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Coherent robust H^{∞} control of linear quantum systems with uncertainties in the Hamiltonian and coupling operators[☆]

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ABSTRACT

This paper considers a class of uncertain linear quantum systems subject to uncertain perturbations both in the system Hamiltonian and in the coupling operators. A sufficient condition is provided such that these uncertain quantum systems can be guaranteed to be robustly strict bounded real with a given disturbance attenuation parameter. A scaled H^{∞} problem is shown to have a connection with the stability of the given uncertain quantum system and can be used as an auxiliary system to design a robust H^{∞} quantum controller. A pair of Riccati equations is used to give explicit formulas for the parameters of a desired controller. Illustrative examples show that for the given uncertain quantum system, the method presented in this paper has improved performance over the existing quantum H^{∞} control results without considering uncertainties.

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1. Introduction

Recent development in quantum technology has shown that quantum information technology and quantum control engineering have the potential to be important future technologies (Dowling & Milburn, 2003; Wiseman & Milburn, 2010). Quantum computation and quantum communication (Nielsen & Chuang, 2000) have many advantages over their classical (non-quantum) counterparts. Quantum control theory has received considerable attention in the development of practical quantum technology (Altafini & Ticozzi, 2012; Doherty & Jacobs, 1999; Dong & Petersen, 2010; Lloyd, 2000; Mirrahimi & van Handel, 2007; Nurdin, James, & Petersen, 2009; Qi, Pan, & Guo, 2013; Yanagisawa & Kimura, 2003; Zhang & James, 2011; Zhang, Wu, Li, & Tarn, 2010). Similarly to the classical control area, feedback control is also widely used in quantum control due to its ability to improve control performance such as in quantum error correction, preparing squeezed states and generating quantum entanglement (Wiseman & Milburn, 2010).

Practical applications of quantum information technology are unavoidably subject to all kinds of disturbances and uncertainties (Dong, Chen, Qi, Petersen, & Nori, 2015; James, Nurdin, & Petersen, 2008; Qi, 2013). Robustness is counted as one of the most important issues in quantum control systems (Petersen, 2013; Petersen, Ugrinovskii, & James, 2012; Shaiju, Petersen, & James, 2007; Xiang, Petersen, & Dong, 2014a,b). To deal with uncertainties in quantum systems, the small gain theorem in classical control theory has been extended to the quantum domain and has been used to analyze the stability of quantum feedback networks (D'Helon & James, 2006). An optimal control method has been presented to solve a risk-sensitive control problem for quantum systems (James, 2004). To attenuate the influence of disturbance signals, the authors in James et al. (2008) and Maalouf and Petersen (2011) focused on finding a controller to bound the influence of disturbance input signals on the performance of output signals using H^{∞} synthesis. In particular, the quadrature form of the quantum system variables was considered and a systematic method was presented to design quantum H^{∞} controllers for a class of linear stochastic quantum systems (James et al., 2008). The connection between H^{∞} control and robustness for classical systems has been widely studied via the small gain theorem (Zhou, Doyle, & Glover, 1996). Similar arguments also apply for quantum linear systems (D'Helon & James, 2006; James et al., 2008). This enables the desired controller to be robustly stabilizing for the system, and deal with plant modeling errors and unknown disturbances. In Maalouf and Petersen (2011), a coherent H^{∞} control problem was investigated for a class of 'passive' quantum systems defined in terms of annihilation operators only.

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However, little attention has been paid to quantum control design that deals with quantum system models with both input disturbances and parameter uncertainties. These unknown factors inevitably exist in realistic environments and laboratory conditions. Hence, in this paper, we consider a class of linear quantum systems subject to input disturbances, uncertain perturbations in the system Hamiltonian (Xiang, Petersen, & Dong, 2015) as well as in the system coupling operators. The aim of this paper is to design a quantum controller to robustly stabilize this class of uncertain quantum systems as well as to guarantee a prescribed level of disturbance attenuation in the H^∞ sense for the closed-loop system with all of the admissible uncertainties.

One of the main contributions in this paper is to systematically consider uncertainties in the system Hamiltonian as well as in the system coupling operators for quantum controller design. Most of the existing results either focused on uncertainties in the Hamiltonian (e.g., Dong & Petersen, 2012; Petersen et al., 2012) or uncertainties in the coupling operator (e.g., Petersen, 2013). Instead of directly adding uncertainties $\hat{\Delta}$ in the quantum stochastic differential equations (QSDEs) (e.g., Shaiju et al., 2007), we describe uncertainties in a triple (S, L, H) where S is a scattering matrix, L is a coupling operator and H describes the system Hamiltonian (for details, see James & Gough, 2010). When the uncertainties are added into QSDEs, the physical realizability (James et al., 2008) of the overall uncertain quantum system should be considered, which is a very challenging task. However, the triple (S, L, H) naturally describes a physically realizable quantum system. Hence, we do not need to worry about whether the addition of uncertainties is physically realizable or not.

In the classical case, robust H^∞ control for uncertain linear systems has been widely addressed, e.g., Xie, Fu, and de Souza (1992), and the relationship between H^∞ optimization and the robust stabilization of uncertain linear systems has been established (Khargonekar, Petersen, & Zhou, 1990). In this paper, we introduce a robustly strict bounded real condition for uncertain quantum systems and use a similar idea as in Xie et al. (1992) to build a relationship between a coherent robust H^∞ control problem and a scaled H^∞ control problem without parameter uncertainties. We aim to design a coherent controller. Here a coherent controller refers to the fact that the controller itself is a quantum system. In order to obtain a desired controller, we need to solve the scaled H^∞ control problem using a pair of Riccati equations. The solution to the Riccati equations will give us some of the parameters in the controller and the undetermined parameters can be appropriately chosen to make the required control system physically realizable. Although some of the results in this paper were presented in the conference paper (Xiang et al., 2015), that paper considered only the case where quantum systems are subject to uncertainties in the Hamiltonian. However, this paper considers quantum systems with uncertainties in both the Hamiltonian and the coupling operators and provides additional illustrative numerical results. Also, that conference paper did not include any proofs of the stated results.

The remainder of the paper proceeds as follows. In Section 2, we introduce a general class of linear quantum stochastic system models and present the correspondence between two different kinds of parameterization. Also, uncertainties in the Hamiltonian and in the coupling operators are described in Section 2. In Section 3, we introduce theory to establish a connection between a robustly strict bounded real lemma for the uncertain quantum system and a strict bounded real lemma for the scaled H^∞ problem. In Section 4, we provide a systematic method to design a dynamic quantum controller that satisfies the specified H^∞ requirement. In Section 5, we provide examples to demonstrate the theory developed in this paper and compare the H^∞ performance using the method in this paper with the results in James et al. (2008). Conclusions are presented in Section 6.

1.1. Notation

The notation used in this paper is as follows: $i = \sqrt{-1}$, the asterisk $*$ represents the adjoint of a linear operator as well as the conjugate of a complex number. If $A = [a_{jk}]$ is a matrix of linear operators or complex numbers, then the conjugate of matrix A is defined $A^\# = [a^*_{jk}]$, the transpose of matrix A is $A^T = [a_{kj}]$ and the adjoint of matrix A is $A^\dagger = (A^\#)^T = (A^T)^\#$. In addition, we denote that $\Re(A) = (A + A^\#)/2$ and $\Im(A) = (A - A^\#)/2i$. $I_{n\times n}$ refers to an $n \times n$ identity matrix and $0_{m\times n}$ denotes an $m \times n$ zero matrix. In cases in which the dimension is clear from the context, the notation I and 0 will be used. Also, $\Xi_{N_aN_b}$ denotes that $\Xi_{N_aN_b} = \begin{bmatrix} \Sigma_{N_a} & 0_{N_a\times N_b} \\ 0_{N_a\times N_b} & \Sigma_{N_a} \end{bmatrix}$, where $\Sigma_{N_a} = \begin{bmatrix} I_{N_a\times N_a} & 0_{N_a\times(N_b-N_a)} \end{bmatrix}$.

2. Quantum system model

In this section, we introduce linear quantum stochastic systems (James et al., 2008) as the system model, which commonly arises as an idealized model of open quantum harmonic oscillators coupled to boson fields (Walls & Milburn, 1994; Wiseman & Milburn, 2010). Two different approaches (i.e., QSDEs and (S, L, H) approaches) to parameterize the given quantum system are presented in Sections 2.1 and 2.2, respectively. A class of perturbations in the system Hamiltonian is presented in Section 2.3 and a class of perturbations in the system coupling operator is introduced in Section 2.4.

2.1. Linear quantum stochastic systems

We consider a class of linear quantum system models, which is described by the following non-commutative quantum stochastic differential equations (James et al., 2008):

$$\begin{aligned} dx(t) &= Ax(t)dt + Bdw(t); \quad x(0) = x_0 \\ dy(t) &= Cx(t)dt + Ddw(t) \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_w}$, $C \in \mathbb{R}^{n_y\times n}$ and $D = \begin{bmatrix} I_{n_y\times n_y} & 0_{n_y\times(n_w-n_y)} \end{bmatrix} \in \mathbb{R}^{n_y\times n_w}$. Also, n, n_w, n_y are even numbers and $n_w \geq n_y$. $x(t) = [q_1(t) \;\; p_1(t) \;\; q_2(t) \;\; p_2(t) \ldots q_{\frac{n}{2}}(t) \;\; p_{\frac{n}{2}}(t)]^T$ is a vector of self-adjoint possibly non-commutative system variables, where $q_j(t)$ and $p_j(t)$ are position operators and momentum operators, respectively, on an appropriate Hilbert space.

The vector $x(t)$ is required to satisfy the canonical commutation relations

$$[x(t), x(t)^T] = x(t)x(t)^T - (x(t)x(t)^T)^T = 2i\Theta, \tag{2}$$

where Θ is a real skew-symmetric matrix and $\Theta = \Theta_n = \mathrm{diag}_{\frac{n}{2}}(J) = \mathrm{diag}\,\underbrace{(J, J, \ldots, J)}_{\frac{n}{2}}$ for an even number n. Here, J denotes the real skew-symmetric 2×2 matrix $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

The vector quantity w denotes the input signals and is assumed to satisfy the decomposition $dw(t) = \beta_w(t)dt + d\tilde{w}(t)$, where $\beta_w(t)$ is a self-adjoint adapted process and $\tilde{w}(t)$ is the noise part of $w(t)$ (see Parthasarathy, 1992). The process $\beta_w(t)$ represents variables of other systems which may be passed to the system (1) via an interaction. The noise $\tilde{w}(t)$ is a vector of quantum Wiener processes with Ito table (Parthasarathy, 1992) $d\tilde{w}(t)d\tilde{w}^T(t) = F_{\tilde{w}}dt$, where $F_{\tilde{w}} = \mathrm{diag}_{\frac{n_w}{2}}(I + iJ)$ is a non-negative definite Hermitian Ito matrix. Hence, the commutation relations for the noise components are shown in the following

$$[d\tilde{w}(t), d\tilde{w}^T(t)] = d\tilde{w}(t)d\tilde{w}^T(t) - (d\tilde{w}(t)d\tilde{w}^T(t))^T = 2T_{\tilde{w}}dt$$

where we define $S_{\tilde{w}} = 1/2(F_{\tilde{w}} + F_{\tilde{w}}^T)$, $T_{\tilde{w}} = 1/2(F_{\tilde{w}} - F_{\tilde{w}}^T)$ so that $F_{\tilde{w}} = S_{\tilde{w}} + T_{\tilde{w}}$. The noise processes can be represented as operators on an appropriate Fock space.

Since $\beta_w(t)$ represents an adapted process, we require $\beta_w(0)$ is an operator on a Hilbert space distinct from that of x_0 and the noise processes. We assume that $\beta_w(t)$ commutes with $x(t)$ for all $t \geq 0$. Also, we denote $\beta_w(t)$ commutes with $d\tilde{w}(t)$ for all $t \geq 0$. Moreover, a property of the Ito increments is that $d\tilde{w}(t)$ commutes with $x(t)$.

Now we present physical realizability conditions for a quantum system, that is, the system (1) represents the dynamics of a physically meaningful quantum system if and only if (see James et al., 2008; Miao, Hush, & James, 2015):

$$A\Theta_n + \Theta_n A^T + B\Theta_{n_w} B^T = 0; \tag{3}$$

$$BD^T = \Theta_n C^T \Theta_{n_y} \tag{4}$$

and $D = \begin{bmatrix} I_{n_y \times n_y} & 0_{n_y \times (n_w - n_y)} \end{bmatrix}$. Here, (3) enables the preservation of the commutation relation; i.e., for $[x_j(0), x_k(0)] = 2i\Theta_{jk}$, we have $[x_j(t), x_k(t)] = 2i\Theta_{jk}$ for all $t \geq 0$.

