Lecture 1 – Feb. 2012

# General linear model: theory of linear model & basic applications in statistics

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# Overview

- Linearity
- Linear models
- Linear algebra
- Regressions as linear models
- 1 way ANOVA

# What is linearity?

# Linearity

- Means created by lines
- In maths it refers to equations or functions that satisfy 2 properties: additivity (also called superposition) and homogeneity of degree 1 (also called scaling)
- Additivity → $y = x1 + x2$ (output y is the sum of inputs xs)
- Scaling → $y = \beta\ x1$ (output y is proportional to input x)

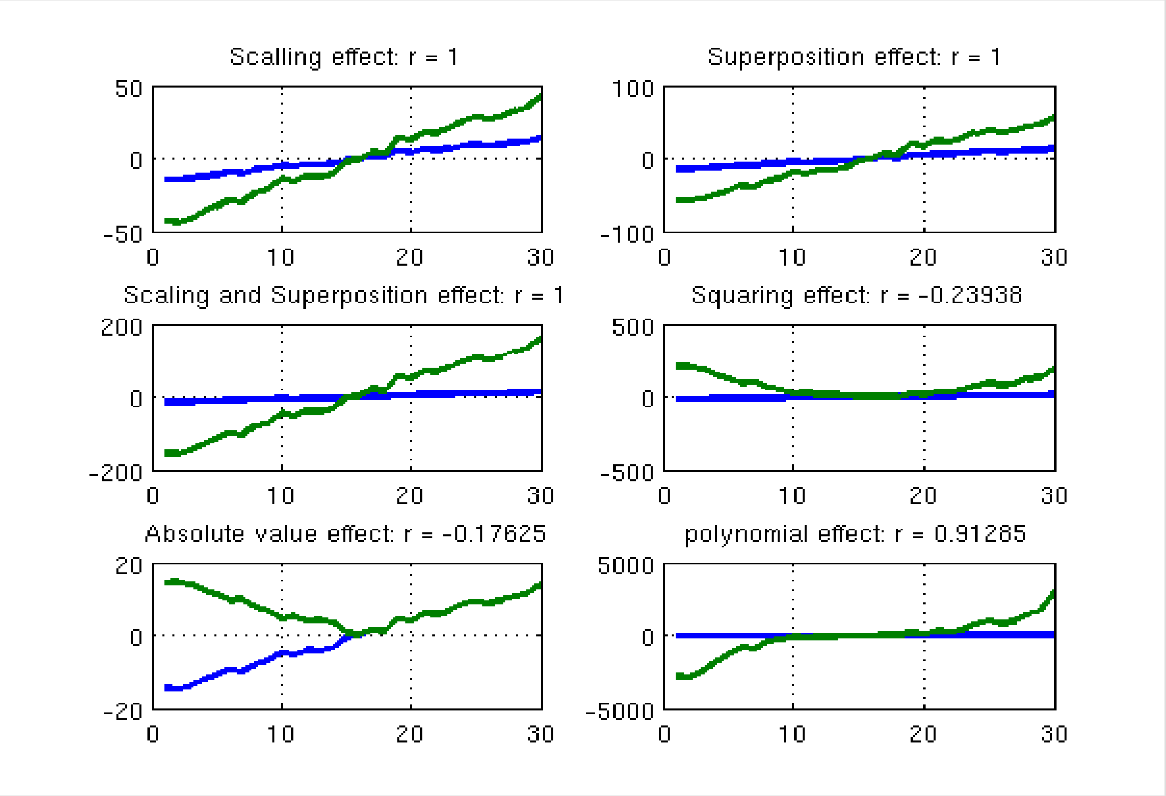
http://en.wikipedia.org/wiki/Linear

# Examples of linearity – non linearity

<u>Matlab exercise 1.</u>

x1 = [-15:14]' + randn(30,1);

try linear/non-linear effects on x1 and compute the correlations



# What is a linear model?

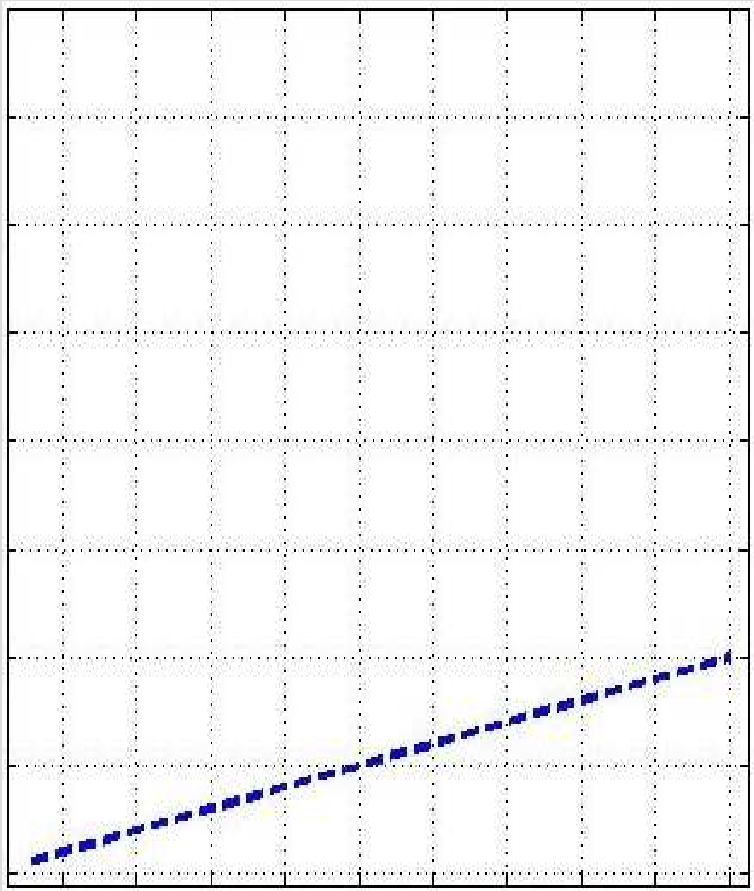
# What is a linear model?

