Bayesian inference

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Overview of the talk

1 Probabilistic modelling and representation of uncertainty

1.1 Bayesian paradigm

1.2 Hierarchical models

1.3 Frequentist versus Bayesian inference

2 Numerical Bayesian inference methods

2.1 Sampling methods

2.2 Variational methods (ReML, EM, VB)

3 SPM applications

3.1 aMRI segmentation

3.2 Decoding of brain images

3.3 Model-based fMRI analysis (with spatial priors)

3.4 Dynamic causal modelling

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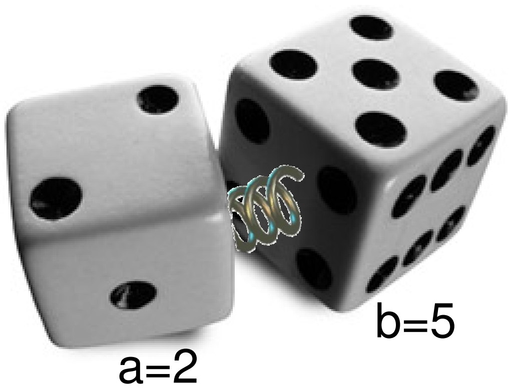
Bayesian paradigm

probability theory: basics

Degree of plausibility desiderata:

- should be represented using real numbers (D1)
- should conform with intuition (D2)
- should be consistent (D3)

• normalization: $$\sum_a P(a) = 1$$

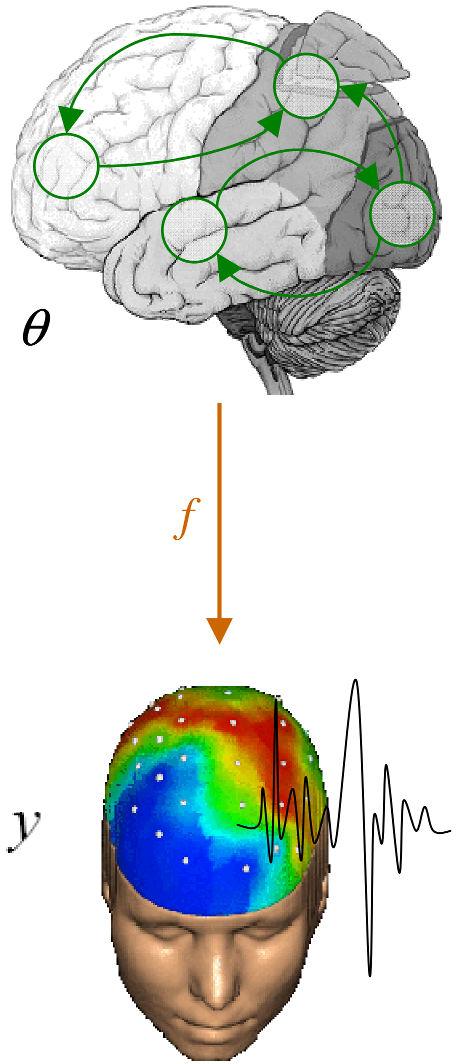


• marginalization: $$P(b) = \sum_a P(a,b)$$

• conditioning : (*Bayes rule*) $$P(a,b) = P(a|b)\,P(b) \\ = P(b|a)\,P(a)$$

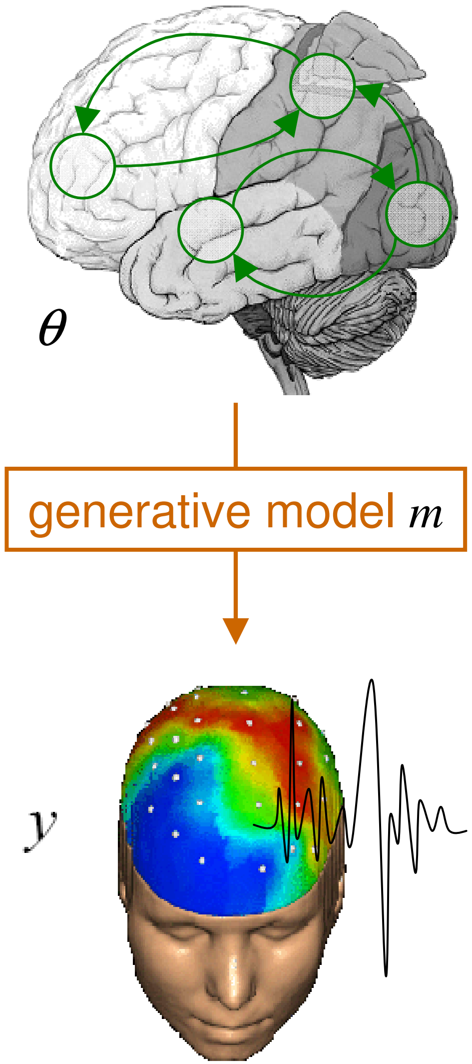
Bayesian paradigm

deriving the likelihood function



Bayesian paradigm

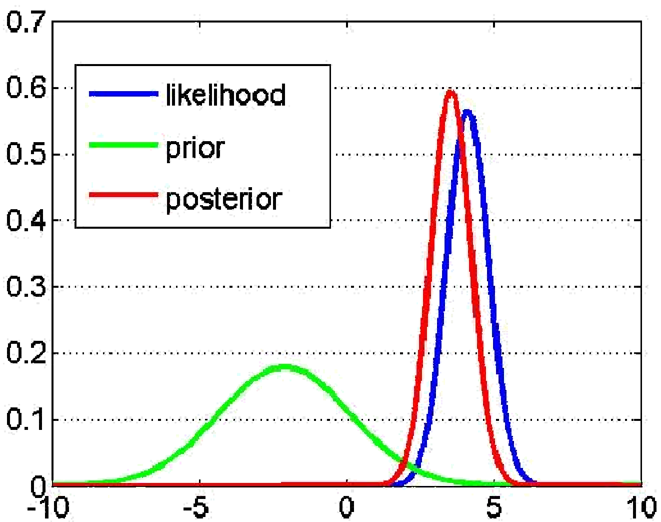
likelihood, priors and the model evidence



