Quantum Robust Optimal Control for Linear Complex Quantum Systems With Uncertainties

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***Abstract*—The aim of this note is to study the quantum robust optimal control problem for a class of linear quantum systems with uncertainties. It is shown that such problem can be converted into a mixed linear quadratic Gaussian and H^∞ quantum control problem. A physical model described by complex quantum stochastic differential equations (CQSDEs) is presented for the quantum system with uncertainties. The quantum system described by CQSDEs is called a complex quantum system in terms of annihilation and creation operators (Maalouf and Petersen, 2011). For an uncertain quantum system, we build a connection with a scaled linear system without uncertainty. The desired quantum robust optimal control results are derived based on the connection. We provide one numerical procedure for quantum robust optimal controller synthesis and then provide an example. The resulting controller presents robustness and optimal performance.**

***Index Terms*—Complex system, H^∞ control, linear quadratic Gaussian (LQG) control, quantum feedback control, quantum linear stochastic system, quantum robust optimal control, scaled system, uncertainty.**

I. Introduction

Nowadays, quantum technology presents powerful capabilities in computational processing, sensing and measurement, and security transmission as well as information data communication [2], [3], [4]. Great advances have been made in the theoretical investigation and the applications of quantum science over the past few decades [5], [6]. Quantum control is becoming a significant branch of control theory [7], [8], [9]. Unique characteristics (e.g., entanglement and coherence) of quantum systems require novel principles of quantum control theory, which provides a great impetus for development in the area of quantum control systems [10], [11]. Quantum systems are unavoidably subjected to some kinds of uncertainties due to practical applications of quantum technology. Research on quantum systems with uncertainties has practical significance and attracted more attention [2], [12], [13], [14], [15], [16]. However, most of the existing research in quantum control cannot directly solve the problem of optimal robustness about uncertainties in the quantum system model described by complex quantum stochastic differential equations (CQSDEs).

In classical control theory, the robust optimal control problem can be converted into a mixed linear quadratic Gaussian (LQG) and H^∞ problem [17]. When the control system with uncertainties is corrupted by both Gaussian noises and disturbance signals, we need to consider not only optimal performance in terms of H_2 norm but also robustness specifications measured in H^∞ norm [18]. The mixed LQG and H^∞ control issue has been studied widely in classical feedback control systems [19]. In last two decades, classical control methods have been extended into the quantum domain [20], [21], [22]. Robustness is considered as one of the most important problems in quantum control systems. James et al. [10] established a general framework of H^∞ control for a class of linear quantum stochastic systems, where necessary and sufficient conditions for quantum physical models were derived and a quantum version of bounded real lemma was investigated. The LQG control problem for linear stochastic quantum systems was investigated in [20]. Shaiju et al. [23] investigated a guaranteed cost control problem of linear stochastic quantum systems with uncertainties. Coherent robust H^∞ control of linear uncertain quantum systems described by QSDEs was discussed in [13].

Nevertheless, few papers have considered the quantum robust optimal controller design for a class of linear uncertain quantum systems described by CQSDEs. Therefore, in this note, we study the mixed LQG and H^∞ control for a class of linear quantum systems described by CQSDEs subjected to uncertainties. The representation of CQSDEs is a natural description for a class of quantum optical systems [1]. To this end, we intend to design a coherent quantum controller to handle a quantum robust optimal control problem for linear complex quantum systems with uncertainties in Hamiltonian. The calculations of our approach to the quantum robust optimal control are carried out in the complex domain. Relations between the CQSDEs under consideration and the corresponding real QSDEs (defined in terms of quadrature variables) are presented. We also build a relationship between a quantum system with uncertainties and an equivalent scaled system without uncertainty. Quantum robust optimal controller synthesis encounters limitations including the nonconvex constraints involved in design procedures. We thus develop a numerical procedure for quantum robust optimal controller designs.

The rest of this note is organized as follows. Section II provides a brief overview of quantum systems with real quadrature representation and annihilation–creation operator representation, respectively, and presents a connection between them. We also describe uncertainties appeared in the Hamiltonian in this section. In Section III, we investigate performance characteristics such as stability, LQG performance, and bounded real lemma for linear quantum systems. Section IV describes the setup of the closed-loop system and proposes a quantum robust optimal control problem for linear quantum complex systems subjected to uncertainties. Section V presents the quantum robust optimal controller synthesis for the proposed model and also gives a numerical example. Finally, Section VI concludes this note.

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A. Notation

Some notations adopted in this note are listed as follows. The commutator is denoted by $[A,B]=AB-BA$. Let $X=[x_{jk}]$ be a matrix of complex numbers or linear operators, then $X^{\#}=[x_{jk}^{*}]$ represents the operation of taking the adjoint of each element of X, and $X^{\dagger}=[x_{jk}^{*}]^{T}$. For a column vector $x=[\,x_1 \quad \cdots \quad x_n\,]^T$, we suppose $x^{\#}=[\,x_1^{*} \quad \cdots \quad x_m^{*}\,]^T$, where the asterisk $*$ represents complex conjugation. The notation $\langle\cdot\rangle$ is used for both classical and quantum expectations. The Kronecker delta function is denoted by δ_{jk}. The doubled-up column vector is described by $\breve{x}=[\,x^T \quad (x^{\#})^T\,]^T$. For any two matrices $E_1, E_2\in\mathbb{C}^{r\times k}$, a doubled-up matrix $\Pi(E_1,E_2)$ is assumed to be $\Pi(E_1,E_2):=[\,E_1 \quad E_2; \quad E_2^{\#} \quad E_1^{\#}\,]$. Define $J_n=\mathrm{diag}(I_{\frac{n}{2}},-I_{\frac{n}{2}})$ and $\tilde{J}_n=\begin{bmatrix}0 & I_{\frac{n}{2}}\\ -I_{\frac{n}{2}} & 0\end{bmatrix}$, where I_n represents the $n\times n$ identity matrix. For a matrix $X\in\mathbb{C}^{n\times m}$ (n and m are even), define $X^{\flat}:=J_m X^{\dagger} J_n$.

II. Linear Quantum Models

A. Annihilation–Creation Representation for Linear Quantum Stochastic Systems

In the Heisenberg picture, the evolution of the annihilation $a=[a_1,\ldots,a_m]^T$ and creation operators $a^*=[a_1^*,\ldots,a_m^*]^T$ interacting with quantum white noise $b_{\text{in}}(t)$ is defined by

$$a_j(t)=U^*(t)a_jU(t),\ a_j^*(t)=U^*(t)a_j^*U(t). \tag{1}$$

Here, $U(t)$ is an unitary operator satisfying

$$dU(t)=\left\{LdB_{\text{in}}^{\dagger}-L^{\dagger}dB_{\text{in}}-\left(\frac{1}{2}L^{\dagger}L+iH\right)dt\right\}U(t),\ U(0)=I$$

where H is a Hamiltonian operator and L is a coupling operator [24]. Vacuum bosonic fields are defined as $B_{\text{in},m}(t)=\int_0^t b_{\text{in},m}(\tau)d\tau$ and their associated nonzero Itō products are given by $dB_{\text{in},j}(t)dB_{\text{in},k}^{*}(t)=\delta_{jk}dt$. The Itō matrix F_{in} is described by

$$F_{\text{in}}dt=(d\breve{B}_{\text{in}}^{\#}(t)d\breve{B}_{\text{in}}^{T}(t))^T=\begin{bmatrix}0_m & 0\\ 0 & I_m\end{bmatrix}dt \tag{2}$$

where m is the dimension of $B_{\text{in}}(t)$.

A complex differential equation defined in terms of annihilation and creation operators $\breve{a}=[a^T\ (a^{\#})^T]^T$ (*annihilation–creation representation*) is derived from (1) as follows:

$$d\breve{a}(t)=A\breve{a}(t)dt+Bd\breve{B}_{\text{in}}(t) \tag{3}$$

where $A\in\mathbb{C}^{n\times n}, B\in\mathbb{C}^{n\times n_\omega}, [\breve{a}_j,\breve{a}_k^*]=(\Theta_n)_{jk}$, and $n=2\,l$. $\Theta_n=\Theta_n^{\dagger}=TJ_nT^{\dagger}$ with $T=\Pi(E_1,E_2)$ is a commutation complex matrix. The *canonical* commutation relation for $\breve{B}_{\text{in}}(t)$ is written as $[d\breve{B}_{\text{in}}(t),\ d\breve{B}_{\text{in}}^{\dagger}(t)]=\Theta_\omega=J_{n_\omega}$ with the dimension $n_\omega=2\,m$ of $\breve{B}_{\text{in}}(t)$.