2.2. Correspondence between the system matrices (A, B, C, D) and the parameters (S, L, H)

Linear quantum stochastic systems can also be described and parameterized by a triple (S, L, H) (Gough & James, 2009). The quadratic Hamiltonian operator H represents self-energy of the system and is of the form

$$H = 1/2\, x^T R x \tag{5}$$

where R is a real symmetric Hamiltonian matrix with dimension $n \times n$. The coupling operator L describes the interface between the system and the fields. It couples to the external bosonic field and is of the form $L = \Lambda x$ where Λ is an $N_w \times n$ complex-valued coupling matrix with $N_w = n_w/2$. In most cases, no interaction between different fields is concerned (i.e., no scattering is involved). Hence, without loss of generality, we can assume the scattering matrix is of the form $S = I$ (James et al., 2008).

Since a linear quantum stochastic system can be described by linear stochastic differential equations with system matrices (A, B, C, D), as well as by the (S, L, H) framework, an equivalence between these two kinds of parameterization approaches can be established. The matrices A, B, C, D are given in terms of H and L in the following form (James et al., 2008):

$$\begin{aligned} &A = 2\Theta(R + \Im(\Lambda^\dagger \Lambda)); \qquad B = 2i\Theta[-\Lambda^\dagger \quad \Lambda^T]\Gamma_w; \\ &C = P_{N_y}^T \Xi_{N_y N_w} \begin{bmatrix} \Lambda + \Lambda^\# \\ -i\Lambda + i\Lambda^\# \end{bmatrix}; \\ &D = P_{N_y}^T \Xi_{N_y N_w} P_{N_w} = \begin{bmatrix} I_{n_y \times n_y} & 0_{n_y \times (n_w - n_y)} \end{bmatrix} \end{aligned} \tag{6}$$

where

$$\Gamma_w = P_{N_w} \mathrm{diag}_{N_w}(M); \; N_y = n_y/2; \quad M = \frac{1}{2}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}. \tag{7}$$

Here, the symbol P_m denotes a $2m \times 2m$ permutation matrix. A $2m \times 2m$ permutation matrix is a full-rank real matrix whose columns consist of standard basis vectors for $\mathbb{R}^{2m}$ such that $P_m^T [\, a_1\; a_2 \cdots a_{2m}\,]^T = [\, a_1\; a_{m+1}\; a_2\; a_{m+2} \cdots a_m\; a_{2m}\,]^T$.

2.3. Uncertainty in the Hamiltonian matrix

When a quantum system is subject to uncertain perturbations in the system Hamiltonian, we assume that the quadratic perturbation Hamiltonian is in the following form

$$H_{\text{perturbation}} = 1/2\, x^T E^T \hat{\Delta} E x, \tag{8}$$

where $E \in \mathbb{R}^{m \times n}$, and $\hat{\Delta} \in \mathbb{R}^{m \times m}$ is an uncertain norm bounded real matrix satisfying

$$\hat{\Delta}^T = \hat{\Delta}, \qquad \hat{\Delta}^2 \leq I. \tag{9}$$

The uncertain linear quantum system is described as follows

$$\begin{aligned} dx(t) &= (A + \Delta A)x(t)dt + Bdw(t); \\ dy(t) &= Cx(t)dt + Ddw(t). \end{aligned}$$

It follows from (6) that $\Delta A = 2\Theta E^T \hat{\Delta} E$.

2.4. Uncertainty in the coupling operators

When a quantum system is subject to perturbations in the system coupling operators, we assume that the coupling operator vector L has uncertainties of the form $L = (\Lambda + \Omega)x$ where nominal coupling matrix Λ and uncertain coupling matrix Ω are complex-valued coupling matrices with dimension $N_w \times n$ and $\Omega = \Re(\Omega) + i\Im(\Omega)$. Also, we assume that Ω satisfies the following bound:

$$\Re(\Omega)^T \Re(\Omega) + \Im(\Omega)^T \Im(\Omega) \leq r I_{n \times n},$$

where $r > 0$ is a known parameter. The dynamics of this class of uncertain quantum systems are described as follows:

$$\begin{aligned} dx(t) &= (A + \Delta A)x(t)dt + (B + \Delta B)dw(t); \\ dy(t) &= (C + \Delta C)x(t)dt + Ddw(t) \end{aligned}$$

where matrices A, B, C, D are given in (6) and $\Delta A = 2\Theta\Im[\Lambda^\dagger \Omega + \Omega^\dagger \Lambda + \Omega^\dagger \Omega]$; $\Delta B = 2i\Theta[-\Omega^\dagger \quad \Omega^T]\Gamma_w$; $\Delta C = 2P_{N_y}^T \Xi_{N_y N_w} \begin{bmatrix} \Re(\Omega) \\ \Im(\Omega) \end{bmatrix}$.

3. Robust H^∞ analysis

To proceed with the robust H^∞ analysis, we first recall the strict bounded real lemma for a general quantum system. Let us consider the following quantum system:

$$\begin{aligned} dx(t) &= Ax(t)dt + [B \quad G][dw(t)^T \quad dv(t)^T]^T; \\ dz(t) &= Cx(t)dt + Ddv(t) \end{aligned} \tag{10}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_w}$, $G \in \mathbb{R}^{n \times n_v}$, $C \in \mathbb{R}^{n_z \times n}$ and $D \in \mathbb{R}^{n_z \times n_v}$. Also, $x(t)$ is a vector of plant variables, $w(t)$ is a disturbance signal, $v(t)$ is additional quantum noise, and $z(t)$ is the controlled output. From Corollary 4.5 in James et al. (2008), we know that this quantum stochastic system (10) is strictly bounded real with disturbance attenuation $g > 0$ if and only if there exists a positive definite symmetric matrix $X > 0$ such that

$$A^T X + XA + C^T C + g^{-2} XBB^T X < 0. \tag{11}$$

In this paper, we consider the robust H^∞ performance for an uncertain quantum system subject to perturbations not only in the system Hamiltonian, but also in the coupling operator. As denoted in (5) and (8), the total Hamiltonian for the uncertain quantum system is described by $H = 1/2\, x^T(R + E^T \hat{\Delta} E)x$. As for the coupling matrix L, we divide it as the noise part L_v and the disturbance part L_w. Uncertainties that exist in the noise part and the disturbance part of the coupling matrix are denoted by Ω_v and Ω_w, respectively. Consequently, we rewrite the total coupling operator L for the uncertain quantum system in the form $L = \begin{bmatrix} L_w \\ L_v \end{bmatrix} = \begin{bmatrix} \Lambda_w + \Omega_w \\ \Lambda_v + \Omega_v \end{bmatrix} x = (\Lambda + \Omega)x$, where $\Lambda \in N_{wv} \times n$ with $N_{wv} = \frac{n_w+n_v}{2}$. Therefore, when the nominal quantum system (10) is subject to perturbations in the Hamiltonian and coupling operator, the stochastic differential equation for the uncertain quantum system is in the form

$$\begin{aligned} dx(t) &= (A + \Delta A)x(t)dt + [B + \Delta B \quad G + \Delta G] \\ &\quad \times [dw(t)^T \quad dv(t)^T]^T; \\ dz(t) &= (C + \Delta C)x(t)dt + Ddv(t) \end{aligned} \tag{12}$$

with the corresponding parameters

$$\begin{aligned} A &= 2\Theta(R + \Im(\Lambda^\dagger \Lambda)); \qquad \Delta A = \Delta A_1 + \Delta A_2; \\ \Delta A_1 &= 2\Theta E^T \hat{\Delta} E; \qquad \Delta A_2 = 2\Theta \Im[\Lambda^\dagger \Omega + \Omega^\dagger \Lambda + \Omega^\dagger \Omega]; \\ B &= 2i\Theta[-\Lambda_w^\dagger \quad \Lambda_w^T]\Gamma_w; \qquad \Delta B = 2i\Theta[-\Omega_w^\dagger \quad \Omega_w^T]\Gamma_w; \\ G &= 2i\Theta[-\Lambda_v^\dagger \quad \Lambda_v^T]\Gamma_v; \qquad \Delta G = 2i\Theta[-\Omega_v^\dagger \quad \Omega_v^T]\Gamma_v; \\ C &= 2P_{N_z}^T \Xi_{N_z N_v} \begin{bmatrix} \Re(\Lambda_v) \\ \Im(\Lambda_v) \end{bmatrix}; \qquad \Delta C = 2P_{N_z}^T \Xi_{N_z N_v} \begin{bmatrix} \Re(\Omega_v) \\ \Im(\Omega_v) \end{bmatrix}; \\ N_z &= n_z/2; \qquad N_v = n_v/2. \end{aligned} \tag{13}$$

We also assume the following bound condition:

$$\Re(\Omega)^T \Re(\Omega) + \Im(\Omega)^T \Im(\Omega) \le rI_{n\times n}. \tag{14}$$

This implies that the following relationships hold:

$$\begin{aligned} &\Re(\Omega_w)^T \Re(\Omega_w) + \Im(\Omega_w)^T \Im(\Omega_w) \le rI_{n\times n}; \\ &\Re(\Omega_v)^T \Re(\Omega_v) + \Im(\Omega_v)^T \Im(\Omega_v) \le rI_{n\times n}. \end{aligned} \tag{15}$$

In order to guarantee H^∞ performance for uncertain quantum system (12) with the above parameter uncertainties, we incorporate uncertain parameters into (11) and introduce the following definition.

Definition 1. The quantum stochastic system (12) is robustly strict bounded real with disturbance attenuation $g > 0$ if there exists a positive definite symmetric matrix $X > 0$ such that

$$\begin{aligned} &(A + \Delta A)^T X + X(A + \Delta A) + (C + \Delta C)^T (C + \Delta C) \\ &+ g^{-2} X(B + \Delta B)(B + \Delta B)^T X < 0 \end{aligned} \tag{16}$$

for all the parameter uncertainties ΔA, ΔB and ΔC defined as in (13).

Now, we introduce the following theorem that can be used to establish a connection between robust H^∞ analysis for the uncertain quantum system and H^∞ analysis for a corresponding system without parameter uncertainties.

Theorem 2. *Let the constant $g > 0$ be given. Then there exists a matrix $X > 0$ such that*

$$\begin{aligned} &(A + \Delta A)^T X + X(A + \Delta A) + (C + \Delta C)^T (C + \Delta C) \\ &+ g^{-2} X(B + \Delta B)(B + \Delta B)^T X < 0 \end{aligned} \tag{17}$$

for $A, B, C, \Delta A, \Delta B, \Delta C$ as in (13), Hamiltonian uncertainty $\hat{\Delta}$ satisfying (9) and coupling uncertainty satisfying (14), if there exist $\varepsilon_i > 0$ $(i = 1, \ldots, 6)$ such that

$$A^T X + XA + XG_1X + g^{-2} XG_3X + G_4 < 0,$$

where G_i $(i = 1, \ldots, 4)$ are defined as follows:

$$\begin{aligned} G_1 &= 4\varepsilon_1 \Theta E^T E \Theta^T + 4\varepsilon_2 \Theta(\Re(\Lambda)^T \Re(\Lambda) + \Im(\Lambda)^T \Im(\Lambda))\Theta^T \\ &\quad + 4(\varepsilon_3 + \varepsilon_4) rI; \\ G_2 &= \frac{1}{\varepsilon_1} E^T E + \left(\frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_4}\right) rI \\ &\quad + \frac{1}{\varepsilon_3}(\Re(\Lambda)^T \Re(\Lambda) + \Im(\Lambda)^T \Im(\Lambda)); \\ G_3 &= (1 + \varepsilon_5) BB^T + (4/\varepsilon_5 + 4) rI; \\ G_4 &= G_2 + (1 + 4\varepsilon_6) C^T C + (4 + 1/\varepsilon_6) rI. \end{aligned}$$

In order to prove Theorem 2, the following preliminary results are required.

Fact 1. *For any real matrices X and Y with appropriate dimensions, we have the following inequality:*

$$X^T Y + Y^T X \le \varepsilon X^T X + Y^T Y/\varepsilon, \tag{18}$$

where $\varepsilon > 0$ is a free parameter.

