Robust Control Performance for Open Quantum Systems

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Abstract*—Robust performance of control schemes for open quantum systems is investigated under classical uncertainties in the generators of the dynamics and nonclassical uncertainties due to decoherence and initial state preparation errors. A formalism is developed to measure performance based on the transmission of a dynamic perturbation or initial state preparation error to the quantum state error. This makes it possible to apply tools from classical robust control, such as structured singular value analysis. A difficulty arising from the singularity of the closed-loop Bloch equations for the quantum state is overcome by introducing the #-inversion lemma, a specialized version of the matrix inversion lemma. Under some conditions, this guarantees continuity of the structured singular value at $s = 0$. Additional difficulties occur when symmetry gives rise to multiple open-loop poles, which under symmetry-breaking unfold into single eigenvalues. The concepts are applied to systems subject to pure decoherence and a general dissipative system example of two qubits in a leaky cavity under laser driving fields and spontaneous emission. A nonclassical performance index, steady-state entanglement quantified by the concurrence, a nonlinear function of the system state, is introduced. Simulations confirm a conflict between entanglement, its log-sensitivity and stability margin under decoherence.

***Index Terms*—H-infinity control, quantum information and control, robust control, uncertain systems.**

I. INTRODUCTION

QUANTUM control offers techniques to steer the dynamics of quantum systems. This is essential for enabling a

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wide range of applications for quantum technologies. However, uncertainties arising from limited knowledge of Hamiltonians, decoherence processes and initial state preparation errors impact the effectiveness of the control schemes. While classical robust control has developed effective solutions for such situations that apply relatively easily to quantum optics [1], they do not apply straightforwardly to other areas of quantum control, such as spin systems. To consider the robustness of quantum control strategies in the presence of uncertainties, we develop a formalism, where the performance is measured by the transmission $T_{z,w}(s, \delta)$ from the dynamic perturbation w (including state preparation errors) to the error z on the quantum state when such transmission is subject to structured uncertainties of strength δ. It is tacitly assumed that this response has been made H^∞-small by the control design under nominal values of the parameters in the Hamiltonian and decoherence. Robust performance is, therefore, defined as the ability of $T_{z,w}(\iota\omega, \delta)$ to remain within identifiable bounds for $\delta \neq 0$. Since uncertainties in the Hamiltonians and Lindbladians are often *structured*, it is natural to quantify robustness of the performance using structured singular values. A generic difficulty that arises for quantum systems is that trace conservation of the density matrix ρ imposes a closed-loop pole at $s = 0$ in the $T_{z,w}(s, \delta)$ dynamics. This creates a singularity in the dynamics at low frequencies, $\omega \approx 0$, mandating some revision of the traditional machinery of structured singular values and a special *matrix #-inversion lemma*, similar to, but distinct from the matrix pseudoinversion lemma [2], [3]. Other difficulties addressed by our formalism include multiple poles, either structurally stable like the pole at $s = 0$ or removable by perturbation of physically meaningful parameters. We further demonstrate applicability of the various concepts to two cases that have no classical counterparts: pure dephasing acting in the Hamiltonian basis (i.e., an eigenbasis of the Hamiltonian) and dissipative cavity dynamics. A deeper underlying issue is whether the classical limitation of conflict between performance and sensitivity of performance to uncertainties holds in coherent quantum control and in the presence of decoherence.

After reviewing quantum dynamics in Section II, the general error dynamics with transfer matrix $T_{z,w}$ that should be robust against uncertainties in the Hamiltonian [4] and decoherence [5] is introduced in Section III. Preparation error response requires a different formulation departing from classical robust performance as in [6] and [7]. In Section IV, the case of pure dephasing in an eigenbasis of the Hamiltonian is developed and analytic bounds for the error transmission $T_{z,w}$

are derived. Section V deals with generic dissipative quantum systems and develops a generalized framework to deal with the $s = 0$ singularity. In Section VI, robust performance for generic dissipative dynamics is illustrated by the case study of two qubits in a cavity. This simple example allows the formulation of another novelty in robust control: a nonlinear performance index in the form of the concurrence, a measure of entanglement. We note, however, that the analysis here can also be applied to linear measures, such as fidelity.

II. QUANTUM DYNAMICS AND UNCERTAINTIES

A. Schrödinger and Lindblad Equations

The dynamics of a quantum system whose pure states $|\Psi(t)\rangle$ are wavefunctions in a Hilbert space $\mathbf{H} = \mathbb{C}^N$, are typically described by the Schrödinger equation, $\frac{d}{dt}|\Psi(t)\rangle = -\imath H|\Psi(t)\rangle$ (in a system of units, where the reduced Planck constant $\hbar = 1$), or the Liouville-von Neumann equation for density operators ρ

$$\frac{d}{dt}\rho(t) = -\imath[H, \rho(t)]. \tag{1}$$

Here, H is the Hamiltonian of the system, ρ is a (bounded) Hermitian operator on $\mathbf{H}$ with $\mathrm{Tr}(\rho) = 1$, and $[A, B] = AB - BA$ is the usual matrix commutator. For pure states, the density operator is simply the projector onto $|\Psi\rangle$, i.e., $\rho = |\Psi\rangle\langle\Psi|$. If $\dim \mathbf{H} = N < \infty$, H and ρ can be represented by $N \times N$ Hermitian matrices. The Liouville-von Neumann formulation can easily be extended to describe open system dynamics by adding Lindbladian terms to the right-hand side (RHS) of (1)

$$\mathfrak{L}(V_k)\rho = V_k\rho V_k^\dagger - \frac{1}{2}(V_k^\dagger V_k\rho + \rho V_k^\dagger V_k). \tag{2}$$

This results in the Lindblad master equation

$$\frac{d}{dt}\rho(t) = -\imath[H, \rho(t)] + \sum_k \gamma_k^2\mathfrak{L}(V_k)\rho(t) \tag{3}$$

where the V_k's are the jump operators [8] and the γ_k^2's are the decoherence rates, which can be interpreted as the *strengths* of the *structured* perturbations defined by $\mathfrak{L}(V_k)$.

The most natural formulation of quantum control is through a control-dependent Hamiltonian $H = H[\boldsymbol{f}]$, which, in its simplest form $H[\boldsymbol{f}] = H_0 + \sum_{m=1}^M f_m(t)H_m$, is linear in the controls $f_m(t)$, leading to a bilinear system. The system Hamiltonian is modified by $\boldsymbol{f}$, which can also alter the system-bath interaction, leading, in the simplest case, to a control-dependent Lindbladian

$$\frac{d}{dt}\rho(t) = -\imath[H[\boldsymbol{f}], \rho(t)] + \sum_k \gamma_k^2\mathfrak{L}(V_k[\boldsymbol{f}])\rho(t). \tag{4}$$

For general system-bath interactions, the control $\boldsymbol{f}$ may result in a structurally different master equation [9], [10].

In the rest of the article, we adopt (4) with time-independent control fields f_m.

B. Initial State Preparation Errors Versus Dephasing

When studying quantum dynamics subject to structured perturbations, note that some perturbations may be indistinguishable, e.g., initial state preparation error and decoherence. If we measure the difference between an actual state $\tilde{\rho}$ and a desired state ρ at a certain time, we cannot determine if the error is the result of dynamic dephasing of a perfectly prepared pure initial state, a mixed initial state evolving perfectly according to unitary dynamics, or a combination of both. This is illustrated with a simple example that also serves as a "warm-up exercise."

Example 1: Consider a two-level system with $|0\rangle = [1, 0]^T$ and $|1\rangle = [0, 1]^T$. Assume the system is prepared in a pure state $|\psi_0\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and evolves under the Hamiltonian $H = \omega\sigma_z$ with $\sigma_z = \mathrm{diag}(-1, 1)$, while dephasing $V = \sigma_z$ acts in the Hamiltonian basis ($[H, V] = 0$) at a rate $\delta = \gamma^2$. In this case, (3) gives

$$\rho_0 = \rho(0) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \rho(t) = \frac{1}{2}\begin{bmatrix} e^{2\imath\omega t} & e^{-2\delta t} \\ e^{-2\delta t} & e^{-2\imath\omega t} \end{bmatrix}.$$

Alternatively, a mixed initial state $\tilde{\rho}_0$ evolving under the same Hamiltonian $H = \omega\sigma_z$ without dephasing results in

$$\tilde{\rho}_0 = \tilde{\rho}(0) = \frac{1}{2}\begin{bmatrix} 1 & e^{-\tau\delta} \\ e^{-\tau\delta} & 1 \end{bmatrix}, \quad \tilde{\rho}(t) = \frac{1}{2}\begin{bmatrix} e^{2\imath\omega t} & e^{-\tau\delta} \\ e^{-\tau\delta} & e^{-2\imath\omega t} \end{bmatrix}.$$

Clearly, the two solutions are indistinguishable at $t = \tau/2$.

C. Bloch Equation

To simplify the analysis we reformulate the dynamics (4) as a linear ODE for a real state vector $\boldsymbol{r}$, derived from the classical technique [11]–[14] of expanding ρ and the Hamiltonian with respect to a suitable basis for the operators on the Hilbert space $\mathbf{H}$, such as the generalized Pauli matrices [15]. Let $\{\sigma_n\}_{n=1}^{N^2}$ be an orthonormal basis for the $N \times N$ Hermitian matrices with $\sigma_{N^2} = \frac{1}{\sqrt{N}}I$. Then defining $\boldsymbol{r} = (r_n)_{n=1}^{N^2}$ with $r_n = \mathrm{Tr}(\sigma_n\rho)$ and

$$(\boldsymbol{A}_H[\boldsymbol{f}])_{mn} = Tr(\imath H[\boldsymbol{f}][\sigma_m, \sigma_n]) \tag{5a}$$

$$(\boldsymbol{L}_k[\boldsymbol{f}])_{mn} = Tr\left(V_k^\dagger[\boldsymbol{f}]\sigma_m V_k[\boldsymbol{f}]\sigma_n\right.$$

$$\left. - \frac{1}{2}V_k^\dagger[\boldsymbol{f}]V_k[\boldsymbol{f}]\{\sigma_m, \sigma_n\}\right) \tag{5b}$$

leads to the state-space equation

$$\frac{d}{dt}\boldsymbol{r}(t) = \left(\boldsymbol{A}_H[\boldsymbol{f}] + \gamma^2\boldsymbol{A}_L[\boldsymbol{f}]\right)\boldsymbol{r}(t) \tag{6}$$

where $\gamma^2\boldsymbol{A}_L[\boldsymbol{f}]$ is short for $\sum_k \gamma_k^2\boldsymbol{L}_k[\boldsymbol{f}]$. Observe that $\boldsymbol{A}_H$ is real and antisymmetric, while $\boldsymbol{A}_L$ is symmetric.

From $r_{N^2} = \mathrm{Tr}(N^{-1/2}\rho) = N^{-1/2}$, it follows that $\frac{d}{dt}r_{N^2}(t) \equiv 0$ and the last row of $\boldsymbol{A}_H + \gamma^2\boldsymbol{A}_L$ vanishes for all γ. Thus, in general, $\mathrm{rank}(\boldsymbol{A}_H + \gamma^2\boldsymbol{A}_L) \leq N^2 - 1 \,\forall\gamma$. Moreover, $\mathrm{rank}(\boldsymbol{A}_H) \leq N^2 - 1$ and $\mathrm{rank}(\boldsymbol{A}_L) \leq N^2 - 1$, indicating that this rank deficiency is a consequence of the choice of the basis operator σ_{N^2} and independent of the dynamical generators.

If the linearity of (6) relative to $\boldsymbol{f}$ is preserved, the RHS of the master equation can be rewritten as

$$\frac{d}{dt}\boldsymbol{r}(t) = \left(\boldsymbol{A}_H[0] + \gamma^2\boldsymbol{A}_L[0] + \sum_{m=1}^M f_m(t)\boldsymbol{A}_m\right)\boldsymbol{r}(t) \tag{7}$$

where the $\boldsymbol{A}_m$'s are obtained by adaptation of (5), and

$$\left[\sum_{m=1}^{M} f_m(t)\boldsymbol{A}_m\right]\boldsymbol{r}(t) =: \boldsymbol{u}(\boldsymbol{r}(t),t) \tag{8}$$

can be interpreted as state feedback. Since the f_m are time-independent controls, the resulting system is a linear time-invariant (LTI) system with autonomous state feedback [4]. Note, however, that controlling such systems by choosing some f_m to produce specified eigenvalues for $\boldsymbol{A}_H[0] + \gamma^2 \boldsymbol{A}_L[0] + \sum_m f_m \boldsymbol{A}_m$ does not reduce to the well-known controllability pole placement method [16].

D. Genericity of the Eigenstructure

Quantum systems often have multiple eigenvalues and degeneracies. It is useful to distinguish two cases. *Structurally stable* degeneracies are those that cannot be eliminated by invoking parameter drift. An example of this is the state transition matrix $\boldsymbol{A}_H + \gamma^2 \boldsymbol{A}_L$ of the Bloch equation, which always has at least one zero eigenvalue due to trace conservation. An N-level system subject to pure dephasing in the Hamiltonian basis always has N zero-eigenvalues, regardless of perturbations of the Hamiltonian or dephasing rates. Other degeneracies are *structurally unstable* and can be lifted by perturbations. Consider a quantum system with Hilbert-space dimension N. Let $\{\Pi_k\}_{k=1}^{\bar{N}\leq N}$ be the family of projectors onto the respective eigenspaces of the Hamiltonian associated with the eigenvalues $\lambda_k(H)$ with $\sum_{k=1}^{\bar{N}} n_k = N$. For many systems subject to symmetries, such as the Hamiltonians of chains and rings [17], [18] under uniform J-coupling, the eigenvalues will have multiplicities $n_k = \text{rank}(\Pi_k) > 1$. Such multiplicities can be removed by modifying the J-couplings or other relevant parameters to break the symmetry. In general, three parameters are necessary to split a structurally unstable eigenvalue with multiplicity greater than one [19]. This statement is made rigorous in Appendix A. Conversely, under parameter drift, we cannot rule out eigenvalues of H crossing at a particular time, which raises the question whether a continuous eigenbasis for their respective eigenspaces exists. This question was answered affirmatively by Doležal's theorem and its generalization [20], [21], details of which can be found in Appendix B.