The output field $\breve{Y}(t)=U^*(t)\breve{B}_{\text{in}}(t)U(t)$ evolves according to the following equation:

$$d\breve{Y}(t)=C\breve{a}(t)dt+Dd\breve{B}_{\text{in}}(t) \tag{4}$$

where C and D are complex matrices.

A linear noncommutative complex quantum system of the form (3) and (4) is physically realizable if and only if the following conditions (*physical realizability conditions*) hold [24]:

$$A+A^{\flat}+B^{\flat}B=0,\ BD^{\flat}=-C^{\flat},\ DD^{\flat}=I. \tag{5}$$

B. Relations Among Quadrature Representation, Annihilation–Creation Representation and the Triple (S,L,H) Representation

An equivalent *quadrature representation* results in a linear quantum system with real matrices described by

$$\begin{aligned} dx(t)&=\tilde{A}x(t)dt+\tilde{B}dw(t)\\ dy(t)&=\tilde{C}x(t)dt+\tilde{D}dw(t) \end{aligned} \tag{6}$$

where $x=[q^T\ p^T]^T$ with $q=[q_1,\ldots,q_l]$ and $p=[p_1,\ldots,p_l]$; q_j and p_j represent position and momentum operators, respectively. The commutation relation for x is given by

$$x(t)x^T(t)-(x(t)x^T(t))^T=2i\tilde{\Theta}_n \tag{7}$$

with $\tilde{\Theta}_n=\tilde{J}_n$. $w=[w_{q_1},\ldots,w_{q_m},w_{p_1},\ldots,w_{p_m}]^T$ is the quantum fields and its Itō products is expressed as $dw(t)dw^T(t)=\tilde{F}_\omega dt$. Here, $\tilde{F}_\omega=I_{2\,m}+i\mathrm{diag}_m(\tilde{J})$ is the *canonical* case. The commutation relation of w is given by

$$[dw(t),dw^T(t)]=\tilde{F}_\omega-\tilde{F}_\omega^T=2i\tilde{\Theta}_\omega dt \tag{8}$$

with $\tilde{\Theta}_\omega=\tilde{J}_{n_\omega}$. The output y satisfies

$$[dy(t),dy^T(t)]=\tilde{F}_y-\tilde{F}_y^T=2i\tilde{\Theta}_y dt \tag{9}$$

with $\tilde{\Theta}_y=\tilde{J}_{n_y}$. A physical quantum system (6) is required to satisfy the constraints as follows [11]:

$$\tilde{A}\tilde{\Theta}_n+\tilde{\Theta}_n\tilde{A}^T+\tilde{B}\tilde{\Theta}_\omega\tilde{B}^T=0 \tag{10}$$

$$\tilde{B}\tilde{\Theta}_\omega\tilde{D}^T=-\tilde{\Theta}_n\tilde{C}^T,\tilde{D}\tilde{\Theta}_\omega\tilde{D}^T=\tilde{\Theta}_y. \tag{11}$$

Now we define a matrix Ω as

$$\Omega=\frac{1}{\sqrt{2}}\begin{bmatrix}I & I\\ -iI & iI\end{bmatrix}. \tag{12}$$

Let Ω_n, Ω_{in}, and Ω_{out} be of the form (12) of suitable dimension. Then the following relations hold

$$\Omega_n\breve{a}=x,\ \Omega_{\text{in}}\breve{B}_{\text{in}}=w,\ \Omega_{\text{out}}\breve{Y}=y. \tag{13}$$

So the system matrices of (6) may be rewritten as $\tilde{A}=\Omega_nA\Omega_n^{\dagger}$, $\tilde{B}=\Omega B\Omega_{\text{in}}^{\dagger}$, $\tilde{C}=\Omega_{\text{out}}C\Omega_n^{\dagger}$, $\tilde{D}=\Omega_{\text{out}}D\Omega_{\text{in}}^{\dagger}$, respectively.

The annihilation–creation model described by (3) and (4) can be equivalently reparameterized in terms of the triple (S,L,H), where the scattering operator S satisfies $S^{\dagger}S=SS^{\dagger}=I$. The coupling operator L is given by

$$L=\Upsilon\breve{a}=\Upsilon_-a+\Upsilon_+a^{\#} \tag{14}$$

with a complex matrix $\Upsilon=[\Upsilon_-\quad\Upsilon_+]$. H is the Hamiltonian operator representing the self-energy of the system defined by

$$H=\frac{1}{2}\breve{a}^{\dagger}\Pi\left(M_-,M_+\right)\breve{a} \tag{15}$$

where M_- and M_- are complex matrices with $M_-=M_-^{\dagger}$ and $M_+=M_+^T$.

Similarly, the *quadrature representation* (6) can also be redescribed by the same triple (S,L,H), where S is an unitary matrix; the coupling operator (14) and the Hamiltonian operator (15) can be equally rewritten as $L=\tilde{\Lambda}x$ with a coupling matrix $\tilde{\Lambda}$ and $H=\frac{1}{2}x^T\tilde{R}x$ with a real symmetric Hamiltonian matrix $\tilde{R}$, respectively.

C. Complex Quantum Model With Uncertainties

Given a linear complex quantum system described by (3) and (4) subjected to perturbations in the system Hamiltonian operator, the perturbation Hamiltonian is defined as

$$H_{\text{perturbation}} = \frac{1}{2}\breve{a}^{\dagger}K^{\dagger}\Pi\left(E_{-},E_{+}\right)K\breve{a} \tag{16}$$

where $K \in \mathbb{C}^{l\times n}$ and $\Pi(E_{-},E_{+}) \in \mathbb{C}^{l\times l}$ is a Hermitian matrix satisfying

$$\Pi^{2}\left(E_{-},E_{+}\right) \leq I. \tag{17}$$

Now we present a general model for an open uncertain quantum system

$$\begin{aligned} d\breve{a}(t) &= (A+\Delta A)\breve{a}(t)dt + Bd\breve{B}_{\text{in}}(t) \\ d\breve{Y}(t) &= C\breve{a}(t)dt + Dd\breve{B}_{\text{in}}(t) \end{aligned} \tag{18}$$

with the following parameters:

$$\begin{aligned} &A = -iJ_{n}\Pi(M_{-},M_{+}) - \frac{1}{2}C^{\flat}C,\ \Delta A = -iJ_{n}K^{\dagger}\Pi(E_{-},E_{+})K \\ &B = -C^{\flat}\Pi(S,0),\ C = \Pi\left(\Upsilon_{-},\Upsilon_{+}\right),\ D = \Pi(S,0) \end{aligned} \tag{19}$$

where the total Hamiltonian for the uncertain system (18) is represented by $H = \frac{1}{2}\breve{a}^{\dagger}[\Pi(M_{-},M_{+}) + K^{\dagger}\Pi(E_{-},E_{+})K]\breve{a}$ and the perturbation Hamiltonian (16) can be modeled by ΔA in (18).

III. Performance Characteristics of Quantum Stochastic Systems

In this section, we investigate stability, LQG performance and bounded real lemma, suitably adapted to the models presented in Section II.

A. Stability

Lemma 1: If there exists a nonnegative function $r(t)$ satisfying the following inequality:

$$\frac{dr(t)}{dt} + cr(t) \leq \lambda \tag{20}$$

where c and λ are positive real numbers, then inequality

$$r(t) \leq e^{-ct}r(0) + \frac{\lambda}{c} \tag{21}$$

holds, which means that $r(t)$ is bounded for all $t > 0$.

The proof of Lemma 1 is similar to that in [19], so it is omitted.

Definition 1: The nominal quantum system described by (3) and (4) is said to be bounded stable if a real-valued function $r(t) = \langle V(t)\rangle$ satisfies inequality (20), where $V(t) = \breve{a}^{\dagger}(t)P\breve{a}(t)$ is a nonnegative operator valued quadratic form, and $\langle V(t)\rangle$ describes an abstract energy of time t.