Likelihood: $p(y|\theta, m)$

Prior: $p(\theta|m)$

Bayes rule: $$p(\theta|y, m) = \frac{p(y|\theta, m)\, p(\theta|m)}{p(y|m)}$$



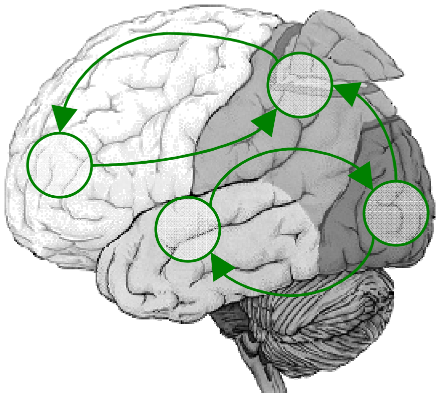
Bayesian paradigm

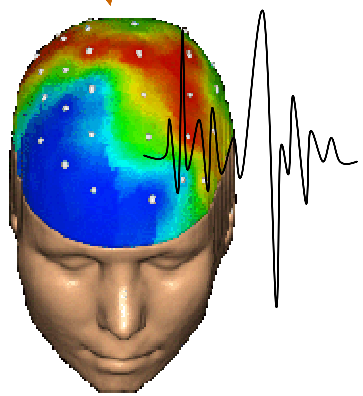
forward and inverse problems

forward problem

$$p\left(y|\vartheta,m\right)$$

likelihood





posterior distribution

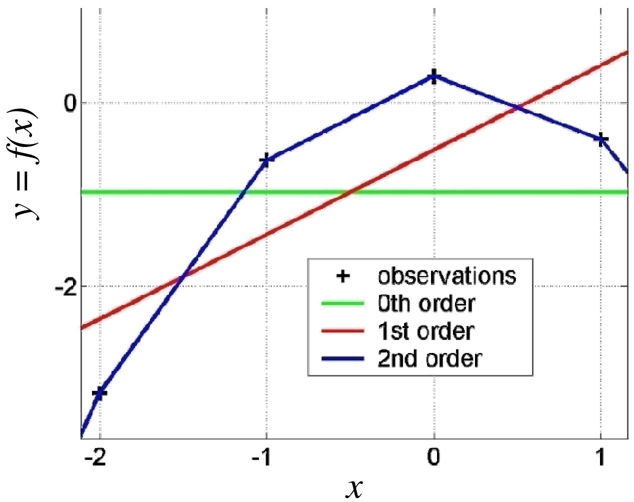
$$p\left(\vartheta|y,m\right)$$

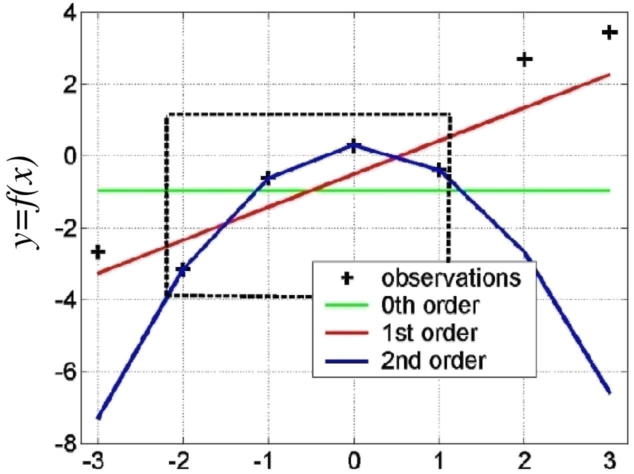
inverse problem

Bayesian paradigm

model comparison

Principle of parsimony :
« plurality should not be assumed without necessity »

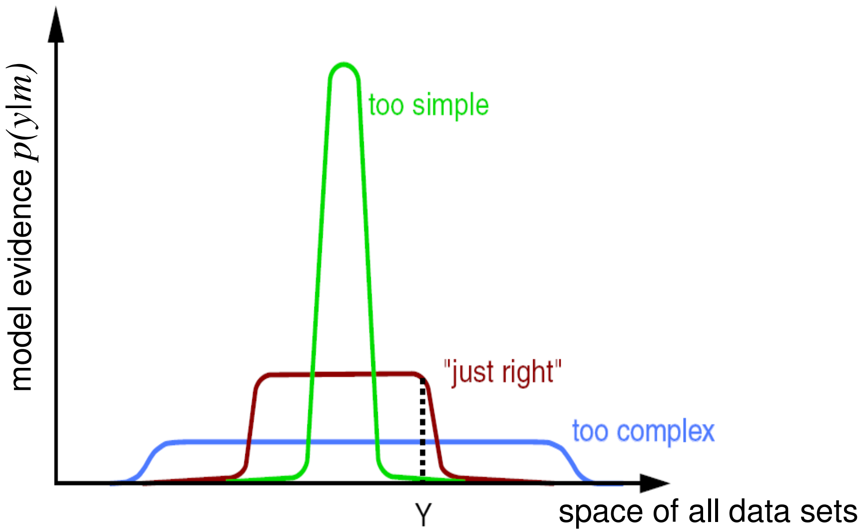




Model evidence:

$$p(y|m) = \int p(y|\theta, m)\, p(\theta|m)\, d\theta$$

“Occam’s razor” :

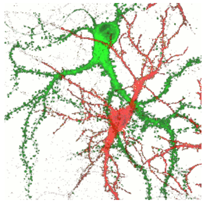


Hierarchical models

principle

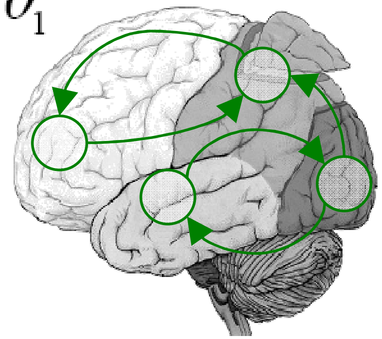
$\vdots$

θ_2



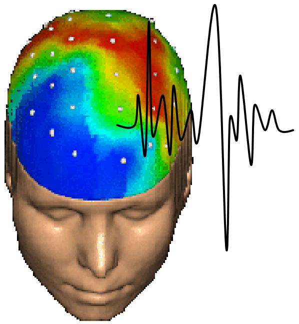
$$p(\theta_2 \mid \theta_3, m)$$

θ_1



$$p(\theta_1 \mid \theta_2, m)$$

y



$$p(y \mid \theta_1, m)$$

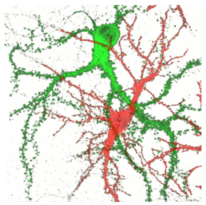
inference

causality

Hierarchical models

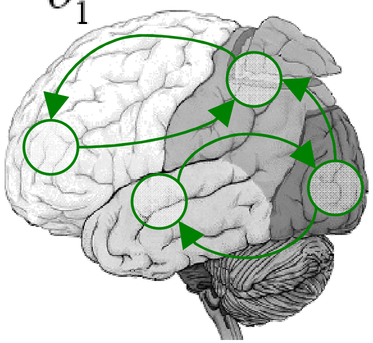
directed acyclic graphs (DAGs)