III. ROBUST PERFORMANCE IN OPEN SYSTEMS

Our goal is to quantify robust performance of open quantum systems in the presence of *structured* perturbations. Decoherence can be treated as a structured perturbation for a closed quantum system. Many other uncertainties in the system or control Hamiltonian can also be treated as structured perturbations. For example, we can model uncertainty as H_θ, where H_θ depends affinely on a parameter $\theta \in \Theta \subset \mathbb{R}$, that is, $H_\theta = H_{\theta_0} + (\theta - \theta_0)S = H_{\theta_0} + \delta S$, where θ_0 is the nominal value. Such a decomposition maps, via (5), to the Bloch state transition matrix.

We can quantify closed-loop performance of a quantum system operating under uncertainties by a δ-dependent disturbance-to-error transmission function $\boldsymbol{T}_{\boldsymbol{z},\boldsymbol{w}}(s,\delta)$. Consider an unperturbed and a perturbed version of (7) with state vectors $\boldsymbol{r}_u$ and $\boldsymbol{r}_p$, respectively, evolving according to

$$\frac{d}{dt}\boldsymbol{r}_u(t) = \boldsymbol{A}\boldsymbol{r}_u(t) \tag{9a}$$

$$\frac{d}{dt}\boldsymbol{r}_p(t) = (\boldsymbol{A} + \delta\boldsymbol{S})\boldsymbol{r}_p(t) \tag{9b}$$

where $\boldsymbol{A}$ is the Bloch operator for the ideal unperturbed system and $\delta\boldsymbol{S}$ is a perturbation to the dynamics of structure $\boldsymbol{S}$ and magnitude δ in the Bloch representation. The error vector $\boldsymbol{z}(t) = \boldsymbol{r}_p(t) - \boldsymbol{r}_u(t)$ satisfies either dynamical representation

$$\frac{d}{dt}\boldsymbol{z}(t) = (\boldsymbol{A} + \delta\boldsymbol{S})\boldsymbol{z}(t) + \delta\boldsymbol{S}\boldsymbol{w}_u(t) \tag{10a}$$

$$\frac{d}{dt}\boldsymbol{z}(t) = \boldsymbol{A}\boldsymbol{z}(t) + \delta\boldsymbol{S}\boldsymbol{w}_p(t) \tag{10b}$$

driven by $\boldsymbol{w}_u(t) = \boldsymbol{r}_u(t)$ and $\boldsymbol{w}_p(t) = \boldsymbol{r}_p(t)$, respectively. Projecting $\boldsymbol{z}$ on some desired subspace yields the error in *fidelity* due to uncertainties. Here, we adopt the *unperturbed* formulation of (10a). The perturbed formulation is available in [22].

Taking the Laplace transform of (10a) yields

$$(s\boldsymbol{I} - \boldsymbol{A} - \delta\boldsymbol{S})\hat{\boldsymbol{z}}(s) = \delta\boldsymbol{S}\hat{\boldsymbol{w}}_u(s) + \boldsymbol{z}(0). \tag{11}$$

If $s\boldsymbol{I} - \boldsymbol{A} - \delta\boldsymbol{S}$ is invertible and there is no initial state preparation error, $\boldsymbol{z}(0) = \boldsymbol{0}$, then

$$\hat{\boldsymbol{z}}(s) = \boldsymbol{T}_{\boldsymbol{z},\boldsymbol{w}_u}^u(s,\delta)\hat{\boldsymbol{w}}_u(s) \tag{12}$$

where the transfer matrix $\boldsymbol{T}_{\boldsymbol{z},\boldsymbol{w}_u}^u$ is given by

$$\boldsymbol{T}_{\boldsymbol{z},\boldsymbol{w}_u}^u := (s\boldsymbol{I} - \boldsymbol{A} - \delta\boldsymbol{S})^{-1}\delta\boldsymbol{S}$$
$$= \left[(s\boldsymbol{I}-\boldsymbol{A}-\delta\boldsymbol{S})^{-1}-(s\boldsymbol{I} - \boldsymbol{A})^{-1}\right]\left[(s\boldsymbol{I}-\boldsymbol{A})^{-1}\right]^{-1} \tag{13}$$

scaling the error relative to the known system. If there are initial state preparation errors, i.e., if $\boldsymbol{z}(0) \neq 0$ in (10a), but $s\boldsymbol{I} - \boldsymbol{A} - \delta\boldsymbol{S}$ is invertible then

$$\hat{\boldsymbol{z}}(s) = \boldsymbol{T}_{\boldsymbol{z},\boldsymbol{w}_u}^u(s,\delta)\hat{\boldsymbol{w}}_u(s) + \boldsymbol{T}_{\boldsymbol{z},\boldsymbol{z}_0}^u(s,\delta)\boldsymbol{z}(0) \tag{14}$$

if we set $\boldsymbol{T}_{\boldsymbol{z},\boldsymbol{z}_0}^u(s,\delta) = (s\boldsymbol{I} - \boldsymbol{A} - \delta\boldsymbol{S})^{-1}$ as in [6]. The transfer function quantifies how errors due to imperfect dynamics or initial state preparation [(14)] scale as functions of frequency and what the system's critical frequencies are.

The formulation of (12) also enables structured singular value analysis [6], [7]. If $\boldsymbol{\Phi}(s) := s\boldsymbol{I} - \boldsymbol{A}$ is invertible then

$$(\boldsymbol{\Phi}(s) - \delta\boldsymbol{S})^{-1}\delta\boldsymbol{S} = [\boldsymbol{\Phi}(s)(\boldsymbol{I} - \boldsymbol{\Phi}(s)^{-1}\delta\boldsymbol{S})]^{-1}\delta\boldsymbol{S}$$
$$= (\boldsymbol{I} - \boldsymbol{\Phi}^{-1}(s)\delta\boldsymbol{S})^{-1}\boldsymbol{\Phi}^{-1}(s)\delta\boldsymbol{S}. \tag{15}$$

This reveals that the error response $\boldsymbol{T}_{\boldsymbol{z},\boldsymbol{w}_u}^u$ is obtained from

$$\begin{pmatrix}\hat{\boldsymbol{v}} \\ \hat{\boldsymbol{z}}(s)\end{pmatrix} = \underbrace{\begin{pmatrix}\boldsymbol{\Phi}^{-1}(s)\boldsymbol{S} & \boldsymbol{\Phi}^{-1}(s)\boldsymbol{S} \\ \boldsymbol{I} & \boldsymbol{0}\end{pmatrix}}_{G_{\boldsymbol{z},\boldsymbol{w}_u}(s)}\begin{pmatrix}\hat{\boldsymbol{\eta}} \\ \hat{\boldsymbol{w}}_u(s)\end{pmatrix} \tag{16}$$

with feedback $\hat{\boldsymbol{\eta}} = (\delta\boldsymbol{I})\hat{\boldsymbol{v}}$. Following standard robust control, $\|\boldsymbol{T}_{\boldsymbol{z},\boldsymbol{w}_u}^u(s,\delta)\|$ can be computed as $1/\min\{\|\Delta_f\| : \det(\boldsymbol{I} +$

$T^u_{z,w_u}(s,\delta)\Delta_f)=0\}$, where $w_u=\Delta_f z$ is a *fictitious* feedback wrapped around G_{z,w_u}, and Δ_f is a fully populated complex matrix. Both feedbacks can be combined in a matrix $\Delta=\begin{pmatrix}\delta I & 0 \\ 0 & \Delta_f\end{pmatrix}$, which defines the set $\mathcal{D}$ of block-diagonal matrices $\mathrm{diag}(M_1,M_2)$, where M_1 and M_2 are $N^2\times N^2$ matrices with M_1 real scalar and M_2 fully populated and complex. The structured singular value specific to $\mathcal{D}$,

$$\mu_{\mathcal{D}}(G_{z,w_u}(s))=\frac{1}{\min\{\|\Delta\in\mathcal{D}\|:\det(I+G_{z,w_u}(s)\Delta)=0\}}$$

is a measure of robust performance [7, Th. 10.8].

Theorem 1: If $\Phi(s)$ is invertible then $\|T^u_{z,w}(s,\delta)\|\leq\mu_{\mathcal{D}}(G_{z,w_u}(s))$ for all $\delta<[\mu_{\mathcal{D}}(G_{z,w_u}(s))]^{-1}$.

Similarly, $\hat{z}_{z_0}=T^u_{z,z_0}(s,\delta)z(0)$ can be obtained from

$$\begin{pmatrix}\hat{v}_1 \\ \hat{z}_{z_0}(s)\end{pmatrix}=\underbrace{\begin{pmatrix}S\Phi^{-1}(s) & S\Phi^{-1}(s) \\ \Phi^{-1}(s) & \Phi^{-1}(s)\end{pmatrix}}_{G_{z,z_0}(s)}\begin{pmatrix}\hat{\eta} \\ z_0\end{pmatrix} \quad (17)$$

with feedback $\hat{\eta}=(\delta I)\hat{v}_1$ if $\Phi(s)$ is invertible. Moreover, $\|T^u_{z,z_0}(s,\delta)\|$ is simultaneously computed via the compound feedback Δ, as defined above, and the related robust performance theorem is a straightforward adaptation of Theorem 1.

However, these results are *not* directly useful for quantum systems because the Bloch matrix A of an open quantum system that enforces constancy of the trace always has an eigenvalue at 0, so that $\Phi^{-1}(s)$ has a pole at $s=0$, invalidating Theorem 1. The heuristic approach $s\approx 0$ has been attempted [5], [6], but proper treatment requires a modification of the matrix inversion lemma. Some matrix pseudoinversion lemmas [2], [3] have been proposed but do not apply in this context because their application is restricted to symmetric matrices. To address this issue, a specialized *matrix #-inversion lemma* is proposed in Section V-A and V-B.

IV. Pure Dephasing in the Hamiltonian Basis

In this section we study the performance of controlled quantum systems subject to dephasing, a typically undesired behavior commonly encountered for quantum systems interacting weakly with an environment. In this special case, we can assume that dephasing acts in the eigenbasis of the Hamiltonian, i.e., $[H_\theta,V_\theta]=0$, where θ is a real uncertain parameter affecting the Hamiltonian and hence the decoherence. We treat dephasing as a perturbation to the closed system dynamics, emphasizing the decomposition $A_\theta+\delta S_\theta$, where A_θ is the Bloch matrix corresponding to H_θ and δS_θ is the Bloch matrix corresponding to $\gamma^2\mathfrak{L}(V_\theta)$, where $\delta=\gamma^2$.

A. Pure Dephasing in the Hamiltonian Formulation

By Doležal's theorem in Appendix B, eigenvectors $\{u_k(\theta)\}_{k=1}^N$ of H_θ can be chosen to form an orthonormal basis such that the unitary operator $U_\theta=(u_1(\theta)\cdots u_N(\theta))$ depends continuously on θ. From U_θ, in accordance with Lemma 3.1, we construct

$$V_\theta=U_\theta\,\mathrm{diag}(\lambda_1(V_\theta),\ldots,\lambda_N(V_\theta))U_\theta^\dagger$$

which secures $[H_\theta,V_\theta]=0$. The eigenvalues $\lambda_k(V_\theta)$ can be chosen arbitrarily provided they are real, positive, and their multiplicities are consistent with those of H_θ.

From the eigenvectors $u_k(\theta)$, one can construct a set of projectors $\{\Pi_k(H_\theta)\}_{k=1}^{\bar{N}\leq N}$ onto the (orthogonal) simultaneous eigenspaces of H_θ and V_θ such that $\sum_{k=1}^{\bar{N}}\Pi_k(H_\theta)=I_{\mathbb{C}^N}$ is a resolution of the identity on the full Hilbert space $\mathbf{H}$ and

$$H_\theta=\sum_{k=1}^{\bar{N}}\lambda_k(H_\theta)\Pi_k(H_\theta),\quad V_\theta=\sum_{k=1}^{\bar{N}}\lambda_k(V_\theta)\Pi_k(H_\theta).$$

$\lambda_k(H_\theta)$ and $\lambda_k(V_\theta)$ are the respective real eigenvalues of H_θ and V_θ. $\bar{N}\leq N$ is the number of *distinct* eigenvalues of H_θ. To simplify the notation we now drop the θ-dependency.

Pre/postmultiplying the master equation (3) with Lindblad term (2) by $\Pi_k(H)$ and $\Pi_\ell(H)$, respectively, yields

$$\Pi_k(H)\dot{\rho}(t)\Pi_\ell(H)=(-\imath\omega_{k\ell}+\delta\gamma_{k\ell})\Pi_k(H)\rho(t)\Pi_\ell(H) \quad (18)$$

with $\omega_{k\ell}=\lambda_k(H)-\lambda_\ell(H)$ and $\gamma_{k\ell}=-\frac{1}{2}(\lambda_k(V)-\lambda_\ell(V))^2\leq 0$. The solution to this equation is

$$\Pi_k(H)\rho(t)\Pi_\ell(H)=e^{-t(\imath\omega_{k\ell}-\delta\gamma_{k\ell})}\Pi_k(H)\rho_0\Pi_\ell(H).$$

Since $\sum_{k=1}^{\bar{N}}\Pi_k(H)=I$, the full solution is found as $\rho(t)=\sum_{k,\ell=1}^{\bar{N}}\Pi_k(H)\rho(t)\Pi_\ell(H)$, which yields

$$\rho(t)=\sum_{k,\ell=1}^{\bar{N}}e^{-t(\imath\omega_{k\ell}-\delta\gamma_{k\ell})}\Pi_k(H)\rho_0\Pi_\ell(H). \quad (19)$$

Moreover, from the above it is easily verified that $\Pi_k\rho(t)\Pi_k=\Pi_k\rho_0\Pi_k$ for $k=1,\ldots,\bar{N}$. Therefore, as $\sum_{k=1}^{\bar{N}}\Pi_k(H)=I_{\mathbb{C}^N}$, the solution $\rho(t)$ has $\bar{N}$ invariant subspaces.