Theorem 1: The nominal quantum system described by (3) and (4) is bounded stable in the sense of Definition 1 with

$$r(t) = \langle V(t)\rangle = \langle \breve{a}^{\dagger}(t)P\breve{a}(t)\rangle \tag{22}$$

if there exist $P \geq 0$ and $Q \geq cP$ ($c > 0$) such that

$$A^{\dagger}P + PA + Q \leq 0. \tag{23}$$

Proof: Applying quantum Itō rule to the Lyapunov function (22) yields

$$\begin{aligned} d\left\langle V(t)\right\rangle &= \left\langle d\breve{a}^{\dagger}(t)P\breve{a}(t) + \breve{a}^{\dagger}(t)Pd\breve{a}(t) + d\breve{a}^{\dagger}(t)Pd\breve{a}(t)\right\rangle dt \\ &= \left\langle \breve{a}^{\dagger}(t)[A^{\dagger}P+PA]\breve{a}(t) + \text{Tr}(B^{\dagger}PBF)\right\rangle dt \end{aligned} \tag{24}$$

where $\left\langle d\breve{B}_{\text{in}}^{\dagger}(t)\right\rangle dt = \left\langle d\breve{B}_{\text{in}}(t)\right\rangle dt = 0$ and F is defined in (2). From (24), we obtain

$$\begin{aligned} \frac{d\langle V(t)\rangle}{dt} + c\langle V(t)\rangle &= \left\langle \breve{a}^{\dagger}[A^{\dagger}P+PA]\breve{a} + \lambda\right\rangle + c\left\langle \breve{a}^{\dagger}P\breve{a}\right\rangle \\ &= \left\langle \breve{a}^{\dagger}[A^{\dagger}P+PA+cP]\breve{a} + \lambda\right\rangle \end{aligned} \tag{25}$$

where $\lambda = \text{Tr}(B^{\dagger}PBF)$.

Now suppose that (23) holds. Then, we get

$$\frac{d\langle V(t)\rangle}{dt} + c\langle V(t)\rangle \leq \left\langle \breve{a}^{\dagger}[A^{\dagger}P+PA+Q]\breve{a} + \lambda\right\rangle \leq \lambda. \tag{26}$$

The proof is completed. ■

B. LQG Performance

The performance variable $\breve{z}_{l}(t)$ of the nominal quantum system described by (3) and (4) is written as

$$\breve{z}_{l}(t) = C_{z}\breve{a}. \tag{27}$$

A quadratic performance index for the nominal quantum system is given by

$$J\left(t_{f}\right) = \int_{0}^{t_{f}} \frac{1}{2}\left\langle \breve{z}_{l}^{\dagger}(t)\breve{z}_{l}(t) + \breve{z}_{l}^{T}(t)\breve{z}_{l}^{\#}(t)\right\rangle dt. \tag{28}$$

In this note, we concentrate on the infinite horizon case $t\to\infty$. Along the line of [20], for the nominal quantum system described by (3) and (4), the infinite-horizon LQG performance index is defined as

$$\begin{aligned} J_{\infty} &= \lim_{t_{f}\to\infty}\frac{1}{t_{f}}\int_{0}^{t_{f}}\text{Tr}\left\{C_{z}P_{l}(t)C_{z}^{\dagger}\right\}dt \\ &= \text{Tr}\left\{C_{z}P_{l}C_{z}^{\dagger}\right\} \end{aligned} \tag{29}$$

where A is assumed to be asymptotically stable and $P_{l} = \lim_{t_{f}\to\infty}P_{l}(t)$ is a Hermitian matrix satisfying

$$AP_{l} + P_{l}A^{\dagger} + \frac{1}{2}BB^{\dagger} = 0. \tag{30}$$

C. Bounded Real Lemma

In this section, we consider the following quantum system:

$$\begin{aligned} d\breve{a}(t) &= A\breve{a}(t)dt + Bd\breve{B}_{\omega}(t) + Gd\breve{B}_{v}(t) \\ d\breve{z}_{h}(t) &= C\breve{a}(t)dt + Dd\breve{B}_{\omega}(t) + Hd\breve{B}_{v}(t) \end{aligned} \tag{31}$$

where $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times n_{\omega}}$, $C \in \mathbb{C}^{n_{y}\times n}$, $D \in \mathbb{C}^{n_{y}\times n_{\omega}}$, $G \in \mathbb{C}^{n\times n_{v}}$, and $H \in \mathbb{C}^{n_{y}\times n_{v}}$. $\breve{B}_{\omega}$ represents a disturbance signal of the form

$$d\breve{B}_{\omega}(t) = \breve{\beta}_{\omega}(t)dt + d\breve{B}_{\tilde{\omega}}(t) \tag{32}$$

where $\breve{\beta}_{\omega}(t)$ is the signal part and $\breve{B}_{\tilde{\omega}}(t)$ is quantum noise part of $\breve{B}_{\omega}(t)$. Quantum noises $\breve{B}_{v}(t)$ are in vacuum state and represent additional noise sources. The performance variable $\breve{z}_{h}$ is given by

$$\beta_{z}(t) = C\breve{a}(t) + D\breve{\beta}_{\omega}(t). \tag{33}$$

Now we define $\|C(sI-A)^{-1}B+D\|_{\infty} < g$ as the H^{∞} norm of (31).

Definition 2: Given a supply rate

$$r(\beta_{z}(t),\breve{\beta}_{w}(t)) = [\breve{a}^{\dagger}(t)\quad \breve{\beta}_{\omega}^{\dagger}(t)]R\begin{bmatrix}\breve{a}(t)\\ \breve{\beta}_{\omega}(t)\end{bmatrix} \tag{34}$$

with a Hermitian matrix

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^{\dagger} & R_{22} \end{bmatrix} \tag{35}$$

and a candidate storage $V(t)$, the quantum system (31) is said to be dissipative if there exists a real number $\lambda \geq 0$ such that

$$\langle V(t)\rangle - \langle V(0)\rangle + \int_0^t \langle r(\bar{\beta}_z(s), \breve{\beta}_\omega(s))\rangle ds \leq \lambda t \tag{36}$$

for all $t > 0$.

Lemma 2 ([10]): The quantum system (31) is dissipative in terms of the supply rate (34) if there exists a positive definite Hermitian matrix P such that the following matrix inequality holds:

$$\begin{bmatrix} A^\dagger P + PA + R_{11} & R_{12} + PB \\ B^\dagger P + R_{12}^\dagger & R_{22} \end{bmatrix} \leq 0. \tag{37}$$

Furthermore, the quantum system (31) is strictly dissipative if the relation (37) holds with the symbol "$\leq$" replaced by the symbol "$<$".

Definition 3 ([10]): The system (31) is said to be bounded real if the system (31) is dissipative with the supply rate

$$\begin{aligned} r(\beta_z(t), \breve{\beta}_\omega(t)) &= \beta_z^\dagger(t)\beta_z(t) - g^2 \breve{\beta}_\omega^\dagger(t)\breve{\beta}_\omega(t) \\ &= [\breve{a}^\dagger(t) \quad \breve{\beta}_\omega^\dagger(t)] R \begin{bmatrix} \breve{a}(t) \\ \breve{\beta}_\omega(t) \end{bmatrix} \end{aligned} \tag{38}$$

where the Hermitian matrix R is of the form (35) and $g > 0$ is a disturbance attenuation scalar.

Theorem 2: Based on Definition 3, the following three statements are equivalent.

1) For a disturbance attenuation scalar $g > 0$, the quantum system (31) is strictly bounded real.

2) $\|C(sI - A)^{-1}B + D\|_\infty < g$ and A is stable.

3) $g^2 I - D^\dagger D_p > 0$, there exists a Hermitian matrix $P_h > 0$ satisfying inequality

$$A^\dagger P_h + P_h A + P_h B(D^\dagger D - g^2 I)^{-1} B^\dagger P_h + C^\dagger C < 0 \tag{39}$$

or, equivalently

$$\begin{bmatrix} A^\dagger P_h + P_h A & P_h B & C^\dagger \\ B^\dagger P_h & -gI & D^\dagger \\ C & D & -gI \end{bmatrix} < 0. \tag{40}$$

The similar proof of Theorem 2 can be found in [10], so it is omitted.

IV. Quantum Robust Optimal Control Analysis

In this section, we first investigate closed-loop systems in Section IV-A. We also build a connection between the uncertain closed-loop system and the scaled closed-loop system without uncertainty in Section IV-B. Then, we formulate a quantum robust optimal control problem for the uncertain model in Section IV-C.