- An equation or a set of equations that models data and which corresponds geometrically to straight lines, plans, hyperplans and satisfy the properties of additivity and scaling.

- Simple regression: $y = \beta 1x+\beta 2+\varepsilon$
- Multiple regression: $y = \beta 1x1+\beta 2x2+\beta 3+\varepsilon$
- One way ANOVA: $y = u+\alpha i+\varepsilon$
- Repeated measure ANOVA: $y=u+\alpha i+\varepsilon$
- ...

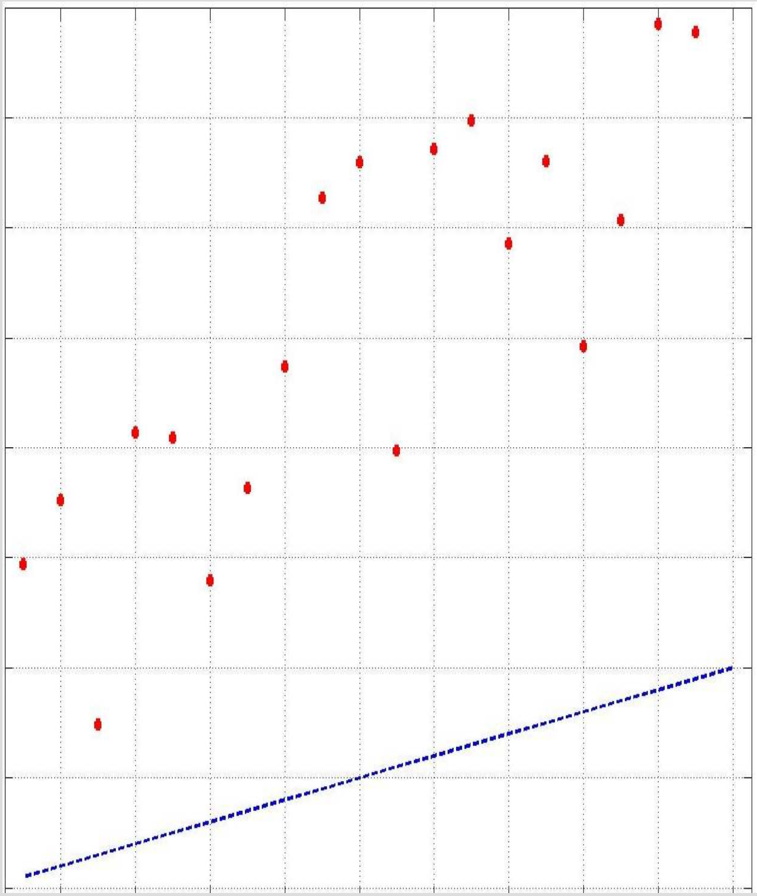
# A regression is a linear model

- We have an experimental measure x (e.g. stimulus intensity from 0 to 20)



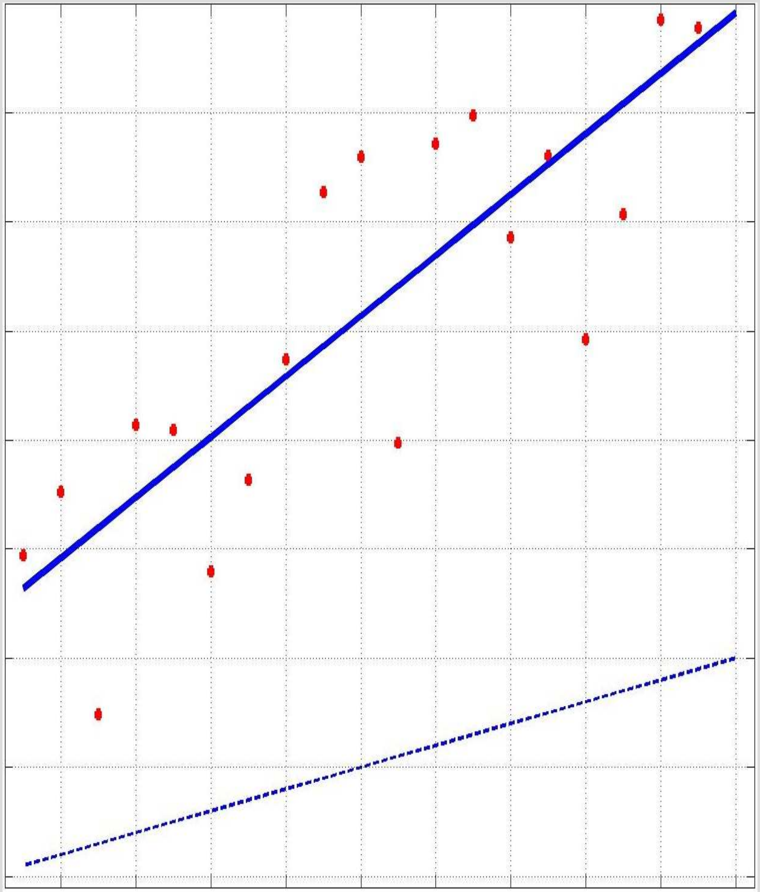
# A regression is a linear model

- We have an experimental measure x (e.g. stimulus intensity from 0 to 20)
- We then do the expe and collect data y (e.g. RTs)



# A regression is a linear model

- We have an experimental measure x (e.g. stimulus intensity from 0 to 20)
- We then do the expe and collect data y (e.g. RTs)
- Model: $y = \beta 1 x + \beta 2$
- Do some maths / run a software to find $\beta 1$ and $\beta 2$
- $y\hat{} = 2.7x + 23.6$



# A regression is a linear model

- The error is the distance between the data and the model
- F = (SSeffect / df) / (SSerror / df_error)