Proof. For any real matrices X and Y, it is easy to see that

$$(\sqrt{\varepsilon} X - Y/\sqrt{\varepsilon})^T (\sqrt{\varepsilon} X - Y/\sqrt{\varepsilon}) \ge 0. \tag{19}$$

The conclusion in Fact 1 follows straightforwardly from the inequality (19). □

Lemma 3. *Let Θ be a real skew-symmetric matrix in (2), Γ_w be defined in (7) and Ω_w be bounded as in (15), then*

$$-\Theta[-\Omega_w^\dagger \quad \Omega_w^T]\Gamma_w \Gamma_w^T \begin{bmatrix} -\Omega_w^\# \\ \Omega_w \end{bmatrix} \Theta^T \le rI.$$

Proof. According to the definition of Γ_w, it follows via straightforward algebraic manipulations that $\Gamma_w \Gamma_w^T = \frac{1}{2}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. The fact that $2\Re(\Omega_w) = \Omega_w + \Omega_w^\#$ implies $2\Re(\Omega_w^\dagger \Omega_w) = \Omega_w^\dagger \Omega_w + \Omega_w^T \Omega_w^\#$. Hence, the following equality is obtained: $-\frac{1}{2}\left[-\Omega_w^\dagger \ \Omega_w^T\right]\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} -\Omega_w^\# \\ \Omega_w \end{bmatrix} = \frac{1}{2}(\Omega_w^\dagger \Omega_w + \Omega_w^T \Omega_w^\#) = \Re(\Omega_w^\dagger \Omega_w)$. Based on the definition of $\Re$, we also have $\Re(\Omega_w^\dagger \Omega_w) = \Re(\Omega_w)^T \Re(\Omega_w) + \Im(\Omega_w)^T \Im(\Omega_w)$. Also, Ω_w satisfies the condition (15). In addition, note that there exist $\Theta\Theta^T = I$ for the real skew-symmetric matrix defined in (2). The result then follows straightforwardly from the above conditions. □

Proof of Theorem 2. To prove the theorem, we firstly focus on dealing with uncertain Hamiltonian part. As given in (13), $\Delta A = \Delta A_1 + \Delta A_2$ where $\Delta A_1 = 2\Theta E^T \hat{\Delta} E$. By applying (18), we have the following upper bound which is independent of the uncertain parameters:

$$(2\Theta E^T \hat{\Delta} E)^T X + X(2\Theta E^T \hat{\Delta} E) \le 4\varepsilon_1 X \Theta E^T E \Theta^T X + \frac{1}{\varepsilon_1} E^T E.$$

We now consider the uncertain coupling operator. Let ΔA_2 be given as in (13) and be expanded in the following way:

$$\begin{aligned} \Delta A_2 &= 2\Theta(\Re(\Lambda)^T \Im(\Omega) - \Im(\Lambda)^T \Re(\Omega) + \Re(\Omega)^T \Im(\Lambda) \\ &\quad - \Im(\Omega)^T \Re(\Lambda) - \Im(\Omega)^T \Re(\Omega) + \Re(\Omega)^T \Im(\Omega)) \\ &= 2\Theta([\Re(\Lambda)^T \quad -\Im(\Lambda)^T][\Im(\Omega)^T \quad \Re(\Omega)^T]^T \\ &\quad + [\Re(\Omega)^T \quad -\Im(\Omega)^T][\Im(\Lambda)^T \quad \Re(\Lambda)^T]^T \\ &\quad + [\Im(\Omega)^T \quad \Re(\Omega)^T][-\Re(\Omega)^T \quad \Im(\Omega)^T]^T) \end{aligned}$$

where $[\Im(\Omega)^T \quad \Re(\Omega)^T]^T = \begin{bmatrix}\Im(\Omega)\\ \Re(\Omega)\end{bmatrix}$. Hence, we have the following:

$$\begin{aligned}
&\Delta A_2^T X + X\Delta A_2\\
&\quad = 2(\Theta[\Re(\Lambda)^T \quad -\Im(\Lambda)^T][\Im(\Omega)^T \quad \Re(\Omega)^T]^T)^T X\\
&\qquad + 2X\Theta[\Re(\Lambda)^T \quad -\Im(\Lambda)^T][\Im(\Omega)^T \quad \Re(\Omega)^T]^T\\
&\qquad + 2(\Theta[\Re(\Omega)^T \quad -\Im(\Omega)^T][\Im(\Lambda)^T \quad \Re(\Lambda)^T]^T)^T X\\
&\qquad + 2X\Theta[\Re(\Omega)^T \quad -\Im(\Omega)^T][\Im(\Lambda)^T \quad \Re(\Lambda)^T]^T\\
&\qquad + 2(\Theta[\Im(\Omega)^T \quad \Re(\Omega)^T][-\Re(\Omega)^T \quad \Im(\Omega)^T]^T)^T X\\
&\qquad + 2X\Theta[\Im(\Omega)^T \quad \Re(\Omega)^T][-\Re(\Omega)^T \quad \Im(\Omega)^T]^T. \qquad (20)
\end{aligned}$$

By applying (18) and (14), the first two terms on the right side of (20) are bounded in the following way:

$$\begin{aligned}
&2(\Theta[\Re(\Lambda)^T \quad -\Im(\Lambda)^T][\Im(\Omega)^T \quad \Re(\Omega)^T]^T)^T X\\
&\qquad + 2X\Theta[\Re(\Lambda)^T \quad -\Im(\Lambda)^T][\Im(\Omega)^T \quad \Re(\Omega)^T]^T\\
&\quad \le 4\varepsilon_2 X\Theta[\Re(\Lambda)^T \quad -\Im(\Lambda)^T][\Re(\Lambda)^T \quad -\Im(\Lambda)^T]^T\Theta^T X\\
&\qquad + [\Im(\Omega)^T \quad \Re(\Omega)^T][\Im(\Omega)^T \quad \Re(\Omega)^T]^T/\varepsilon_2\\
&\quad \le 4\varepsilon_2 X\Theta(\Re(\Lambda)^T\Re(\Lambda) + \Im(\Lambda)^T\Im(\Lambda))\Theta^T X + rI/\varepsilon_2.
\end{aligned}$$

The third and fourth terms on the right side of (20) have the following bound:

$$\begin{aligned}
&2(\Theta[\Re(\Omega)^T \quad -\Im(\Omega)^T][\Im(\Lambda)^T \quad \Re(\Lambda)^T]^T)^T X\\
&\qquad + 2X\Theta[\Re(\Omega)^T \quad -\Im(\Omega)^T][\Im(\Lambda)^T \quad \Re(\Lambda)^T]^T\\
&\quad \le 4\varepsilon_3 X\Theta[\Re(\Omega)^T \quad -\Im(\Omega)^T][\Re(\Omega)^T \quad -\Im(\Omega)^T]^T\Theta^T X\\
&\qquad + [\Im(\Lambda)^T \quad \Re(\Lambda)^T][\Im(\Lambda)^T \quad \Re(\Lambda)^T]^T/\varepsilon_3\\
&\quad \le 4\varepsilon_3 rXX + (\Re(\Lambda)^T\Re(\Lambda) + \Im(\Lambda)^T\Im(\Lambda))/\varepsilon_3. \qquad (21)
\end{aligned}$$

The last inequality in (21) follows since $\Theta\Theta^T = I$. By applying the same manner as in (21), the fifth and the sixth terms on the right side of (20) yield the following bound:

$$\begin{aligned}
&2(\Theta[\Im(\Omega)^T \quad \Re(\Omega)^T][-\Re(\Omega)^T \quad \Im(\Omega)^T]^T)^T X\\
&\qquad + 2X\Theta[\Im(\Omega)^T \quad \Re(\Omega)^T][-\Re(\Omega)^T \quad \Im(\Omega)^T]^T\\
&\quad \le 4\varepsilon_4 X\Theta[\Im(\Omega)^T \quad \Re(\Omega)^T][\Im(\Omega)^T \quad \Re(\Omega)^T]^T\Theta^T X\\
&\qquad + [-\Re(\Omega)^T \quad \Im(\Omega)^T][-\Re(\Omega)^T \quad \Im(\Omega)^T]^T/\varepsilon_4\\
&\quad \le 4\varepsilon_4 rXX + rI/\varepsilon_4.
\end{aligned}$$

Therefore, we derive the following bound which only depends on the known parameters:

$$\Delta A^T X + X\Delta A \le XG_1X + G_2, \qquad (22)$$

where

$$\begin{aligned}
G_1 &= 4\varepsilon_1\Theta E^T E\Theta^T + 4\varepsilon_2\Theta(\Re(\Lambda)^T\Re(\Lambda) + \Im(\Lambda)^T\Im(\Lambda))\Theta^T\\
&\quad + 4(\varepsilon_3 + \varepsilon_4)rI;\\
G_2 &= \frac{1}{\varepsilon_1}E^T E + \left(\frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_4}\right) rI\\
&\quad + \frac{1}{\varepsilon_3}(\Re(\Lambda)^T\Re(\Lambda) + \Im(\Lambda)^T\Im(\Lambda)).
\end{aligned}$$

Then, we move to the term $(B+\Delta B)(B+\Delta B)^T$ on the left side of (17) and derive a bound that does not depend on uncertainties. By substituting the corresponding parameters in (13), we obtain the following terms:

$$\begin{aligned}
&B\Delta B^T + \Delta BB^T + \Delta B\Delta B^T\\
&\quad = B\left(2i\Gamma_w^T\begin{bmatrix}-\Omega_w^\#\\ \Omega_w\end{bmatrix}\Theta^T\right) + 2i\Theta[-\Omega_w^\dagger \quad \Omega_w^T]\Gamma_w B^T\\
&\qquad - 4\Theta[-\Omega_w^\dagger \quad \Omega_w^T]\Gamma_w\Gamma_w^T\begin{bmatrix}-\Omega_w^\#\\ \Omega_w\end{bmatrix}\Theta^T. \qquad (23)
\end{aligned}$$

It follows from (18) and Lemma 3 that the first two terms on the right side of (23) satisfy the following bound:

$$\begin{aligned}
&B\left(2i\Gamma_w^T\begin{bmatrix}-\Omega_w^\#\\ \Omega_w\end{bmatrix}\Theta^T\right) + 2i\Theta[-\Omega_w^\dagger \quad \Omega_w^T]\Gamma_w B^T\\
&\quad \le \varepsilon_5 BB^T - \frac{4}{\varepsilon_5}\Theta[-\Omega_w^\dagger \quad \Omega_w^T]\Gamma_w\Gamma_w^T\begin{bmatrix}-\Omega_w^\#\\ \Omega_w\end{bmatrix}\Theta^T\\
&\quad \le \varepsilon_5 BB^T + 4rI/\varepsilon_5. \qquad (24)
\end{aligned}$$

Based on Lemma 3, the last term on the right side of (23) is bounded as follows:

$$\Delta B\Delta B^T = -4\Theta[-\Omega_w^\dagger \quad \Omega_w^T]\Gamma_w\Gamma_w^T\begin{bmatrix}-\Omega_w^\#\\ \Omega_w\end{bmatrix}\Theta^T \le 4rI. \qquad (25)$$

Hence, inequalities (24) and (25) imply that the last term on the left side of (17) is bounded as

$$\begin{aligned}
&X(B+\Delta B)(B+\Delta B)^T X\\
&\quad \le X\left((1+\varepsilon_5)BB^T + \left(\frac{4}{\varepsilon_5} + 4\right) rI\right) X. \qquad (26)
\end{aligned}$$

Next, we need to construct a bound on $(C+\Delta C)^T(C+\Delta C)$, which is the third term on the left side of (17). By substituting the parameters from (13), we obtain the following expression:

$$\begin{aligned}
&C^T\Delta C + \Delta C^T C + \Delta C^T\Delta C\\
&\quad = C^T\left(2P_{N_v}^T\Xi_{N_zN_v}\begin{bmatrix}\Re(\Omega_v)\\ \Im(\Omega_v)\end{bmatrix}\right) + \left(2P_{N_v}^T\Xi_{N_zN_v}\begin{bmatrix}\Re(\Omega_v)\\ \Im(\Omega_v)\end{bmatrix}\right)^T C\\
&\qquad + \left(2P_{N_v}^T\Xi_{N_zN_v}\begin{bmatrix}\Re(\Omega_v)\\ \Im(\Omega_v)\end{bmatrix}\right)^T 2P_{N_v}^T\Xi_{N_zN_v}\begin{bmatrix}\Re(\Omega_v)\\ \Im(\Omega_v)\end{bmatrix}. \qquad (27)
\end{aligned}$$

By using (18), (15) and the fact that the permutation matrix P_{N_v} has the property $P_{N_v}P_{N_v}^T = I$, the following result is achieved:

$$\begin{aligned}
&C^T\Delta C + \Delta C^T C\\
&\quad \le 4\varepsilon_6 C^T P_{N_v}^T P_{N_v} C\\
&\qquad + \frac{1}{\varepsilon_6}\left[\Re(\Omega_v)^T \quad \Im(\Omega_v)^T\right]\Xi_{N_zN_v}^T\Xi_{N_zN_v}\begin{bmatrix}\Re(\Omega_v)\\ \Im(\Omega_v)\end{bmatrix}\\
&\quad \le 4\varepsilon_6 C^T C + rI/\varepsilon_6. \qquad (28)
\end{aligned}$$

We know that $\Xi_{N_zN_v}\begin{bmatrix}\Re(\Omega_v)\\ \Im(\Omega_v)\end{bmatrix} = \begin{bmatrix}\Re(\Omega_{N_z})\\ \Im(\Omega_{N_z})\end{bmatrix}$, where Ω_{N_z} is a matrix with the first N_z rows of Ω_v. It is clear that $\Re(\Omega_{N_z})^T\Re(\Omega_{N_z}) + \Im(\Omega_{N_z})^T\Im(\Omega_{N_z}) \le rI_{n\times n}$. Hence, the last inequality of (28) is satisfied. Similarly, the last term of (27) has the following bound:

$$\begin{aligned}
\Delta C^T\Delta C &= 4\left[\Re(\Omega_v)^T \quad \Im(\Omega_v)^T\right]\\
&\qquad \times \Xi_{N_zN_v}^T P_{N_v}P_{N_v}^T\Xi_{N_zN_v}\begin{bmatrix}\Re(\Omega_v)\\ \Im(\Omega_v)\end{bmatrix}\\
&\quad \le 4rI. \qquad (29)
\end{aligned}$$