A. Closed-Loop System

Consider a quantum plant described by a noncommutative stochastic model of the following form:

$$\begin{aligned} d\breve{a}_p(t) &= (A + \Delta A)\breve{a}_p(t)dt + B_0 d\breve{B}_v(t) \\ &\quad + B_1 d\breve{B}_\omega(t) + B_2 d\breve{B}_u(t) \\ d\breve{Y}_{\text{out}}(t) &= C_2 \breve{a}_p(t)dt + D_v d\breve{B}_v(t) + D_\omega d\breve{B}_\omega(t) \\ d\breve{z}_h(t) &= C_1 \breve{a}_p(t)dt + D_u d\breve{B}_u(t) \\ \breve{z}_l(t) &= C_l \breve{a}_p(t) + D_l \breve{\beta}_u(t) \end{aligned} \tag{41}$$

with the following parameters:

$$\begin{aligned} A &= -iJ_n \Pi(M_-, M_+) - \frac{1}{2} C^\flat C, \ B_0 = -C_0^\flat \Pi(S_v, 0) \\ B_1 &= -C_1^\flat \Pi(S_\omega, 0), \ B_2 = -C_2^\flat \Pi(S_u, 0), \Upsilon_v = [\Upsilon_-^v \ \Upsilon_+^v] \\ \Upsilon_\omega &= [\Upsilon_-^w \ \ \Upsilon_+^w], \ \Upsilon_u = [\Upsilon_-^u \ \Upsilon_+^u], \ \Psi_u = [\Psi_-^u \ \ \Psi_+^u] \\ \Upsilon_{vw} &= [\Upsilon_v^T \ \ \Upsilon_\omega^T]^T = [\Upsilon_-^{vw} \ \Upsilon_+^{vw}], \ C_1 = \Pi\left(\Upsilon_-^u, \Upsilon_+^u\right) \\ C_2 &= \Pi\left(\Upsilon_-^{vw}, \Upsilon_+^{vw}\right), \ D_v = \Pi(S_v, 0), D_\omega = \Pi(S_u, 0) \\ D_u &= \Pi(S_\omega, 0), \ L = [\Upsilon_v^T \ \Upsilon_\omega^T \ \Upsilon_u^T]^T \breve{a}_p \end{aligned} \tag{42}$$

where $A \in \mathbb{C}^{n\times n}$, $B_0 \in \mathbb{C}^{n\times n_v}$, $B_1 \in \mathbb{C}^{n\times n_\omega}$, $B_2 \in \mathbb{C}^{n\times n_u}$, $C_1 \in \mathbb{C}^{n_h\times n}$, $C_2 \in \mathbb{C}^{n_y\times n}$, $C_l \in \mathbb{C}^{n_l\times n}$, $D_v \in \mathbb{C}^{n_y\times n_v}$, $D_\omega \in \mathbb{C}^{n_y\times n_\omega}$, and $D_u \in \mathbb{C}^{n_h\times n_u}$, ($n, n_\omega, n_u, n_y, n_h$, and n_l are even). ΔA is defined in Section II-C. S_v, S_ω and S_u are scattering matrices. $\breve{a}_p$ represents a vector of plant variables and satisfies $[\breve{a}_p(t), \ \breve{a}_p^\dagger(t)] = \Theta_n = J_n$. $\breve{B}_\omega(t)$ and $\breve{B}_v(t)$ are the same as defined in (31). $\breve{B}_u(t)$ represents the control input of the form $d\breve{B}_u(t) = \breve{\beta}_u(t)dt + d\breve{B}_{\tilde{u}}(t)$, where $\breve{\beta}_u(t)$ is the control signal part and $\breve{B}_u(t)$ is quantum noise part of $\breve{B}_u(t)$. The vectors $\breve{B}_v(t)$, $\breve{B}_{\tilde{\omega}}(t)$, and $\breve{B}_{\tilde{u}}(t)$ correspond to their Itô matrices F_v, F_ω, and F_u of the nonnegative Hermitian form (2), respectively. The *canonical* commutation relation for $\breve{Y}_{\text{out}}(t)$ is given by $[d\breve{Y}_{\text{out}}(t), \ d\breve{Y}_{\text{out}}^\dagger(t)] = \Theta_y = J_{n_y}$, where n_y is the dimension of $\breve{Y}_{\text{out}}$. $Z_h(t)$ and $Z_l(t)$ describe the control outputs, which are referred to as H^∞ and LQG performances, respectively.

To develop a relationship between quantum robust H^∞ control for the quantum system with uncertainties and H^∞ control for a scaled system without uncertainty, we introduce the following lemmas.

Lemma 3: Given two complex matrices X and Y, the following inequality holds:

$$X^\dagger Y + Y^\dagger X \leq \alpha X^\dagger X + Y^\dagger Y/\alpha \tag{43}$$

where $\alpha > 0$ is a scaling parameter.

Lemma 4: For $g > 0$, there exists a positive Hermitian matrix P_h such that

$$(A + \Delta A)^\dagger P_h + P_h(A + \Delta A) + g^{-2} P_h B_1 B_1^\dagger P_h + C_1^\dagger C_1 < 0 \tag{44}$$

for A, B_1, and C_1 defined in the system (41) and $\Pi(E_-, E_+)$ satisfying (17), if there exists a scaling parameter $\alpha > 0$ such that

$$\begin{aligned} &A^\dagger P_h + P_h A + g^{-2} P_h B_1 B_1^\dagger P_h + C_1^\dagger C_1 \\ &+ \frac{K^\dagger K}{\alpha} + \alpha P_h J_n K^\dagger K J_n^\dagger P_h < 0 \end{aligned} \tag{45}$$

where $\Delta A = -iJ_n K^\dagger \Pi(E_-, E_+) K$.

Proof: It follows immediately from the fact that for $\Pi(E_-, E_+)$ satisfying (17) and $\alpha > 0$, we have that

$$\begin{aligned} &\Delta A^\dagger P_h + P_h \Delta A \\ &= [-iJ_n K^\dagger \Pi(E_-, E_+) K]^\dagger P_h + P_h[-iJ_n K^\dagger \Pi(E_-, E_+) K] \\ &= (-iK)^\dagger \Pi(E_-, E_+) K J_n P_h + [\Pi(E_-, E_+) K J_n P_h]^\dagger (-iK). \end{aligned} \tag{46}$$

By Lemma 3, we have

$$\begin{aligned} &(-iK)^\dagger \Pi(E_-, E_+) K J_n P_h + [\Pi(E_-, E_+) K J_n P_h]^\dagger (-iK) \\ &\leq \frac{K^\dagger K}{\alpha} + \alpha P_h J_n K^\dagger K J_n^\dagger P_h \end{aligned} \tag{47}$$

where $X = -iK$ and $Y = \Pi(E_-, E_+) K J_n P_h$. From (47), we have

$$\begin{aligned} &(A + \Delta A)^\dagger P_h + P_h(A + \Delta A) + g^{-2} P_h B_1 B_1^\dagger P_h \\ &+ C_1^\dagger C_1 \leq A^\dagger P_h + P_h A \end{aligned}$$

$$+\,g^{-2}P_hB_1B_1^\dagger P_h+C_1^\dagger C_1+\frac{K^\dagger K}{\alpha}+\alpha P_hJ_nK^\dagger KJ_n^\dagger P_h<0.$$