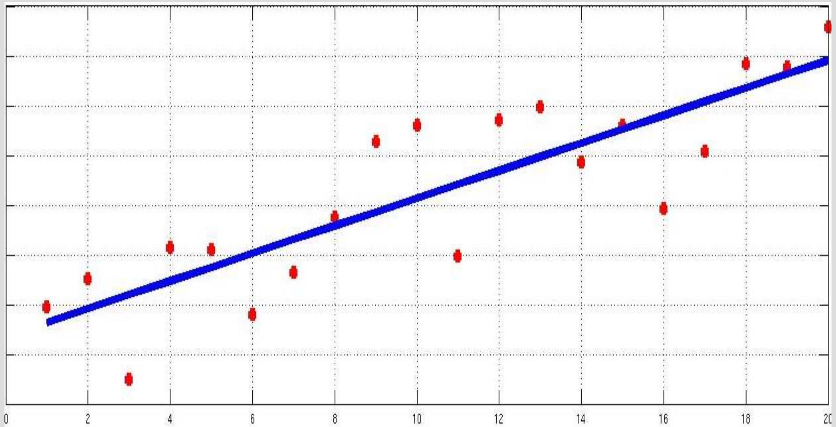
$$F = \frac{(SSeffect / df)}{(SSerror / df\_error)}$$

- <u>SSeffect</u> =

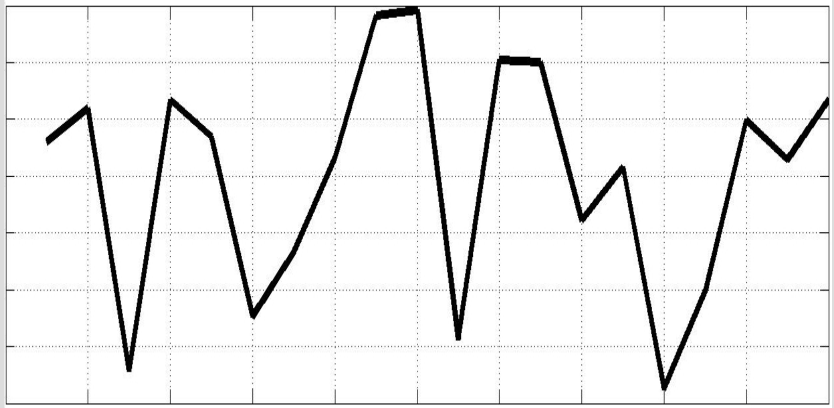
Sum(Yi^ - mean(Y^).^2);

- <u>SSerror</u> =

Sum(Ei^ - mean(E).^2);



Error = Y – Xβ



# Linear algebra

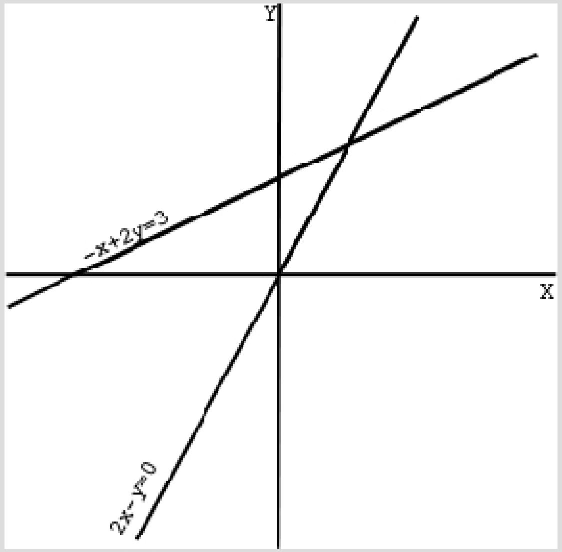
Change from rows to columns

# Linear algebra

- Linear algebra has to do with solving linear systems, i.e. a set of linear equations
- For instance we have observations (y) for a stimulus characterized by its properties $x_1$ and $x_2$ such as $y = x_1 \beta_1 + x_2 \beta 2$

$$2\beta 1 - \beta 2 = 0$$
$$-\beta 1 + 2\beta 2 = 3$$

$$\beta 1 = 1 \; ; \; \beta 2 = 2$$



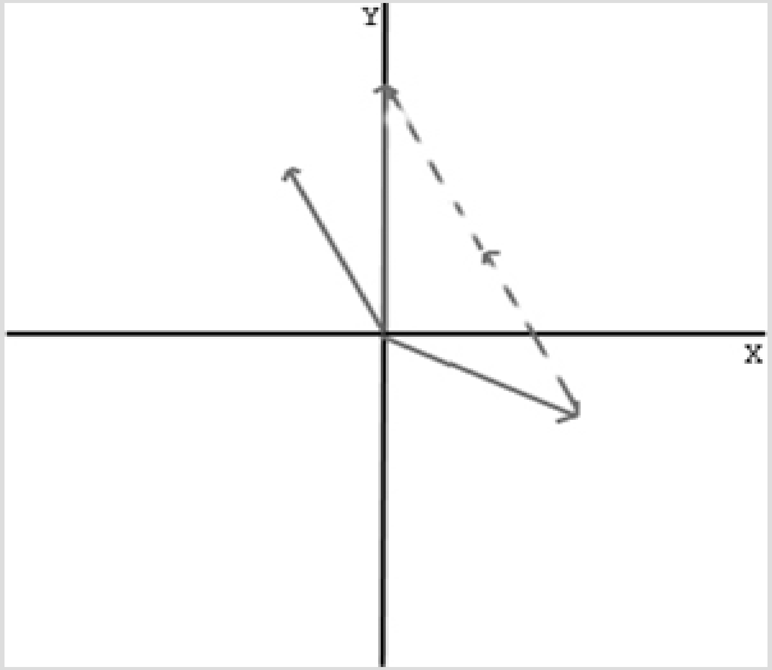
# Linear algebra

- With matrices, we change the perspective and try to combine columns instead of rows, i.e. we look for the coefficients with allow the linear combination of vectors

$$2\beta 1 - \beta 2 = 0$$
$$-\beta 1 + 2\beta 2 = 3$$

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} [\beta 1 \beta 2] = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\beta 1 = 1 \; ; \; \beta 2 = 2$$