Using (28) and (29), we obtain the following bound:

$$(C+\Delta C)^T(C+\Delta C) \le (1+4\varepsilon_6)C^T C + \left(4 + \frac{1}{\varepsilon_6}\right) rI. \qquad (30)$$

To conclude, inequalities (22), (26) and (30) lead to the following result:

$$
\begin{aligned}
&(A+\Delta A)^T X + X(A+\Delta A) + (C+\Delta C)^T(C+\Delta C)\\
&\quad + g^{-2}X(B+\Delta B)(B+\Delta B)^T X\\
&\le A^T X + XA + XG_1X + g^{-2}XG_3X + G_4 < 0
\end{aligned}
\tag{31}
$$

where $G_3 = (1+\varepsilon_5)BB^T + \left(\frac{4}{\varepsilon_5}+4\right)rI$; $G_4 = G_2 + (1+4\varepsilon_6)C^TC + (4+\frac{1}{\varepsilon_6})rI$. Therefore, the conclusion of Theorem 2 follows from (31). □

Now we are in a position to introduce a scaled system for establishing a connection between the robust H^∞ analysis of the uncertain quantum system and the H^∞ analysis of the scaled system. The scaled system without parameter uncertainties has the following form:

$$
\begin{aligned}
dx(t) &= Ax(t)dt + [[J_1 \quad g^{-1}J_2] \quad G]\\
&\quad \times [d\overline{w}(t)^T \quad dv(t)^T]^T;\\
d\overline{z}(t) &= J_3x(t)dt + [0 \quad D^T]^T dv(t)
\end{aligned}
\tag{32}
$$

where $\overline{w}(t)$ is the disturbance input and $\overline{z}(t)$ is the controlled output. Also,

$$
\begin{aligned}
J_1 &= [2\sqrt{\varepsilon_1}\Theta E^T \quad 2\sqrt{\varepsilon_2}\Theta \Re(\Lambda)^T \quad 2\sqrt{\varepsilon_2}\Theta\Im(\Lambda)^T\\
&\quad 2\sqrt{(\varepsilon_3+\varepsilon_4)r}I];\\
J_2 &= \left[\sqrt{1+\varepsilon_5}B \quad \sqrt{\left(\frac{4}{\varepsilon_5}+4\right)rI}\right];\\
J_3 &= \left[\sqrt{\frac{1}{\varepsilon_1}}E^T \quad \sqrt{\left(\frac{1}{\varepsilon_2}+\frac{1}{\varepsilon_4}\right)rI} \quad \sqrt{\frac{1}{\varepsilon_3}}\Re(\Lambda)^T\right.\\
&\quad \left.\sqrt{\frac{1}{\varepsilon_3}}\Im(\Lambda)^T\sqrt{1+4\varepsilon_6}C^T \quad \sqrt{\left(4+\frac{1}{\varepsilon_6}\right)rI}\right]^T
\end{aligned}
$$

and $\varepsilon_i > 0$ $(i = 1,\dots,6)$ are scaling parameters. Note that $J_1J_1^T = G_1$, $J_2J_2^T = G_3$ and $J_3^TJ_3 = G_4$. According to Definition 1, Theorem 2 and (11), we have the following corollary.

Corollary 4. *The system (12) is robustly strict bounded real with disturbance attenuation $g > 0$ if there exist scaling parameters $\varepsilon_i > 0$ $(i = 1,\dots,6)$ such that the system (32) is strictly bounded real with unitary disturbance attenuation.*

4. Robust H^∞ controller synthesis

In this section, we design a coherent robust H^∞ controller for the class of uncertain quantum systems considered in Section 3. In Section 4.1, the closed-loop system consisting of the plant and the controller is described and the H^∞ control objective is introduced. In Section 4.2, we apply the strict bounded real inequality to build a connection between the uncertain quantum system and the scaled H^∞ problem. Then we provide a solution to design a coherent H^∞ controller in Section 4.3.

4.1. Closed-loop system and H^∞ control objective

4.1.1. Closed-loop system

We consider a quantum plant subject to perturbations in both the Hamiltonian and the coupling operator, and describe it by a noncommutative stochastic model in the following form

$$
\begin{aligned}
dx(t) &= (A+\Delta A)x(t)dt\\
&\quad + [B_0+\Delta B_0 \quad B_1+\Delta B_1 \quad B_2+\Delta B_2]\\
&\quad \times [dv(t)^T \quad dw(t)^T \quad du(t)^T]^T; \quad x(0)=x_0\\
dz(t) &= (C_1+\Delta C_1)x(t)dt + D_{12}du(t);\\
dy(t) &= (C_2+\Delta C_2)x(t)dt + [D_{20} \quad D_{21} \quad 0]\\
&\quad \times [dv(t)^T \quad dw(t)^T \quad du(t)^T]^T
\end{aligned}
\tag{33}
$$

where $A \in \mathbb{R}^{n\times n}$, $B_0 \in \mathbb{R}^{n\times n_v}$, $B_1 \in \mathbb{R}^{n\times n_w}$, $B_2 \in \mathbb{R}^{n\times n_u}$, $C_1 \in \mathbb{R}^{n_z\times n}$, $C_2 \in \mathbb{R}^{n_y\times n}$, $D_{12} \in \mathbb{R}^{n_z\times n_u}$, $D_{20} \in \mathbb{R}^{n_y\times n_v}$, $D_{21} \in \mathbb{R}^{n_y\times n_w}$. Here, $w(t)$ represents a disturbance signal and $v(t)$ represents any additional quantum noise. The signal $u(t)$ is a control input of the form $du(t) = \beta_u(t)dt + d\tilde{u}(t)$ where $\tilde{u}(t)$ is the noise part of $u(t)$ and $\beta_u(t)$ is an adapted process. The quantity $z(t)$ describes the control output and the quantity $y(t)$ describes the measured output.

The uncertain perturbation in the system Hamiltonian is in the form of (8) satisfying the condition (9). As for the coupling matrix L, we divide it as the control input part L_u, the noise part L_v and the disturbance part L_w. Uncertainties existing in the control input part, the noise part and the disturbance part of the coupling operator are described by Ω_u, Ω_v and Ω_w, respectively.

$$
L = \begin{bmatrix} L_u \\ L_v \\ L_w \end{bmatrix} = \begin{bmatrix} \Lambda_u + \Omega_u \\ \Lambda_v + \Omega_v \\ \Lambda_w + \Omega_w \end{bmatrix} x = (\Lambda + \Omega)x,
$$

where $\Lambda \in \mathbb{C}^{N_{uvw}\times n}$ with $N_{uvw} = \frac{n_u+n_v+n_w}{2}$. Also, the uncertainties satisfy the following bound condition:

$$
\Re(\Omega)^T\Re(\Omega) + \Im(\Omega)^T\Im(\Omega) \le rI_{n\times n}. \tag{34}
$$

The condition (34) also implies that the following bound conditions hold:

$$
\begin{aligned}
&\Re(\Omega_u)^T\Re(\Omega_u) + \Im(\Omega_u)^T\Im(\Omega_u) \le rI_{n\times n};\\
&\Re(\Omega_v)^T\Re(\Omega_v) + \Im(\Omega_v)^T\Im(\Omega_v) \le rI_{n\times n};\\
&\Re(\Omega_w)^T\Re(\Omega_w) + \Im(\Omega_w)^T\Im(\Omega_w) \le rI_{n\times n}.
\end{aligned}
\tag{35}
$$

Coherent controllers are assumed to be noncommutative stochastic systems of the form

$$
\begin{aligned}
d\xi(t) &= A_K\xi(t)dt + [B_{K1} \quad B_K]\\
&\quad \times [dv_K(t)^T \quad dy(t)^T]^T; \quad \xi(0)=\xi_0\\
du(t) &= C_K\xi(t)dt + [B_{K0} \quad 0][dv_K(t)^T \quad dy(t)^T]^T
\end{aligned}
\tag{36}
$$

where $A_K \in \mathbb{R}^{n_K\times n_K}$, $B_{K0} \in \mathbb{R}^{n_u\times n_{v_K}}$, $B_{K1} \in \mathbb{R}^{n_K\times n_{v_K}}$, $B_K \in \mathbb{R}^{n_K\times n_y}$, $C_K \in \mathbb{R}^{n_u\times n_K}$ and $\xi(t) = [\xi_1(t) \quad \dots \quad \xi_{n_K}(t)]^T$ is a vector of self-adjoint controller variables.

We assume that $x(0)$ commutes with $\xi(0)$ at time $t = 0$. By interconnecting (33) and (36), and making the identification $\beta_u(t) = C_K\xi(t)$, the closed loop system is of the form

$$
\begin{aligned}
d\eta(t) &= \begin{bmatrix} A+\Delta A & (B_2+\Delta B_2)C_K \\ B_K(C_2+\Delta C_2) & A_K \end{bmatrix}\eta(t)dt\\
&\quad + \begin{bmatrix} (B_0+\Delta B_0) & (B_2+\Delta B_2)B_{K0} \\ B_KD_{20} & B_{K1} \end{bmatrix}\begin{bmatrix} dv(t) \\ dv_K(t) \end{bmatrix}\\
&\quad + \begin{bmatrix} B_1+\Delta B_1 \\ B_KD_{21} \end{bmatrix}dw(t);\\
dz(t) &= \left[(C_1+\Delta C_1) \quad D_{12}C_K\right]\eta(t)dt\\
&\quad + \left[0_{n_z\times n_v} \quad D_{12}B_{K0}\right]\begin{bmatrix} dv(t) \\ dv_K(t) \end{bmatrix}
\end{aligned}
\tag{37}
$$

where we denote $\eta(t) = \left[x(t)^T\ \xi(t)^T\right]^T$. This state equation can also be rewritten as

$$\begin{aligned} d\eta(t) &= (\tilde{A}+\Delta\tilde{A})\eta(t)dt + (\tilde{B}+\Delta\tilde{B})dw(t) \\ &\quad + (\tilde{G}+\Delta\tilde{G})d\zeta(t); \\ dz(t) &= (\tilde{C}+\Delta\tilde{C})\eta(t)dt + \tilde{H}d\zeta(t) \end{aligned} \tag{38}$$

where

$$\begin{aligned}
&\zeta(t) = \begin{bmatrix} v(t) \\ v_K(t) \end{bmatrix}; \qquad \tilde{A} = \begin{bmatrix} A & B_2C_K \\ B_KC_2 & A_K \end{bmatrix}; \\
&\tilde{B} = \begin{bmatrix} B_1 \\ B_KD_{21} \end{bmatrix}; \qquad \tilde{G} = \begin{bmatrix} B_0 & B_2B_{K0} \\ B_KD_{20} & B_{K1} \end{bmatrix}; \\
&\tilde{C} = \begin{bmatrix} C_1 & D_{12}C_K \end{bmatrix}; \qquad \tilde{H} = \begin{bmatrix} 0 & D_{12}B_{K0} \end{bmatrix}; \\
&\Delta\tilde{A} = \begin{bmatrix} \Delta A & \Delta B_2C_K \\ B_K\Delta C_2 & 0_{n_K\times n_K} \end{bmatrix} \quad \text{with } \Delta A \text{ as in (13)}; \\
&\Delta B_2 = 2i\Theta[-\Omega_u^\dagger \quad \Omega_u^T]\Gamma_u; \quad \Gamma_u = P_{N_u}\mathrm{diag}_{N_u}(M); \\
&\Omega_{vw} = \begin{bmatrix} \Omega_v \\ \Omega_w \end{bmatrix}; \qquad \Delta C_2 = 2P_{N_y}^T \Xi_{N_yN_{vw}} \begin{bmatrix} \Re(\Omega_{vw}) \\ \Im(\Omega_{vw}) \end{bmatrix}; \\
&N_u = n_u/2; \qquad N_{vw} = \frac{n_v+n_w}{2}; \\
&\Delta\tilde{B} = E_1\Delta B_1; \qquad \Delta\tilde{C} = \Delta C_1E_2; \\
&\Delta\tilde{G} = \begin{bmatrix} \Delta B_0 & \Delta B_2B_{K0} \\ 0_{n_K\times n_v} & 0_{n_K\times n_{v_K}} \end{bmatrix}; \qquad \Delta C_1 = 2P_{N_z}^T\Xi_{N_zN_u}\begin{bmatrix} \Re(\Omega_u) \\ \Im(\Omega_u) \end{bmatrix}; \\
&\Delta B_0 = 2i\Theta[-\Omega_v^\dagger \quad \Omega_v^T]\Gamma_v; \qquad \Delta B_1 = 2i\Theta[-\Omega_w^\dagger \quad \Omega_w^T]\Gamma_w; \\
&E_1 = \begin{bmatrix} I_{n\times n} & 0_{n\times n_K} \end{bmatrix}^T; \qquad E_2 = \begin{bmatrix} I_{n\times n} & 0_{n\times n_K} \end{bmatrix}.
\end{aligned} \tag{39}$$

The closed-loop system (38) is in a similar form to the uncertain system (12).