B. Pure Dephasing in the Bloch Representation

Lemma 1: Let P and Q be $N\times N$ Hermitian operators with $\mathrm{Tr}(P^\dagger Q)=0$. Then their Bloch vectors $p=(\mathrm{Tr}(P^\dagger\sigma_n))_{n=1}^{N^2}$, $q=(\mathrm{Tr}(Q^\dagger\sigma_n))_{n=1}^{N^2}$ are orthogonal, $p^T q=0$.

Proof: Expand P and Q in terms of the basis $\{\sigma_n\}_{n=1}^{N^2}$ of the set of Hermitian $N\times N$ operators. ∎

Theorem 2: Let $\bar{N}=N$. If $[H,V]=0$ in the quantum master (3), then $[A,S]=0$ in the Bloch equation, and the kernels of A and S coincide and are both N-dimensional.

Proof: See Appendix C. ∎

1) Simultaneous Diagonalization Approach: We first consider the case where H does not depend on θ. Recalling $[H,V]=0$, then by Theorem 2, A and S are $N^2\times N^2$ matrices of rank $\leq N^2-N$ with equality in the generic case. More specifically, by Lemma 3.1, A and S are simultaneously diagonalizable by a complex unitary matrix U

$$U^\dagger AU=\mathrm{diag}(\Omega,0) \quad (20a)$$

$$U^\dagger SU=\mathrm{diag}(\Gamma,0) \quad (20b)$$

where Ω and Γ are diagonal matrices of rank N^2-N in the generic case, with purely imaginary diagonal entries $\imath\omega_{k\neq\ell}=\imath(\lambda_k(H)-\lambda_\ell(H))$ for Ω and purely real, negative diagonal entries $\gamma_{k\neq\ell}=-\frac{1}{2}(\lambda_k(V)-\lambda_\ell(V))^2$ for Γ (see [5]). This allows

us to rewrite (10a) as

$$\boldsymbol{U}^\dagger \dot{\boldsymbol{z}} = \boldsymbol{U}^\dagger (\boldsymbol{A} + \delta \boldsymbol{S}) \boldsymbol{z} + \delta \boldsymbol{U}^\dagger \boldsymbol{S} \boldsymbol{w}_u(t). \tag{21}$$

Setting $\boldsymbol{\zeta} = \boldsymbol{U}^\dagger \boldsymbol{z}$, $\boldsymbol{v}_u = \boldsymbol{U}^\dagger \boldsymbol{w}_u$ we obtain

$$\frac{d}{dt}\boldsymbol{\zeta} = \mathrm{diag}(\boldsymbol{\Omega} + \delta\boldsymbol{\Gamma}, 0)\boldsymbol{\zeta} + \delta\,\mathrm{diag}(\boldsymbol{\Gamma}, 0)\boldsymbol{v}_u. \tag{22}$$

Note that, despite the *real* form of the Bloch equations, $\boldsymbol{\zeta}$ and $\boldsymbol{v}_u$ are complex, as $\boldsymbol{U}^\dagger$ is in general a complex unitary operator, although we could easily define an equivalent real form. Finally, we can partition the vectors $\boldsymbol{\zeta}$ and $\boldsymbol{v}_u$ such that

$$\begin{pmatrix} \boldsymbol{\Omega} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\zeta}_1 \\ \boldsymbol{\zeta}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Omega}\boldsymbol{\zeta}_1 \\ 0 \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Gamma} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{v}_{u,1} \\ \boldsymbol{v}_{u,2} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Gamma}\boldsymbol{v}_{u,1} \\ 0 \end{pmatrix}.$$

We clearly have $\dot{\boldsymbol{\zeta}}_2 = 0$, i.e., $\boldsymbol{\zeta}_2(t)$ is constant. Therefore, the dynamics of the system are completely determined by $\boldsymbol{\zeta}_1(0)$ and the reduced model Bloch equation

$$\dot{\boldsymbol{\zeta}}_1 = (\boldsymbol{\Omega} + \delta\boldsymbol{\Gamma})\boldsymbol{\zeta}_1 + \delta\boldsymbol{\Gamma}\boldsymbol{v}_{u,1} \tag{23}$$

a particular manifestation of (10a). Generally, $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega} + \delta\boldsymbol{\Gamma}$ are invertible, and taking the Laplace transform yields

$$\widehat{\boldsymbol{\zeta}}_1(s) = (s\boldsymbol{I} - \boldsymbol{\Omega} - \delta\boldsymbol{\Gamma})^{-1}\delta\boldsymbol{\Gamma}\hat{\boldsymbol{v}}_{u,1}(s). \tag{24}$$

Therefore, $\hat{\boldsymbol{\zeta}}_1 = \boldsymbol{T}^u_{\boldsymbol{\zeta}_1, \boldsymbol{v}_{u,1}}(s, \delta)\hat{\boldsymbol{v}}_{u,1}$, and the transfer matrix from the disturbance input $\hat{\boldsymbol{v}}_{u,1}(s)$ to the error state $\hat{\boldsymbol{\zeta}}_1(s)$ is

$$\boldsymbol{T}^u_{\boldsymbol{\zeta}_1, \boldsymbol{v}_{u,1}}(s, \delta) = (s\boldsymbol{I} - \boldsymbol{\Omega} - \delta\boldsymbol{\Gamma})^{-1}\delta\boldsymbol{\Gamma}. \tag{25}$$

Taking $\boldsymbol{\Omega} = \mathrm{diag}(\imath\omega_{k\neq\ell})$ and $\boldsymbol{\Gamma} = \mathrm{diag}(\gamma_{k\neq\ell}) < 0$ we obtain

$$\boldsymbol{T}^u_{\boldsymbol{\zeta}_1, \boldsymbol{v}_{u,1}}(\imath\omega, \delta) = \mathrm{diag}((\imath\omega - \imath\omega_{k\ell} - \delta\gamma_{k\ell})^{-1})\,\mathrm{diag}(\delta\gamma_{k\ell})$$

$$= \mathrm{diag}\left(\frac{\delta\gamma_{k\ell}}{\imath\omega - \imath\omega_{k\ell} - \delta\gamma_{k\ell}}\right).$$

Taking the norm to be the largest singular value yields

$$\|\boldsymbol{T}^u_{\boldsymbol{\zeta}_1, \boldsymbol{v}_{u,1}}(\imath\omega, \delta)\|_\infty = \max_{\omega,(k,\ell)}\left|\frac{\delta\gamma_{k\ell}}{\imath(\omega - \omega_{k\ell}) - \delta\gamma_{k\ell}}\right| = 1 \tag{26}$$

where the bound is obtained for $\omega = \omega_{k,\ell}$, i.e., if ω is an eigenfrequency of the system, for all δ, including $\delta \to 0$.

2) Simultaneous Diagonalization Under Errors in the Hamiltonian:
Let $\omega_{k\ell}(\theta) = \lambda_k(H_\theta) - \lambda_\ell(H_\theta)$ and $\gamma_{k\ell}(\theta) = -\frac{1}{2}(\lambda_k(V_\theta) - \lambda_\ell(V_\theta))^2$. Assume the $k \neq \ell$ eigenvalues $-(\imath\omega_{k\ell}(\theta) + \delta\gamma_{k\ell}(\theta))$ of $H_\theta + \delta S_\theta$ have constant multiplicities, and let $\boldsymbol{A}_\theta$ and $\boldsymbol{S}_\theta$ be the Bloch representations of H_θ and $\mathfrak{L}(V_\theta)$, resp. By the proof of Theorems 2, the corresponding eigenvalues of $\boldsymbol{A}_\theta + \delta\boldsymbol{S}_\theta$ do not cross, so we can simultaneously diagonalize $\boldsymbol{A}_\theta$ and $\boldsymbol{S}_\theta$. Hence, as $\boldsymbol{U}_\theta$ is unitary and, under the no-crossing hypothesis, depends continuously on θ, we get, for all θ

$$\boldsymbol{U}_\theta^\dagger(\boldsymbol{A}_\theta + \delta\boldsymbol{S}_\theta)\boldsymbol{U}_\theta = \mathrm{diag}(\boldsymbol{\Omega}_\theta + \delta\boldsymbol{\Gamma}_\theta, 0) \tag{27}$$

where $\boldsymbol{\Omega}_\theta$, $\boldsymbol{\Gamma}_\theta$ display the perturbed eigenfrequencies, dampings, resp., on their diagonals. Proceeding as before, we set

$$\boldsymbol{\zeta}(\theta) = \boldsymbol{U}_\theta^\dagger\boldsymbol{z}(\theta), \quad \boldsymbol{v}(\theta) = \boldsymbol{U}_\theta^\dagger\boldsymbol{w}(\theta) \tag{28}$$

to obtain, in the unperturbed case of (10a),

$$\frac{d}{dt}\boldsymbol{\zeta}(\theta) = \boldsymbol{U}_\theta^\dagger(\boldsymbol{A}_\theta + \delta\boldsymbol{S}_\theta)\boldsymbol{U}_\theta\boldsymbol{U}_\theta^\dagger\boldsymbol{z}(\theta) + \delta\boldsymbol{U}_\theta^\dagger\boldsymbol{S}_\theta\boldsymbol{U}_\theta\boldsymbol{U}_\theta^\dagger\boldsymbol{w}_u(\theta)$$

$$= \mathrm{diag}(\boldsymbol{\Omega}_\theta + \delta\boldsymbol{\Gamma}_\theta, 0)\boldsymbol{\zeta}(\theta) + \mathrm{diag}(\delta\boldsymbol{\Gamma}_\theta, 0)\boldsymbol{v}_u(\theta) \tag{29}$$

under the assumption that the θ-variation is slower than the dynamics as it is common practice in robust control. Equation (29) is now used in lieu of (22). If eigenvalues cross, under analyticity conditions, we can still proceed with block-diagonalization invoking the generalization [21] of Doležal's theorem [20] (see Appendix B).

3) Robust Performance Approach:
Finally, we show that the above method is consistent with, and a simplification of, the robust performance approach of Section III culminating in Theorem 1, but here restricted to $s \neq 0$. Setting $\boldsymbol{\Phi}(s) = s\boldsymbol{I} - \boldsymbol{\Omega}$, (16) becomes

$$G_{\boldsymbol{\zeta}_1, \boldsymbol{v}_{u,1}} = \begin{pmatrix} \boldsymbol{\Phi}^{-1}(s)\boldsymbol{\Gamma} & \boldsymbol{\Phi}^{-1}(s)\boldsymbol{\Gamma} \\ \boldsymbol{I} & 0 \end{pmatrix}, \quad \boldsymbol{\Delta} = \mathrm{diag}(\delta\boldsymbol{I}, \Delta_f).$$

It follows that

$$\det(\boldsymbol{I} + G_{\boldsymbol{\zeta}_1, \boldsymbol{v}_{u,1}}\boldsymbol{\Delta}) = \det[\boldsymbol{I} + \delta\boldsymbol{\Phi}^{-1}(s)\boldsymbol{\Gamma} - \delta\boldsymbol{\Phi}^{-1}(s)\boldsymbol{\Gamma}\Delta_f].$$

Nominally from Theorem 1, Δ_f should be complex and fully populated to capture $\|\boldsymbol{T}^u_{\boldsymbol{\zeta}_1, \boldsymbol{v}_{u,1}}\|$. However, in the open quantum system situation, the following holds.

Lemma 2: If the fictitious feedback Δ_f is diagonal, $\mathrm{diag}\{\Delta_{f_{ii}}\}$, the optimal Δ_f yields $\|\boldsymbol{T}^u_{\boldsymbol{\zeta}_1, \boldsymbol{v}_{u,1}}\| = 1/\|\Delta_f\|$.

Proof: We prove that a diagonal Δ_f captures $\|\boldsymbol{T}^u_{\boldsymbol{\zeta}_1, \boldsymbol{v}_{u,1}}\|$. To simplify the notation, the super/subscripts of $\boldsymbol{T}^u_{\boldsymbol{\zeta}_1, \boldsymbol{v}_{u,1}}$ are dropped and the subscript f of Δ_f is dropped. Consider

$$1/\min_\Delta\{\|\,\mathrm{diag}\{\Delta_{ii}\}\| : \det(I + \mathrm{diag}\{T_{ii}\}\,\mathrm{diag}\{\Delta_{ii}\}) = 0\}. \tag{30}$$

Assume $\min_\Delta$ is achieved for Δ^* and that $\|\,\mathrm{diag}\{\Delta^*_{ii}\}\|$ is achieved for $|\Delta^*_{i_*i_*}|$, where i_*i_* might contain many indices. (It is always possible to take $|\Delta^*_{i_*i_*}|$ strictly dominating all other $|\Delta^*_{ii}|$ unless all $|T_{ii}|$'s are equal in which case the lemma is trivial.) It is claimed that $1 + T_{i_*i_*}\Delta^*_{i_*i_*} = 0$. It suffices to show that $1 + T_{jj}\Delta^*_{jj} \neq 0 \ \forall j \notin i_*$. Assume by contradiction that $1 + T_{jj}\Delta^*_{jj} = 0$ for some $j \notin i_*$. Then

$$\det(\boldsymbol{I} + \mathrm{diag}\{T_{ii}\}\,\mathrm{diag}\{\Delta^*_{jj}\ldots\Delta^*_{jj}\}) = 0$$

while $\|\,\mathrm{diag}\{\Delta^*_{jj}\ldots\Delta^*_{jj}\}\| = |\Delta^*_{jj}| < |\Delta^*_{i_*i_*}|$, which is a contradiction to $\|\Delta^*\|$ achieved for $|\Delta^*_{i_*i_*}|$. From $1 + T_{i_*i_*}\Delta^*_{i_*i_*} = 0$ it follows that $\|T\| = |T_{i_*i_*}| = 1/|\Delta^*_{i_*i_*}|$, which equals (30). ∎

With this diagonal structure, $\det(\boldsymbol{I} + G\boldsymbol{\Delta})$ vanishes if

$$1 + \delta\frac{\gamma_{k\ell}}{s - \imath\omega_{k\ell}} - \delta\frac{\gamma_{k\ell}}{s - \imath\omega_{k\ell}}(\Delta_f)_{k\ell,k\ell} = 0$$

for some $k\ell$ indexing the diagonal. The above, solved for $(\Delta_f)_{k\ell,k\ell}$, gives

$$(\Delta_f)_{k\ell,k\ell} = \frac{s - \imath\omega_{k\ell}}{\delta\gamma_{k\ell}} + 1.$$

This assumes its minimum of 1 for $s = \imath\omega_{k\ell}$ for all $\delta > 0$. Choosing $\delta \leq 1$ yields $\min \|\mathbf{\Delta}\| = 1$. Thus, $\mu_{\mathcal{D}}(G(\imath\omega \neq 0)) = 1$, which is consistent with (26).