# Linear algebra

- A simple solution to linear equation / system is to multiple by the inverse of x ($x\beta = y \rightarrow \beta = 1/x*y$)
- e.g. $3\beta 1 = 15 \text{ --> } 1/3*3\beta 1 = 1/3*15 \text{ --> } \beta 1 = 5$
- We multiply each side by 1/3; 1/3 is the inverse of 3 because 1/3 * 3 = 1

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} [\beta 1 \beta 2] = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

X B = Y

# Linear algebra

- XB = Y --> inv(X)XB = inv(X)Y --> B = inv(X)Y
- Just as 1/3 is the inverse of 3, we can define a matrix A being the inverse of X such as AX=I with I being the identity matrix (=1 on diagonal and zeros everywhere else)

$$inv\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0.66 & 0.33 \\ 0.33 & 0.66 \end{bmatrix} \longrightarrow inv\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} * \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\left[\beta 1 \beta 2\right] = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \longrightarrow inv\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} * \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

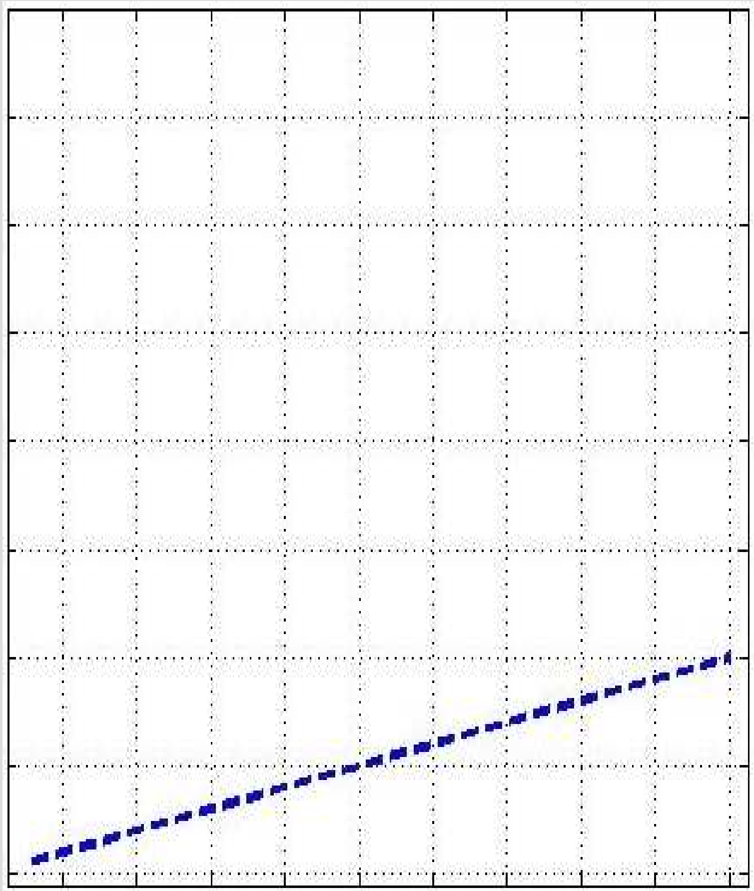
Matlab exercise 2. Linear algebra

# Regression analyses

Solving using linear algebra

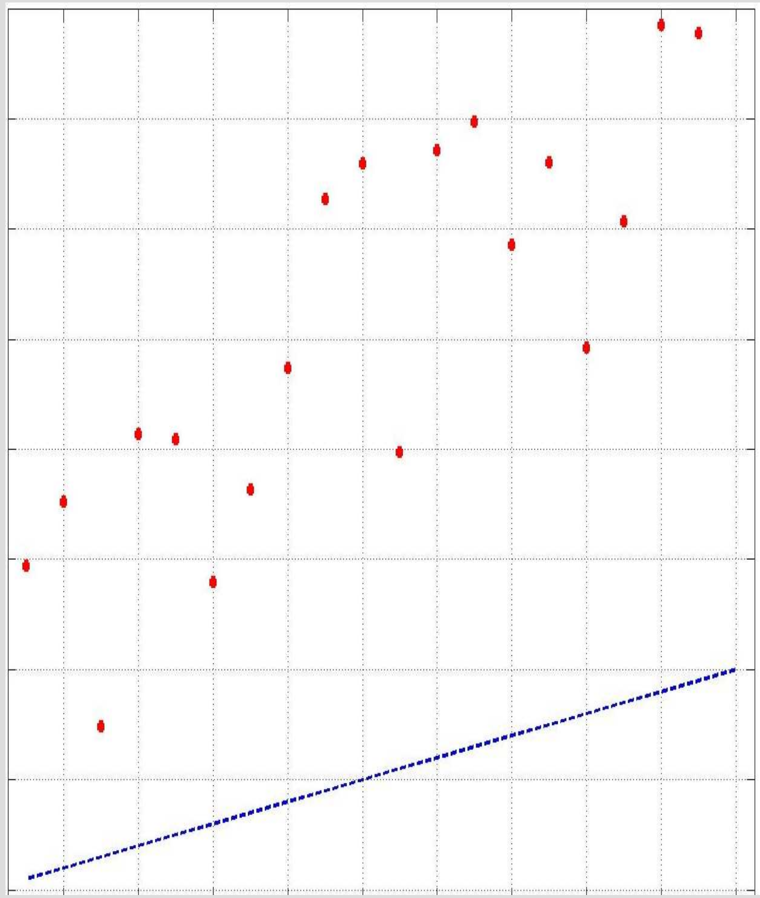
# Simple regression

- We have an experimental measure x (e.g. stimulus intensity from 1 to 20)



# Simple regression

- We have an experimental measure x (e.g. stimulus intensity from 1 to 20)
- We then do the expe and collect data y (e.g. RTs)



