Primer on matrices:
Basic stuffs for neuroscientists

Cyril Pernet, PhD.

Matrices are used mainly to solve sets of equation though, as we shall see, it is a mathematical object on its own. What I describe here is the minimal requirement to perform some matrix operations with an emphasis on solving a set of equations.

1. Linear equations and matrices

Let's start by a set of 2 equations:
2x - y = 0
-x + 2y = 3

this can be represented in a matrix form as $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

Solving a system of equations consists usually in multiplying by a non-zero constant, interchange two equations and add a multiple of one equation to the other. As rows of a matrix represent the coefficient of the equation the same can be applied on row. One have however to do it on the ***augmented matrix***

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 3 \end{bmatrix}$$

One way to solve this would be: 2x – y = 0 gives 2x = y, then one interchanges in equation 2, i.e. –x + 2y = 3 gives –x + 2(2x) = 3 and finally we get x = 1 and therefore y = 2.

This equivalent to

2x - y = 0
-x + 2y = 3

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 3 \end{bmatrix}$$

Add 2 times the first equation to the second one

Add 2 times the first row to the second one

2x - y = 0
3x + 0y = 3

$$\begin{bmatrix} 2 & -1 & 0 \\ 3 & 0 & 3 \end{bmatrix}$$

Multiply by 1/3 the second equation

Multiply by 1/3 the second row

$$\begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

2x - y = 0
1x + 0y = 1

Add -2 times the second equation to the first one

0x - y = -2
1x + 0y = 1

Add -2 times the second equation to the first one

$$\begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \end{bmatrix}$$

Multiply the first equation by -1

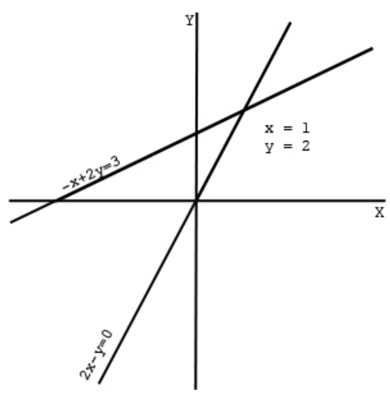
0x + y = 2
1x + 0y = 1

Multiply the first row by -1

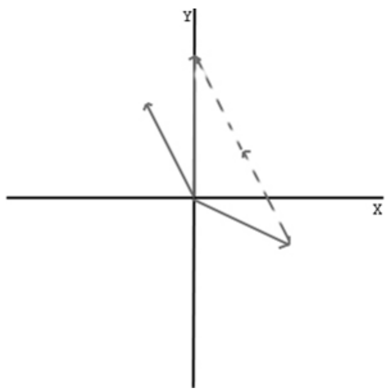
$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

Geometrical perspective

In the XY plane, each equation represents a straight line and their intersection is the solution of the system



Another way to visualize this is to 'read' the column of the matrix instead of the row, then using a vector approach one has x times the vector [2 -1] and y times the vector [-1 2] and this particular linear combination gives a vector [0 3]



If one combines 1 time the vector [2 1] + 2 times [-1 2] one obtains [0 3]

Solving a system of linear equations using elimination (again)

Let our linear system be:

$$
\begin{aligned}
4y + z &= 2\\
3x + 8y + z &= 12\\
x + 2y + z &= 2
\end{aligned}
$$

Each equation represents a plane, if there is a solution then 2 equations meets on a line and the 3 equations on a point x,y,z

Gaussian elimination = row-echelon form

In a matrix form the above system can be written as

$$\begin{bmatrix} 0 & 4 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

Before doing anything one have to make sure that one always starts by a non-zero i.e. $cell_{i,j} \neq 0$. Then, add a multiple of the above row the next one and repeat till the last row to get zeros above the pivot ($cell_{i,j}$). For instance here one starts by exchanging the rows then add to the second row -3 times the first row and to the 3rd row, zeros times the first row …

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Now if we go back to the equations, i.e. we do a *back substitution*
x + 2y + z = 2 → x + 2y + z = 2
3x + 8y + z = 12 → y + -z = 3
4y + z = 2 → z = -2

and we find the solution [2 1 -2]

Gauss-Jordan elimination = row-echelon form

From the last matrix, one can also find the solution such as not only the entries bellow the pivot are 0 but also the ones above. This time one works in the reverse order from the last row going up: add 1 time row 3 to row 2 and -1 time row 3 to row 2; then add -2 times row 2 to row 1

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

<u>2. Matrix operations</u>

If A and B are i x j matrices then (A+B)ij = (A)ij + (B)ij
For a scalar c, (cA)ij = c(A)ij

For A a m x n matrix and B a n x p matrix, one has C a m x p matrix = A x B
As one can foresee from the geometrical perspective presented above, multiplying matrices is like adding multiple (defining by the right hand side matrix) of the columns of the first matrix (one left hand side). Therefore, AB ≠ BA.

