Why is a contrast 'orthogonal'?

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1. Vector product

Say we have a vector x = $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ then $x^T x$ is simply the sum square of its elements i.e. $x^T x = 13$.

Interestingly, if we project x into a 2D space, the distance between the origin (0,0) and the point (2,3) is equal to sqrt(2^2+3^2) = sqrt(13) (Pythagoras' theorem) i.e. sqrt($x^T x$) – this distance is called the norm of the vector. The same can be applied to matrices in which one works on many dimensions.

Suppose now another vector y = $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$ then $x^T y = y^T x = 5*2 + 6*3 = 28$

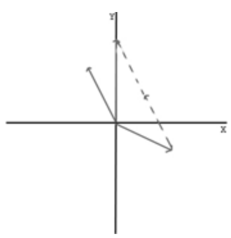
In terms of geometry, if we trace these the two lines (0,0 ; 2,3) and (0,0 ; 5,6), we found that
Cos θ = $x^T y$ / sqrt(($x^T x$)*($y^T y$))

2. Contrasts in a linear system

If we take a simple linear system such as 2x - y = 0 and -x + 2y = 3

We can rewrite it as $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

And we search for the coefficients x and y such as the linear combination of $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ gives $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$



The answer is 1 time $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ + 2 times $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ = $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$

Now, for a more complex linear system with many ys and xs, we may want to combine the coefficients in a certain way – for instance such as the sum of the coefficient = 0 - then y is a contrast in the xs.

$y_1 = x_{11}c_{11} + x_{12}c_{12} + \ldots + x_{1n}c_{1n}$
$y_2 = x_{21}c_{21} + x_{22}c_{22} + \ldots + x_{2n}c_{2n}$
$y_m = x_{m1}c_{m1} + x_{m2}c_{m2} + \ldots + x_{mn}c_{mn}$

$Y = X_{m,n}\, c_{m,1}$

3. 'Orthogonal' contrast

For $y_m = x_{m1}c_{m1} + x_{m2}c_{m2} + \ldots + x_{mn}c_{mn}$, we can have, say, two linear combinations of coefficients c like $y_k = c_k^T x$ and $y_j = c_j^T x$. Remember that the sum of coefficients for a contrast = 0. But if in addition $c_j^{T\,*} c_k = 0$ then the two contrasts are orthogonal (and if their norm is 1, they are said orthonormal).

Geometrically speaking, we have two vectors y_k and y_j such as corr(y_k, y_j) = $c_j^T c_k$ / sqrt(($c_j^T c_j$)*(c_k^T c_k)). If $c_j^T c_k$ = 0, corr = θ = 90° i.e. y_k and y_j are uncorrelated.