θ_2

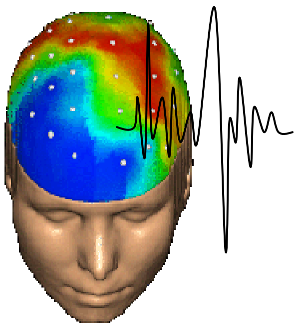


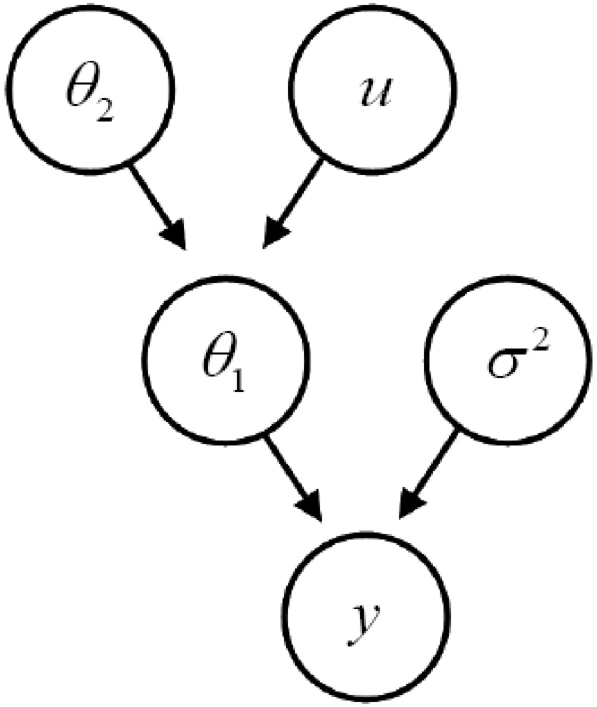
u

θ_1



y





$$p(\theta_1 | \theta_2, u, m)$$

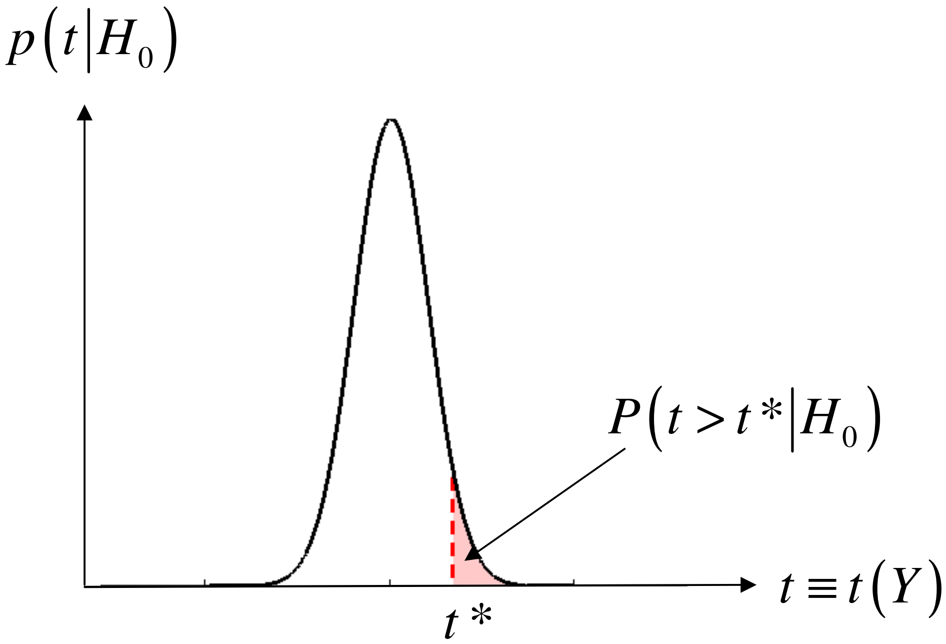
$$p(y | \theta_1, \sigma^2, m)$$

$$p(\theta | m) = \prod_j p(\theta_j | par(\theta_j), m)$$

Frequentist versus Bayesian inference

a (quick) note on hypothesis testing

• define the null, e.g.: $H_0 : \theta = 0$



• estimate parameters (obtain test stat.)

• apply decision rule, i.e.:

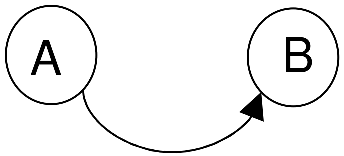
if $P\left(t > t^* \middle| H_0\right) \le \alpha$ then reject H0

classical (null) hypothesis testing

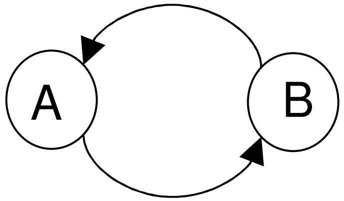
Family-level inference

trading model resolution against statistical power

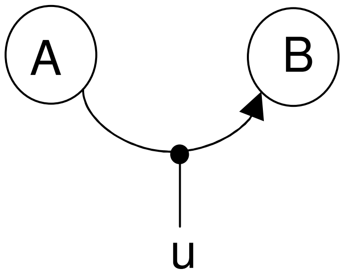
$P(m_1|y) = 0.04$



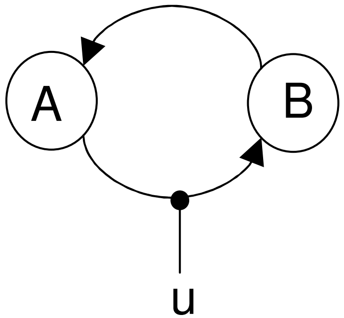
$P(m_2|y) = 0.25$



$P(m_2|y) = 0.01$



$P(m_2|y) = 0.7$

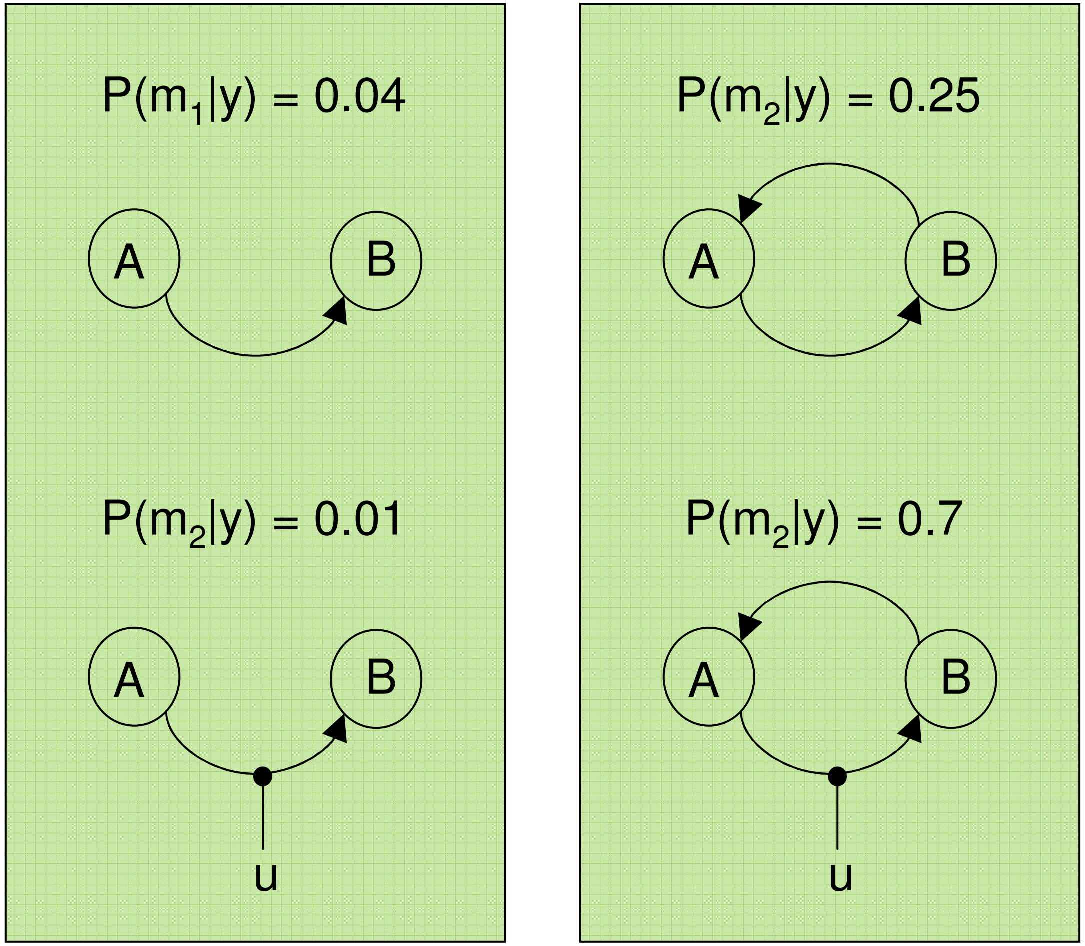


model selection error risk:

$$P\left(e=1\middle|y\right) = 1 - \max_m P\left(m\middle|y\right)$$
$$= 0.3$$

Family-level inference

trading inference resolution against statistical power



$P(f_1|y) = 0.05$

$P(f_2|y) = 0.95$

model selection error risk:

$$P\left(e=1\middle|y\right) = 1 - \max_m P\left(m\middle|y\right)$$
$$= 0.3$$

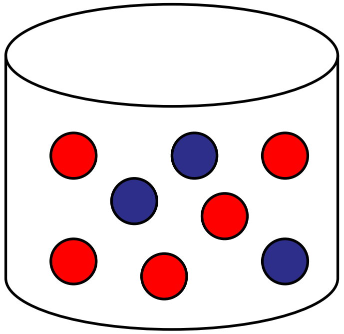
family inference
(pool statistical evidence)

$$P\left(f\middle|y\right) = \sum_{m \in f} P\left(m\middle|y\right)$$

$$P\left(e=1\middle|y\right) = 1 - \max_f P\left(f\middle|y\right)$$
$$= 0.05$$

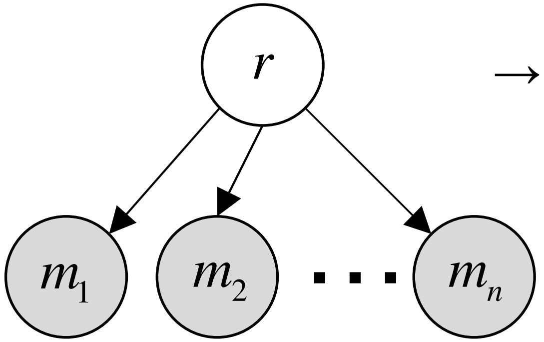
Group-level model comparison

preliminary: Polya's urn



$$\begin{cases} m_i = 1 & \rightarrow i^{\text{th}} \text{ marble is blue} \\ m_i = 0 & \rightarrow i^{\text{th}} \text{ marble is purple} \end{cases}$$

r = proportion of blue marbles in the urn



→ (binomial) probability of drawing a set of n marbles:

$$p(m|r) = \prod_{i=1}^{n} r^{m_i} (1-r)^{1-m_i}$$

Thus, our belief about the proportion of blue marbles is:

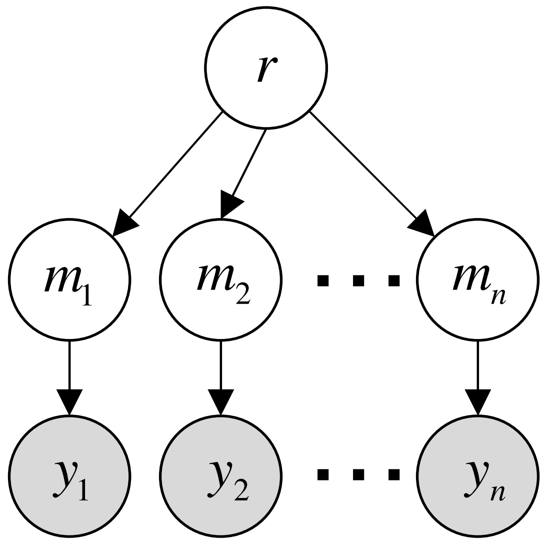
$$p(r|m) \propto p(r) \prod_{i=1}^{n} r^{m_i} (1-r)^{1-m_i} \quad \overset{p(r)\propto 1}{\Rightarrow} \quad E[r|m] = \frac{1}{n} \sum_{i=1}^{n} m_i$$

Group-level model comparison

what if we are colour blind?

At least, we can measure how likely is the i^{th} subject's data under each model!

$$p\left(y_1 \mid m_1\right) \quad p\left(y_2 \mid m_2\right) \quad \cdots \quad p\left(y_i \mid m_i\right) \quad \cdots \quad p\left(y_n \mid m_n\right)$$



$$p\left(r,m \mid y\right) \propto p\left(r\right) \prod_{i=1}^{n} p\left(y_i \mid m_i\right) p\left(m_i \mid r\right)$$

Our belief about the proportion of models is:

$$p\left(r \mid y\right) = \sum_{m} p\left(r,m \mid y\right)$$

Exceedance probability: $\varphi_k = P\left(r_k > r_{k' \neq k} \mid y\right)$

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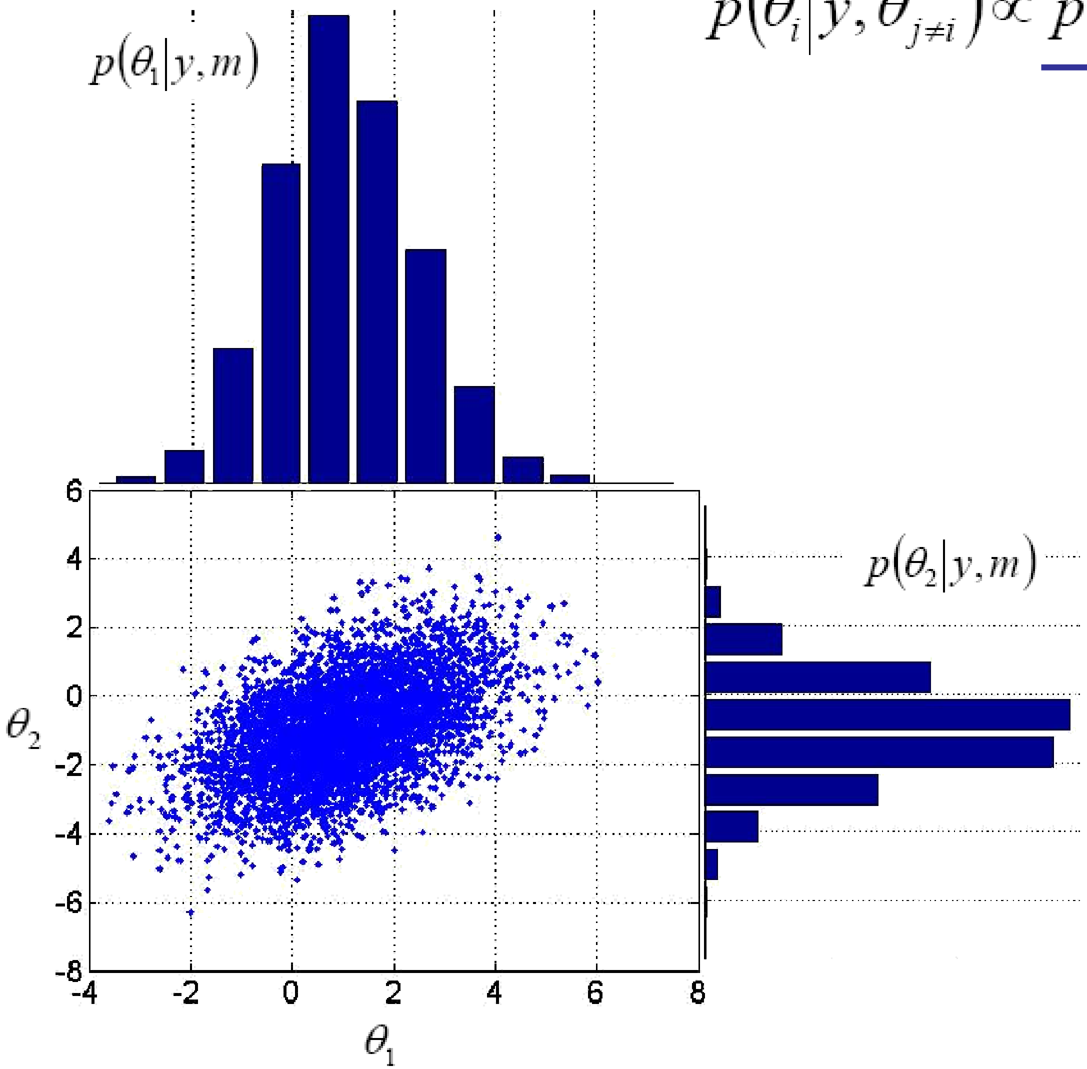
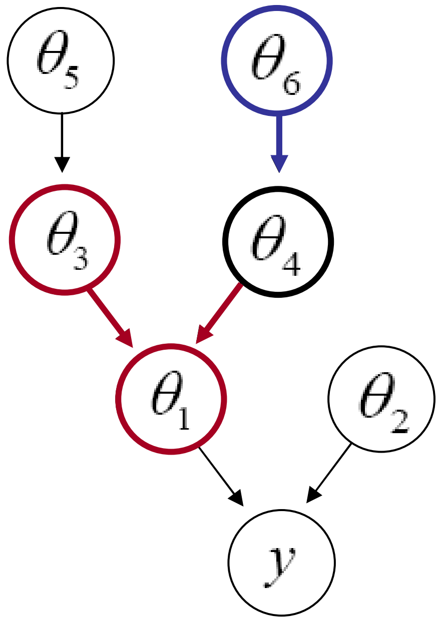
Sampling methods

MCMC example: Gibbs sampling

$$p(\theta_i | y, \theta_{j \neq i}) \propto \underline{p(\theta_i | par(\theta_i))} \underline{\prod_{j=ch(i)} p(\theta_j | par(\theta_j))}$$