This completes the proof. ■

By Lemma 4, we now introduce the corresponding scaled system for (41)

$$\begin{aligned}
d\breve{a}_p(t)&=A\breve{a}_p(t)dt+B_0d\breve{B}_v(t)+B_{\bar\omega}d\breve{B}_{\bar\omega}(t)+B_2d\breve{B}_u(t)\\
d\breve{z}_{\bar h}(t)&=\begin{bmatrix}\frac{K}{\sqrt{\alpha}}\\ C_1\end{bmatrix}\breve{a}_p(t)dt+\begin{bmatrix}0_{l\times n_u}\\ D_u\end{bmatrix}d\breve{B}_u(t)\\
d\breve{Y}_{\rm out}(t)&=C_2\breve{a}_p(t)dt+D_vd\breve{B}_v(t)+\begin{bmatrix}0_{n_y\times l} & g^{-1}D_\omega\end{bmatrix}d\breve{B}_{\bar\omega}(t)\\
\breve{z}_l(t)&=C_l\breve{a}_p(t)+D_l\breve{\beta}_u(t)
\end{aligned}\tag{48}$$

where $B_{\bar\omega}=[\sqrt{\alpha}J_nK^\dagger \;\; g^{-1}B_1]$ and $\breve{B}_{\bar\omega}(t)$ is the disturbance input. Consider the following quantum controller G with complex matrices as

$$\begin{aligned}
d\breve{a}_k(t)&=A_k\breve{a}_k(t)dt+B_{k1}d\breve{B}_{k1}(t)+B_{k2}d\breve{B}_{k2}(t)+B_{k3}d\breve{Y}_{\rm out}(t)\\
d\breve{B}_u(t)&=C_k\breve{a}_k(t)dt+d\breve{B}_{k1}(t)
\end{aligned}\tag{49}$$

where $A_k\in\mathbb{C}^{n_k\times n_k}$, $B_{k1}\in\mathbb{C}^{n_k\times n_{v1}}$, $B_{k2}\in\mathbb{C}^{n_k\times n_{v2}}$, $B_{k3}\in\mathbb{C}^{n_k\times n_{v3}}$, and $C_k\in\mathbb{C}^{n_u\times n_k}$ (n_k, n_{v1}, n_{v2}, n_{v3}, and n_u are even). $\breve{a}_k(t)$ represents a vector of quantum controller variables and has the same order as the plant state variable $\breve{a}_p(t)$. Quantum noises $\breve{B}_{k1}(t)$ and $\breve{B}_{k2}(t)$ are independent in the vacuum state. The following relations for the matrices in the quantum controller G are satisfied:

$$A_k+A_k^\flat+B_{k1}^\flat B_{k1}+B_{k2}^\flat B_{k2}+B_{k3}^\flat B_{k3}=0,\; B_{k1}f=-C_k^\flat.\tag{50}$$

Interconnecting systems (41) and (49) yields

$$\begin{aligned}
d\mathbf{a}(t)&=(A_{\rm cl}+\Delta A_{\rm cl})\mathbf{a}(t)dt+B_{\rm cl}d\mathbf{B}_{\rm cl}(t)\\
d\breve{z}_h(t)&=C_{\rm cl}\mathbf{a}(t)dt+D_{\rm cl}d\mathbf{B}_{\rm cl}(t)\\
\breve{z}_l(t)&=C_z\mathbf{a}(t)
\end{aligned}\tag{51}$$

where the all matrices are given by

$$\begin{aligned}
A_{\rm cl}&=\begin{bmatrix}A & B_2C_k\\ B_{k3}C_2 & A_k\end{bmatrix},\; \Delta A_{\rm cl}=\begin{bmatrix}\Delta A & 0\\ 0 & 0\end{bmatrix}\\
B_{\rm cl}&=\begin{bmatrix}B_0 & B_1 & B_2 & 0\\ B_{k3}D_v & B_{k3}D_\omega & B_{k1} & B_{k2}\end{bmatrix},C_{\rm cl}=[C_1\;D_uC_k]\\
D_{\rm cl}&=[0\;\;0\;\;D_u\;\;0],\;C_z=[C_l\;\;D_lC_k]
\end{aligned}\tag{52}$$

$\mathbf{a}(t)=[\breve{a}_p^T(t)\;\;\breve{a}_k^T(t)]^T$, and $\mathbf{B}_{\rm cl}(t)=[\breve{B}_v^T(t)\;\;\breve{B}_\omega^T(t)\;\;\breve{B}_{k1}^T(t)\;\;\breve{B}_{k2}^T(t)]^T$. Similarly, the closed-loop system including the scaled system (48) and the controller (49) can be described by

$$\begin{aligned}
d\mathbf{a}(t)&=A_{\rm cl}\mathbf{a}(t)dt+B_{\rm cl1}d\breve{B}_{\bar\omega}(t)+B_{\rm cl2}\begin{bmatrix}d\breve{B}_v(t)\\ d\breve{B}_{k1}(t)\\ d\breve{B}_{k2}(t)\end{bmatrix}\\
d\breve{z}_{\bar h}(t)&=\bar{C}_{\rm cl}\mathbf{a}(t)dt+\bar{D}_{\rm cl}d\begin{bmatrix}d\breve{B}_v(t)\\ d\breve{B}_{k1}(t)\\ d\breve{B}_{k2}(t)\end{bmatrix}\\
\breve{z}_l(t)&=C_z\mathbf{a}(t)
\end{aligned}\tag{53}$$

where the matrices $B_{\rm cl1}=\begin{bmatrix}\sqrt{\alpha}J_nK^\dagger & g^{-1}B_1\\ 0 & g^{-1}B_{k3}D_\omega\end{bmatrix}$, $B_{\rm cl2}=\begin{bmatrix}B_0 & B_2 & 0\\ B_{k3}D_v & B_{k1} & B_{k2}\end{bmatrix}$, $\bar{C}_{\rm cl}=\left[\begin{bmatrix}\frac{K}{\sqrt{\alpha}}\\ C_1\end{bmatrix}\;\begin{bmatrix}0\\ D_u\end{bmatrix}C_k\right]$, and $\bar{D}_{\rm cl}=\left[0\;\begin{bmatrix}0\\ D_u\end{bmatrix}\;0\right]$. An infinite-horizon quadratic cost function for the system (53) is given by

$$\begin{aligned}
J_\infty&=\lim_{t_f\to+\infty}\frac{1}{t_f}\int_{t_0}^{t_f}\frac{1}{2}\left\langle \mathbf{a}^\dagger(t)\mathbf{a}(t)+\mathbf{a}^T(t)\mathbf{a}^\#(t)\right\rangle dt\\
&={\rm Tr}\left\{C_zP_lC_z^\dagger\right\}
\end{aligned}\tag{54}$$

where P_l satisfies

$$A_{\rm cl}P_l+P_lA_{\rm cl}^\dagger+B_{\rm cl2}B_{\rm cl2}^\dagger=0.\tag{55}$$

Here, $\|G_{\breve{B}_{\bar\omega}}\to\breve{z}_h\|_\infty=\|\bar{D}_{cl}+\bar{C}_{cl}(sI-A_{cl})^{-1}B_{cl1}\|_\infty$ is defined as the H^∞ norm of the closed-loop system (53).

B. Relation Between the Uncertain Closed-Loop System and the Scaled Closed-Loop System Without Uncertainty

To build the equivalence between quantum robust optimal control of the quantum system (41) with uncertainties through a coherent controller (49) and robust optimal control of a scaled system (48) without uncertainty via the same coherent controller (49), we develop Lemma 5 and Theorem 3 as follows.

Lemma 5: For a given constant $g>0$, there exists a Hermitian matrix $P_h>0$ satisfying

$$\begin{aligned}
&(A_{\rm cl}+\Delta A_{\rm cl})^\dagger P_h+P_h(A_{\rm cl}+\Delta A_{\rm cl})\\
&\quad+C_{\rm cl}^\dagger C_{\rm cl}+g^{-2}P_hB_{\rm cl}B_{\rm cl}^\dagger P_h<0
\end{aligned}\tag{56}$$

if there exists a scaling parameter $\alpha>0$ such that

$$\begin{aligned}
&A_{\rm cl}^\dagger P_h+P_hA_{\rm cl}+C_{\rm cl}^\dagger C_{\rm cl}+g^{-2}P_hB_{\rm cl}B_{\rm cl}^\dagger P_h\\
&\quad+\alpha P_hS_1J_nK^\dagger KJ_n^\dagger S_1^\dagger P_h+\frac{S_2^\dagger K^\dagger KS_2}{\alpha}<0
\end{aligned}\tag{57}$$

where $S_1=[I\;\;0]^T$ and $S_2=[I\;\;0]$.