V. GENERAL DISSIPATIVE DYNAMICS

We apply and extend the formalism of Section III to the general case of dissipative systems. Contrary to Section IV, the decoherence no longer acts in the Hamiltonian basis, that is, $[H, V] \neq 0$, and the uncertainty manifests itself as decoherence and/or uncertainties in the parameters of the Hamiltonian. The corresponding Bloch operator $\mathbf{A}$ comprises *both* the nominal Hamiltonian and the nominal decoherence dynamics, and all uncertainties are relegated to $\delta \mathbf{S}$. One obstacle to applying robust performance results to this case is the rank deficiency of the $N^2 \times N^2$ Bloch matrix $\mathbf{A} + \delta \mathbf{S}$, which generically has rank $N^2 - 1$. One of the main objectives of this section is to address this difficulty.

Despite its inconvenience, the rank deficiency of $\mathbf{A} + \delta \mathbf{S}$ can be exploited. It is common in physics to define a reduced $(N^2 - 1) \times (N^2 - 1)$ Bloch matrix $\overline{\mathbf{A}} + \delta \overline{\mathbf{S}}$ of full rank. This leads to an inhomogeneous Bloch equation for the reduced Bloch vector $\mathbf{s}$, where the trace component of $\mathbf{r}$ in (6) has been removed

$$\frac{d}{dt}\mathbf{s}(t) = (\overline{\mathbf{A}} + \delta \overline{\mathbf{S}})\mathbf{s}(t) + \mathbf{c}. \tag{31}$$

This equation is useful in some regards. If $\overline{\mathbf{A}} + \delta \overline{\mathbf{S}}$ is invertible (generic case) then the system has a unique steady-state $\mathbf{s}_{\mathrm{ss}} = -(\overline{\mathbf{A}} + \delta \overline{\mathbf{S}})^{-1}\mathbf{c}$, which can be shown to be *globally* asymptotically stable [12]. Therefore, the steady state is independent of the initial state and robust to initial state preparation errors. However, the control is sensitive to uncertainty in the Hamiltonian and dissipative processes.

A. Dynamic Disturbance Transmission

Consider (11) in the generalized uncertainty structure, but with no initial state preparation error. The singularity of $\mathbf{\Phi}(s) := s\mathbf{I} - \mathbf{A}$ at $s = 0$ raised in Section III leads to the question of solvability of the equation

$$(\mathbf{\Phi}(s) - \delta \mathbf{S})\hat{\mathbf{z}}(s) = \delta \mathbf{S}\hat{\mathbf{w}}_u(s) \tag{32}$$

for $\hat{\mathbf{z}}(s)$ when invertibility of $\mathbf{\Phi}(s) - \delta \mathbf{S}$ is not guaranteed. From the Bloch representation we know that the state vector is partitioned as $\mathbf{r} = [\mathbf{r}_1^T, c]^T$, where $\mathbf{r}_1$ is a column vector of length $N^2 - 1$ and c is a constant, the value of which depends on the choice of basis, $c = \mathrm{Tr}[\boldsymbol{\sigma}_{N^2}\rho]$. Since both $\mathbf{r}_u$ and $\mathbf{r}_p$ have this structure, it follows that $\mathbf{z} = [\mathbf{z}_1^T; 0]^T$, where $\mathbf{z}_1$ is a column vector of length $N^2 - 1$. Similarly, the Bloch matrix $\mathbf{A}$ and its structured perturbation $\mathbf{S}$ are of the form

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ 0 & 0 \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ 0 & 0 \end{bmatrix} \tag{33}$$

where $\mathbf{A}_{11}$ and $\mathbf{S}_{11}$ are square matrices of size $N^2 - 1$, and

$$\mathbf{\Phi}(s) = s\mathbf{I} - \mathbf{A} = \begin{bmatrix} \mathbf{\Phi}_{11}(s) & \mathbf{\Phi}_{12} \\ 0 & s \end{bmatrix}$$

with $\mathbf{\Phi}_{11}(s) = s\mathbf{I}' - \mathbf{A}_{11}$, $\mathbf{\Phi}_{12} = -\mathbf{A}_{12}$, and $\mathbf{I}$ and $\mathbf{I}'$ the identity matrix in dimension N^2 and $N^2 - 1$, respectively. Thus (32) can be written as

$$\begin{bmatrix} \mathbf{\Phi}_{11}(s) - \delta \mathbf{S}_{11} & \mathbf{\Phi}_{12} - \delta \mathbf{S}_{12} \\ 0 & s \end{bmatrix} \begin{bmatrix} \hat{\mathbf{z}}_1 \\ 0 \end{bmatrix} = \delta \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{w}}_1 \\ \hat{c} \end{bmatrix}$$

where $\hat{c} = c/s$. The above yields the equation for $\hat{\mathbf{z}}_1$

$$(\mathbf{\Phi}_{11}(s) - \delta \mathbf{S}_{11})\hat{\mathbf{z}}_1 = \delta(\mathbf{S}_{11}\hat{\mathbf{w}}_1 + \hat{c}\mathbf{S}_{12}). \tag{34}$$

This shows that if $\mathbf{\Phi}_{11}(s) - \delta \mathbf{S}_{11}$ is invertible then there is a unique solution $\hat{\mathbf{z}}_1 = (\mathbf{\Phi}_{11}(s) - \delta \mathbf{S}_{11})^{-1}\delta(\mathbf{S}_{11}\hat{\mathbf{w}}_1 + \hat{c}\mathbf{S}_{12})$, which converges to an asymptotically stable steady state. If $\mathbf{\Phi}_{11}(s) - \delta \mathbf{S}_{11}$ is not invertible then there is a manifold of steady states and stability is not guaranteed.

Definition 1: For $\mathbf{A}$ and $\mathbf{S}$ of the form of (33) with $\mathbf{\Phi}_{11}(s) - \delta \mathbf{S}_{11}$ invertible, we define the *#-inverse* as

$$(\mathbf{\Phi}(s) - \delta \mathbf{S})^{\#} = \begin{bmatrix} (\mathbf{\Phi}_{11}(s) - \delta \mathbf{S}_{11})^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

We can verify that this definition ensures that $\hat{\mathbf{z}}$ defined as

$$\hat{\mathbf{z}} := (\mathbf{\Phi}(s) - \delta \mathbf{S})^{\#}\delta \mathbf{S}\hat{\mathbf{w}}_u(s)$$

$$= \begin{bmatrix} (\mathbf{\Phi}_{11}(s) - \delta \mathbf{S}_{11})^{-1}\delta \mathbf{S}_{11} & (\mathbf{\Phi}_{11}(s) - \delta \mathbf{S}_{11})^{-1}\delta \mathbf{S}_{12} \\ 0 & 0 \end{bmatrix}$$

$$\cdot \begin{bmatrix} \hat{\mathbf{w}}_1 \\ \hat{c} \end{bmatrix} = \begin{bmatrix} (\mathbf{\Phi}_{11}(s) - \delta \mathbf{S}_{11})^{-1}\delta(\mathbf{S}_{11}\hat{\mathbf{w}}_1 + \mathbf{S}_{12}\hat{c}) \\ 0 \end{bmatrix} \tag{35}$$

is of the correct form $[\hat{\mathbf{z}}_1^T, 0]^T$ and $\hat{\mathbf{z}}_1$ satisfies (34). With this concept, in lieu of (13), we *define*

$$\mathbf{T}_{\mathbf{z},\mathbf{w}_u}^{u,\#}(s,\delta) = (\mathbf{\Phi}(s) - \delta \mathbf{S})^{\#}\delta \mathbf{S}. \tag{36}$$

Lemma 3: We have, in lieu of (15)

$$(\mathbf{\Phi}(s) - \delta \mathbf{S})^{\#}\delta \mathbf{S} = (\mathbf{I} - \mathbf{\Phi}(s)^{\#}\delta \mathbf{S})^{\#}\mathbf{\Phi}(s)^{\#}\delta \mathbf{S}$$

if $\mathbf{\Phi}_{11}(s)$ is invertible.

Proof: This can be verified by direct computation. The LHS of the equation gives

$$(\mathbf{\Phi} - \delta \mathbf{S})^{\#}\delta \mathbf{S} = \begin{bmatrix} (\mathbf{\Phi}_{11} - \delta \mathbf{S}_{11})^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta \mathbf{S}_{11} & \delta \mathbf{S}_{12} \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (\mathbf{\Phi}_{11} - \delta \mathbf{S}_{11})^{-1}\delta \mathbf{S}_{11} & (\mathbf{\Phi}_{11} - \delta \mathbf{S}_{11})^{-1}\delta \mathbf{S}_{12} \\ 0 & 0 \end{bmatrix}.$$

For the RHS we note that

$$\mathbf{\Phi}^{\#}\delta \mathbf{S} = \begin{bmatrix} \mathbf{\Phi}_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta \mathbf{S}_{11} & \delta \mathbf{S}_{12} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_{11}^{-1}\delta \mathbf{S}_{11} & \mathbf{\Phi}_{11}^{-1}\delta \mathbf{S}_{12} \\ 0 & 0 \end{bmatrix}.$$

Thus, we have

$$\mathbf{I} - \mathbf{\Phi}^{\#}\delta \mathbf{S} = \begin{bmatrix} \mathbf{I}' - \mathbf{\Phi}_{11}^{-1}\delta \mathbf{S}_{11} & -\mathbf{\Phi}_{11}^{-1}\delta \mathbf{S}_{12} \\ 0 & 1 \end{bmatrix}.$$

Furthermore, $(\boldsymbol{I}' - \boldsymbol{\Phi}_{11}^{-1}\delta\boldsymbol{S}_{11})^{-1}\boldsymbol{\Phi}_{11}^{-1} = (\boldsymbol{\Phi}_{11} - \delta\boldsymbol{S}_{11})^{-1}$ as $B^{-1}A^{-1} = (AB)^{-1}$, and, thus,

$$
\begin{aligned}
&(\boldsymbol{I} - \boldsymbol{\Phi}^{\#}\delta\boldsymbol{S})^{\#}\boldsymbol{\Phi}^{\#}\delta\boldsymbol{S} \\
&= \begin{bmatrix} (\boldsymbol{I}' - \boldsymbol{\Phi}_{11}^{-1}\delta\boldsymbol{S}_{11})^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Phi}_{11}^{-1}\delta\boldsymbol{S}_{11} & \boldsymbol{\Phi}_{11}^{-1}\delta\boldsymbol{S}_{12} \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} (\boldsymbol{\Phi}_{11} - \delta\boldsymbol{S}_{11})^{-1}\delta\boldsymbol{S}_{11} & (\boldsymbol{\Phi}_{11} - \delta\boldsymbol{S}_{11})^{-1}\delta\boldsymbol{S}_{12} \\ 0 & 0 \end{bmatrix}
\end{aligned}
$$

which agrees with the LHS. ∎

With the #-inverse concept, we can now rewrite (16) as

$$
G_{\boldsymbol{z},\boldsymbol{w}_u}(s) = \begin{pmatrix} \boldsymbol{\Phi}(s)^{\#}\boldsymbol{S} & \boldsymbol{\Phi}(s)^{\#}\boldsymbol{S} \\ \boldsymbol{I} & 0 \end{pmatrix}. \tag{37}
$$

Along with the feedback $\hat{\boldsymbol{\eta}} = (\delta I)\hat{\boldsymbol{v}}$ this reproduces the #-inverse version (36) of $\boldsymbol{T}_{\boldsymbol{z},\boldsymbol{w}_u}^u(s,\delta)$, which coincides with its classical inverse version of (13)–(15).

Theorem 3: If $\boldsymbol{\Phi}_{11}^{-1}(s)$ exists at $s = 0$, $G_{\boldsymbol{z},\boldsymbol{w}_u}(s)$ is continuous, including at $s = 0$. Moreover, $\mu_{\mathcal{D}}(G_{\boldsymbol{z},\boldsymbol{w}_u}(s))$ is continuous, including at $s = 0$, provided 0 is not a critical value of the s-parameterized mapping $f_s : \boldsymbol{\Delta} \mapsto \det(I + G_{\boldsymbol{z},\boldsymbol{w}_u}(s)\boldsymbol{\Delta})$.

Proof: The first statement is trivial. The proof of the second statement follows the same lines as [23, Th. 23.25]. From the implicit function theorem, it follows that $f_{\omega}^{-1}(0)$ is a differentiable manifold; call it X_{ω}. $1/\mu_{\mathcal{D}}(\imath\omega)$ is the minimum $\|\boldsymbol{\Delta}\|$ such that $\boldsymbol{\Delta} \cap X_{\omega} \neq \emptyset$. Moreover, under variation of ω, X_{ω} is deformed by an isotopy. Hence, $1/\mu_{\mathcal{D}}(\imath\omega)$ and, therefore, $\mu_{\mathcal{D}}(\imath\omega)$ is continuous relative to ω. ∎

Continuity of $G_{\boldsymbol{z},\boldsymbol{w}_u}(s)$ and $\boldsymbol{T}_{\boldsymbol{z},\boldsymbol{w}_u}^u(s,\delta)$ is salvaged by defining the #-inverse to generate (35) as the solution of (32).