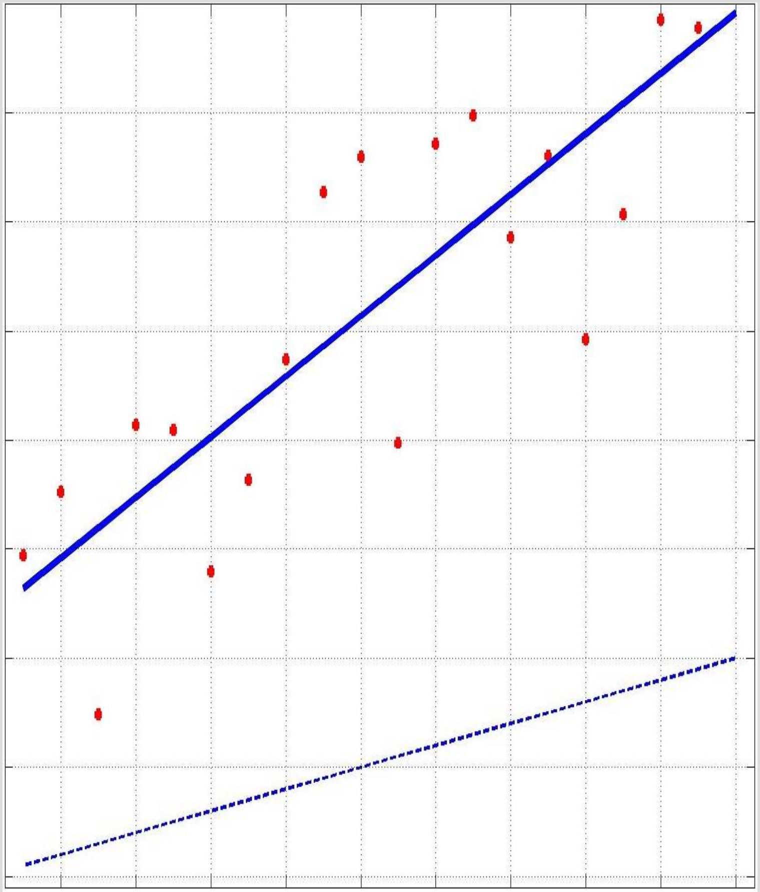
# Simple regression

- Model: Y = XB
- X=[[1:20]' ones(20,1)];

→ Note the ones = intercept

- B=inv(X'X)X'Y --> y^ = XB^

We use inv(X'X)X'Y rather than inv(X)Y because we need a square matrix X
Y = XB → X'Y = X'XB → inv(X'X) X'Y = inv(X'X) X'XB since inv(X)X=I this is the same inv(X'X) X'X = I



# Simple regression

$Y = XB \rightarrow X'Y = X'XB \rightarrow inv(X'X)X'Y = B$

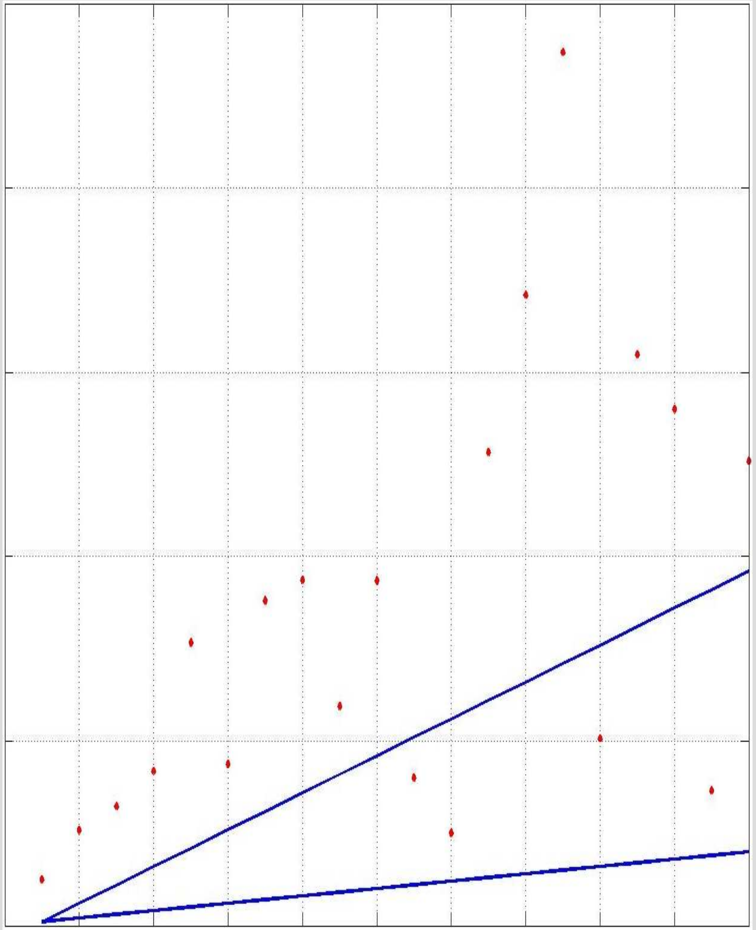
Matlab exercise 3:

```
x = rand(20,1)*5;
y = 3*x+25+rand(20,1)*4
```

--> find B using the inv operator
--> F = (SS effect * df) / (SS error * dfe)
--> SS effect = sum((Yhat – mean(Y)).^2)
--> find Yhat, SS error (hint: compute error first), df, dfe, F
--> find p value (see fcdf function)