Proof: Given the property (17), the following inequality is obtained:

$$\begin{aligned}
&\Delta A_{\rm cl}^\dagger P_h+P_h\Delta A_{\rm cl}\\
&=(S_1\Delta AS_2)^\dagger P_h+P_hS_1\Delta AS_2\\
&=(-iKS_2)^\dagger\Pi(E_-,E_+)KJ_nS_1^\dagger P_h\\
&\quad+[\Pi(E_-,E_+)KJ_nS_1^\dagger P_h]^\dagger(-iKS_2)\\
&\le\frac{S_2^\dagger K^\dagger KS_2}{\alpha}+\alpha P_hS_1J_nK^\dagger KJ_n^\dagger S_1^\dagger P_h.
\end{aligned}\tag{58}$$

From (58), we have

$$\begin{aligned}
&(A_{\rm cl}+\Delta A_{\rm cl})^\dagger P_h+P_h(A_{\rm cl}+\Delta A_{\rm cl})\\
&\quad+C_{\rm cl}^\dagger C_{\rm cl}+g^{-2}P_hB_{\rm cl}B_{\rm cl}^\dagger P_h\\
&\le A_{\rm cl}^\dagger P_h+P_hA_{\rm cl}+C_{\rm cl}^\dagger C_{\rm cl}+g^{-2}P_hB_{\rm cl}B_{\rm cl}^\dagger P_h\\
&\quad+\alpha P_hS_1J_nK^\dagger KJ_n^\dagger S_1^\dagger P_h+\frac{S_2^\dagger K^\dagger KS_2}{\alpha}<0.
\end{aligned}$$

This completes the proof. ■

C. Quantum Robust Optimal Control Problem

Now we consider a relaxed problem of finding a controller that achieves the cost bound $J_\infty<\gamma$ for some prespecified bound γ [20] and H^∞ performance index $\|G_{\breve{B}_{\bar\omega}}\to\breve{z}_h\|_\infty<g$ with a prespecified

constant $g > 0$ [10]. Therefore, we may formulate our quantum robust optimal problem as follows.

Problem 1: The quantum robust optimal problem is to find a quantum controller G for the system (41) such that the following conditions hold.

1) G can stabilize the system (41).
2) G can achieve $J_\infty = \mathrm{Tr}\{C_z P_l C_z^\dagger\} < \gamma$ and $\|G_{\breve{B}_{\bar{\omega}}} \to \breve{z}_h\|_\infty < g$ simultaneously, where $\gamma, g > 0$.
3) The matrices of G satisfy (50).

Theorem 3: Problem 1 can be addressed via robust optimal control for the closed-loop system (53) consisting of the scaled system (48) and the coherent controller (49).

Proof: By Lemma 5, it can be easily shown that the system (41) is robustly strict bounded real with a disturbance attenuation scalar $g > 0$ via the coherent controller (49) if there exists a scaling parameter $\alpha > 0$ such that the system (48) is strictly bounded real with disturbance attenuation via (49). The transfer function for (51) from $\breve{B}_\omega$ to $\breve{z}_l$ is defined as $\Xi_{w\breve{z}_l}(s)$ while the transfer function for (53) from $\breve{B}_{\bar{\omega}}$ to $\breve{z}_l$ is denoted by $\Xi_{\bar{w}\breve{z}_l}(G,\Delta)$. It can be easily checked from (51) and (53) that when $\Delta A = 0$, $\|\Xi_{w\breve{z}_l}(s)\|_2 = \|\Xi_{\bar{w}\breve{z}_l}(G,0)\|_2$. Combining the above statements, we have the desired result. ■

Theorem 3 shows that the robust optimal control problem for the system (41) can be recast into a mixed LQG and H^∞ control problem.

V. Quantum Robust Optimal Controller Synthesis

In order to effectively solve the matrix inequalities using available matrix analysis software toolbox in MATLAB, we should first transform the system (48) and the controller (49) into real matrix representation (*quadrature representation*) according to the relations mentioned in Section II-B. The scaled plant (48) may be rewritten as

$$\begin{aligned}
dx_p(t) &= \tilde{A}x_p(t)dt + \tilde{B}'dw'(t) + \tilde{B}_2\tilde{\beta}_u(t)\\
d\tilde{z}_{\bar{h}}(t) &= \begin{bmatrix} \frac{\tilde{K}}{\sqrt{\alpha}} \\ \tilde{C}_1 \end{bmatrix} x_p(t)dt + \begin{bmatrix} 0 \\ \tilde{D}_u \end{bmatrix} du(t)\\
d\tilde{y}'(t) &= \tilde{C}'x_p(t)dt + \tilde{D}'dw'(t)\\
\tilde{z}_l(t) &= \tilde{C}_l x_p(t) + \tilde{D}_l\tilde{\beta}_u(t)
\end{aligned} \tag{59}$$

where the matrices $\tilde{B}' = \left[\begin{array}{cc|ccc} \sqrt{\alpha}\tilde{J}_n\tilde{K}^T & g^{-1}\tilde{B}_1 & \tilde{B}_0 & \tilde{B}_2 & 0 \end{array}\right]$, $\tilde{C}' = \begin{bmatrix} \tilde{C}_2 \\ 0 \\ 0 \end{bmatrix}$, $\tilde{D}' = \left[\begin{array}{cc|ccc} 0 & g^{-1}\tilde{D}_\omega & \tilde{D}_v & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{array}\right]$, $w'(t) = \begin{bmatrix} w_{\bar{\omega}}(t) \\ w_v(t) \\ w_{k1}(t) \\ w_{k2}(t) \end{bmatrix}$.

Correspondingly, the controller is redefined as

$$dx_k(t) = \tilde{A}_k x_k(t)dt + \tilde{B}_k d\tilde{y}'(t),\ \tilde{\beta}_u(t) = \tilde{C}_k x_k(t)dt \tag{60}$$

where $\tilde{B}_k = [\tilde{B}_{k3}\ \tilde{B}_{k1}\ \tilde{B}_{k2}]$. The quadrature representation of (50) can be rewritten as

$$\begin{aligned}
&\tilde{A}_k J_{n_k} + \tilde{J}_{n_k}\tilde{A}_k^T + \tilde{B}_{k1}\tilde{J}_{n_{v1}}\tilde{B}_{k1}^T + \tilde{B}_{k2}\tilde{J}_{n_{v2}}\tilde{B}_{k2}^T + \tilde{B}_{k3}\tilde{J}_{n_{v3}}\tilde{B}_{k3}^T = 0\\
&\tilde{B}_{k1} = \tilde{J}_{n_k}\tilde{C}_k^T\tilde{J}_{n_u}.
\end{aligned} \tag{61}$$

Let $x(t) = [x_p(t)^T\ x_k(t)^T]^T$, the closed-loop system obtained below consisting of (59) and (60) is the quadrature form of (53)

$$\begin{aligned}
dx(t) &= \tilde{A}_{\mathrm{cl}}x(t)dt + \tilde{B}_{\mathrm{cl}}dw'(t)\\
d\tilde{z}_{\bar{h}}(t) &= \tilde{\bar{C}}_{\mathrm{cl}}x(t)dt + \tilde{D}_{\mathrm{cl}}dw'(t)\\
\tilde{z}_l(t) &= \tilde{C}_z x(t)
\end{aligned} \tag{62}$$

where the matrices $\tilde{A}_{\mathrm{cl}} = \begin{bmatrix} \tilde{A} & \tilde{B}_2\tilde{C}_k \\ \tilde{B}_{k3}\tilde{C}_2 & \tilde{A}_k \end{bmatrix}$, $\tilde{B}_{\mathrm{cl1}} = \begin{bmatrix} \sqrt{\alpha}\tilde{J}_n\tilde{K}^T & g^{-1}\tilde{B}_1 \\ 0 & g^{-1}\tilde{B}_{k3}\tilde{D}_\omega \end{bmatrix}$, $\tilde{B}_{\mathrm{cl2}} = \begin{bmatrix} \tilde{B}_0 & \tilde{B}_2 & 0 \\ \tilde{B}_{k3}\tilde{D}_v & \tilde{B}_{k1} & \tilde{B}_{k2} \end{bmatrix}$, $\tilde{\bar{C}}_{\mathrm{cl}} = \left[\begin{bmatrix} \frac{\tilde{K}}{\sqrt{\alpha}} \\ \tilde{C}_1 \end{bmatrix} \begin{bmatrix} 0 \\ \tilde{D}_u \end{bmatrix}\tilde{C}_k\right]$, $\tilde{B}_{\mathrm{cl}} = [\tilde{B}_{\mathrm{cl1}}\ \ \tilde{B}_{\mathrm{cl2}}]$, $\tilde{C}_z = \left[\tilde{C}_l\ \tilde{D}_l\tilde{C}_k\right]$, and $\tilde{D}_{\mathrm{cl}} = \left[0\ \begin{bmatrix} 0 \\ \tilde{D}_u \end{bmatrix}\ 0\right]$. All matrices appeared in (59)–(62) are real. Similarly, the cost function (54) may be rewritten as $J_\infty = \mathrm{Tr}\{\tilde{C}_z\tilde{P}_l\tilde{C}_z^T\}$, where $\tilde{P}_l$ satisfies

$$\tilde{A}_{\mathrm{cl}}\tilde{P}_l + \tilde{P}_l\tilde{A}_{\mathrm{cl}}^T + \tilde{B}_{\mathrm{cl2}}\tilde{B}_{\mathrm{cl2}}^T = 0. \tag{63}$$

The condition (63) and $J_\infty = \mathrm{Tr}\{\tilde{C}_z\tilde{P}_l\tilde{C}_z^T\} < \gamma$ can be expressed as the following LMI constraints [25]:

$$\begin{bmatrix} \tilde{A}_{\mathrm{cl}}^T\tilde{P}_l + \tilde{P}_l\tilde{A}_{\mathrm{cl}} & \tilde{P}_l\tilde{B}_{\mathrm{cl}} \\ \tilde{B}_{\mathrm{cl}}^T\tilde{P}_l & -I \end{bmatrix} < 0,\ \begin{bmatrix} \tilde{P}_l & \tilde{C}_z^T \\ \tilde{C}_z & Q \end{bmatrix} > 0,\ \mathrm{Tr}(Q) < \gamma \tag{64}$$

where $\tilde{P}_l$, $Q > 0$ are both real symmetric matrices. By Theorem 1, condition (64) guarantees the closed system (62) stable.