B. Initial State Preparation Error Response

As before, the last (N^2th) row of $\boldsymbol{A}$ and $\boldsymbol{S}$ vanishes and $z_{N^2}(0) = 0$ as any prepared state must be represented by a density of trace 1. Therefore, (14) gives $\hat{\boldsymbol{z}}(s) = \boldsymbol{T}_{\boldsymbol{z},\boldsymbol{w}_u}^{u,\#}(s)\hat{\boldsymbol{w}}_u + \boldsymbol{T}_{\boldsymbol{z},\boldsymbol{z}_0}^{u,\#}\boldsymbol{z}(0)$ and $\boldsymbol{T}_{\boldsymbol{z},\boldsymbol{z}_0}^{u,\#}(s) := (\boldsymbol{\Phi} - \delta\boldsymbol{S})^{\#}$.

Lemma 4: Under the same assumptions as in Lemma 3 the following matrix #-inversion lemma holds:

$$
(\boldsymbol{\Phi} - \delta\boldsymbol{S})^{\#} = \boldsymbol{\Phi}^{\#} + \boldsymbol{\Phi}^{\#}\delta \left(\boldsymbol{I} - \boldsymbol{S}\boldsymbol{\Phi}^{\#}\delta\right)^{\#} \boldsymbol{S}\boldsymbol{\Phi}^{\#}. \tag{38}
$$

Proof: We need to show that the matrix inversion lemma holds when we replace the regular inverse by the #-inverse. $\boldsymbol{A}^{\#}\boldsymbol{B}^{\#} = (\boldsymbol{B}\boldsymbol{A})^{\#} = \text{diag}((\boldsymbol{B}_{11}\boldsymbol{A}_{11})^{-1}, 0)$ still holds and

LHS: $(\boldsymbol{\Phi} - \delta\boldsymbol{S})(\boldsymbol{\Phi} - \delta\boldsymbol{S})^{\#} = \boldsymbol{I}'$

RHS: $(\boldsymbol{\Phi} - \delta\boldsymbol{S})\boldsymbol{\Phi}^{\#} + (\boldsymbol{\Phi} - \delta\boldsymbol{S})(\boldsymbol{\Phi} - \delta\boldsymbol{S})^{\#}\delta\boldsymbol{S}\boldsymbol{\Phi}^{\#}$

$\qquad = \boldsymbol{I}' - \delta\boldsymbol{S}\boldsymbol{\Phi}^{\#} + \boldsymbol{I}'\delta\boldsymbol{S}\boldsymbol{\Phi}^{\#} = \boldsymbol{I}'$. ∎

It follows from the lemma that $T_{\boldsymbol{z},\boldsymbol{z}_0}^{u,\#}(s)$ can be represented as

$$
\begin{pmatrix} \hat{\boldsymbol{v}} \\ \hat{\boldsymbol{z}}_{\boldsymbol{z}_0}(s) \end{pmatrix} = \underbrace{\begin{pmatrix} \boldsymbol{S}\boldsymbol{\Phi}(s)^{\#} & \boldsymbol{S}\boldsymbol{\Phi}(s)^{\#} \\ \boldsymbol{\Phi}(s)^{\#} & \boldsymbol{\Phi}(s)^{\#} \end{pmatrix}}_{=:G_{\boldsymbol{z},\boldsymbol{z}_0}^{(\#)}(s)} \begin{pmatrix} \hat{\boldsymbol{\eta}} \\ \boldsymbol{z}_0 \end{pmatrix} \tag{39}
$$

with the feedback $\hat{\boldsymbol{\eta}} = (\delta\boldsymbol{I})\hat{\boldsymbol{v}}$ wrapped around it. Equation (39) provides the substitute for (17) for all s and leads to $\mu_{\mathcal{D}}^{(\#)}$, as shown in Fig. 3(c).

Remark 1: Note that $(\cdot)^{\#}$ is not the Moore–Penrose pseudoinverse since $(\boldsymbol{\Phi} - \delta\boldsymbol{S})^{\#}(\boldsymbol{\Phi} - \delta\boldsymbol{S})$ is not Hermitian.

Remark 2: The *matrix pseudoinversion lemma* [2], [3], $(\boldsymbol{\Phi} - \delta\boldsymbol{S})^{+} = \boldsymbol{\Phi}^{+} + \boldsymbol{\Phi}^{+}\delta(\boldsymbol{I} - \boldsymbol{S}\boldsymbol{\Phi}^{+}\delta)^{+}\boldsymbol{S}\boldsymbol{\Phi}^{+}$, where $(\cdot)^{+}$ is the Moore–Penrose pseudoinverse, is not applicable here as it requires $\boldsymbol{A}$ and $\boldsymbol{S}$ to be Hermitian, nonnegative and satisfy restrictive range conditions.

VI. APPLICATION: TWO QUBITS IN A CAVITY

We apply the method of perturbation-to-error transmission to two two-level atoms in a lossy cavity designed to maximize entanglement generation between the atoms [24], or more broadly between two quantum devices in the quantum Internet [25]. After adiabatic elimination of the cavity via a unitary transformation [24], the dynamics can be described by

$$
\frac{d}{dt}\rho(t) = -\imath[H_{\alpha,\Delta}, \rho(t)] + \sum_k \gamma_k^2 \mathfrak{L}\left(\sigma_-^{(k)}\right)\rho(t). \tag{40}
$$

Denoting the raising operator by $\sigma_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, the lowering operator by $\sigma_- := \sigma_+^{\dagger}$, and defining the operators $\sigma_{\pm}^{(1)} = \sigma_{\pm} \otimes I_{2\times 2}$ and $\sigma_{\pm}^{(2)} = I_{2\times 2} \otimes \sigma_{\pm}$ as usual, the Hamiltonian is

$$
\begin{aligned}
H_{\alpha,\Delta} &= \sum_{n=1}^2 \left(\alpha_n^* \sigma_+^{(n)} + \alpha_n \sigma_-^{(n)} + \Delta_n \sigma_+^{(n)}\sigma_-^{(n)}\right) \\
&= \begin{pmatrix} 0 & \alpha_2 & \alpha_1 & 0 \\ \alpha_2^* & \Delta_2 & 0 & \alpha_1 \\ \alpha_1^* & 0 & \Delta_1 & \alpha_2 \\ 0 & \alpha_1^* & \alpha_2^* & \Delta_1 + \Delta_2 \end{pmatrix}
\end{aligned} \tag{41}
$$

where α_1 and α_2 are the driving fields and Δ_1 and Δ_2 are the detuning parameters. The decoherence in (40) will be generalized by replacing it by the super-operator $\mathfrak{L}(V_{\gamma})$, where

$$
V_{\gamma} = \sum_{n=1}^2 \gamma_n \sigma_-^{(n)} = \begin{pmatrix} 0 & \gamma_2 & \gamma_1 & 0 \\ 0 & 0 & 0 & \gamma_1 \\ 0 & 0 & 0 & \gamma_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{42}
$$

Note that $[H_{\alpha,\Delta}, V_{\gamma}] \neq 0$. To examine the system's robustness, we consider the system dynamics in the Bloch formulation

$$
\frac{d}{dt}\boldsymbol{r}_p(t) = (\boldsymbol{A}_{\alpha,\Delta,\gamma} + \delta\boldsymbol{S}(\alpha_1, \alpha_2; \Delta_1, \Delta_2; \gamma_1, \gamma_2))\boldsymbol{r}_p(t). \tag{43}
$$
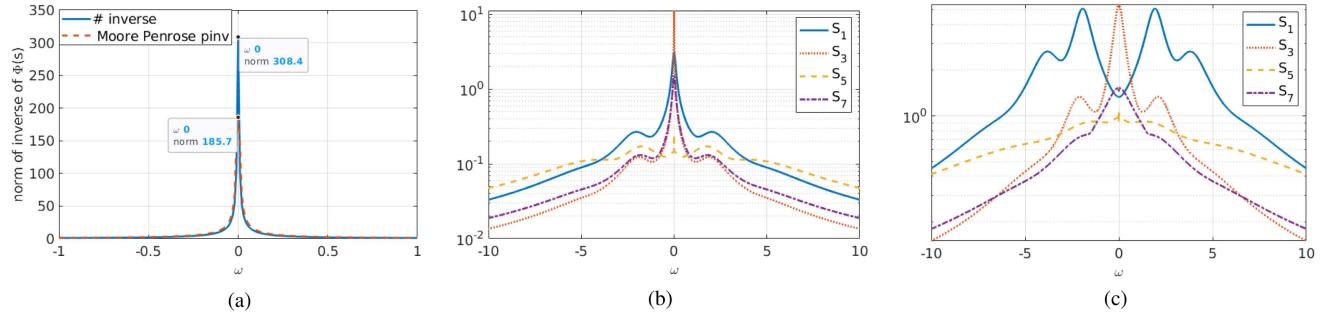
Fig. 1. (a) Comparison of the norm of the inverse of $\boldsymbol{\Phi}(s)$ for the #-inverse and the Moore–Penrose pseudoinverse $((\cdot)^+)$. (b) and (c) Error gains $\|T_{\boldsymbol{z},\boldsymbol{w}_u}^u(\imath\omega,\delta\boldsymbol{S}_k)\|$ as a function of frequency for the structured uncertainties in (45) and different sizes $\delta \in \{0.1, 1\}$. Due to the pairwise similarities $\boldsymbol{S}_1 \sim \boldsymbol{S}_2$, $\boldsymbol{S}_3 \sim \boldsymbol{S}_4$, and $\boldsymbol{S}_6 \sim \boldsymbol{S}_7$, the cases $\boldsymbol{S}_2$, $\boldsymbol{S}_4$, and $\boldsymbol{S}_6$ are not plotted. (a) $\|\boldsymbol{\Phi}(\imath\omega)^\#\|$ vs $\|\boldsymbol{\Phi}^+(\imath\omega)\|$. (b) $\|T_{\boldsymbol{z},\boldsymbol{w}_u}^u(\imath\omega,\delta\boldsymbol{S}_k)\|$ for $\delta = 0.1$. (c) $\|T_{\boldsymbol{z},\boldsymbol{w}_u}^u(\imath\omega,\delta\boldsymbol{S}_k)\|$ for $\delta = 1$.

This is relative to the Pauli basis $\{e_k \otimes e_\ell : k, \ell = 1, \ldots, 4\}$, where $(e_1, e_2, e_3, e_4) = \frac{1}{\sqrt{2}}(I_{2\times2}, \sigma_x, \sigma_y, \sigma_z)$, with

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & \imath \\ -\imath & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (44)$$

The Pauli operators and components are ordered so that $\boldsymbol{z}_{16}$ is the error on $\mathrm{Tr}(\rho)$ and, hence, vanishes. $\boldsymbol{A}_{\alpha,\Delta,\gamma}$ is a real 16×16 matrix whose last row vanishes and whose last column depends exclusively on γ but does not vanish for $\gamma > 0$. It can further be verified that for $\alpha \neq 0$, $\gamma \neq 0$, and $\Delta \neq 0$, the rank of $\boldsymbol{A}_{\alpha,\Delta,\gamma}$ is 15, and the eigenvalues of $\boldsymbol{A}_{\alpha,\Delta,\gamma}$ have negative real parts except for one 0 eigenvalue due to the trace constraint for ρ. Generally, we have stability for nonzero detuning but for $\Delta = 0$ the rank drops to 14, implying the existence of a 1-D subspace of steady states.

$\boldsymbol{A}_{\alpha,\Delta,0}$ corresponds to unitary evolution and hence its eigenvalues are purely imaginary. Generically, its rank is 12, except for special cases, such as $\alpha_1 = \alpha_2$ and $\Delta_1 = -\Delta_2$ when $\mathrm{rank}(\boldsymbol{A}_{\alpha,\Delta,0}) = 10$. Thus the decoherence $\boldsymbol{A}_{0,0,\gamma}$ acts as a stabilizing controller for the plant $\boldsymbol{A}_{\alpha,\Delta,0}$ with state feedback control $\boldsymbol{u}(r(t)) = \boldsymbol{A}_{0,0,\gamma}\boldsymbol{r}(t)$, in accordance with (8).

A. Structured Uncertainties and Frequency Response

The *structure* of the perturbation, $\boldsymbol{S}(\alpha_1, \alpha_2; \Delta_1, \Delta_2; \gamma_1, \gamma_2)$, of (43) is normalized as $\alpha_k = 0, 1$, $\Delta_k = 0, \pm1$, $\gamma_k = 0, 1$. Specifically, we distinguish the following cases:

$$\boldsymbol{S}_1 = \boldsymbol{S}(1,0;0,0;0,0), \quad \boldsymbol{S}_2 = \boldsymbol{S}(0,1;0,0;0,0) \quad (45a)$$

$$\boldsymbol{S}_3 = \boldsymbol{S}(0,0;1,0;0,0), \quad \boldsymbol{S}_4 = \boldsymbol{S}(0,0;0,-1;0,0) \quad (45b)$$

$$\boldsymbol{S}_5 = \boldsymbol{S}(0,0;0,0;1,1), \quad (45c)$$

$$\boldsymbol{S}_6 = \boldsymbol{S}(0,0;0,0;1,0), \quad \boldsymbol{S}_7 = \boldsymbol{S}(0,0;0,0;0,1). \quad (45d)$$

$\boldsymbol{S}_5$ corresponds to *collective* dissipation, while $\boldsymbol{S}_6$ and $\boldsymbol{S}_7$ are structured perturbations corresponding to *single* qubit spontaneous emission. For structures $\boldsymbol{S}_6$ and $\boldsymbol{S}_7$, the sizes are $\delta =$

TABLE I
REAL, FINITE GENERALIZED EIGENVALUES OF THE PAIR $(\boldsymbol{A}, -\boldsymbol{S}_k)$

coefficient matrices	real, finite generalized eigenvalues
$\boldsymbol{A} + \delta\boldsymbol{S}_1$, $\boldsymbol{A} + \delta\boldsymbol{S}_2$	none
$\boldsymbol{A} + \delta\boldsymbol{S}_3$, $\boldsymbol{A} + \delta\boldsymbol{S}_4$	$\delta = -0.2$ (double)
$\boldsymbol{A} + \delta\boldsymbol{S}_5$	$\delta = -1$ (double)
$\boldsymbol{A} + \delta\boldsymbol{S}_6$, $\boldsymbol{A} + \delta\boldsymbol{S}_7$	$\delta = -0.0057, -0.6346, -1.0462, -2.6465$

For our nominal system parameters any value of δ is admissible for $k = 1, \ldots, 4$. To avoid negative decoherence rates we must have $\delta \geq -1$ for $k = 5$ and $\delta \geq 0$ for $k = 6, 7$.