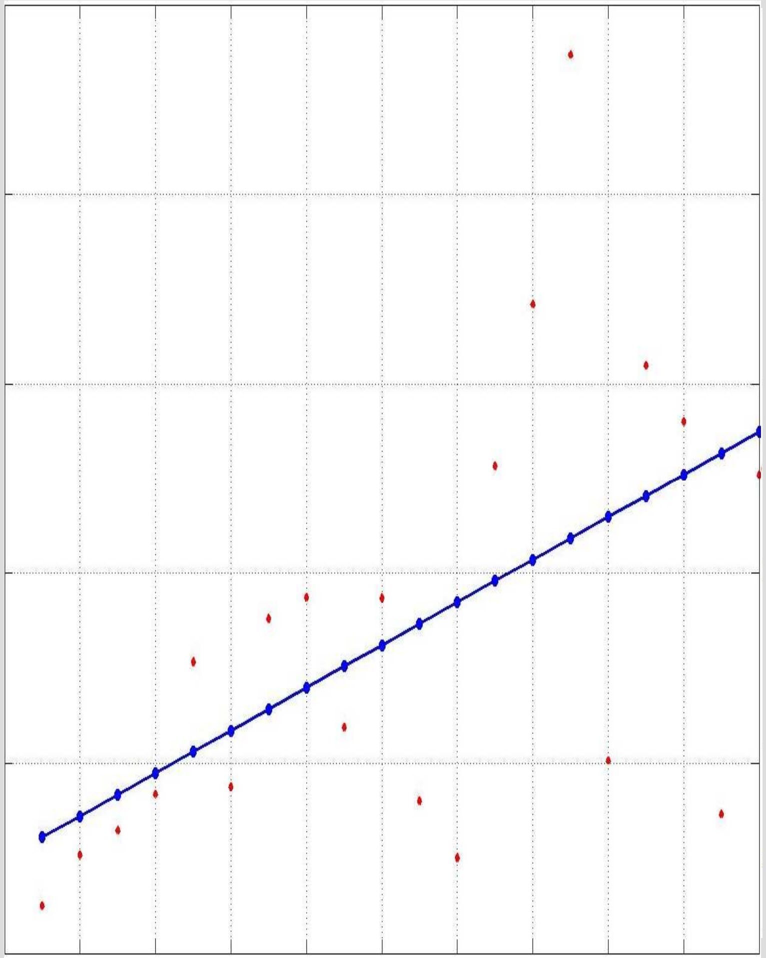
# Multiple regression

- Now we have several experimental measures X (e.g. stimulus intensity from 1 to 20 and stimulus duration from 1 to 96) and still one data set Y (e.g. RTs)



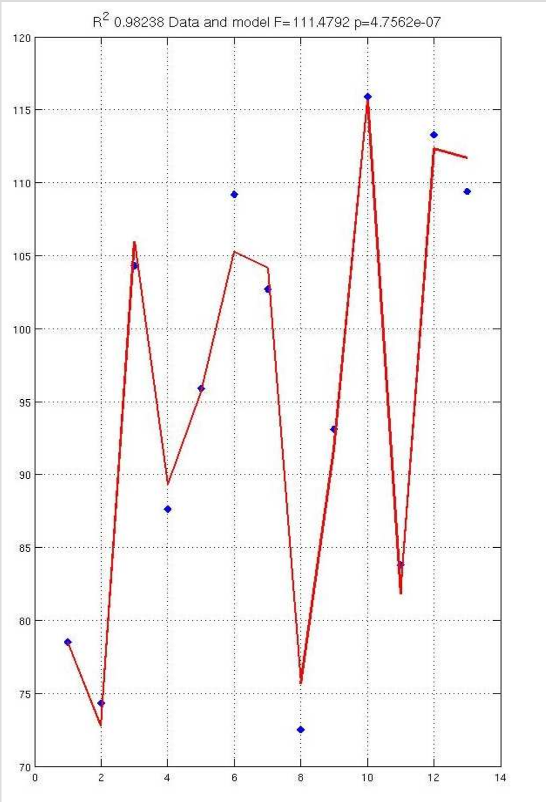
# Multiple regression

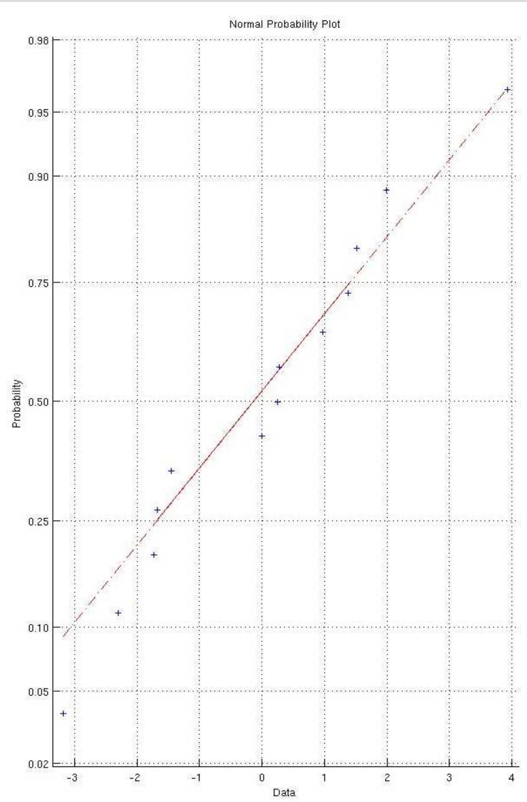
- Now we have several experimental measures X (e.g. stimulus intensity from 1 to 20 and stimulus duration from 1 to 96) and still one data set Y (e.g. RTs)
- Model: Y = XB
- X=[[1:20;1:5:96]' ones(20,1)];
- B=inv(X'X)X'Y --> yhat = XB^