Based on *strict bounded real lemma* as introduced in Section III-C, we have

$$\begin{bmatrix} \tilde{A}_{\mathrm{cl}}^T\tilde{P}_h + \tilde{P}_h\tilde{A}_{\mathrm{cl}} & \tilde{P}_h\tilde{B}_{\mathrm{cl1}} & \tilde{\bar{C}}_{\mathrm{cl}}^T \\ \tilde{B}_{\mathrm{cl1}}^T\tilde{P}_h & -gI & 0 \\ \tilde{\bar{C}}_{\mathrm{cl}} & 0 & -gI \end{bmatrix} < 0 \tag{65}$$

where $g > 0$ and the symmetric matrix $\tilde{P}_h > 0$ is real.

Now, we need to design a controller of the form (60) that minimizes LQG and H^∞ performance indices of the system (62), while the coefficient matrices of (60) satisfy the physical realizability constraints (61) and the inequality constraints (64)–(65) hold.

A. Algorithm of Controller Design

To solve the nonlinear conditions (61), we adopt the method proposed in [25] by introducing additional variables $\mathbf{N}$, $\mathbf{M}$, $\mathbf{X}$, and $\mathbf{Y}$, where $\mathbf{M}\mathbf{N}^T + \mathbf{X}\mathbf{Y} = I$; $\mathbf{N}$ and $\mathbf{M}$ are invertible; and $\mathbf{X}$ and $\mathbf{Y}$ are symmetric. Define $\tilde{P}_l = \tilde{P}_h = \tilde{P}^{-1}$, $\tilde{P} = \begin{bmatrix} \mathbf{Y} & \mathbf{N} \\ \mathbf{N}^T & * \end{bmatrix}$, and $\Pi = \begin{bmatrix} \mathbf{X} & I \\ \mathbf{M}^T & 0 \end{bmatrix}$, then we have $\tilde{P}\Pi = \begin{bmatrix} I & \mathbf{Y} \\ 0 & \mathbf{N}^T \end{bmatrix}$. Now we introduce new variables $\hat{\mathbf{N}} = \mathbf{N}J$, $\tilde{\mathbf{B}}_{k1} = \mathbf{N}\tilde{B}_{k1}$, $\tilde{\mathbf{B}}_{k2} = \mathbf{N}\tilde{B}_{k2}$, and $\tilde{\mathbf{B}}_{k3} = \mathbf{N}\tilde{B}_{k3}$. Performing congruence transformations on the both sides of (64)–(65) with transformation matrices $\Gamma_l = \mathrm{diag}(\Pi^T, I)$ and $\Gamma_h = \mathrm{diag}(\Pi^T, I, I)$ as well as their own transpose, respectively, we have (66), shown at the bottom of the next page, where $\hat{A} = \mathbf{N}A_k\mathbf{M}^T + \mathbf{N}\tilde{B}_k\tilde{C}'\mathbf{X} + \mathbf{Y}\tilde{B}_2\tilde{C}_k\mathbf{M}^T + \mathbf{Y}\hat{A}\mathbf{X}$, $\hat{B} = [\tilde{\mathbf{B}}_{k1}\ \ \tilde{\mathbf{B}}_{k2}\ \ \tilde{\mathbf{B}}_{k3}]$, $\bar{B} = [\sqrt{\alpha}\tilde{J}_n\tilde{K}^T\ \ g^{-1}\tilde{B}_1]$, $\hat{C} = \tilde{C}_k\mathbf{M}^T$, and $\bar{D} = \begin{bmatrix} 0 & g^{-1}\tilde{D}_\omega \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

For simplicity, we choose $\mathbf{M} = I$, and hence, $\mathbf{N} = I - \mathbf{Y}\mathbf{X}$. Let $Z_{x_1} = \mathbf{X}$, $Z_{x_2} = \mathbf{Y}$, $Z_{x_3} = \hat{A}$, $Z_{x_4} = \tilde{\mathbf{B}}_{k1}$, $Z_{x_5} = \tilde{\mathbf{B}}_{k2}$, $Z_{x_6} = \tilde{\mathbf{B}}_{k3}$, $Z_{x_7} = \tilde{C}_k$, $Z_{x_8} = \mathbf{N}$, and $Z_{x_9} = \hat{\mathbf{N}}$. Define a symmetric matrix Z of dimension $24n \times 24n$ as $Z = \mathbf{V}\mathbf{V}^T$, where $\mathbf{V} = [I\ \ Z_{x_1}^T\ \ \cdots\ \ Z_{x_9}^T\ \ Z_{v_1}^T\ \ \cdots\ \ Z_{v_{14}}^T]^T$, $Z_{v_1} = \mathbf{Y}\mathbf{X}$, $Z_{v_2} = \mathbf{Y}\tilde{B}_2$, $Z_{v_3} = \hat{\mathbf{N}}\tilde{C}_k^T$, $Z_{v_4} = \hat{\mathbf{N}}\mathbf{X}$, $Z_{v_5} = \hat{A}\hat{\mathbf{N}}^T$, $Z_{v_6} = \tilde{\mathbf{B}}_{k1}\tilde{J}_{n_{v1}}$, $Z_{v_7} = \tilde{\mathbf{B}}_{k2}\tilde{J}_{n_{v2}}$, $Z_{v_8} = \tilde{\mathbf{B}}_{k3}\tilde{J}_{n_{v3}}$, $Z_{v_9} = Z_{v_6}\tilde{\mathbf{B}}_{k1}^T$, $Z_{v_{10}} = Z_{v_7}\tilde{\mathbf{B}}_{k2}^T$, $Z_{v_{11}} = Z_{v_8}\tilde{\mathbf{B}}_{k3}^T$, $Z_{v_{12}} = \tilde{\mathbf{B}}_{k3}\tilde{C}_2 + \mathbf{Y}\tilde{A}$, $Z_{v_{13}} = \mathbf{Y}\tilde{B}_2\tilde{C}_k\hat{\mathbf{N}}^T$, and $Z_{v_{14}} = (\tilde{\mathbf{B}}_{k3}\tilde{C}_2 + \mathbf{Y}\tilde{A})\mathbf{X}\hat{\mathbf{N}}^T$. Then, the following set of additional constraints hold:

$$\begin{array}{ll}
Z \geq 0, & Z_{0,0} - I = 0 \\
Z_{x_1} - Z_{x_1}^T = 0, & Z_{x_2} - Z_{x_2}^T = 0 \\
Z_{v_1} - Z_{x_2} Z_{x_1}^T = 0, & Z_{v_2} - Z_{x_2} \tilde{B}_2 = 0 \\
Z_{x_8} + Z_{v_1} - I = 0, & Z_{x_9} - Z_{x_8} J_{n_k} = 0 \\
Z_{v_3} - Z_{x_{10}} Z_{x_8}^T = 0, & Z_{v_4} - Z_{x_{10}} Z_{x_1}^T = 0 \\
Z_{v_5} - Z_{x_3} Z_{x_{10}}^T = 0, & Z_{v_6} - Z_{x_4} \tilde{J}_{n_{v1}} = 0 \\
Z_{v_7} - Z_{x_5} \tilde{J}_{n_{v2}} = 0, & Z_{v_8} - Z_{x_6} \tilde{J}_{n_{v3}} = 0 \\
Z_{v_9} - Z_{v_6} Z_{x_4}^T = 0, & Z_{v_{10}} - Z_{v_7} Z_{x_5}^T = 0 \\
Z_{v_{11}} - Z_{v_8} Z_{x_6}^T = 0, & Z_{v_{12}} - Z_{x_6} \tilde{C}_2 - Z_{x_2} \tilde{A} = 0 \\
Z_{v_{13}} - Z_{v_2} Z_{v_3}^T = 0, & Z_{v_{14}} - Z_{v_{12}} Z_{v_4}^T = 0.
\end{array} \tag{67}$$