γ_1^2, γ_2^2, resp. Due to the structure of $\boldsymbol{S}_5$ this does not reduce to the case of a single, lossy qubit, but rather a special case of a two-qubit system with coupling mediated by a cavity.

The $\|\boldsymbol{\Phi}^\#(s)\|$ and the error gain $\|T_{\boldsymbol{z},\boldsymbol{w}_u}^u(s, \delta\boldsymbol{S}_k)\|$ for the selected perturbations as functions of frequency $s = \imath\omega$ are shown in Fig. 1 for nominal plant and controller parameters $\alpha_1 = \alpha_2 = 1$, $\gamma_1 = \gamma_2 = 1$, and $\Delta_1 = -\Delta_2 = \Delta = 1$ and different values of δ. Due to symmetry, the effects of $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are the same, and similarly for $\boldsymbol{S}_3$ and $\boldsymbol{S}_4$, and $\boldsymbol{S}_6$ and $\boldsymbol{S}_7$, respectively. Hence, it suffices to consider four perturbations. Except for $\boldsymbol{S}_5$, the bound of (26) for dephasing in the Hamiltonian basis is violated here, as expected, as this system is not a system with dephasing in the Hamiltonian basis. We focus on sensitivity to low-frequencies as we expect low-frequency drift in the detuning and coupling parameters and $1/f$ noise, e.g., due to laser flicker noise in atomic clock systems [26] or magnetic flux noise in superconducting qubits [27].

Plotting $\|\boldsymbol{\Phi}^\#(\imath\omega)\|$ versus frequency in Fig. 1(a) shows it is maximal for $\omega = 0$. The figure also shows that the #-inverse gives different values from the Moore–Penrose inverse at $s = 0$. For nonzero δ, the gain $\|T_{\boldsymbol{z},\boldsymbol{w}_u}^u(\imath\omega, \delta\boldsymbol{S}_k)\|$ need not be maximal at $\omega = 0$, as illustrated in Fig. 1(b) and (c). Although, Fig. 2(b) suggests that for most perturbations the maximum gain is still achieved for $\omega = 0$. The maximum of $\|T_{\boldsymbol{z},\boldsymbol{w}_u}^u(\imath\omega, \delta\boldsymbol{S}_k)\|$ depends on the perturbation $\boldsymbol{S}_k$ and strength δ. Fig. 2(a) suggests that the system is most sensitive to perturbations $\boldsymbol{S}_3$ in the detuning. Other work [24] suggests that robust solutions can
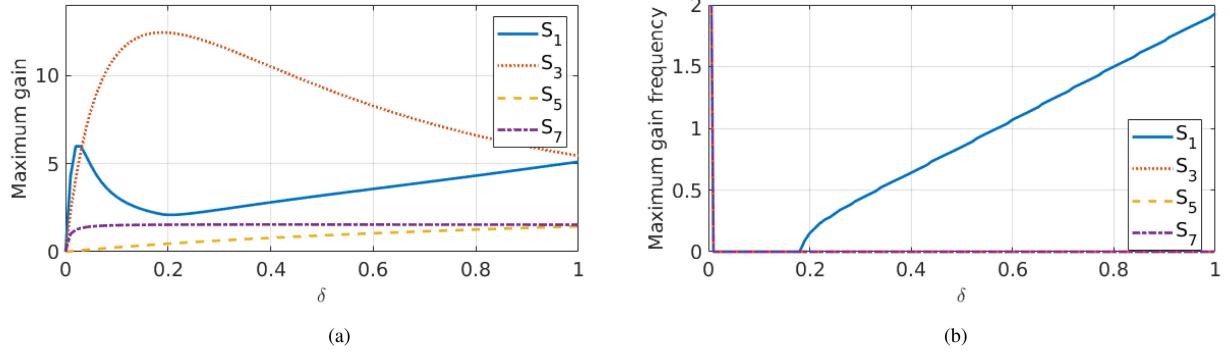
Fig. 2. Maximum gain for the structured uncertainties in (45) suggests that for small δ ($\delta < 0.1$) the system is most sensitive to perturbations of type $\boldsymbol{S}_1$, while for larger δ sensitivity to $\boldsymbol{S}_3$ dominates. (a) Maximum gain $\|\boldsymbol{T}^u_{\boldsymbol{z},\boldsymbol{w}_u}(\imath\omega,\delta\boldsymbol{S})\|_\infty$ (b) Frequency where maximum gain $\|\boldsymbol{T}^u_{\boldsymbol{z},\boldsymbol{w}_u}(\imath\omega,\delta\boldsymbol{S})\|$ is achieved.
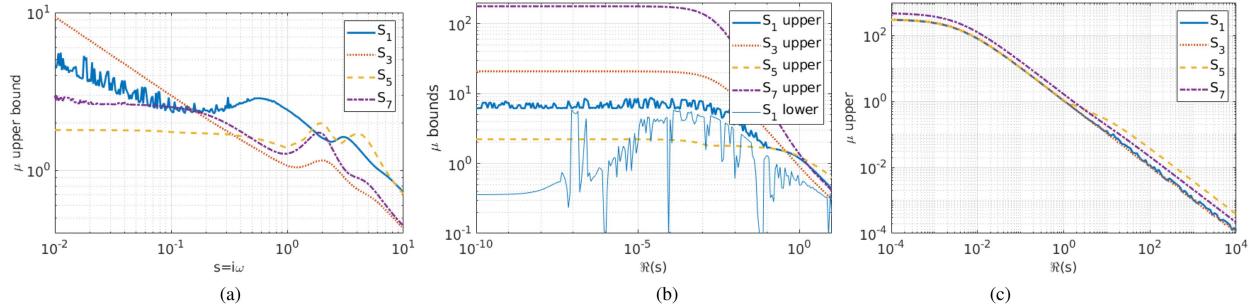
Fig. 3. (a) Upper bounds on the $\mu_\mathcal{D}$ bounding the error transmission $\boldsymbol{T}^u_{\boldsymbol{z},\boldsymbol{w}_u}(\imath\omega,\delta\boldsymbol{S}_k)$ for the structured uncertainties in (45) under frequency sweep $s = \imath\omega$. (b) Upper and lower bounds for $s \downarrow 0$ along the real axis. Upper and lower bounds in (b) coincide for $\boldsymbol{S}_3$, $\boldsymbol{S}_5$, and $\boldsymbol{S}_7$. $\boldsymbol{S}_1$ displays aberrant behavior under frequency and real axis sweep. (c) Upper bounds on $\mu_\mathcal{D}^{(\#)}$ of initial state error transmission $\bar{\boldsymbol{T}}^u_{\boldsymbol{z},\boldsymbol{z}_0}(s,\delta\boldsymbol{S}_k)$ for the structured uncertainties in (45) for $s \downarrow 0$ along the real axis. (a) Frequency sweep. (b) $\Re(s)$ sweep. (c) $\Re(s)$ sweep for initial state error.

be found outside of the regime, where $\Delta_1 = -\Delta_2$, but such investigations are outside of the scope of this work.

B. Bounding Dynamic Perturbation Transmission

Another way to assess robustness against parameter variation is to examine the structured singular value $\mu_\mathcal{D}$ bounding the error transmission as $\|\boldsymbol{T}^u_{\boldsymbol{zw}}(s,\delta\boldsymbol{S})\| \leq \mu_\mathcal{D}(s)$, for $\delta < 1/\mu_\mathcal{D}(s)$ as made precise by Theorem 1. The results of Section V-A apply if the submatrices $\boldsymbol{\Phi}_{11}$ and $\boldsymbol{S}_{11}$ of $\boldsymbol{\Phi}(0)$ and $\boldsymbol{S}$ are invertible. The 16×16 matrices $\boldsymbol{A}_{\alpha,\Delta,\gamma}$ and $\boldsymbol{A}_{\alpha,\Delta,\gamma} + \delta\boldsymbol{S}_k$ of (43), evaluated at $\alpha_1 = \alpha_1 = 1$, $\Delta_1 = -\Delta_2 = 0.1$, and $\gamma_1 = \gamma_1 = 1$, are singular with rank 15. δ-generically $\boldsymbol{S}_k$ has rank 15 for $k \neq 5$, and rank 14 for $k = 5$. The nongeneric δ-values are computed as generalized eigenvalues of the pair $(\boldsymbol{A}, -\boldsymbol{S}_k)$ and are displayed in Table I.

The minimum structured "destabilizing" perturbation $\boldsymbol{\Delta}$ need not be unique as it is easily seen from (33) that $G_{\boldsymbol{z},\boldsymbol{w}_u}(0)$ has one vanishing row and one vanishing column, which causes the solution to $\det(I + G_{\boldsymbol{z},\boldsymbol{w}_u}(0)\boldsymbol{\Delta}) = 0$ to have a completely arbitrary row and column but the size $\|\boldsymbol{\Delta}\|$ is uniquely defined.

Fig. 3(a) shows simulation results for the structured uncertainties $\boldsymbol{S}_1, \boldsymbol{S}_3, \boldsymbol{S}_5$ and $\boldsymbol{S}_7$ as a function of frequency on
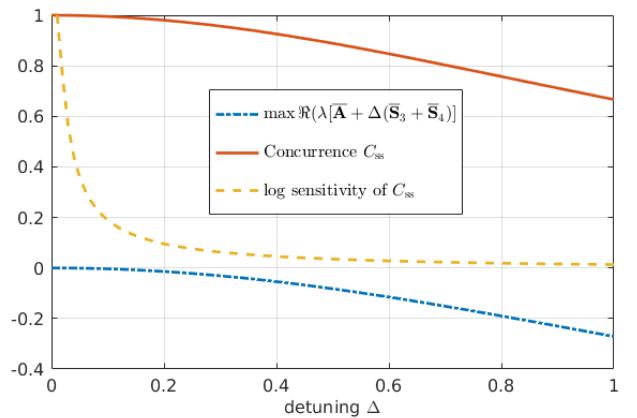
Fig. 4. Maximum of the real part of the eigenvalues λ of $\overline{\boldsymbol{A}} + \Delta(\overline{\boldsymbol{S}}_3 + \overline{\boldsymbol{S}}_4)$, concurrence of steady state and log-sensitivity of steady-state concurrence as function of detuning Δ for $\alpha = \gamma = 1$. All three figures of merit are concordant, i.e., they decrease with increasing detuning.

a frequency scale comparable with that of Fig. 1. Simulation results for the structured singular value μ as s decreases to 0 along the real axis are shown in Fig. 3(b). Except for $\boldsymbol{S}_1$ they show continuity of $\mu_\mathcal{D}$ and the discrepancy between

the upper and lower bounds is very mild (not visible on a log-scale). The sensitivity for S_1 indicates that asymmetric perturbation of the driving fields is detrimental to entanglement generation. This argument is strengthened as the behavior disappears if S_1 is replaced by a symmetric perturbation of the driving fields, $S(1, 1; 0, 0; 0, 0)$ (not shown). Fig. 3(c) shows the upper bounds on $\mu_{\mathcal{D}}^{(\#)}$ for initial state preparation error transmission (Section V-B) for $s \to 0$ along the real axis.

C. Concurrence and Its Log-Sensitivity

Two-qubit entanglement can be measured by the *concurrence* of the two-qubit density operator ρ [28]

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\} \tag{46}$$

where λ_k are the eigenvalues, in decreasing order, of $\sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$ with $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\text{conj}(\rho)(\sigma_y \otimes \sigma_y)$, and σ_x, σ_y, and σ_z are the Pauli operators.

For antisymmetric detuning, $\Delta_1 = -\Delta_2 = \Delta$, symmetric driving, $\alpha_1 = \alpha_2 = \alpha$, symmetric dissipation $\gamma_1 = \gamma_2 = \gamma$, and Δ, α and γ real, i.e., in the case considered here

$$|\Psi_{\text{ss}}\rangle = \left(1/\sqrt{\Delta^2 + 2\alpha^2}\right)[\Delta, \alpha, -\alpha, 0]^T$$

is a steady state of the system: $H_{\alpha,\Delta}|\Psi_{\text{ss}}\rangle = \mathbf{0}$ and $V_\gamma|\Psi_{\text{ss}}\rangle = \mathbf{0}$. Thus, $\rho_{\text{ss}} = |\Psi_{\text{ss}}\rangle\langle\Psi_{\text{ss}}|$ satisfies $\frac{d}{dt}\rho_{\text{ss}} = 0$. The concurrence of this steady state is [29]

$$C_{\text{ss}} := C(\rho_{\text{ss}}) = \left[\tfrac{1}{2}(\Delta/\alpha)^2 + 1\right]^{-1}.$$

Since this steady state is generically globally attractive, any initial state converges to it, and its concurrence determines the performance. To maximize the concurrence, we want Δ/α as small as possible, but in the limit of no detuning, $\Delta \to 0$, the attractivity of the steady state is lost. So there are tradeoffs in the speed of convergence and robustness.