# Multiple regression

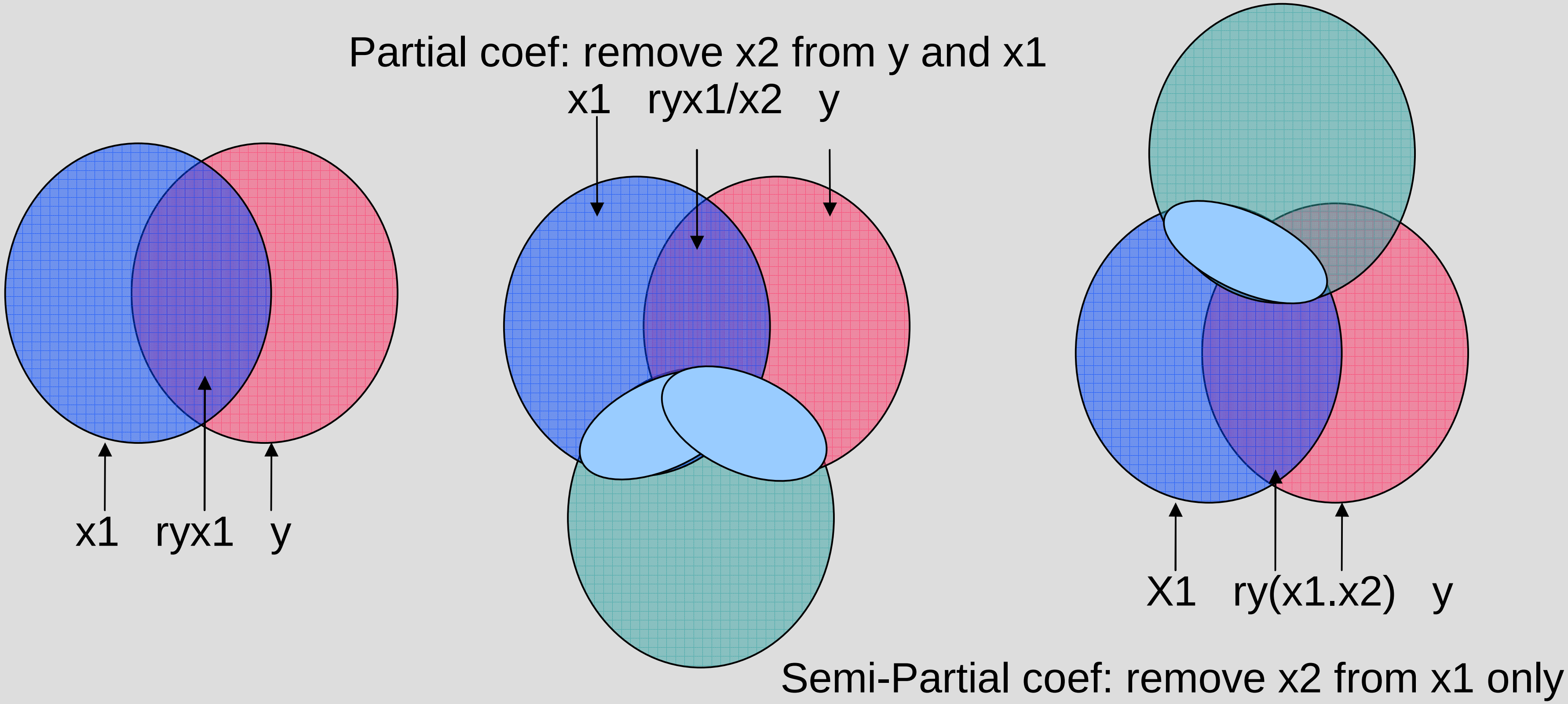
- Matlab exercise 4: load hald, Y=hald(:,5); X=hald(:,1:4);
- Conpute B, Yhat, Res, R2, F and p as before



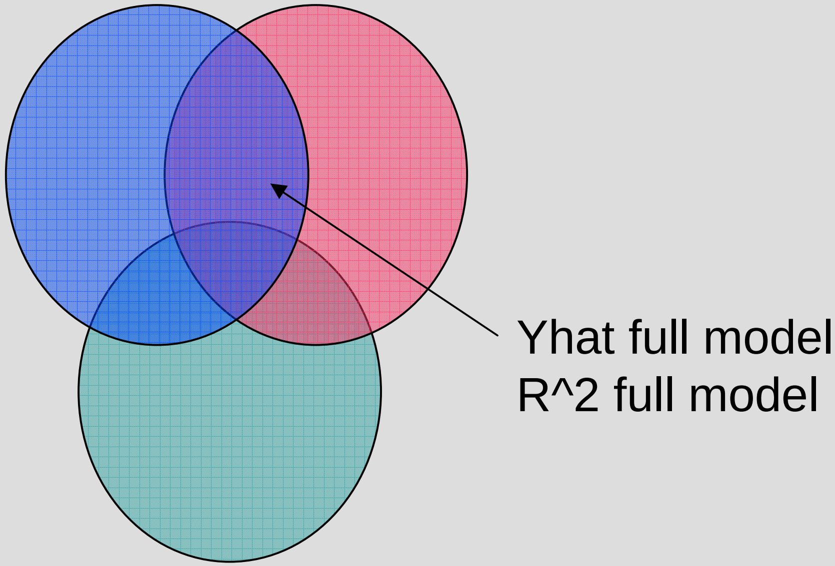
# Multiple regression

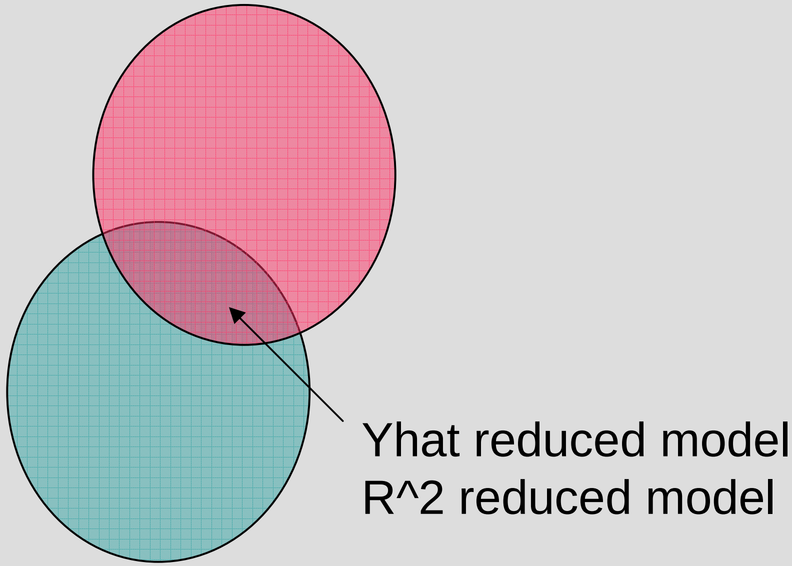
- What is the contribution of each regressor to the model (partial correlation coefficient) and to the data (semi-partial correlation coefficient)



