Bi/BE/CS 183 Set 6

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Problem 1)

A)

All we need to do here is expand out the first two terms of the Taylor series for $f(x)$

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \approx f(\mu) + f'(\mu)(x-\mu)$

B) Using the Taylor expansion above, we will now show that $Var[f(x)] \approx f'(\mu)^2 Var[x]$

To do this we will need two properties of variance:

1) Multiplying the rv by some constant multiplies the variance by that value squared.

let $a \in \mathbb{R}$

$var(ax) = E[a^2x^2] - (E[ax])^2 = a^2E[x^2] - (aE[x])^2 = a^2(E[x^2] - (aE[x])^2$

2) the variance is invariant to added terms:

Adding any constant to the random variable shifts its mean, but does not change the spread of the variable.

With these two properties of variances we can reduce the variance of the Taylor expansion to our desired form.

$Var[f(x)] \approx var[f(\mu) + f'(\mu)(x-\mu)]$

first we can remove the $f(\mu)$ term as it is an added term per property 2.

$Var[f(x)] \approx var[f'(\mu)(x-\mu)]$

Next, multiplying by $f'(\mu)$ will multiply the variance by $f'(\mu)^2$

$Var[f(x)] \approx f'(\mu)^2 var[(x-\mu)]$

finally, using property 2 again, $var[(x-\mu)] = var[(x)]$. thus:

$Var[f(x)] \approx f'(\mu)^2 var[(x)]$

As desired!

Problem 2)

A)

Given that we know that $K|Q$ is Poisson distributed with constant q, the expectation and variance of $K|Q$ will both be q from the definition of a Poisson distribution. Formally this is written as $E[K|Q] = var[K|Q] = q$

To find the expectation of K We must find $E_Q[E[k|Q]]$

$E[K] = E_Q[E[k|Q]] = E_Q[q] = k\theta$

as desired!

We can now use the law of total variance to solve for the variance of K

Law of total variance: $var[K] = E_Q[var[k|Q]] + var_Q[E[k|Q]]$

We know, from our initial solving, $E[K|Q]$ and $var[K|Q]$ which we can substitute into the law of total variance. $var[K] = E_Q[q] + var_Q[q] = k\theta + k\theta^2$ as desired!

B)

We can use the same fact as above that a Poisson distribution has the same variance and mean to find that: $E[K'|Q,s] = var[K'|Q,s] = sq$

To find the expectation of K' We must find $E_Q[E[k'|Q,a]]$

$E[K'] = E_Q[E[K'|Q,s]] = E_Q[sq] = s\mu = \mu'$

We can once again use the law of total variance.

$var[K'] = E_Q[var[K'|Q,s]] + var_Q[E[K'|Q,s]]$

$var[K'] = E_Q[sq] + var_Q[sq]$

$var[K'] = s\mu + s^2\mu^2\phi$

$var[K'] = \mu' + (\mu')^2\phi$

C)

To find $E[Y]$ we can find $E[\frac{K'}{s}]$:

$E[\frac{K'}{s}] = \frac{1}{s}E[K'] = \frac{1}{s}s\mu = \mu$

To find $var[Y]$ We can use the result from the law of total variance (before substituting $s\mu = \mu'$):

$var[Y] = var[\frac{K'}{s}] = \frac{1}{s}var[K'] = \frac{1}{s^2}(s\mu + s^2\mu^2\phi) = \frac{\mu}{s} + \mu^2\phi$

D)

The biggest issue with the variance stabilization using a transformation for Gamma-Poisson (negative-binomial) random variable derived by the Delta method is that the variance is a function of the size factor s. This factor is different for each of the cells and therefore must be computed for each cell. In order to do so, certain assumptions must be made about the size factor and the cells which may not always apply and may give weird results if they do not apply.

Problem 3)

Google Colab notebook:

`https://colab.research.google.com/github/CPukszta/BI-BE-CS-183-2023/blob/main/HW6/Problem3.ipynb`