If we measure the performance of the control scheme by the concurrence C_{ss} of the steady state and compare it with the other measures shown in Fig. 4, there is concordance [30] between C_{ss} and its log-sensitivity, i.e., they both decrease with increasing detuning. The concurrence error $1 - C_{\text{ss}}$ increases while its log-sensitivity decreases. This is in agreement with the classical conflict between the sensitivity function S and its log-sensitivity, the complementary sensitivity T, for which $S + T = 1$ holds. The concordance between C_{ss} and the stability margin measured as $|\max_n\{\Re\lambda_n(\overline{A} + \Delta(\overline{S}_3 + \overline{S}_4))\}|$ is also classical as the lower the concurrence performance, the higher the stability margin.

VII. Conclusion

We have developed a robust performance formalism for controlled open quantum systems subject to a variety of structured uncertainties, ranging from uncertain parameters in the Hamiltonian to uncertainties caused by initial state preparation errors and decoherence. The existence of closed-loop poles at 0 in the Bloch equation violates the traditional closed-loop stability requirement. The formalism addresses these issues and allows quantification of the transmission of the dynamic disturbance to quantum state error subject to structured uncertainties, while addressing the continuity of $\mu_{\mathcal{D}}$. Proceeding from the general Lindblad equation gives the formalism wide physical applicability. The cavity case-study reveals that quantum control requires enlarging the concept of a "performance measure" to include novel measures, such as entanglement concurrence, which may have properties unconventional in control, such as being nonlinear in the state. While robustness of a nonlinear performance remains to be developed, the cavity example suggests a tradeoff between the concurrence and its log-sensitivity due to decoherence, as expected in conventional robust control. Earlier results already show that coherent control escapes some of the classical limitations [30], but under decoherence, classicality reemerges [5]. It is an open question in quantum control whether coherent control allows objectives that are traditionally conflicting to coexist in general [4], [30]–[32]. Additionally, the development of more general tools and methods for quantum control that are not derived from existing classical methods requires further work.

Appendix

A. Multiple Eigenvalues

Let $\mathcal{H}(N)$ be the set of $N \times N$ Hermitian matrices.

Definition 1.1: A property $\mathfrak{P}$ of a set of matrices $\mathcal{M}$ is *generic* if the subset of matrices, where it holds is open and dense in $\mathcal{M}$ for a relevant topology on $\mathcal{M}$. A submanifold $\mathfrak{M}$ is $\mathbb{R}^*$-homogeneous if $\mathbb{R}^*\mathfrak{M} = \mathfrak{M}$.

Theorem 1.1 [33, Corollary 4.12]: For $N \geq 2$, the subset $\mathfrak{M}_{n_1,n_2,\ldots,n_{\bar{N}}}$ of $\mathcal{H}(N)$ with eigenvalue multiplicities $n_1, n_2, \ldots, n_{\bar{N}}$ is a $\mathbb{R}^*$-homogeneous submanifold of codimension $(\sum_{k=1}^{\bar{N}} n_k^2) - \bar{N}$ in $\mathcal{H}(N)$. The subset $\mathfrak{V}$ of $\mathcal{H}(N)$ with multiple eigenvalues is a real algebraic variety of codimension 3 in $\mathcal{H}(N)$.

Corollary 1.1: The property "no multiple eigenvalues" is generic in $\mathcal{H}(N)$.

Corollary 1.2: Let H_θ be a family of Hermitian matrices in $\mathcal{H}(N)$ that depends continuously on the real parameter θ. Let $H_0 \in \mathcal{H}(N) \setminus \mathfrak{V}$ be the subset of matrices that has no multiple eigenvalues. Then, generically, a 3-D (real) perturbation $\theta = (\theta_1, \theta_2, \theta_3)$ is necessary to reach multiple eigenvalues. Under nongeneric conditions, more parameters are needed, unless there exists a unique k^* such that $n_{k^*} = 2$ and $n_{k \neq k^*} = 1$, in which case three parameters still suffice.

Proof: Consider $H_{\theta^*} = \arg\min_{H_\theta \in \mathfrak{V}} d(H_\theta, H_0)$, defining a projection $\pi : \mathcal{H}(N) \to \mathfrak{V}$ orthogonal to the stratum of $\mathfrak{V}$ that contains H_0. Assume H_{θ^*} is a differentiable point of $\mathfrak{V}$. Let $\boldsymbol{\sigma}_\mathfrak{V}$ be an orthonormal basis of the tangent space of $\mathfrak{V}$ at H_{θ^*} and complete it to an orthonormal basis $\{e_\mathfrak{V}, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ of $\mathcal{H}(N)$. Then the coordinates of $(H_{\theta^*} - H_0)$ relative to $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ are the three parameters necessary to reach multiple eigenvalues. This situation is generic in $\mathfrak{V}$. Since H_{θ^*} is differentiable

in $\mathfrak{V}$, there exists a neighborhood $\mathcal{N}_{H_{\theta^*}}$, where $\mathfrak{V}$-genericity remains valid. Since π is continuous, $\pi^{-1}(\mathcal{N}_{H_{\theta^*}})$ is a neighborhood of H_0, where the projection is differentiable; hence $\mathfrak{V}$-genericity. If H_{θ^*} is a singular point, it belongs to a manifold $\mathfrak{M}_{n_1,n_2,\ldots,n_{\bar{N}}}$ of the algebraic variety $\mathfrak{V}$. Construct a basis $\boldsymbol{\sigma}_{\mathfrak{M}}$ of the tangent space, and complete it to a basis of $\mathcal{H}(N)$, viz., $\boldsymbol{\sigma}_{\mathfrak{M}}, \boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_{(\sum_{k=1}^{\bar{N}} n_k^2)-\bar{N}}$. Clearly, $(\sum_{k=1}^{\bar{N}} n_k^2) - \bar{N} \geq 3$ parameters are needed, with equality only if a unique $k_* n_{k^*} = 1$; for all other $n_k = 1$ (see [33, p. 162] for details regarding that last inequality). ■

This result can be traced back to [19] but we have clarified the "in general" on [19, p. 553]. The most general singularities in $\mathfrak{V}$ are those of the smallest codimension, i.e., 3. Such an eigenstructure is *in general* unstable under perturbation as eigenvalues with multiplicity >1 split into lower multiplicity eigenvalues under universal unfolding [34]. Securing stability of the eigenstructure of H (and V) requires $\bar{N} = N$, which can be justified invoking *genericity*. See also [33] and [35].

The argument can be reversed to split a multiple eigenvalue into simple eigenvalues under the $\mathfrak{V}$-generic condition if we have three uncertain parameters. The coupled four-qubit system has 16 parameters in its 4×4 Hamiltonian, all of which are uncertain to some degree. If we consider the simplest case of exactly one double eigenvalue, then $\mathrm{codim}\,(\mathfrak{M}_{n_1=2,n_2=n_3=1}) = 3$, i.e., the double eigenvalue can be split with three parameters. Any higher multiplicity structure of the eigenvalues (precisely, $\bar{N} < 3$) would create $\mathrm{codim}\,(\mathfrak{M}_{n_1,n_2,\ldots,n_{\bar{N}}}) > 3$ and more than three parameters would be needed to achieve an arbitrary splitting. Simply put, given the high number of uncertain parameters in quantum systems, the "no multiple eigenvalues" assumption is reasonable.

B. Doležal's Theorem and Its Extension

The 16×16 Bloch state transition matrix $\boldsymbol{A}_\theta$ in the cavity example in Section VI has generic rank 15, dropping for some θs, raising the question of whether the eigendecomposition can be made continuous in $\theta = [\alpha_1, \alpha_2, \Delta, \Delta_2, \gamma_1, \gamma_2]$.

Theorem 2.1 (Doležal [20]): If $\boldsymbol{A}_\theta$ is continuous in θ with constant rank r for a subset $\Theta_0 \subset \Theta$, there exist M_θ, B_θ continuous in θ with M_θ nonsingular such that $\boldsymbol{A}_\theta M_\theta = (B_\theta, 0)$ where B_θ has r columns.

Corollary 2.1: Under the same conditions as in Theorem 1.1, the basis of the null space can be continuous and orthonormal.

Proof: Let $\{e_1, \ldots, e_n\}$ be a basis depending continuously on a parameter θ. Take the first vector e_1, $\|e_1\| = 1$, of the orthonormalized basis and consider $P_{e_1^\perp}\{e_2, \ldots, e_n\}$. Assume the latter $n - 1$ vectors are not independent. Then for some $\alpha_2, \ldots, \alpha_n$ not identically vanishing, we would have $P_{e_1^\perp}(\sum_{i=2}^n \alpha_i e_i) = 0$. Define β_i such that $\beta_i e_1 = (I - P_{e_1^\perp})\alpha_i e_i$. Hence, $P_{e_1^\perp}(\sum_{i=2}^n \alpha_i e_i) + (\sum_{i=2}^n \beta_i)e_1 = \sum_{i=2}^n \alpha_i e_i$, and finally, $-(\sum_{i=2}^n \beta_i)e_1 + \sum_{i=2}^n \alpha_i e_i = 0$, which contradicts the linear independence of $\{e_1, \ldots, e_n\}$. Orthonormalization proceeds by induction on the dimension. Continuity in θ follows from continuity of $P_{e_1(\theta)^\perp}$ on θ. ■

When the rank of A_θ changes, we have the following.

Theorem 2.2 (Silverman and Bucy [21]): If $\boldsymbol{A}_\theta$ is analytic in Θ_ω where $\mathrm{rank}(A_\theta) \leq r$, there exist analytic M_θ, B_θ with M_θ nonsingular such that $\boldsymbol{A}_\theta M_\theta = (B_\theta, 0)$, and B_θ has r columns.

This result does not hold in general without analyticity. Doležal's theorem was originally proved over a real interval but can be extended to multiple parameters by a partition of unity argument [36, p. 72]. Partitions of unity do not exist in the analytic category [36, pp. 174–175] so analytic extension is restricted to a neighborhood of θ_* where a rank changes. Such results can be extended in the continuous case to a basis of the invariant subspace of any eigenvalue by replacing $\boldsymbol{A}_\theta$ by $\boldsymbol{A}_\theta - \lambda_\theta I$, invoking continuity of the eigenvalues. In the analytic case, Weierstrass' preparation theorem [36, Section 6.1], [23, Section D.1] shows λ_θ is analytic along the analytic branches through the multiple eigenvalue with guaranteed analytic basis of the eigenspace.

C. Dephasing in the Hamiltonian Eigenbasis

Lemma 3.1: Let P, Q be Hermitian operators in $\mathbf{H}$ that commute. If the (orthonormal) bases of the eigenspaces of P or Q associated with the multiple eigenvalues are freely adjustable then P and Q are simultaneously diagonalizable by a unitary transformation. If they are constrained then P and Q are only simultaneously *block* diagonalizable via a unitary. The results remain valid if P and Q depend continuously on a parameter θ subject to eigenvalues of constant multiplicities.

Proof: Let $P\mathcal{V}_{\lambda_i} = \lambda_i \mathcal{V}_{\lambda_i}$ and $Q\mathcal{W}_{\mu_j} = \mu_j \mathcal{W}_{\mu_j}$ be the eigenvalues λ_i, μ_j and orthonormalized eigenbases $\mathcal{V}_{\lambda_i}, \mathcal{W}_{\mu_j}$ of P and Q. $PQ = QP$ yields

$$Q\mathcal{V}_{\lambda_i} \subseteq \{\mathcal{V}_{\lambda_i}\}, \quad P\mathcal{W}_{\mu_j} \subseteq \{\mathcal{W}_{\mu_j}\} \qquad (47)$$

where $\{\mathcal{V}_{\lambda_i}\}$ and $\{\mathcal{W}_{\mu_j}\}$ denote the *subspaces* spanned by the (orthonormal) columns of $\mathcal{V}_{\lambda_i}$ and $\mathcal{W}_{\mu_j}$, respectively. It follows from (47) that $\mathcal{W}_{\mu_j}$ is an invariant subspace of P and, therefore, must consist of eigensubspaces of P. To express $\mathcal{W}_{\mu_j}$ in such eigensubspaces of P, choose a set $I(j)$ of i-indices such that $\{\mathcal{W}_{\mu_j}\} \subseteq \oplus_{i \in I(j)}\{\mathcal{V}_i\}$. In each $\{\mathcal{V}_{\lambda_i}\}$, choose a (possibly reduced) basis $\bar{\mathcal{V}}_{\lambda_i}$ such that

$$\{\mathcal{W}_{\mu_j}\} = \oplus_{i \in I(j)}\{\bar{\mathcal{V}}_{\lambda_i}\}. \qquad (48)$$

$\oplus_j\{\mathcal{W}_{\mu_j}\} = \mathbf{H}$ implies $\oplus_j \oplus_{i \in I(j)} \{\bar{\mathcal{V}}_{\lambda_i}\} = \mathbf{H}$. To transform the *subspace* equality to a *basis* equality, let $\boxplus_i \mathcal{V}_{\lambda_i} = (\mathcal{V}_{\lambda_1}, \mathcal{V}_{\lambda_2}, \ldots)$ denote *basis* rather than *subspace* addition.