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$$

AB = C with the rows of C a linear combination of the rows of B and the columns of C a linear combination of the rows of A

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 7 & 4 \\ 1 & 2 \end{bmatrix} = 7 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 5 \\ 3 \end{bmatrix} \text{ and } 4 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 19 & 18 \\ 10 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 47 \\ 4 & 11 \end{bmatrix}$$

Cell(2,2) = row2 of A x column2 of B = a2,1b12+a2,1b22

When matrices are too large, one can work one a ***partitioned matrix***, such as each multiplication concerns only ***submatrices***

$$\begin{bmatrix} A1 & A2 \\ A3 & A4 \end{bmatrix}\begin{bmatrix} B1 & B2 \\ B3 & B4 \end{bmatrix} = \begin{bmatrix} A1B1 + A2B3 & A1B2 + A2B4 \\ A3B1 + A4B3 & A4B2 + A4B4 \end{bmatrix}$$

Although the commutative law does not hold for matrices, the ***associative law*** does A(BC)=(AB)C. In addition, A(B+C)=AB+AC and (A+B)C = AC + BC.

Other useful operations are the transpose and the trace. For a matrix A n x m, the transpose A^T exchanges rows and column such as one obtains a m x n matrix.

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}^T = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$$

$(A^T)^T = A$

$(A+B)^T = A^T + B^T$

$(kA)^T = kA^T$

$(AB)^T = B^T A^T$

Matrix arithmetic: identity matrix and inverse

Some matrices are ‘special’; one of them is the ***zero matrix* 0**. For any **0** matrix of a suitable size A+**0** = **0**+A = A. Because of this property, if AB = AC and A ≠ **0** then B = C and if AB =0 then A or B is = **0**. Another fact is that A**0 = 0**A **= 0.**

Another ‘special’ matrix is the ***identity matrix*** I. I represent a matrix with ones on its main diagonal and zeros anywhere else. For a square matrix AI=IA=A and for a m x n matrix AIn=A and ImA=A.

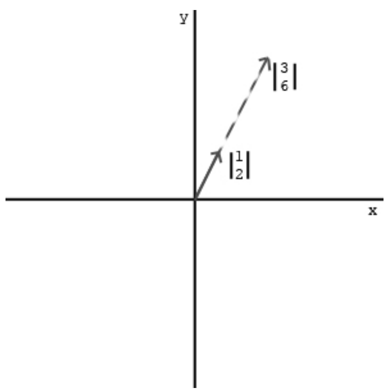
$$\begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1\begin{bmatrix} 3 \\ 5 \end{bmatrix} + 0\begin{bmatrix} 4 \\ 7 \end{bmatrix} \text{ and } 0\begin{bmatrix} 3 \\ 5 \end{bmatrix} + 1\begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$$

$$\begin{bmatrix} a11 & a12 & a13 & a14 \\ a21 & a22 & a23 & a24 \\ a31 & a32 & a33 & a34 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a11 & a12 & a13 & a14 \\ a21 & a22 & a23 & a24 \\ a31 & a32 & a33 & a34 \end{bmatrix}$$

If A is a square matrix and it exists a matrix B of the same size such as AB=I, then A is said ***invertible*** and B is the ***inverse*** of A. If B does not exist A is said ***singular***.

A matrix is not invertible is it is **rank deficient**, i.e. some column are multiple of the others. A good way to think about this is to go back to the geometrical perspective. Since one column, i.e. one vector is a linear combination of the other columns, it is impossible to combine them all to go back to 0.



The matrix $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ is singular since $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ is a multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Some properties of the inverse are:

$AA^{-1} = I = A^{-1}A$
$(A^{-1})^T A^T = I$
$(A^T)^{-1} = (A^{-1})^T$

$(AB)^{-1} = B^{-1}A^{-1}$
$(AB)(B^{-1}A^{-1}) = I$
$A(BB^{-1})A^{-1} = I$

The matrix A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if $ad - bc \neq 0$ i.e. if the ***determinant*** is not null.

Is this case $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}$

One general solution to find A^{-1} is to perform a Gauss-Jordan elimination. Indeed, any operations on the rows of I are equivalent to the operation performed on the row of A, such as A|I will give us $I|A^{-1}$. Say with have a matrix A =

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \text{ the matrix I is therefore } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

Given this possibility to obtain A^{-1} using I, it appears that a linear system which matrix A is singular has no solution. But we can go further and solve a linear system using A^{-1}. Indeed, ***for any linear system Ax = b, the solution is given by x = A^{-1} b.***

The last set of 'special' matrices is the ***diagonal*** and ***symmetric*** matrices. Diagonal matrices are matrices of zeros except on the main diagonal. If all the entries above the main diagonal are non-zeros then this called a upper triangular matrix, which is the opposite of lower triangular matrices (i.e. all non-zeros bellow the man diagonal). Upper and lower triangular matrices are related by the transpose operator ($lower^T$=upper and $upper^T$=lower). A matrix is said symmetric if $A=A^T$.