4.1.2. H^∞ control objective

For a given disturbance attenuation parameter $g > 0$, the H^∞ control objective is to find a coherent quantum controller of the form (36) for the uncertain quantum system (33) such that the closed-loop system (38) satisfies

$$\begin{aligned} &\int_0^t \langle \beta_z(s)^T\beta_z(s) + \tilde{\varepsilon}\eta(s)^T\eta(s)\rangle ds \\ &\le (g^2-\tilde{\varepsilon})\int_0^T \langle \beta_w(s)^T\beta_w(s)\rangle ds + \mu_1 + \mu_2 t, \quad \forall t > 0 \end{aligned} \tag{40}$$

for some real constants $\tilde{\varepsilon}, \mu_1, \mu_2 > 0$. Here, $\beta_z(t)$ is controlled output operator $\beta_z(t) = \tilde{C}\eta(t)$. The control objective (40) was first introduced in James et al. (2008) and can be interpreted as that the desired controller could bound the effect of the 'energy' in the disturbance signal $\beta_w(t)$ on the 'energy' of the controlled signal $z(t)$. We should also notice that if the closed-loop system (38) is robustly strict bounded real with disturbance attenuation g, it then satisfies the H^∞ control synthesis objective (40).

4.2. Relation between the uncertain quantum system and the scaled system

In this section, we develop a relationship between robust H^∞ control for the uncertain quantum system via a coherent controller and H^∞ control for an auxiliary scaled system without parameter uncertainties via the same coherent controller.

The uncertain quantum system is described by (33). We call the auxiliary system to be considered a scaled H^∞ control system. In order to solve the robust H^∞ control problem, we introduce the corresponding scaled H^∞ control system for (33) as follows:

$$\begin{aligned} dx(t) &= Ax(t)dt + [B_0 \quad [J_1 \quad g^{-1}J_2] \quad B_2] \\ &\quad \times [dv(t)^T \quad d\overline{w}(t)^T \quad du(t)^T]^T; \quad x(0) = x_0 \\ d\overline{z}(t) &= J_3x(t)dt + \tilde{D}_{12}du(t); \\ dy(t) &= C_2x(t)dt + [D_{20} \quad \tilde{D}_{21} \quad 0] \\ &\quad \times [dv(t)^T \quad d\overline{w}(t)^T \quad du(t)^T]^T \end{aligned} \tag{41}$$

where

$$\begin{aligned}
\tilde{D}_{12} &= [0^T_{(m+2N_{uvw}+2n)\times n_u} \quad \sqrt{4r/\varepsilon_5}I \quad \sqrt{1+4\varepsilon_8}D_{12}^T]^T; \\
\tilde{D}_{21} &= [0_{n_y\times(m+2N_{uvw}+2n)} \quad 2\sqrt{\varepsilon_6}I \quad 0_{n_y\times n} \quad g^{-1}\sqrt{(1+\varepsilon_7)}D_{21}]; \\
J_1 &= [2\sqrt{\varepsilon_1}\Theta E^T \quad 2\sqrt{\varepsilon_2}\Theta\Re(\Lambda)^T \quad 2\sqrt{\varepsilon_2}\Theta\Im(\Lambda)^T \\
&\quad 2\sqrt{(\varepsilon_3+\varepsilon_4)r}I \quad \sqrt{\varepsilon_5}I \quad 0_{n\times n_y}]; \\
J_2 &= \left[\sqrt{\left(\frac{4}{\varepsilon_7}+4\right)r}I \quad \sqrt{(1+\varepsilon_7)}B_1\right]; \\
J_3 &= \left[\sqrt{\frac{1}{\varepsilon_1}}E^T \quad \sqrt{\left(\frac{1}{\varepsilon_2}+\frac{1}{\varepsilon_4}\right)r}I \quad \sqrt{\frac{1}{\varepsilon_3}}\Re(\Lambda)^T\right. \\
&\quad \left.\sqrt{\frac{1}{\varepsilon_3}}\Im(\Lambda)^T \quad \sqrt{\left(4+\frac{1}{\varepsilon_6}+\frac{1}{\varepsilon_8}\right)r}I \quad 0^T_{n_u\times n} \quad \sqrt{1+4\varepsilon_8}C_1^T\right]^T.
\end{aligned} \tag{42}$$

Here, $A, B_0, B_1, B_2, C_1, C_2, D_{20}, D_{21}$ and D_{22} are the same as in (33), $\varepsilon_i > 0$ $(i = 1, \ldots, 8)$ are scaling parameters to be chosen and $g > 0$ is the disturbance attenuation that is required to be achieved.

The closed-loop system connected by the system (41) and the controller (36) can be written as

$$\begin{aligned} d\eta(t) &= \tilde{A}\eta(t)dt + \left[\tilde{J}_1 \quad g^{-1}\tilde{J}_2\right]d\overline{w}(t) + \tilde{G}d\zeta(t); \\ d\overline{z}(t) &= \tilde{J}_3\eta(t)dt + \tilde{J}_4d\zeta(t) \end{aligned} \tag{43}$$

where

$$\begin{aligned}
\tilde{J}_1 &= \begin{bmatrix} 2\sqrt{\varepsilon_1}E\Theta^T & 0_{m\times n_K} \\ 2\sqrt{\varepsilon_2}\Re(\Lambda)\Theta^T & 0_{N_{uvw}\times n_K} \\ 2\sqrt{\varepsilon_2}\Im(\Lambda)\Theta^T & 0_{N_{uvw}\times n_K} \\ 2\sqrt{(\varepsilon_3+\varepsilon_4)r}I_{n\times n} & 0_{n\times n_K} \\ \sqrt{\varepsilon_5}I_n & 0_{n\times n_K} \\ 0_{n_y\times n} & 2\sqrt{\varepsilon_6}B_K^T \end{bmatrix}^T; \\
\tilde{J}_2 &= \begin{bmatrix} \sqrt{(4/\varepsilon_7+4)r}I_n & 0_{n\times n_K} \\ \sqrt{(1+\varepsilon_7)}B_1^T & \sqrt{(1+\varepsilon_7)}(B_KD_{21})^T \end{bmatrix}^T; \\
\tilde{J}_3 &= \begin{bmatrix} \sqrt{1/\varepsilon_1}E & 0_{m\times n_K} \\ \sqrt{(1/\varepsilon_2+1/\varepsilon_4)r}I & 0_{n\times n_K} \\ \sqrt{1/\varepsilon_3}\Re(\Lambda) & 0_{N_{uvw}\times n_K} \\ \sqrt{1/\varepsilon_3}\Im(\Lambda) & 0_{N_{uvw}\times n_K} \\ \sqrt{(4+1/\varepsilon_6+1/\varepsilon_8)r}I & 0_{n\times n_K} \\ 0_{n_u\times n} & \sqrt{4r/\varepsilon_5}C_K \\ \sqrt{1+4\varepsilon_8}C_1 & \sqrt{1+4\varepsilon_8}D_{12}C_K \end{bmatrix}; \\
\tilde{J}_4 &= \begin{bmatrix} 0_{(m+2N_{uvw}+2n)\times n_v} & 0_{(m+2N_{uvw}+2n)\times n_{v_K}} \\ 0_{n_u\times n_v} & \sqrt{4r/\varepsilon_5}B_{K0} \\ 0_{n_z\times n_v} & \sqrt{1+4\varepsilon_8}D_{12}B_{K0} \end{bmatrix}.
\end{aligned} \tag{44}$$

Here, we develop a connection between the robust H^∞ control problem for the uncertain quantum system (33) and an H^∞ control problem for the scaled system (41) in the following theorem.

Theorem 5. *Let $g > 0$ be a prescribed level of disturbance attenuation and consider a linear dynamic controller of the form (36).*

Then the system (33) is robustly strict bounded real with disturbance attenuation $g > 0$ via the coherent controller (36) if there exist $\varepsilon_i > 0$ $(i = 1, \ldots, 8)$ such that the system (41) is strictly bounded real with unitary disturbance attenuation via the same coherent controller (36).

The following lemmas are required for proving Theorem 5.

Lemma 6. *For $C \in R^{m\times n}$, $C^TC \le rI_{n\times n}$ if and only if $CC^T \le rI_{m\times m}$.*

The proof of Lemma 6 follows straightforwardly from Section 3.8 of Chen (1999).

Lemma 7. *For Γ_u is defined by (39), Θ is defined in (2) and Ω_u is bounded by the condition (35), then the following bound is satisfied*

$$\Gamma_u^T[-\Omega_u^\dagger \quad \Omega_u^T]^T\Theta^T 2i \times 2i\Theta[-\Omega_u^\dagger \quad \Omega_u^T]\Gamma_u \le 4rI.$$

Proof. Let $M_1, M_2, \ldots, M_{N_u}$ be the column vectors of the matrix Ω_u^T.

$$\begin{aligned}
&[-\Omega_u^\dagger \quad \Omega_u^T]\Gamma_u \\
&= [-M_1^\# \quad -M_2^\# \quad \ldots \quad -M_{N_u}^\# \quad M_1 \quad M_2 \quad \ldots \quad M_{N_u}]\Gamma_u \\
&= \frac{1}{2}[-M_1^\# + M_1 \quad -iM_1^\# - iM_1 \quad \ldots \\
&\quad -M_{N_u}^\# + M_{N_u} \quad -iM_{N_u}^\# - iM_{N_u}] \\
&= [i\Im(M_1) \quad -i\Re(M_1) \quad \ldots \quad i\Im(M_{N_u}) \quad -i\Re(M_{N_u})].
\end{aligned}$$

Now, we use the permutation matrix P_{N_u} and have

$$\begin{aligned}
[-\Omega_u^\dagger \quad \Omega_u^T]\Gamma_u P_{N_u}^T &= i[\Im(M_1) \quad \Im(M_2) \quad \ldots \Im(M_{N_u}) \\
&\quad -\Re(M_1) \quad -\Re(M_2) \ldots \quad -\Re(M_{N_u})] \\
&= i[\Im(\Omega_u)^T \quad -\Re(\Omega_u)^T].
\end{aligned} \tag{45}$$

It follows straightforwardly from (45) that $[-\Omega_u^\dagger \quad \Omega_u^T]\Gamma_u = i[\Im(\Omega_u)^T \quad -\Re(\Omega_u)^T]P_{N_u}$. Hence, we obtain the following relation after some algebraic computation:

$$\begin{aligned}
&\Gamma_u^T[-\Omega_u^\dagger \quad \Omega_u^T]^T\Theta^T 2i \times 2i\Theta[-\Omega_u^\dagger \quad \Omega_u^T]\Gamma_u \\
&= 4P_{N_u}^T[\Im(\Omega_u)^T \quad -\Re(\Omega_u)^T]^T[\Im(\Omega_u)^T \quad -\Re(\Omega_u)^T]P_{N_u} \\
&= 4P_{N_u}^T\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}\begin{bmatrix} \Im(\Omega_u) \\ \Re(\Omega_u) \end{bmatrix}[\Im(\Omega_u)^T \ \Re(\Omega_u)^T]\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}P_{N_u}.
\end{aligned}$$

Furthermore, note that $\begin{bmatrix} \Im(\Omega_u)^T & \Re(\Omega_u)^T \end{bmatrix}\begin{bmatrix} \Im(\Omega_u) \\ \Re(\Omega_u) \end{bmatrix} \le rI$. It follows from Lemma 6 that $\begin{bmatrix} \Im(\Omega_u) \\ \Re(\Omega_u) \end{bmatrix}\begin{bmatrix} \Im(\Omega_u)^T & \Re(\Omega_u)^T \end{bmatrix} \le rI$. Using the fact that $\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ and $P_{N_u}^TP_{N_u} = I$, the following inequality is obtained:

$$\Gamma_u^T[-\Omega_u^\dagger \quad \Omega_u^T]^T\Theta^T 2i \times 2i\Theta[-\Omega_u^\dagger \quad \Omega_u^T]\Gamma_u \le 4rI. \quad \square$$

Proof of Theorem 5. The essential feature of the theorem is that we need to show that the closed-loop system (37) consisting of (33) and (36) is robustly strict bounded real with disturbance attenuation $g > 0$ if the closed-loop system (43) consisting of (41) and (36) is strict bounded real with unitary disturbance attenuation. In other words, this theorem can be rephrased as the fact that for a given constant $g > 0$, there exists a matrix $X > 0$ such that

$$\begin{aligned}
&(\tilde{A} + \Delta\tilde{A})^TX + X(\tilde{A} + \Delta\tilde{A}) + (\tilde{C} + \Delta\tilde{C})^T(\tilde{C} + \Delta\tilde{C}) \\
&+ g^{-2}X(\tilde{B} + \Delta\tilde{B})(\tilde{B} + \Delta\tilde{B})^TX < 0,
\end{aligned} \tag{46}$$

if there exist $\varepsilon_i > 0$ $(i = 1, \ldots, 8)$ such that

$$\tilde{A}^TX + X\tilde{A} + XG_1X + g^{-2}XG_3X + G_4 < 0 \tag{47}$$

where

$$\begin{aligned}
G_1 &= \tilde{J}_1\tilde{J}_1^T = 4\varepsilon_1E_1\Theta E^TE\Theta^TE_1^T + 4(\varepsilon_3 + \varepsilon_4)rE_1E_1^T \\
&\quad + 4\varepsilon_2E_1\Theta(\Re(\Lambda)^T\Re(\Lambda) + \Im(\Lambda)^T\Im(\Lambda))\Theta^TE_1^T \\
&\quad + \varepsilon_5E_1E_1^T + 4\varepsilon_6\tilde{B}_K\tilde{B}_K^T; \\
G_2 &= E_2^TE^TEE_2/\varepsilon_1 + (1/\varepsilon_2 + 1/\varepsilon_4)rE_2^TE_2 \\
&\quad + E_2^T(\Re(\Lambda)^T\Re(\Lambda) + \Im(\Lambda)^T\Im(\Lambda))E_2/\varepsilon_3; \\
G_3 &= \tilde{J}_2\tilde{J}_2^T = (1 + \varepsilon_7)\tilde{B}\tilde{B}^T + (4/\varepsilon_7 + 4)rE_1E_1^T; \\
G_4 &= \tilde{J}_3^T\tilde{J}_3 = G_2 + 4r\tilde{C}_K^T\tilde{C}_K/\varepsilon_5 + (1 + 4\varepsilon_8)\tilde{C}^T\tilde{C} \\
&\quad + (4 + 1/\varepsilon_6 + 1/\varepsilon_8)rE_2^TE_2.
\end{aligned}$$

Here, $\tilde{B}_K = \begin{bmatrix} 0 \\ B_K \end{bmatrix}$ and $\tilde{C}_K = [0 \quad C_K]$.