Case 1: Assume both bases $\boxplus_j \mathcal{W}_{\mu_j}$ and $\boxplus_i \mathcal{V}_{\lambda_i}$ are given. To refine the subspace equality to an equality between bases, choose a rotation R_{μ_j} such that

$$\mathcal{W}_{\mu_j} = \left(\boxplus_{i \in I(j)} \bar{\mathcal{V}}_{\lambda_i}\right) R_{\mu_j}. \qquad (49)$$

Rewriting the eigenvector/eigenvalue equations for P and Q in terms of $\bar{\mathcal{V}}_{\lambda_i}$ using (49) yields

$$P\left(\boxplus_j \boxplus_{i \in I(j)} \bar{\mathcal{V}}_{\lambda_i}\right) = \left(\boxplus_j \boxplus_{i \in I(j)} \bar{\mathcal{V}}_{\lambda_i}\right) \mathrm{diag}\{\Lambda_{I(j)}\}$$

$$Q\left(\boxplus_j \left(\boxplus_{i \in I(j)} \bar{\mathcal{V}}_{\lambda_i} R_{\mu_j}\right)\right) = \left(\boxplus_j \left(\boxplus_{i \in I(j)} \bar{\mathcal{V}}_{\lambda_i} R_{\mu_j}\right)\right) \mathrm{diag}\{\mu_j\}.$$

Moving R_{μ_j} to the RHS of the second equation yields

$$P\left(\boxplus_j \boxplus_{i \in I(j)} \bar{\mathcal{V}}_{\lambda_i}\right) = \left(\boxplus_j \boxplus_{i \in I(j)} \bar{\mathcal{V}}_{\lambda_i}\right) \operatorname{diag}\{\Lambda_{I(j)}\}$$

$$Q\left(\boxplus_j \left(\boxplus_{i \in I(j)} \bar{\mathcal{V}}_{\lambda_i}\right)\right) = \left(\boxplus_j \left(\boxplus_{i \in I(j)} \bar{\mathcal{V}}_{\lambda_i}\right)\right) \operatorname{diag}\{\mu_j R_{\mu_j}^\dagger\}.$$

$\boxplus_j \boxplus_{i \in I(j)} \bar{\mathcal{V}}_{\lambda_i}$ simultaneously *block*-diagonalizes P and Q.

Case 2: By choosing the basis $\mathcal{V}_{\lambda_i}$, (49) is simplified to $\mathcal{W}_{\mu_j} = \boxplus_{i \in I(j)} \bar{\mathcal{V}}_{\lambda_i}$. The rotations are no longer needed and P and Q are simultaneously diagonalizable.

Theorem 1.1 guarantees that the results remain valid if P and Q depend continuously on a parameter θ subject to eigenvalues of constant multiplicities. ∎

Corollary 3.1: Under the same conditions as Lemma 2.1, the kernel of one operator equals the direct sum of selected invariant subspaces of the other. If one such invariant subspace corresponds to an eigenvalue $\neq 0$, the kernels of P and Q are not coincidental.

Proof: Setting $\mu_j = 0$ in (48), $\{\mathcal{W}_{\mu_j = 0}\}$ becomes the kernel of Q and $\ker(Q) = \oplus_{i \in I(j)}\{\bar{\mathcal{V}}_{\lambda_{I(j)}}\}$. Therefore, the $\ker(Q)$ is made up of some invariant subspaces of P. If one such invariant subspace has corresponding eigenvalue $\neq 0$, the two kernels are not coincidental. ∎

Lemma 2.1 in its second form is known, proved via minimal polynomial methods [37] rather than via invariant subspaces.

Proof of Theorem 2: For $\boldsymbol{A}$ and $\boldsymbol{S}$ to commute, we have to show that they have the same eigenspaces. Let us begin with the kernel, the eigenspace of the 0 eigenvalue. Recall that the invariant directions are $\{\Pi_k \rho_0 \Pi_k\}_{k=1}^N$. Therefore, in the Bloch representation, we easily find a basis for the kernel of $\boldsymbol{A} + \delta\boldsymbol{S}$

$$\boldsymbol{u}_{N^2-N+k} = \left(\operatorname{Tr}((\Pi_k \rho_0 \Pi_k)\boldsymbol{\sigma}_n)\right)_{n=1}^{N^2}, \quad k = 1, \ldots, N.$$

This kernel basis does not depend on δ and is the common kernel of $\boldsymbol{A}$ and $\boldsymbol{S}$. We further have the freedom to orthonormalize this basis by Corollary 1.1. Next, for generically nonvanishing eigenvalues, elementary manipulations show that

$$(-\imath \operatorname{Ad}_H + \delta\mathfrak{L}(V))(\Pi_k \rho_0 \Pi_\ell) = (-\imath\omega_{k\ell} + \delta\gamma_{k\ell})(\Pi_k \rho_0 \Pi_\ell)$$

i.e., $(\Pi_k \rho_0 \Pi_\ell)$ is an eigenvector of the RHS of (3) associated with the eigenvalue $-\imath\omega_{k\ell} + \delta\gamma_{k\ell} \neq 0$. Thus, in the Bloch representation the eigenvectors associated with the nonvanishing eigenvalues of $\boldsymbol{A} + \delta\boldsymbol{S}$ are

$$\boldsymbol{u}_{k \neq \ell} = \left(\operatorname{Tr}((\Pi_k \rho_0 \Pi_\ell)\boldsymbol{\sigma}_n)\right)_{n=1}^{N^2}, \quad 1 \leq k \neq \ell \leq N. \quad (50)$$

The eigenvalues remain the same, as can be seen from the commutativity of the following diagram:

$$
\begin{array}{ccc}
\mathbf{Herm} & \xrightarrow{-\imath \operatorname{Ad}_H + \delta\mathfrak{L}(V)} & \mathbf{Herm} \\
\downarrow & & \downarrow \\
\mathbb{R}^{N^2} & \xrightarrow{\boldsymbol{A} + \delta\boldsymbol{S}} & \mathbb{R}^{N^2}
\end{array}
$$

together with the linearity of the Bloch representation ↓. Relabel the eigenvectors in (50) as $\{\boldsymbol{u}_n\}_{n=1}^{N^2-N}$. Lemma 1 shows this set is orthonormal, and together with the kernel defines a unitary matrix $(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{N^2-N}, \boldsymbol{u}_{N^2-N+1}, \ldots, \boldsymbol{u}_{N^2})$ that diagonalizes $\boldsymbol{A} + \delta\boldsymbol{S}$ for all δ. U does not depend on δ and *simultaneously* diagonalizes $\boldsymbol{A}$ and $\boldsymbol{S}$: setting $\delta = 0$ implies that U diagonalizes $\boldsymbol{A}$. U also diagonalizes $\frac{1}{\delta+1}\boldsymbol{A} + \frac{\delta}{1+\delta}\boldsymbol{S}$. Setting $\delta \uparrow \infty$ implies

that $\boldsymbol{U}$ diagonalizes $\boldsymbol{S}$. Thus $\boldsymbol{A}$ and $\boldsymbol{S}$ have the same eigenvectors and $[\boldsymbol{A}, \boldsymbol{S}] = 0$. ∎

REFERENCES

[1] I. R. Petersen, *Robustness Issues in Quantum Control*. London, U.K.: Springer, 2013, pp. 1–7.

[2] K. Kohno, Y. Inouye, and M. Kawamoto, "A matrix pseudo-inverse lemma and its applications to block-based adaptive blind deconvolution for MIMO systems," in *Proc. IEEE Int. Symp. Circuits Syst.*, 2007, pp. 3490–3493.

[3] H. Ogawa, "An operator pseudo-inversion lemma," *SIAM J. Appl. Math.*, vol. 48, no. 6, pp. 1527–1531, 1988.

[4] S. Schirmer, E. Jonckheere, and F. Langbein, "Design of feedback control laws for spintronics networks," *IEEE Trans. IEEE Autom. Control*, vol. 63, no. 8, pp. 2523–2536, Aug. 2018.

[5] S. Schirmer, E. Jonckheere, S. O'Neil, and F. Langbein, "Robustness of energy landscape control for spin networks under decoherence," in *Proc. IEEE Conf. Decis. Control*, 2018, pp. 6608–6613.

[6] E. Jonckheere, S. Schirmer, and F. Langbein, "Structured singular value analysis for spintronics network information transfer control," *IEEE Trans. Autom. Control*, vol. 62, no. 12, pp. 6568–6574, Dec. 2017.

[7] K. Zhou and J. C. Doyle, *Essentials of Robust Control*. Englewood Cliffs, NJ, USA: Prentice Hall, 1998.

[8] A. Carollo, I. Fuentes-Guridi, M. F. Santos, and V. Vedral, "Geometric phase in open systems," *Phys. Rev. Lett.*, vol. 90, no. 16, pp. 160402-1–160402-4, 2003.

[9] D. D'Alessandro, E. Jonckheere, and R. Romano, "Control of open quantum systems in a Bosonic bath," in *Proc. IEEE Conf. Decis. Control*, 2015, pp. 6460–6465.

[10] D. P. D'Alessandro, E. Jonckheere, and R. Romano, "On the control of open quantum systems in the weak coupling limit," in *Proc. Int. Symp. Math. Theory Netw. Syst.*, 2014, pp. 1677–1684.

[11] P. Rooney, A. M. Bloch, and C. Rangan, "Flag-based control of quantum purity for $n = 2$ systems," *Phys. Rev. A*, vol. 93, 2016, Art. no. 063424.

[12] S. G. Schirmer and X. Wang, "Stabilizing open quantum systems by Markovian reservoir engineering," *Phys. Rev. A*, vol. 81, 2010, Art. no. 062306.

[13] S. G. Schirmer, T. Zhang, and J. V. Leahy, "Orbits of quantum states and geometry of Bloch vectors for N-level systems," *J. Phys. A*, vol. 37, no. 4, 2004, Art. no. 1389.

[14] F. F. Floether, P. de Fouquieres, and S. Schirmer, "Robust quantum gates for open systems via optimal control: Markovian versus non-Markovian dynamics," *New J. Phys.*, vol. 14, 2012, Art. no. 073023.

[15] R. A. Bertlmann and P. Kramer, "Bloch vectors for qudits," *J. Phys. A: Math. Theor.*, vol. 41, no. 23, 2012, Art. no. 235303.

[16] R. Luesink and H. Nijmeijer, "On the stabilization of bilinear systems via constant feedback," *Linear Algebra Appl.*, vol. 122/124, pp. 457–474, 1989.

[17] E. Jonckheere, S. Schirmer, and F. Langbein, "Quantum networks: The anti-core of spin chains," *Quant. Inf. Process.*, vol. 13, pp. 1607–1637, 2014.

[18] E. Jonckheere, S. Schirmer, and F. Langbein, "Information transfer fidelity in spin networks and ring-based quantum routers," *Quant. Inf. Process.*, vol. 14, no. 10, pp. 4751–4785, 2015.

[19] J. von Neumann and E. Wigner, "Über das Verhalten von Eigenwerten bei adiabatischen Prozessen," *Phys. Zschr.*, vol. 30, pp. 467–470, 1929.

[20] V. Doležal, "The existence of a continuous basis of a certain linear subspace of E_r which depends on a parameter," *Casopis pro Pěstování Matematiky*, vol. 89, pp. 466–469, 1964.

[21] L. Silverman and R. Bucy, "Generalizations of a theorem of Doležal," *Math. Syst. Theory*, vol. 4, pp. 334–339, 1970.

[22] S. G. Schirmer, F. C. langbein, C. A. Weidner, and E. A. Jonckheere, "Robustness of quantum systems subject to decoherence: Structured singular value analysis?," in *Proc. IEEE Conf. Decis. Control*, Austin, TX, USA, 2021, pp. 4158–4163.

[23] E. A. Jonckheere, *Algebraic and Differential Topology of Robust Stability*. Oxford, U.K.: Oxford Univ. Press, 1997.

[24] F. Motzoi, E. Halperin, X. Wang, K. B. Whaley, and S. Schirmer, "Backaction-driven, robust, steady-state long-distance qubit entanglement over lossy channels," *Phys. Rev. A*, vol. 94, Sep. 2016, Art. no. 032313.

[25] R. V. Meter, *Quantum Networking*. Hoboken, NJ, USA: ISTE Wiley, 2014.

[26] E. Oelker *et al.*, "Demonstration of 4.8×10–17 stability at 1/s for two independent optical clocks," *Nature Photon.*, vol. 13, pp. 714–719, 2019.

[27] P. Kumar *et al.*, "Origin and reduction of 1/f magnetic flux noise in superconducting devices," *Phys. Rev. Appl.*, vol. 6, 2016, Art. no. 041001.

[28] W. Wootters, "Entanglement of formation and concurrence," *Quantum Inf. Comput.*, vol. 1, pp. 27–44, 2001.

[29] X. Wang, A. Bayat, S. Bose, and S. Schirmer, "Global control methods for Greenberger–Horne–Zeilinger-state generation on a one-dimensional Ising chain," *Phys. Rev. A*, vol. 82, 2010, Art. no. 012330.

[30] E. Jonckheere, S. Schirmer, and F. Langbein, "Jonckheere–Terpstra test for nonclassical error versus log-sensitivity relationship of quantum spin network controllers," *Int. J. Robust Nonlinear Control*, vol. 28, no. 6, pp. 2383–2403, 2018.

[31] E. Jonckheere, S. Schirmer, and F. Langbein, "Effect of quantum mechanical global phase factor on error versus sensitivity limitation in quantum routing," in *Proc. IEEE Conf. Decis. Control*, 2019, pp. 1139–1344.

[32] S. O'Neil, E. Jonckheere, S. Schirmer, and F. Langbein, "Sensitivity and robustness of quantum rings to parameter uncertainty," in *Proc. IEEE Conf. Decis. Control*, 2017, pp. 6137–6143.

[33] E. Gutkin, E. A. Jonckheere, and M. Karow, "Convexity of the joint numerical range: Topological and differential geometric viewpoints," *Linear Algebra Appl.*, vol. 376C, pp. 143–171, 2003.

[34] D. P. L. Castrigiano and S. A. Hayes, *Catastrophe Theory*. Boston, MA, USA: Addison-Wesley, 1993.

[35] E. Jonckheere, A. T. Rezakhani, and F. Ahmad, "Differential topology of adiabatically controlled quantum processes," *Quant. Inf. Process.*, vol. 12, no. 3, pp. 1515–1538, 2013.

[36] S. G. Krantz and H. R. Parks, *A Primer of Real Analytic Functions*, 2nd ed. Cambridge, MA, USA: Birkhäuser Adv. Texts, 2002.

[37] K. Conrad, "The minimal polynomial and some applications," 2008. [Online]. Available: https://kconrad.math.uconn.edu/blurbs/linmultialg/minpolyandappns.pdf

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