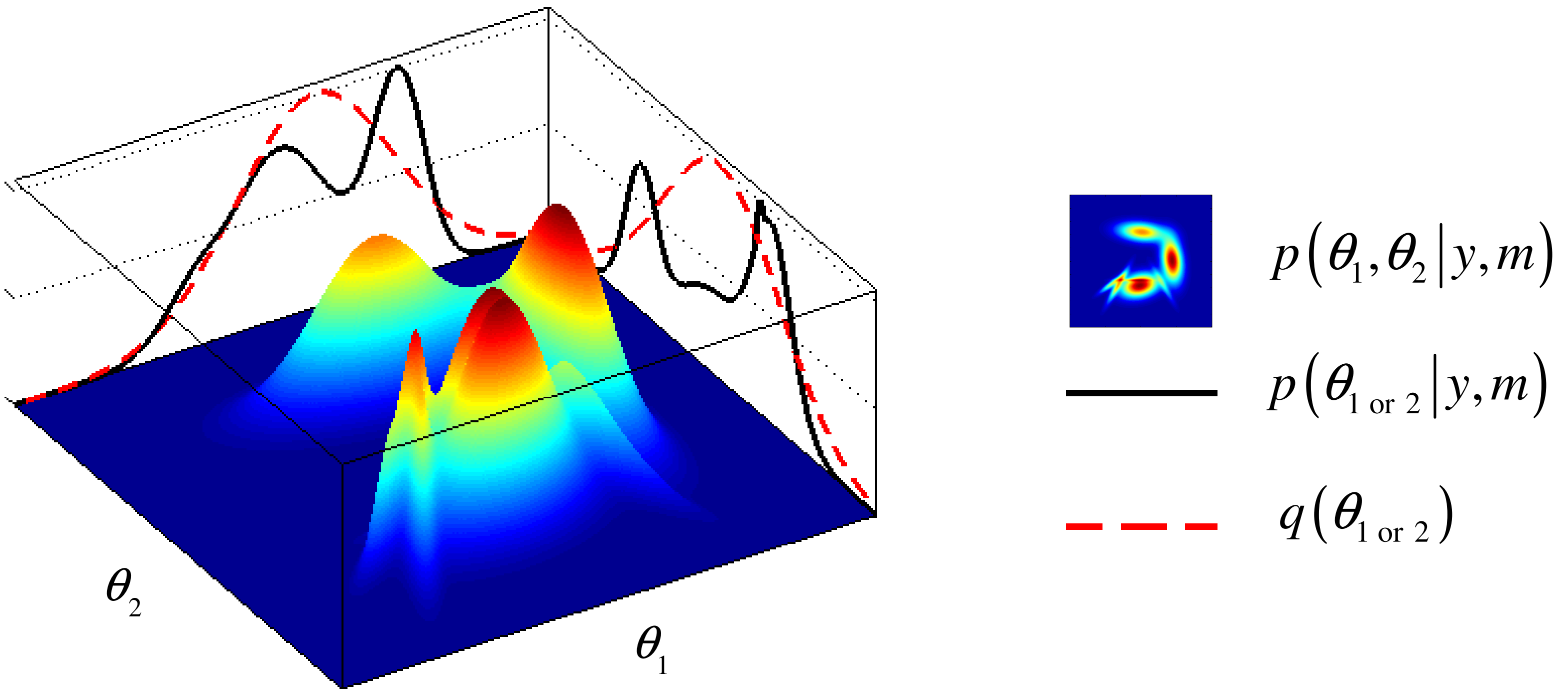
$$\frac{1}{N} \sum_{n=1}^{N} p(y | \theta^{(n)}, m) \approx p(y|m)$$

Variational methods

VB / EM / ReML

$$\ln p(y|m) = \underbrace{\langle \ln p(\theta, y|m) \rangle_q + S(q)}_{\text{free energy } F(q)} + D_{KL}\big(q(\theta); p(\theta|y,m)\big)$$

→ **VB** : maximize the free energy $F(q)$ w.r.t. the approximate posterior $q(\theta)$ under some (e.g., *mean field, Laplace*) simplifying constraint