In order to prove the theorem, first we need to rewrite $\Delta\tilde{A}$ of (39) in the following form

$$\Delta\tilde{A} = E_1\Delta AE_2 + E_1\Delta B_2\tilde{C}_K + \tilde{B}_K\Delta C_2E_2.$$

Hence, the uncertain part of the first two terms on the left side of (46) can be written as:

$$\begin{aligned}
\Delta\tilde{A}^TX + X\Delta\tilde{A} = &(E_1\Delta AE_2)^TX + XE_1\Delta AE_2 \\
&+ (E_1\Delta B_2\tilde{C}_K)^TX + XE_1\Delta B_2\tilde{C}_K \\
&+ (\tilde{B}_K\Delta C_2E_2)^TX + X\tilde{B}_K\Delta C_2E_2.
\end{aligned} \tag{48}$$

By applying a similar method to the proof of Theorem 2, we are able to obtain an upper bound on the first line on right side of (48) as:

$$(E_1\Delta AE_2)^TX + XE_1\Delta AE_2 \le X\tilde{G_1}X + G_2$$

where

$$\begin{aligned}
\tilde{G_1} &= 4\varepsilon_1E_1\Theta E^TE\Theta^TE_1^T + 4(\varepsilon_3 + \varepsilon_4)rE_1E_1^T \\
&\quad + 4\varepsilon_2E_1\Theta(\Re(\Lambda)^T\Re(\Lambda) + \Im(\Lambda)^T\Im(\Lambda))\Theta^TE_1^T; \\
G_2 &= E_2^TE^TEE_2/\varepsilon_1 + (1/\varepsilon_2 + 1/\varepsilon_4)rE_2^TE_2 \\
&\quad + E_2^T(\Re(\Lambda)^T\Re(\Lambda) + \Im(\Lambda)^T\Im(\Lambda))E_2/\varepsilon_3.
\end{aligned}$$

As for the second line on the right side of (48), by substituting corresponding parameters in (39) and applying inequality (18), we obtain the following bound:

$$\begin{aligned}
&(E_1\Delta B_2\tilde{C}_K)^TX + XE_1\Delta B_2\tilde{C}_K \\
&\le \varepsilon_5XE_1E_1^TX + \frac{1}{\varepsilon_5}\tilde{C}_K^T\Gamma_u^T[-\Omega_u^\dagger \quad \Omega_u^T]^T\Theta^T 2i \\
&\quad \times 2i\Theta[-\Omega_u^\dagger \quad \Omega_u^T]\Gamma_u\tilde{C}_K \\
&\le \varepsilon_5XE_1E_1^TX + 4r\tilde{C}_K^T\tilde{C}_K/\varepsilon_5.
\end{aligned} \tag{49}$$

The last line of (49) follows from Lemma 7. In addition, with the property $P_{N_y}^TP_{N_y} = I$, (35) and (18), the following inequalities are obtained:

$$\begin{aligned}
&(\tilde{B}_K\Delta C_2E_2)^TX + X\tilde{B}_K\Delta C_2E_2 \\
&\le 4\varepsilon_6X\tilde{B}_KP_{N_y}^TP_{N_y}\tilde{B}_K^TX \\
&\quad + \frac{1}{\varepsilon_6}E_2^T\left[\Re(\Omega_{vw})^T \quad \Im(\Omega_{vw})^T\right]\Xi_{N_yN_{vw}}^T \\
&\quad \times \Xi_{N_yN_{vw}}\begin{bmatrix} \Re(\Omega_{vw}) \\ \Im(\Omega_{vw}) \end{bmatrix}E_2 \\
&\le 4\varepsilon_6X\tilde{B}_K\tilde{B}_K^TX + \frac{1}{\varepsilon_6}rE_2^TE_2.
\end{aligned} \tag{50}$$

The last inequality in (50) follows in the same manner as the proof in (28). The last term on the left side of (46) can be calculated by replacing the corresponding parameters in (39):

$$\begin{aligned}
&(\tilde{B}+E_1\Delta B_1)(\tilde{B}+E_1\Delta B_1)^T \\
&\quad= \tilde{B}\tilde{B}^T+\tilde{B}\Delta B_1^TE_1^T+E_1\Delta B_1\tilde{B}^T+E_1\Delta B_1\Delta B_1^TE_1^T \\
&\quad= \tilde{B}\tilde{B}^T+\tilde{B}\left(2i\Gamma_w^T\begin{bmatrix}-\Omega_w^{\#}\\ \Omega_w\end{bmatrix}\Theta^TE_1^T\right) \\
&\qquad+2iE_1\Theta[-\Omega_w^{\dagger}\quad \Omega_w^T]\Gamma_w\tilde{B}^T \\
&\qquad-4E_1\Theta[-\Omega_w^{\dagger}\quad \Omega_w^T]\Gamma_w\Gamma_w^T\begin{bmatrix}-\Omega_w^{\#}\\ \Omega_w\end{bmatrix}\Theta^TE_1^T. \qquad (51)
\end{aligned}$$

It follows from (18) and Lemma 3 that the second and third terms on the right side of (51) are bounded as follows:

$$\begin{aligned}
&\tilde{B}\left(2i\Gamma_w^T\begin{bmatrix}-\Omega_w^{\#}\\ \Omega_w\end{bmatrix}\Theta^TE_1^T\right)+2iE_1\Theta[-\Omega_w^{\dagger}\quad \Omega_w^T]\Gamma_w\tilde{B}^T \\
&\quad\le \varepsilon_7\tilde{B}\tilde{B}^T+4rE_1E_1^T/\varepsilon_7. \qquad (52)
\end{aligned}$$

Also, the last term on the right side of (51) is bounded as

$$-4E_1\Theta[-\Omega_w^{\dagger}\quad \Omega_w^T]\Gamma_w\Gamma_w^T\begin{bmatrix}-\Omega_w^{\#}\\ \Omega_w\end{bmatrix}\Theta^TE_1^T\le 4rE_1E_1^T. \qquad (53)$$

Therefore, (51)–(53) imply that

$$(\tilde{B}+E_1\Delta B_1)(\tilde{B}+E_1\Delta B_1)^T\le(1+\varepsilon_7)\tilde{B}\tilde{B}^T+\left(\frac{4}{\varepsilon_7}+4\right)rE_1E_1^T.$$

In the meanwhile, the third term on the left side of (46) can be rewritten as follows:

$$\begin{aligned}
&(\tilde{C}+\Delta\tilde{C})^T(\tilde{C}+\Delta\tilde{C}) \\
&\quad=\tilde{C}^T\tilde{C}+\tilde{C}^T(2P_{N_z}^T\Xi_{N_zN_u}[\Re(\Omega_u)^T\quad \Im(\Omega_u)^T]^TE_2) \\
&\qquad+(2P_{N_z}^T\Xi_{N_zN_u}[\Re(\Omega_u)^T\quad \Im(\Omega_u)^T]^TE_2)^T\tilde{C} \\
&\qquad+\left(2P_{N_z}^T\Xi_{N_zN_u}\begin{bmatrix}\Re(\Omega_u)\\ \Im(\Omega_u)\end{bmatrix}E_2\right)^T2P_{N_z}^T\Xi_{N_zN_u}\begin{bmatrix}\Re(\Omega_u)\\ \Im(\Omega_u)\end{bmatrix}E_2. \qquad (54)
\end{aligned}$$

The fact $P_{N_z}P_{N_z}^T=I$, (18) and (35) imply that

$$\begin{aligned}
&\tilde{C}^T\Delta\tilde{C}+\Delta\tilde{C}^T\tilde{C} \\
&\quad=\tilde{C}^T(2P_{N_z}^T\Xi_{N_zN_u}[\Re(\Omega_u)^T\quad \Im(\Omega_u)^T]^TE_2) \\
&\qquad+(2P_{N_z}^T\Xi_{N_zN_u}[\Re(\Omega_u)^T\quad \Im(\Omega_u)^T]^TE_2)^T\tilde{C} \\
&\quad\le 4\varepsilon_8\tilde{C}^TP_{N_z}^TP_{N_z}\tilde{C}+\frac{1}{\varepsilon_8}E_2^T\left[\Re(\Omega_u)^T\quad \Im(\Omega_u)^T\right] \\
&\qquad\times\Xi_{N_zN_u}^T\Xi_{N_zN_u}\begin{bmatrix}\Re(\Omega_u)\\ \Im(\Omega_u)\end{bmatrix}E_2 \\
&\quad\le 4\varepsilon_8\tilde{C}^T\tilde{C}+rE_2^TE_2/\varepsilon_8, \qquad (55)
\end{aligned}$$

and the following inequality holds

$$\begin{aligned}
&\Delta\tilde{C}^T\Delta\tilde{C} \\
&\quad=\left(2P_{N_z}^T\Xi_{N_zN_u}\begin{bmatrix}\Re(\Omega_u)\\ \Im(\Omega_u)\end{bmatrix}E_2\right)^T\left(2P_{N_z}^T\Xi_{N_zN_u}\begin{bmatrix}\Re(\Omega_u)\\ \Im(\Omega_u)\end{bmatrix}E_2\right) \\
&\quad=4E_2^T\left[\Re(\Omega_u)^T\quad \Im(\Omega_u)^T\right]\Xi_{N_zN_u}^T\Xi_{N_zN_u}\begin{bmatrix}\Re(\Omega_u)\\ \Im(\Omega_u)\end{bmatrix}E_2 \\
&\quad\le 4rE_2^TE_2. \qquad (56)
\end{aligned}$$

By combining (54)–(56), the following inequality is obtained:

$$(\tilde{C}+\Delta\tilde{C})^T(\tilde{C}+\Delta\tilde{C})\le(1+4\varepsilon_8)\tilde{C}^T\tilde{C}+\left(4+\frac{1}{\varepsilon_8}\right)rE_2^TE_2.$$

Therefore, (46) has a solution if there exist $\varepsilon_i>0$ $(i=1,\ldots,8)$ such that (47) holds. Hence, the conclusion in Theorem 5 follows from this result. □

Remark 8. In order to solve the H^∞ controller design problem for the uncertain quantum system (33), we can then solve the H^∞ design problem for the scaled system (41) via existing H^∞ control techniques.

Remark 9. More often than not, quantum systems may only be subject to some of the uncertainties under consideration. For example, uncertainty may exist only in the Hamiltonian or only in noise channel v of the coupling operator. In these cases, only part of uncertainties appear in (33). For these specific situations, it is straightforward to establish a modified scaled H^∞ system by following the proof of Theorem 5.

4.3. A solution to the H^∞ controller design problem

In Section 4.2, we have developed a connection between the uncertain quantum system (33) and the scaled H^∞ system (41). In this section, instead of designing a controller for uncertain system (33) to achieve the H^∞ objective for a given disturbance attenuation g, we focus on constructing a controller for the scaled system (41) to achieve H^∞ performance with unitary attenuation. Necessary and sufficient conditions for the existence of a specific type of required controllers are presented, and detailed expressions for A_K, B_K and C_K are given.