The physical realisability conditions (61) then become

$$\begin{aligned}
&Z_{v_5}^T - Z_{v_5} + Z_{v_{14}} - Z_{v_{14}}^T + Z_{v_{13}} - Z_{v_{13}}^T + Z_{v_9} + Z_{v_{10}} + Z_{v_{11}} = 0 \\
&Z_{x_4} - \tilde{J}_{n_k} Z_{x_8}^T \tilde{J}_{n_u} = 0.
\end{aligned} \tag{68}$$

We also require that Z should satisfy a rank constraint:

$$\operatorname{rank}(Z) \leq n. \tag{69}$$

Once the quantum robust optimal control problem with constraints (66)–(69) is solvable through a semidefinite programming [20], [26], [27], then the matrices of the designed quantum controller are given by

$$\begin{aligned}
&\tilde{B}_{k1} = Z_{x_8}^{-1} Z_{x_4},\ \tilde{B}_{k2} = Z_{x_8}^{-1} Z_{x_5},\ \tilde{B}_{k3} = Z_{x_8}^{-1} Z_{x_6},\ \tilde{C}_k = Z_{x_7} \\
&\tilde{A}_k = Z_{x_8}^{-1} \left(Z_{x_3} - Z_{x_2} \tilde{A} Z_{x_1} - Z_{v_2} Z_{x_7} \right) - \tilde{B}_{k3} \tilde{C}_2 Z_{x_1}.
\end{aligned}$$

B. Illustrative Example

To demonstrate the feasibility of numerically solving this problem, we now show an example to check our numerical procedure proposed in Section V-A. We work in MATLAB R2013b using the Yalmip prototyping environment. The semidefinite program solver used for LMIRank is SeDuMi Version 1.3.4 Release.

Consider a quantum plant with uncertainties as follows:

$$\begin{aligned}
d\breve{a}_p(t) &= \left(\begin{bmatrix} 0 & -0.1i \\ -0.1i & 0 \end{bmatrix} + \Delta A \right) \breve{a}_p(t) dt \\
&\quad + \begin{bmatrix} -0.2 & 0.2i \\ -0.2i & -0.2 \end{bmatrix} d\breve{B}_v(t) \\
&\quad + \begin{bmatrix} -0.2 & 0.2i \\ -0.2i & -0.2 \end{bmatrix} d\breve{B}_\omega(t) + \begin{bmatrix} -0.2 & 0.2i \\ -0.2i & -0.2 \end{bmatrix} d\breve{B}_u(t) \\
dY_{\text{out}}(t) &= \begin{bmatrix} 0.2 & 0.2i \\ -0.2i & 0.2 \end{bmatrix} \breve{a}_p(t) dt + d\breve{B}_w(t) \\
d\breve{z}_h(t) &= \begin{bmatrix} 0.2 & 0.2i \\ -0.2i & 0.2 \end{bmatrix} \breve{a}_p(t) dt + d\breve{B}_u(t) \\
\breve{z}_l(t) &= \breve{a}_p(t) + \breve{\beta}_u(t).
\end{aligned}$$

TABLE I

COMPARISONS OF RESULTS

Cases	LQG index	H^∞ index
Optimization results only for LQG index by the method in [20]	9.3593	–
Optimization results only for H^∞ by the method in [13]	–	1.2653
Optimization results by our method	9.3216	1.2701

Applying the method presented in Section V-A with a cost bound parameter $\gamma = 10$ and a disturbance attenuation constant $g = 1.4513$, we get the real matrices of the quantum controller:

$$\begin{aligned}
\tilde{A}_k &= \begin{bmatrix} -0.1804 & -1.0338 \\ -0.0848 & -0.8930 \end{bmatrix}, \tilde{B}_{k1} = \begin{bmatrix} 0.1207 & -0.1993 \\ 0.0957 & -0.1580 \end{bmatrix} \\
\tilde{B}_{k2} &= \begin{bmatrix} -0.4811 & -1.0869 \\ -0.3856 & -0.8695 \end{bmatrix}, \tilde{B}_{k3} = \begin{bmatrix} 1.5237 & -0.9841 \\ 1.4128 & -0.2076 \end{bmatrix} \\
\tilde{C}_k &= \begin{bmatrix} 0.1580 & -0.1993 \\ 0.0957 & -0.1207 \end{bmatrix}.
\end{aligned}$$

It can be easily checked that the above matrices of the quantum controller satisfy the physical realization conditions (10)–(11). The closed-loop LQG cost achieved by this quantum controller is $J_\infty = 9.3216$. The running time is 634.06 s. All computations of the designed controller were performed on MATLAB running on an ThinkPad workstation laptop configured with 7th Generation Intel Core i7-7820HQ Processor, 16 GB of memory, and 500 GB hard disk capacity.

Now we proceed to make comparisons with other methods proposed in [13] and [20]. The comparison results in Table I show that our proposed method can make LQG and H^∞ performance indexes close to the minimum simultaneously.

$$\begin{bmatrix} \tilde{A}\mathbf{X} + \mathbf{X}\tilde{A}^T + \tilde{B}_2\hat{C} + \left(\tilde{B}_2\hat{C}\right)^T & \hat{A}^T + \tilde{A} & \tilde{B}' \\ \hat{A} + \tilde{A}^T & \tilde{A}^T\mathbf{Y} + \mathbf{Y}\tilde{A} + \tilde{\mathbf{B}}_{k3}\tilde{C}_2 + (\tilde{\mathbf{B}}_{k3}\tilde{C}_2)^T & \mathbf{Y}\tilde{B}' + \hat{B}\tilde{D}' \\ \tilde{B}'^T & \left(\mathbf{Y}\tilde{B}' + \hat{B}\tilde{D}'\right)^T & -I \end{bmatrix} < 0$$

$$\begin{bmatrix} \mathbf{X} & I & \left(\tilde{C}_l\mathbf{X} + \tilde{D}_l\hat{C}\right)^T \\ I & \mathbf{Y} & \tilde{C}_l^T \\ \left(\tilde{C}_l\mathbf{X} + \tilde{D}_l\hat{C}\right) & \tilde{C}_l & Q \end{bmatrix} > 0$$

$$\operatorname{Tr}(Q) < \gamma$$

$$\begin{bmatrix} \tilde{A}\mathbf{X} + \mathbf{X}\tilde{A}^T + \tilde{B}_2\hat{C} + \left(\tilde{B}_2\hat{C}\right)^T & \hat{A}^T + \tilde{A} & * & * \\ \hat{A} + \tilde{A}^T & \tilde{A}^T\mathbf{Y} + \mathbf{Y}\tilde{A} + \tilde{\mathbf{B}}_{k3}\tilde{C}_2 + (\tilde{\mathbf{B}}_{k3}\tilde{C}_2)^T & * & * \\ \bar{B}^T & \left(\mathbf{Y}\bar{B} + \hat{B}\bar{D}\right)^T & -gI & * \\ \begin{bmatrix} \frac{\tilde{K}}{\sqrt{\alpha}} \\ \tilde{C}_1 \end{bmatrix} \mathbf{X} + \begin{bmatrix} 0 \\ \tilde{D}_u \end{bmatrix} \hat{C} & \begin{bmatrix} \frac{\tilde{K}}{\sqrt{\alpha}} \\ \tilde{C}_1 \end{bmatrix} & 0 & -gI \end{bmatrix} < 0. \tag{66}$$

VI. Conclusion

In this note, we have proposed a complex uncertain linear stochastic quantum model that arises in quantum optics. We have investigated quantum robust optimal control problem for a feedback control closed-loop system consisting of a scaled plant without uncertainty and a quantum controller. Quantum robust optimal control theories have been developed. A mixed LQG and H^∞ approach has been presented to solve quantum robust optimal control problem in this note. An example has been provided to demonstrate procedures of the controller design.

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