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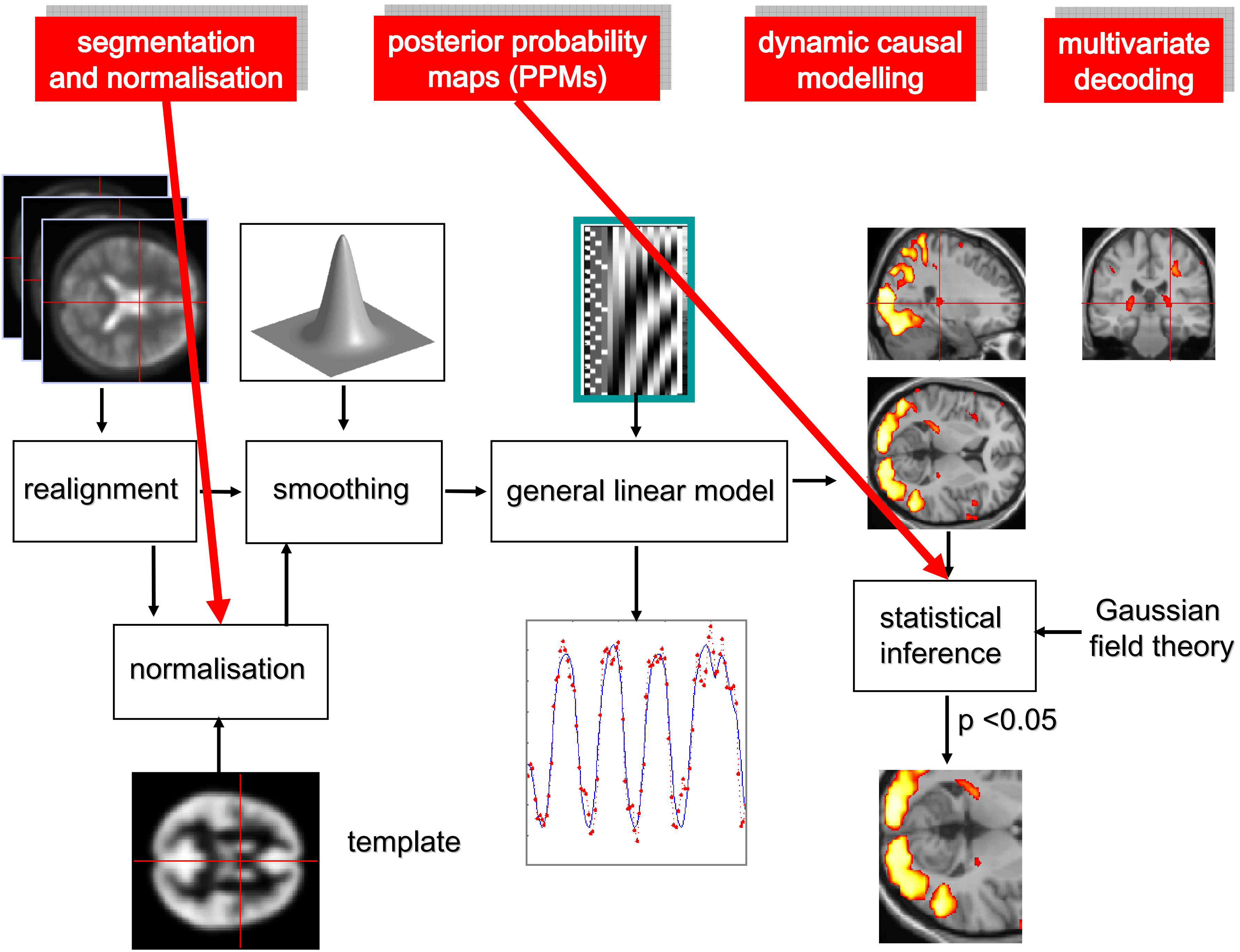
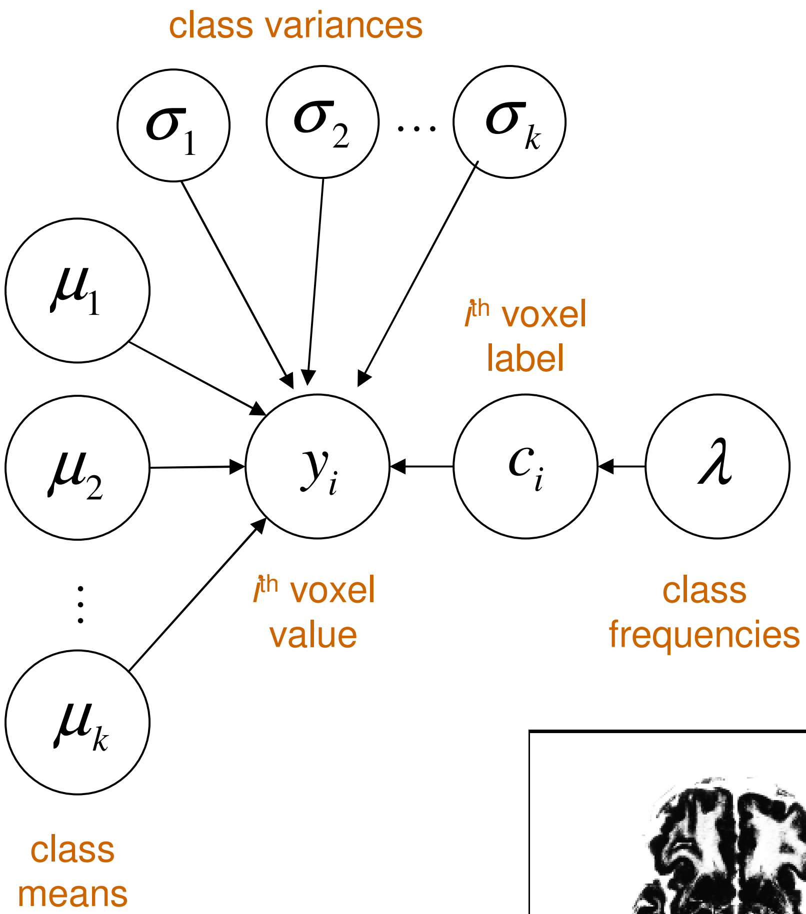
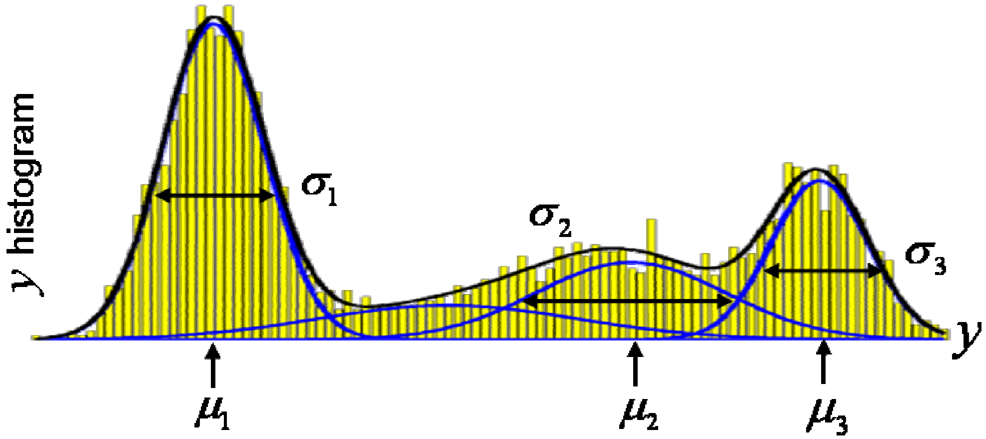
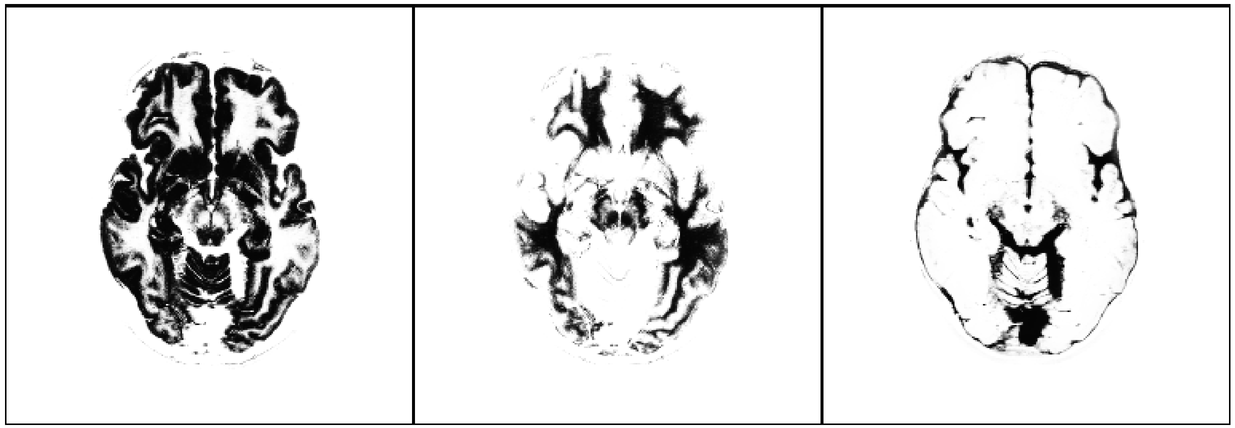

segmentation and normalisation
posterior probability maps (PPMs)
dynamic causal modelling
multivariate decoding
realignment
smoothing
general linear model
normalisation
template
statistical inference
Gaussian field theory
p <0.05