To present the results on quantum H^∞ control for the auxiliary system (41), we use a similar method to the non-singular H^∞ control approach in classical linear systems, e.g., Doyle, Glover, Khargonekar, and Francis (1989) and Petersen, Anderson, and Jonckheere (1991). We know that the following terms are positive definite:

$$\tilde{D}_{12}^T\tilde{D}_{12}=4rI/\varepsilon_5+(1+4\varepsilon_8)D_{12}^TD_{12}=\tilde{E}_1>0;$$
$$\tilde{D}_{21}\tilde{D}_{21}^T=4\varepsilon_6I+g^{-2}(1+\varepsilon_7)D_{21}D_{21}^T=\tilde{E}_2>0.$$

Then, the system (41) is required to satisfy the following assumption:

Assumption 10. (1) The matrix $\begin{bmatrix}A-iwI & B_2\\ J_3 & \tilde{D}_{12}\end{bmatrix}$ is full column rank for all $w\ge 0$.

(2) The matrix $\begin{bmatrix}A-iwI & J_{12}\\ C_2 & \tilde{D}_{21}\end{bmatrix}$ is full row rank for all $w\ge 0$, where $J_{12}=[J_1\quad g^{-1}J_2]$.

The solution to the H^∞ control problem for the scaled system (41) is given in terms of the following pair of algebraic Riccati equations:

$$\begin{aligned}
&(A-(1+4\varepsilon_8)B_2\tilde{E}_1^{-1}D_{12}^TC_1)^TX \\
&\quad+X(A-(1+4\varepsilon_8)B_2\tilde{E}_1^{-1}D_{12}^TC_1) \\
&\quad+X(J_{12}J_{12}^T-B_2\tilde{E}_1^{-1}B_2^T)X \\
&\quad+J_3^TJ_3-(1+4\varepsilon_8)^2C_1^TD_{12}\tilde{E}_1^{-1}D_{12}^TC_1=0; \qquad (57) \\
&(A-g^{-2}(1+\varepsilon_7)B_1D_{21}^T\tilde{E}_2^{-1}C_2)Y \\
&\quad+Y(A-g^{-2}(1+\varepsilon_7)B_1D_{21}^T\tilde{E}_2^{-1}C_2)^T \\
&\quad+Y(J_3^TJ_3-C_2^T\tilde{E}_2^{-1}C_2)Y \\
&\quad+J_{12}J_{12}^T-g^{-4}(1+\varepsilon_7)^2B_1D_{21}^T\tilde{E}_2^{-1}D_{21}B_1^T=0 \qquad (58)
\end{aligned}$$

where X and Y are positive-definite symmetric matrices and $J_{12}J_{12}^T=4\varepsilon_1\Theta E^TE\Theta^T+4\varepsilon_2\Theta(\Re(\Lambda)^T\Re(\Lambda)+\Im(\Lambda)^T\Im(\Lambda))\Theta^T+(4\varepsilon_3r+4\varepsilon_4r+\varepsilon_5+g^{-2}(\frac{4}{\varepsilon_7}+4)r)I+(1+\varepsilon_7)B_1B_1^T$. The solutions to these Riccati equations need to satisfy the following assumption.

Assumption 11. (1) $A-(1+4\varepsilon_8)B_2\tilde{E}_1^{-1}D_{12}^TC_1+(J_{12}J_{12}^T-B_2\tilde{E}_1^{-1}B_2^T)X$ is a stability matrix.
(2) $A-g^{-2}(1+\varepsilon_7)B_1D_{21}^T\tilde{E}_2^{-1}C_2+Y(J_3^TJ_3-C_2^T\tilde{E}_2^{-1}C_2)$ is a stability matrix.
(3) The matrix XY has a spectral radius strictly less than one.

It will be shown that if the solution to the Riccati equations (57) and (58) satisfies Assumption 11, then a controller of the form (36) will achieve the required H^∞ control objective where its system matrices are designed from the Riccati solutions in the following form:

$$\begin{aligned}
A_K &= A+B_2C_K-B_KC_2+J_{12}J_{12}^TX-g^{-2}(1+\varepsilon_7)B_KD_{21}B_1^TX;\\
B_K &= (I-YX)^{-1}(YC_2^T+g^{-2}(1+\varepsilon_7)B_1D_{21}^T)\tilde{E}_2^{-1};\\
C_K &= -\tilde{E}_1^{-1}(B_2^TX+(1+4\varepsilon_8)D_{12}^TC_1).
\end{aligned} \tag{59}$$

Now, we present our main results on coherent robust H^∞ controller synthesis.

Theorem 12. Necessity. *Consider the system (41) and suppose that Assumption 10 is satisfied. If there exists a controller of the form (36) such that the resulting closed-loop system (43) is strictly bounded real with unitary disturbance attenuation, then the Riccati equations (57) and (58) will have stabilizing solutions $X \geq 0$ and $Y \geq 0$ satisfying Assumption 11.*

Sufficiency. *Suppose the Riccati equations (57) and (58) have stabilizing solutions $X \geq 0$ and $Y \geq 0$ satisfying Assumption 11. If the controller (36) is such that the matrices A_K, B_K, C_K are as defined in (58), then the resulting closed-loop system (43) is strictly bounded real with unitary disturbance attenuation.*

Remark 13. As mentioned in Remark 9, when the quantum system is only subject to part of the uncertainties, a modified scaled H^∞ system can be easily obtained. We can apply the same method as Theorem 12 to the corresponding scaled H^∞ system and obtain the required controller matrices. Examples will be provided to demonstrate the procedure in Section 5.

So far, we have obtained the explicit formulas for A_K, B_K, C_K in (58). However, an H^∞ controller defined by the matrices A_K, B_K, C_K is not always physically realizable, that is, A_K, B_K, C_K may not satisfy the relationships (3) and (4). As can be seen from Theorem 12, parameters B_{K0} and B_{K1} can be chosen freely, since the H^∞ controller design is independent of these parameters. The following theorem (James et al., 2008) shows that B_{K0} and B_{K1} can be constructed to guarantee the quantum controller physically realizable.

Theorem 14 (*See Theorem 5.5 of James et al., 2008*). *Assume $F_y = D_{20}F_vD_{20}^T + D_{21}F_wD_{21}^T$ is canonical. Let A_K, B_K, C_K be an arbitrary triple (such as given by (58)), and the controller commutation matrix is canonical Θ_K. Then there exists controller parameters B_{K0}, B_{K1}, and the controller noise v_K such that the controller (36) is physically realizable. In particular, $2i\Theta_K = (\xi(t)\xi(t)^T-(\xi(t)\xi(t)^T))^T$ for all $t \geq 0$ whenever $2i\Theta_K = (\xi(0)\xi(0)^T-(\xi(0)\xi(0)^T))^T$.*

To conclude, Theorem 14 shows that it is always possible to find a physically realizable controller given the matrices A_K, B_K, C_K (Vuglar & Petersen, 2011).

5. Examples from quantum optics

In this section, we illustrate the coherent robust H^∞ controller design method proposed in this paper. The quantum plant under consideration is a quantum optical plant which comprises of optical cavities coupled to optical fields. This quantum optical model has also been experimentally implemented (e.g., Mabuchi, 2008).

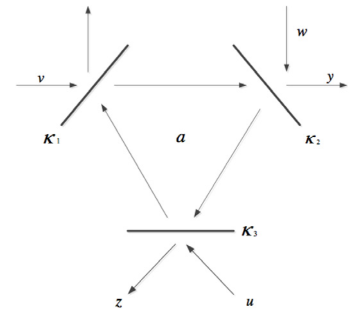

Fig. 1. Optical cavity system.

5.1. Nominal quantum optical cavity plant

In this paper, we consider an optical cavity as a ring cavity which is shown in Fig. 1. This ring optical cavity consists of three partially transmitting mirrors in which a light beam can be trapped and circulates inside the cavity to form a standing wave. This optical mode is described by a harmonic oscillator with annihilation operator a. The partially transmitting mirrors enable this optical mode to interact with an external free field. As can be seen from Fig. 1, there are three optical channels v, w, u coupling to the optical cavity. Our control objective is to attenuate the effect of the disturbance w on the output z. The dynamic of the system is represented by the following state equations

$$\begin{aligned}
da(t) &= -\frac{\gamma}{2}a(t)dt-\sqrt{\kappa_1}dA_1(t)-\sqrt{\kappa_2}dA_2(t)\\
&\quad -\sqrt{\kappa_3}dA_3(t);\\
da^*(t) &= -\frac{\gamma}{2}a^*(t)dt-\sqrt{\kappa_1}dA_1^*(t)-\sqrt{\kappa_2}dA_2^*(t)\\
&\quad -\sqrt{\kappa_3}dA_3^*(t);\\
dB_2(t) &= \sqrt{\kappa_2}a(t)dt+dA_2(t);\\
dB_3(t) &= \sqrt{\kappa_3}a(t)dt+dA_3(t).
\end{aligned} \tag{60}$$

Here $\gamma = \kappa_1+\kappa_2+\kappa_3$. Also, $A_1(t), A_2(t), A_3(t)$ describe the input fields in channels v, w, u, respectively, and $B_2(t), B_3(t)$ represent the output fields in channels w, u, respectively. In this model, we have the commutation relation between annihilation and creation operator

$$\left[\begin{bmatrix} a \\ a^\# \end{bmatrix}, \begin{bmatrix} a \\ a^\# \end{bmatrix}^\dagger\right] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{61}$$

Normally, there often exist complex-valued coefficients with this annihilation/creation operator representation. For convenience, we rewrite (59) in the real-valued quadrature form as in (33) which is represented by position/momentum operators. In the quadrature form, $x_1(t) = q(t) = a(t)+a^*(t)$ is the position operator of the cavity mode (also called the amplitude quadrature) and $x_2(t) = p(t) = (a(t)-a^*(t))/i$ is the momentum operator of the cavity mode (also called the phase quadrature). Consequently, the corresponding coefficients for the system of the form (33) are as follows:

$$\begin{aligned}
&A = -\frac{\gamma}{2}I; \quad B_0 = -\sqrt{\kappa_1}I;\\
&B_1 = -\sqrt{\kappa_2}I; \quad B_2 = -\sqrt{\kappa_3}I;\\
&C_1 = \sqrt{\kappa_3}I; \quad D_{12} = I; \quad C_2 = \sqrt{\kappa_2}I; \quad D_{21} = I.
\end{aligned} \tag{62}$$

The quadrature mode commutation relation for this plant is represented by $[x(t), x(t)^T] = 2iJ$ and the quantum noises $v, \tilde{w}$ have Hermitian Ito matrices $F_v = F_{\tilde{w}} = I+iJ$.

We choose the total cavity decay rate $\gamma = 5.6$ and the coupling coefficients $\kappa_1 = 3.5, \kappa_2 = 2, \kappa_3 = 0.1$. Note that $L =$

$\left[\sqrt{\kappa_1}I \quad \sqrt{\kappa_2}I \quad \sqrt{\kappa_3}I\right]^T a$ is the coupling operator for the given system, where κ_i $(i = 1, 2, 3)$ are the so called mirror coupling coefficients.

5.2. With uncertainty in the Hamiltonian

In quantum optics, the quantum system may be subject to uncertain perturbations in the system Hamiltonian. In this part, we consider an example of a detuned cavity. The system model corresponds to Fig. 1 and the dynamics of this system can be described in the following way:

$$\begin{aligned} da(t) &= \left(-\frac{\gamma}{2} - 2i\hat{\Delta}\right) a(t)dt - \sqrt{\kappa_1}dA_1(t) - \sqrt{\kappa_2}dA_2(t) \\ &\quad - \sqrt{\kappa_3}dA_3(t); \\ da^*(t) &= \left(-\frac{\gamma}{2} + 2i\hat{\Delta}\right) a^*(t)dt - \sqrt{\kappa_1}dA_1^*(t) - \sqrt{\kappa_2}dA_2^*(t) \\ &\quad - \sqrt{\kappa_3}dA_3^*(t); \\ dB_2(t) &= \sqrt{\kappa_2}a(t)dt + dA_2(t); \\ dB_3(t) &= \sqrt{\kappa_3}a(t)dt + dA_3(t). \end{aligned} \tag{63}$$

Here, the uncertainty $\hat{\Delta}$ represents the "detuning" and describes the difference between the nominal external field frequency and the cavity mode frequency. The corresponding quadrature form of the system matrices is: $A = -\frac{\gamma}{2}I + 2J\hat{\Delta}$, and $B_0, B_1, B_2, C_1, C_2, D_{12}, D_{21}$ are the same as in (62). Hence, we have that $H_{\text{perturbation}} = \frac{1}{2}x^T E^T \hat{\Delta} E x$, where $E = I$.