# Multiple regression

- Semi-Partial correlations



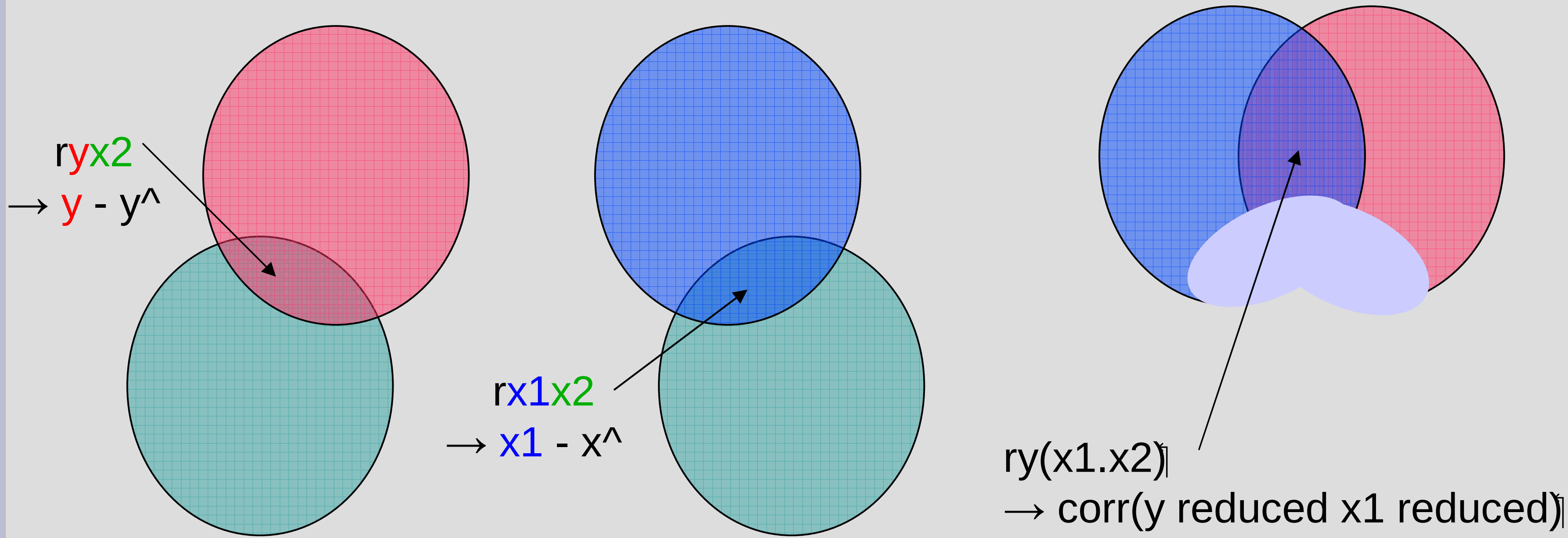




Semi-partial coef = R^2 full – R^2 reduced

Matlab exercise 5: compute Semi-partial coef for x1 (ie X(:,1))

# Multiple regression

- Partial correlations




<u>Matlab exercise 6</u>: compute partial coef for x1 (ie X(:,1))

# 1 way ANOVA

# 1 way ANOVA

- In text books we have $y = u + xi + \varepsilon$, that is to say the data (e.g. RT) = a constant term (grand mean u) + the effect of a treatment (xi) and the error term ($\varepsilon$)
- In a regression xi takes several values like e.g. [1:20]
- In an ANOVA xi is designed to represent groups

# 1 way ANOVA

| Y | Gp |
|---|---|
| 8 | 1 |
| 9 | 1 |
| 7 | 1 |
| 5 | 2 |
| 7 | 2 |
| 3 | 2 |
| 3 | 3 |
| 4 | 3 |
| 1 | 3 |
| 6 | 4 |
| 4 | 4 |
| 9 | 4 |

y(1..3)1= 1x1+0x2+0x3+0x4+c+e11
y(1..3)2= 0x1+1x2+0x3+0x4+c+e12
y(1..3)3= 0x1+0x2+1x3+0x4+c+e13
y(1..3)4= 0x1+0x2+0x3+1x4+c+e13

$$\begin{Bmatrix} 8 \\ 9 \\ 7 \\ 5 \\ 7 \\ 3 \\ 3 \\ 4 \\ 1 \\ 6 \\ 4 \\ 9 \end{Bmatrix} = \begin{Bmatrix} 1&0&0&0&1 \\ 1&0&0&0&1 \\ 1&0&0&0&1 \\ 0&1&0&0&1 \\ 0&1&0&0&1 \\ 0&1&0&0&1 \\ 0&0&1&0&1 \\ 0&0&1&0&1 \\ 0&0&1&0&1 \\ 0&0&0&1&1 \\ 0&0&0&1&1 \\ 0&0&0&1&1 \end{Bmatrix} * \begin{Bmatrix} \beta 1 \\ \beta 2 \\ \beta 3 \\ \beta 4 \\ c \end{Bmatrix} + \begin{Bmatrix} e1 \\ \\ \\ \\ e13 \end{Bmatrix}$$

→ This is like the multiple regression except that we have ones and zeros instead of 'real' values

# 1 way ANOVA

- Now we have rewritten the ANOVA using matrices such Y = XB
- One last thing: X is rank deficient, that is take any 4 columns and you can make the fife one up → inv(X) or inv(X'X) do not exist → solution use pinv
- B = pinv(X)Y
- All other analyses are the same

http://en.wikipedia.org/wiki/Generalized_inverse
http://en.wikipedia.org/wiki/Moore-Penrose_pseudoinverse

# 1 way ANOVA

- Matlab exercise 7: solve an ANOVA just as for regression
- u1 = rand(10,1) + 11.5; u2 = rand(10,1) + 7.2; u3 = rand(10,1) + 5; Y = [u1; u2; u3];
- x1 =[ones(10,1); zeros(20,1)]; x2 =[zeros(10,1); ones(10,1); zeros(10,1)]; x3 =[zeros(20,1); ones(10,1)]; X =[x1 x2 x3 ones(30,1)];
- B = pinv(X)*Y
- Yhat, Res, F and p values as before

# END

Next time, even more general way
to solve even more general designs