aMRI segmentation

mixture of Gaussians (MoG) model



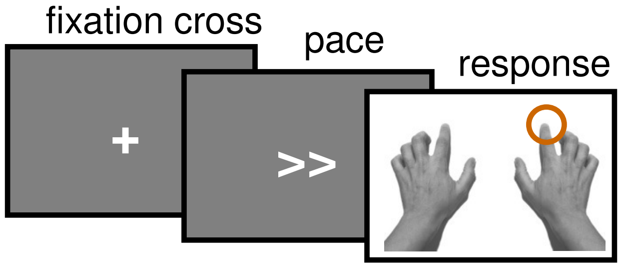


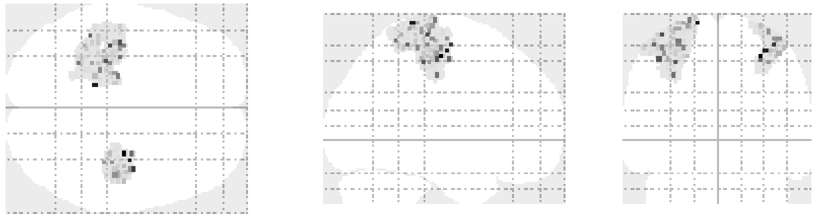


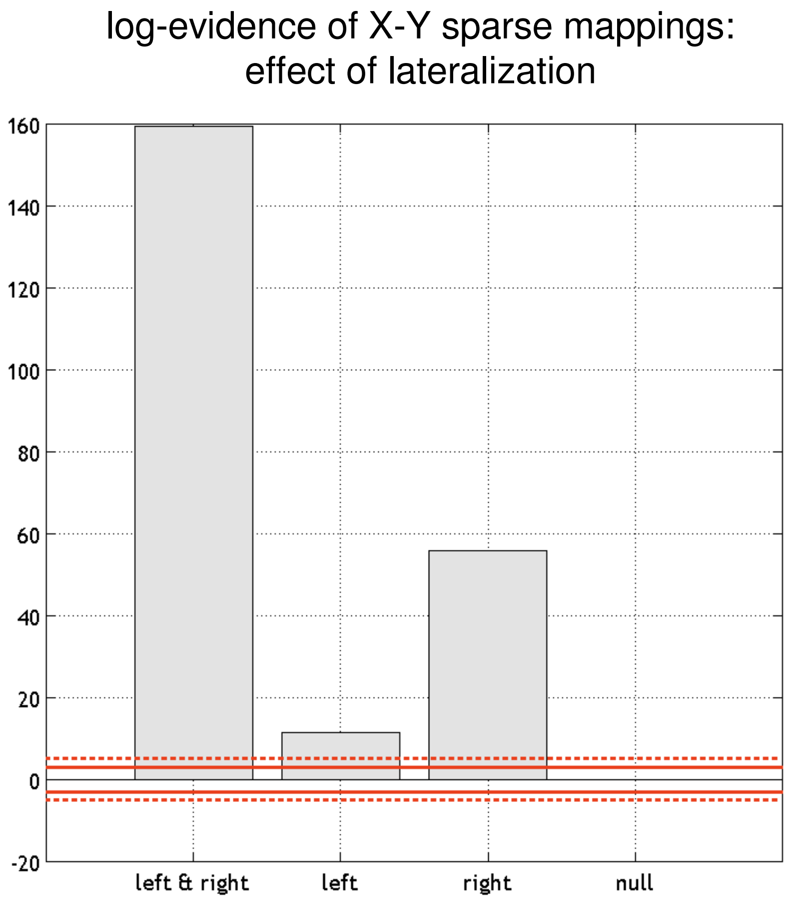
grey matter | white matter | CSF

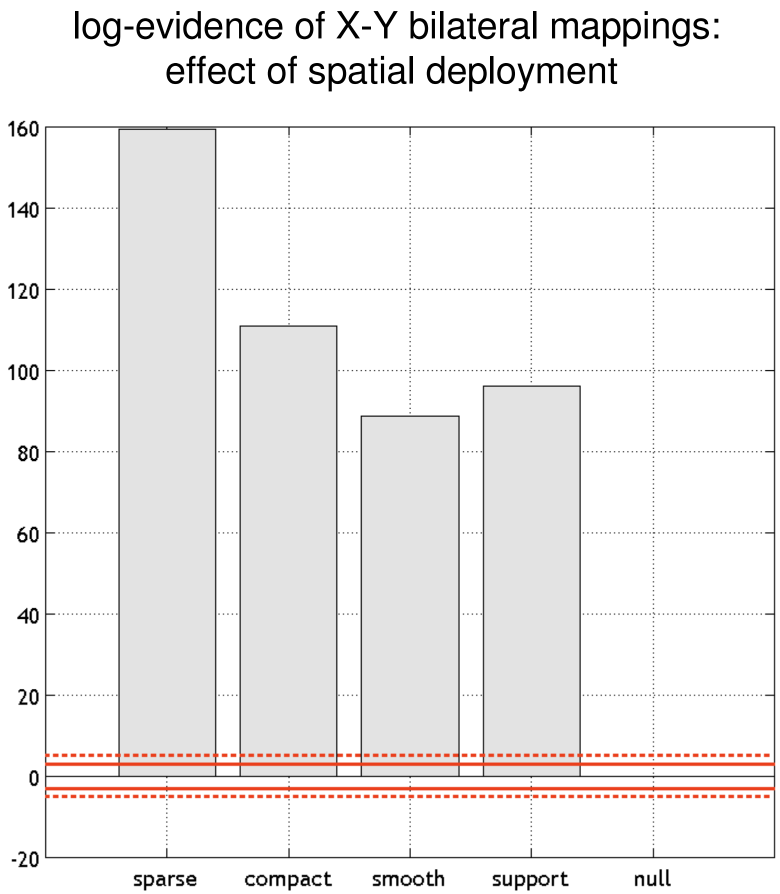
Decoding of brain images

recognizing brain states from fMRI





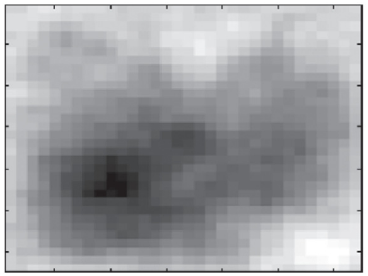


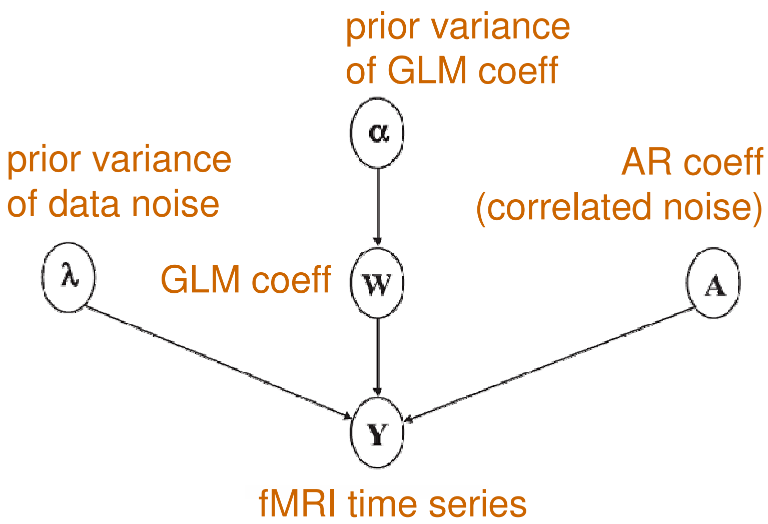


fMRI time series analysis

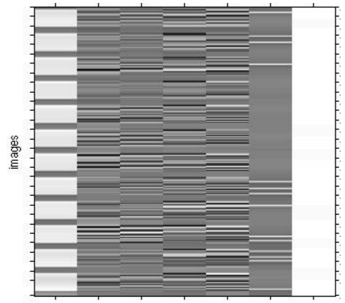
spatial priors and model comparison

		1		
	2	-8	2	
1	-8	20	-8	1
	2	-8	2	
		1		

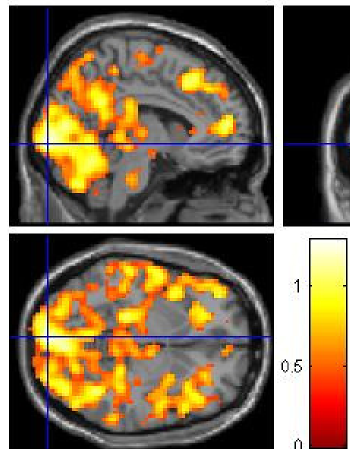




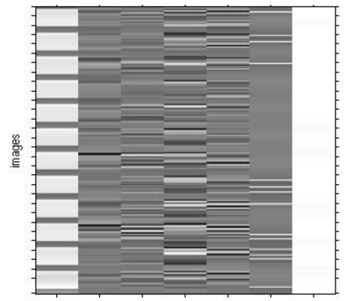
short-term memory
design matrix (X)



PPM: regions best explained
by short-term memory model



long-term memory
design matrix (X)



