PK}\boldsymbol{M}^T & \hat{\mathcal{B}}C + (\hat{\mathcal{B}}C)^T + \boldsymbol{N}\mathcal{B}_{KP} + (\boldsymbol{N}\mathcal{B}_{KP})^T & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} < 0. \end{aligned} \tag{54}$$

By solving (53) and (54), the desired controller can be obtained. According to (51), the corresponding parameters of the controller are defined in the following way:

$$\begin{aligned}\mathcal{A}_K &= \boldsymbol{N}^{-1}(\hat{\mathcal{A}} - \boldsymbol{N}\mathcal{B}_K C X - Y(\mathcal{B}_u C_K \boldsymbol{M}^T + \mathcal{A}X))\boldsymbol{M}^{-T}\\ \mathcal{B}_K &= \boldsymbol{N}^{-1}\hat{\mathcal{B}}\\ C_K &= \hat{C}(\boldsymbol{M}^T)^{-1}.\end{aligned} \tag{55}$$

Notice that (54) is still not a linear matrix inequality due to terms $\boldsymbol{N}\mathcal{B}_{KP}X$ and $Y\mathcal{B}_{PK}\boldsymbol{M}^T$ in (54). Hence, we apply the multi-step optimization method in Section IV of [29] to solve this issue. The method is formulated as follows:

Initialization: Set $\mathcal{B}_{PK} = 0$, $\mathcal{B}_{KP} = 0$, and $E_1 = 0$.

Step 1: Solve linear matrix inequalities (53) and (54) using the LMI technique [40]. Then find nonsingular matrices $\boldsymbol{M}$ and $\boldsymbol{N}$ to satisfy $\boldsymbol{M}\boldsymbol{N}^T = I - XY$. Thus, according to (55), indirect coupling parameters $\mathcal{A}_K$, $\mathcal{B}_K$, and C_K can be obtained.

Step 2: Pertaining to Step 1: Solve (54) to find direct coupling parameters $\mathcal{B}_{PK}$ and $\mathcal{B}_{KP}$ based on the relationship (36).

Step 3: Fix $\mathcal{B}_{PK}$ and $\mathcal{B}_{KP}$ achieved in Step 2, and fix $\boldsymbol{M}$ and $\boldsymbol{N}$ as achieved in Step 1; return to Step 1 to find parameters $\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{C}, X, Y$. Then, go to Step 2.

Using this multi-step optimization and change of variables technique, (53) and (54) can be solved using LMI method and coherent controller parameters $\mathcal{A}_K$, $\mathcal{B}_K$, C_K, $\mathcal{B}_{PK}$, and $\mathcal{B}_{KP}$ can be obtained.

B. Physical Realizability

As explained in Section V-A, the coefficients of a coherent controller (34): $\mathcal{A}_K$, $\mathcal{B}_K$, C_K, $\mathcal{B}_{PK}$, and $\mathcal{B}_{KP}$ can be achieved through numerical procedure. To obtain a physically realizable controller, the desired controller also needs to satisfy the physical realizability conditions (5) and (36).

It is shown in Theorem 5.5 of [25] that for an arbitrarily given $\mathcal{A}_K$, $\mathcal{B}_K$, C_K, it is always possible to find parameters $\mathcal{B}_{Kh}$, $\mathcal{D}_K$ to ensure the resultant indirect coupling physically realizable [44]. Moreover, according to Section IV of [29], a physically realizable controller can be designed using the multi-step optimization. Hence, by using the method in Section V-A and Theorem 5.5 of [25], a fully quantum controller with direct and indirect couplings can be obtained.

C. An Illustrative Example

In this section, we consider a quantum optical cavity system [45]–[47] which contains an optical cavity coupled to three optical channels, u, v, w. Also, this quantum model has been demonstrated experimentally; e.g., see [48]. The aim of the H^∞ control is to attenuate the influence of disturbance w on the output z.