We assume that the uncertainty $\hat{\Delta}$ is in the range of $\hat{\Delta} \in [-1, 1]$ and the required disturbance attenuation constant is $g = 0.25$. Now, we apply the coherent robust H^∞ controller design method to this uncertain quantum system. As indicated in Theorem 12, Remarks 9, and 13, by solving corresponding Riccati equations, we obtain the solution $X = 0.0046I$, $Y = 96.1250I$, which satisfies Assumption 11. The corresponding controller matrices are

$$A_K = -18.315I; \qquad B_K = 12.741I; \qquad C_K = -0.3148I.$$

Since the controller is designed to be a coherent controller, the physical realization conditions need to be satisfied. It follows from Theorem 14 that the following system coefficients are obtained:

$$B_{K1} = \begin{bmatrix} 0.3148 & 0 & 8.1 & -8.1 \\ 0 & 0.3148 & -8.1 & -7.4311 \end{bmatrix};$$

$$B_{K0} = [I \quad 0].$$

To make a performance comparison between the method in this paper and the method proposed in James et al. (2008), we apply the approach in James et al. (2008) and obtain the following results:

$$\begin{aligned} &X = Y = 0_{2\times 2}; \\ &A_K = -0.7I; \qquad B_K = -1.4142I; \qquad C_K = -0.3162I; \\ &B_{K1} = \begin{bmatrix} 0.3162 & 0 & 1 & 1.7 \\ 0 & 0.3162 & 1 & 1 \end{bmatrix}; \qquad B_{K0} = [I \quad 0]. \end{aligned} \tag{64}$$

For the same uncertain quantum system as given before, we can make a performance comparison between the method in James et al. (2008) where the uncertainty was not considered in the controller design and the coherent robust H^∞ controller presented in this paper. Fig. 2 shows how the H^∞ norm of the closed-loop system changes as the uncertainty varies. The dashed line shows the performance of the closed loop system with the coherent controller used in James et al. (2008), while the solid line describes the performance of the closed-loop system with a coherent robust controller designed in this paper. As can be seen from Fig. 2, the controller in James et al. (2008) performs better when the uncertainty is small, while as the uncertainty increases, the robust controller in this paper is better than the controller in James et al. (2008). In the meanwhile, as the uncertainty varies, the H^∞ norm using the method in this paper does not change much and leads to a closed loop system having improved performance.

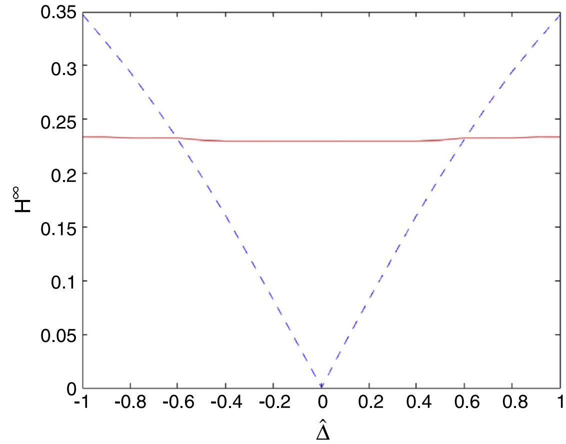

Fig. 2. H^∞ norm of the closed loop systems when there exists uncertainty $\hat{\Delta}$ in the system Hamiltonian.

5.3. With uncertainty in the coupling operator

In this section, we consider the quantum system (59) subject to uncertain perturbations in the coupling operator. The system is formulated by assuming that there is uncertainty in the value of coupling operator corresponding to the optical cavity input v. The dynamics of the system are in the following form

$$\begin{aligned} da(t) &= \left(-\frac{\gamma}{2} - \frac{1}{2}(2\sqrt{\kappa_1}\delta + \delta^2)\right) a(t)dt - (\sqrt{\kappa_1} + \delta)dA_1(t) \\ &\quad - \sqrt{\kappa_2}dA_2(t) - \sqrt{\kappa_3}dA_3(t); \\ da^*(t) &= \left(-\frac{\gamma}{2} - \frac{1}{2}(2\sqrt{\kappa_1}\delta + \delta^2)\right) a^*(t)dt - (\sqrt{\kappa_1} + \delta)dA_1^*(t) \\ &\quad - \sqrt{\kappa_2}dA_2^*(t) - \sqrt{\kappa_3}dA_3^*(t); \\ dB_2(t) &= \sqrt{\kappa_2}a(t)dt + dA_2(t); \\ dB_3(t) &= \sqrt{\kappa_3}a(t)dt + dA_3(t). \end{aligned} \tag{65}$$

Here, δ is the unknown parameter and $\delta^T\delta \le I$. The system (64) can also be represented in real quadrature form (33) with the following system matrices:

$$\Delta A = (-\sqrt{\kappa_1}\delta - \delta^2/2)I; \qquad \Delta B_0 = -\delta I$$

and $A, B_0, B_1, B_2, C_1, C_2, D_{12}, D_{21}$ are the same as in (62).

Note that the coupling operator for this uncertain quantum system is in the form of

$$\begin{aligned} L &= \begin{bmatrix} \sqrt{\kappa_1} + \delta & 0 \\ \sqrt{\kappa_2} & 0 \\ \sqrt{\kappa_3} & 0 \end{bmatrix} \begin{bmatrix} a \\ a^* \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} \sqrt{\kappa_1} + \delta & (\sqrt{\kappa_1} + \delta)i \\ \sqrt{\kappa_2} & \sqrt{\kappa_2}i \\ \sqrt{\kappa_3} & \sqrt{\kappa_3}i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \end{aligned}$$

Therefore, uncertain coupling matrix is $\Omega = \frac{1}{2}\begin{bmatrix} \delta & \delta i \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ and the corresponding $\Re(\Omega)$ and $\Im(\Omega)$ satisfy the condition:

$$\Re(\Omega)^T\Re(\Omega) + \Im(\Omega)^T\Im(\Omega) = \frac{1}{4}\delta^T\delta I_{2\times 2} \le \frac{1}{4}I_{2\times 2}. \tag{66}$$

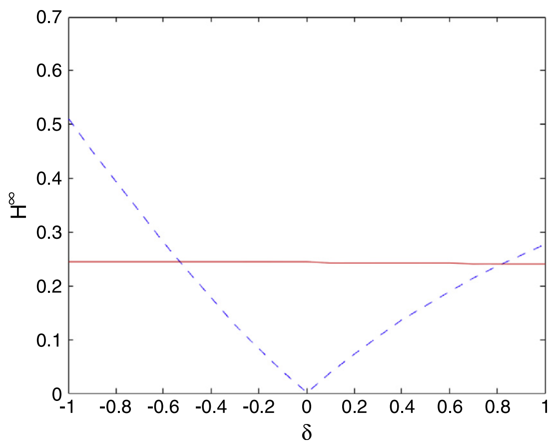

Fig. 3. H^∞ norm of the closed loop systems when there exists uncertainty in the coupling operator.

Now we design a robust quantum H^∞ controller for the uncertain quantum system. The required disturbance attenuation constant is $g = 0.25$ which is the same as in Section 5.2. By applying Theorem 12, Remarks 9, and 13 to the given system, we obtain the required solutions to Riccati equations (57) and (58) that satisfy Assumption 11 with $X = 0.0053I$, $Y = 139.7916I$. Then corresponding controller matrices are given by

$$A_K = -53.8793I; \qquad B_K = 41.1801I; \qquad C_K = -0.3146I.$$

Since the coherent controller needs to be physical realizable, we have the following matrix coefficients

$$B_{K1} = \begin{bmatrix} 0.3146 & 0 & -30 & -30.05 \\ 0 & 0.3146 & -27 & 25.893 \end{bmatrix}; \qquad B_{K0} = [I \quad 0].$$

Fig. 3 shows an H^∞ norm comparison between the closed loop system with a robust coherent controller described in this paper (solid line) and the closed-loop system with a coherent controller obtained in James et al. (2008) (dashed line). A similar result to Fig. 2 is derived. That is, when the coupling uncertainty increases, the robust coherent controller has better H^∞ performance than the coherent controller in James et al. (2008).

5.4. With uncertainties in the Hamiltonian and in the coupling operator

Now, we consider both uncertainties in the Hamiltonian and in the coupling operator, and present an illustrative example. This uncertain quantum system is formulated by assuming that there are uncertainties in the system Hamiltonian as well as in the noise channel v of the optical cavity. The dynamics of the system are in the following form:

$$\begin{aligned} da(t) &= \left(-\frac{\gamma}{2} - 2i\hat{\Delta} - \frac{1}{2}(2\sqrt{\kappa_1}\delta + \delta^2)\right) a(t)dt \\ &\quad - (\sqrt{\kappa_1} + \delta)dA_1(t) - \sqrt{\kappa_2}dA_2(t) - \sqrt{\kappa_3}dA_3(t); \\ da^*(t) &= \left(-\frac{\gamma}{2} + 2i\hat{\Delta} - \frac{1}{2}(2\sqrt{\kappa_1}\delta + \delta^2)\right) a^*(t)dt \\ &\quad - (\sqrt{\kappa_1} + \delta)dA_1^*(t) - \sqrt{\kappa_2}dA_2^*(t) - \sqrt{\kappa_3}dA_3^*(t); \\ dB_2(t) &= \sqrt{\kappa_2}a(t)dt + dA_2(t); \\ dB_3(t) &= \sqrt{\kappa_3}a(t)dt + dA_3(t). \end{aligned} \tag{67}$$

Here, the uncertainty $\hat{\Delta}$ represents the "detuning" of the cavity and $\hat{\Delta} \in [-1, 1]$. δ is the unknown parameter and $\delta^T\delta \le I$. The corresponding quadrature form of the system matrices is:

$$\Delta A = 2J\hat{\Delta} - (\sqrt{\kappa_1}\delta - \delta^2/2)I; \qquad \Delta B_0 = -\delta I$$

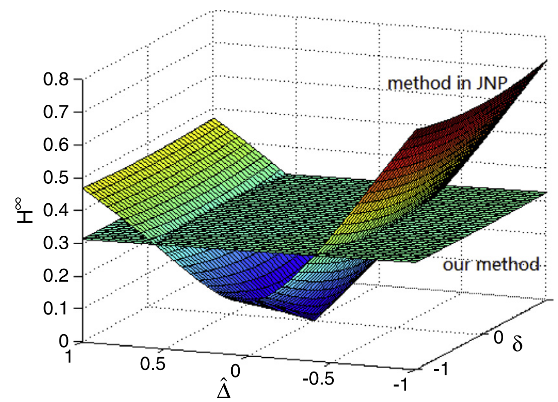

Fig. 4. H^∞ norm of the closed loop systems when there exist uncertainties in the Hamiltonian and the coupling operator, where JNP refers to the paper (James et al., 2008).

and $A, B_0, B_1, B_2, C_1, C_2, D_{12}, D_{21}$ are the same as in (62). Now we apply our robust H^∞ controller design method to this quantum system. The required disturbance attenuation constant is $g = 0.35$. Based on Theorem 12, Remarks 9, and 13, we obtain the solutions to Riccati equations (57) and (58) with $X = 0.0030I$, $Y = 84.6175I$. Following controller formulas (58), the desired controller matrices are constructed as follows:

$$A_K = -25.8035I; \qquad B_K = 17.6663I; \qquad C_K = -0.4459I.$$

Since the coherent controller is required to be physically realizable, it follows from Theorem 14 that the following system coefficients are obtained:

$$B_{K1} = \begin{bmatrix} 0.7058 & 0 & 8 & -8 \\ 0 & 0.7058 & -8 & -6.4062 \end{bmatrix};$$

$$B_{K0} = [I \quad 0].$$

When we apply the existing H^∞ control method in James et al. (2008) to the same uncertain quantum system, we obtain the following results:

$$X = Y = 0_{2\times 2};$$

$$A_K = -0.5500I; \qquad B_K = -1.4142I; \qquad C_K = -0.4472I;$$

$$B_{K1} = \begin{bmatrix} 0.7071 & 0 & -1 & 1 \\ 0 & 0.7071 & 1 & 3.5 \end{bmatrix}; \qquad B_{K0} = [I \quad 0].$$

To make a performance comparison between different methods, Fig. 4 shows the H^∞ norm of the closed loop system with a coherent controller designed as in this paper and as in James et al. (2008). Similar results to those in Figs. 2 and 3 are obtained. As uncertainties in the Hamiltonian as well as in the coupling operator increase, the closed loop system with the coherent robust H^∞ controller has better performance than that using the method in James et al. (2008).

As for the selection of scaling constants ε_j, different approaches could be employed to achieve this task. In this example, the required ε_j can be found from Riccati equations (57) and (58) so that solutions X and Y to Riccati equations (57) and (58) are positive-definite symmetric matrix and also satisfy Assumption 11. In particular, it is not difficult to find scaling constants ε_j that satisfy the requirement for the example under consideration. For more complicated cases, nonlinear searching methods or optimization methods, (e.g., genetic algorithms or other global optimization methods) may be required.

6. Conclusion

In this paper, we have considered a class of uncertain linear quantum systems subject to quadratic perturbations in the system

Hamiltonian and linear perturbations in the system coupling operator. For this class of uncertain quantum systems, we have built a connection between a coherent robust H^{∞} control problem and a scaled H^{∞} control problem without parameter uncertainties. Then, we used Riccati equations to construct a linear dynamic quantum controller for the given quantum system to make the closed loop system robustly stable as well as to satisfy a prescribed level of disturbance attenuation. We provided optical cavity examples to demonstrate the method we presented in this paper and showed that our method has improved robustness performance over the previous result in James et al. (2008). Moreover, this paper developed a systematic way to describe uncertainties in the Hamiltonian and in the coupling operators. The obtained results can also be extended to systems involving a non-identity scattering matrix. This method can possibly be used in other control performance analysis for uncertain quantum system control problems (e.g., robust LQG control).

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