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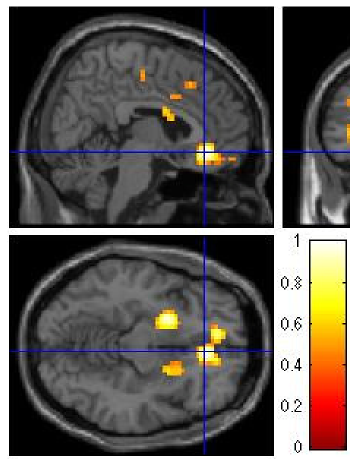
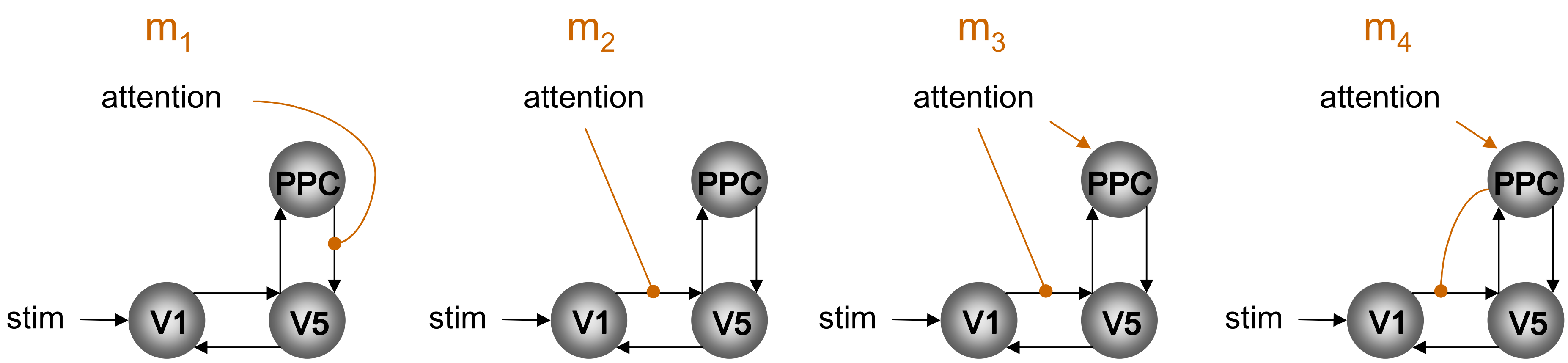
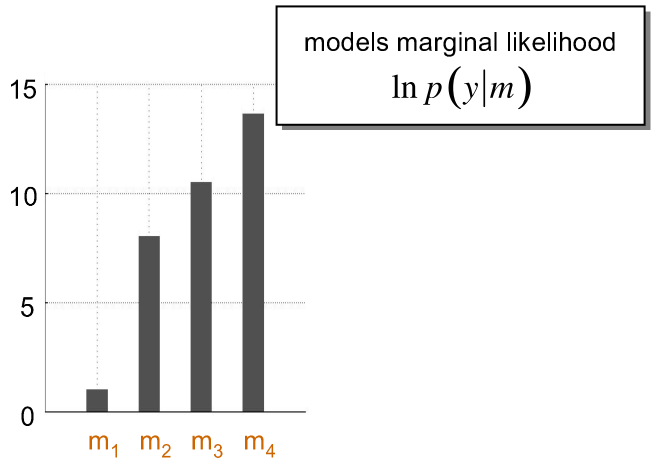
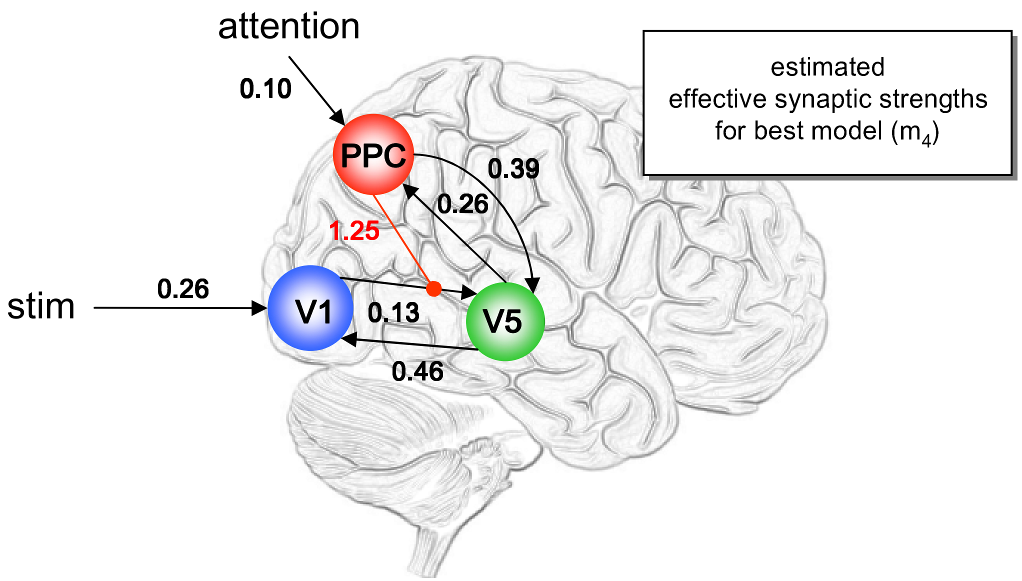


Dynamic Causal Modelling

network structure identification

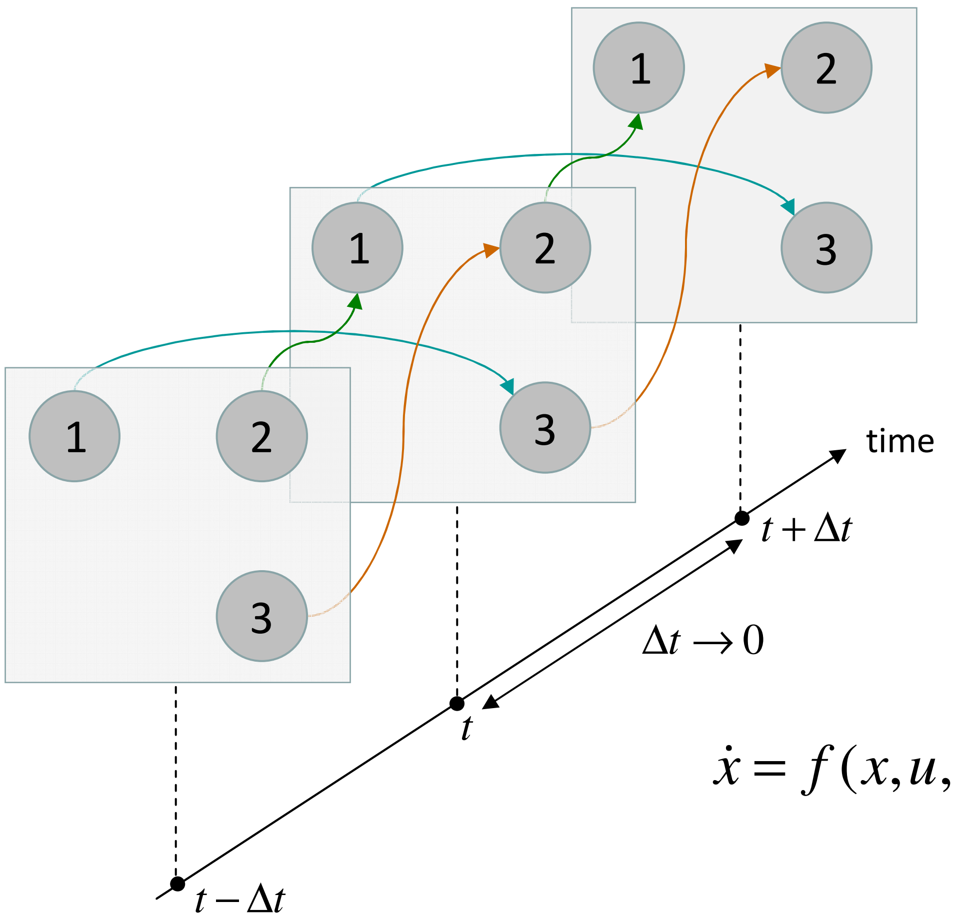




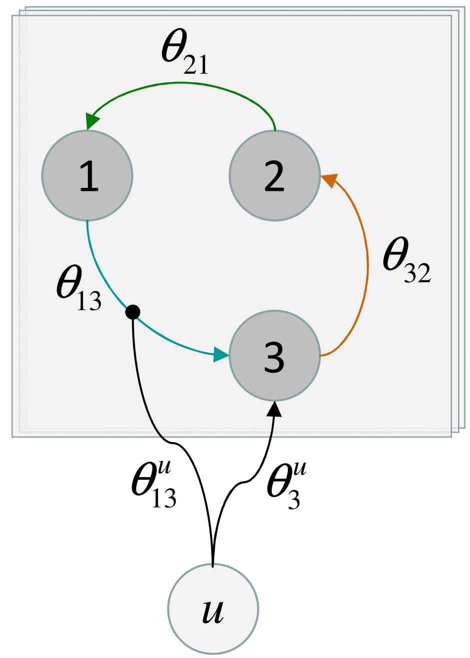


DCMs and DAGs

a note on causality



$$\dot{x} = f(x, u, \theta)$$



I thank you for your attention.

A note on statistical significance
lessons from the Neyman-Pearson lemma

- Neyman-Pearson lemma: the likelihood ratio (or Bayes factor) test

$$\Lambda = \frac{p(y|H_1)}{p(y|H_0)} \geq u$$

is the most powerful test of size $\alpha = p(\Lambda \geq u | H_0)$ to test the null.

- what is the threshold u, above which the Bayes factor test yields a error I rate of 5%?