The dynamics of the system are in the following form:

$$\begin{aligned}da(t) &= -\frac{\gamma}{2}a(t)dt - \sqrt{\kappa_1}dv(t) - \sqrt{\kappa_2}dw(t) - \sqrt{\kappa_3}du(t)\\ da^*(t) &= -\frac{\gamma}{2}a^*(t)dt - \sqrt{\kappa_1}dv^*(t) - \sqrt{\kappa_2}dw^*(t)\\ &\quad - \sqrt{\kappa_3}du^*(t)\\ dy(t) &= \sqrt{\kappa_2}a(t)dt + dw(t)\\ dz(t) &= \sqrt{\kappa_3}a(t)dt + du(t).\end{aligned} \tag{56}$$

Here, $\gamma = \kappa_1 + \kappa_2 + \kappa_3$. The QSDEs (56) are described by the annihilation operator a and creation operator a^*. Since this annihilation/creation representation often results in complex-valued coefficients, we rewrite (56) as in (33) which is real-valued quadrature form. The relationship between the annihilation/creation operators and position/momentum operators is as follows: $q(t) = a(t) + a^*(t)$ and $p(t) = (a(t) - a^*(t))/i$. The corresponding coefficients are shown below:

$$\begin{aligned}&\mathcal{A} = -\frac{\gamma}{2}I, \quad \mathcal{B}_h = -\sqrt{\kappa_1}I\\ &\mathcal{B}_f = -\sqrt{\kappa_2}I, \quad \mathcal{B}_u = -\sqrt{\kappa_3}I\\ &C_p = \sqrt{\kappa_3}I, \quad C = \sqrt{\kappa_2}I\\ &\mathcal{D}_u = I, \quad \mathcal{D}_f = I.\end{aligned} \tag{57}$$

We choose the parameters as follows: $\kappa_1 = 4.6$, $\kappa_2 = 0.3$, $\kappa_3 = 0.2$, the required disturbance attenuation is $g = 0.3$, and the uncertainty in interaction Hamiltonian $\boldsymbol{\Delta}$ satisfies $\boldsymbol{\Delta}^T\boldsymbol{\Delta} \le I$ with $E_1 = I$ and $E_2 = 0.4I$.

To achieve a desired controller, we apply the controller design method in Theorem 2 and solve the corresponding (53) and (54) using LMI techniques and multi-step optimization. The coefficient matrices of the controller (34) can be obtained as follows:

$$\begin{aligned}&\mathcal{A}_K = -6.1207I, \quad \mathcal{B}_K = -1.0532I\\ &C_K = 0.7278I, \quad \mathcal{B}_{PK} = -0.0838I\\ &\mathcal{B}_{KP} = 0.0838I.\end{aligned} \tag{58}$$

As indicated in Section V-B, the resulted coherent controller needs to be fully quantum and the physical realizability conditions need to be satisfied. Based on the physical realizability conditions in (5), we obtain the following matrices:

$$\begin{aligned}\mathcal{B}_{Kh} &= \begin{bmatrix} -0.7278 & 0 & 2 & 3.5\\ 0 & -0.7278 & -4 & -1.699\end{bmatrix}\\ \mathcal{D}_K &= \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}.\end{aligned} \tag{59}$$

Therefore, for quantum system (56) with perturbation in the interaction Hamiltonian, we apply the coherent H^∞ controller method with direct and indirect couplings. Then, the desired coherent controller can be obtained to ensure the required H^∞ performance for the overall system.

VI. Conclusion

This paper has considered a class of quantum systems with uncertainties in the interaction Hamiltonian. This class of systems interact with exosystems through direct and indirect couplings. For a given disturbance attenuation, a robustly strict bounded real condition for this class of uncertain quantum systems is given. A necessary and sufficient condition of H^∞ analysis for the quantum systems is proposed. Moreover, a coherent robust H^∞ controller design method for this class of uncertain quantum systems via direct and indirect couplings is studied. A necessary and sufficient condition was provided to convert this robust H^∞ control problem into a scaled H^∞ problem. To obtain coefficients of a coherent controller, a numerical procedure is presented by using LMI formulation, multi-step optimization and physical realizability conditions. As a future research direction, we may focus on other types of uncertainties and develop new methods for robust controller design of these uncertain systems. Another interesting idea is to extend the robustness results to the framework of the Kalman decomposition for linear quantum systems [49].

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