Motion planning—Probability

1.Baysian Method

(1)Baysian infer

We can make use of Baysian formula:

thod:

$$p(w|Data) = p(w|X,Y) = \frac{p(w,Y|X)}{p(Y|X)} = \frac{p(Y|w,X)\,p(w)}{\int p(Y|w,X)\,p_w\,dw}$$

postprior; likelihood; prior

Suppose:

$$w \sim N(0, \Sigma_p)$$

That is to say, w is prior normal distribution.

According to some calculation,we can get posterior distribution:

$$p(w|Data) \sim N(\sigma^{-2}A^{-1}X^TY, A^{-1}) \quad (A = \sigma^{-2}X^TX + \Sigma_p^{-1})$$

If the process include error ε : $y = f(\mathbf{x}) + \varepsilon$, $f(\mathbf{x}) = \mathbf{x}^\top \mathbf{w}$

$$\varepsilon \sim \mathcal{N}(0, \sigma_n^2).$$

If the process doesn't contain error. So the posterior distribution doesn't contain error covariance.

(2)Prediction

- predict:
  - idea: noise-free: $f(x^*) = x^{*T}w \rightsquigarrow$ postprior, short as $N(\mu_w, \Sigma_w)$ $\Longrightarrow f(x^*) \sim N(x^{*T}\mu_w, x^{*T}\Sigma_w x^*)$

    noise: $y^* = f(x^*) + \varepsilon$, $\varepsilon \sim (0, \sigma^2)$ $\Rightarrow p(y^*|Data, x^*) \sim N(x^{*}\mu_w, x^{*T}\Sigma_w x^* + \sigma^2)$

Posterior is expressed as followed for simplity.

$$Posterior \sim N(\mu_w, \Sigma_w)$$

2 Gaussian Process for motion planning

A Gaussian process is a collection of random variables, any Gaussian process finite number of which have a joint Gaussian distribution.

$$f(\mathbf{x}) \sim \mathcal{GP}\big(m(\mathbf{x}), k(\mathbf{x}, \mathbf{x}')\big)$$

Gaussian process prediction formula:

consider. $f(x) \sim N(\mu(x), k(x,x))$ ⇒ noise free.

$Y = f(x) + \varepsilon \sim N(\mu(x), K(x,x) + \sigma^2 I)$ ⇒ noise

prediction: Given $X^* = (x_1^*, x_2^*, \dots, x_n^*)$

$Y^* = f(X^*) + \varepsilon$

$$\begin{pmatrix} Y \\ f(X^*) \end{pmatrix} \sim N\left[\begin{pmatrix} \mu(x) \\ \mu(x^*) \end{pmatrix}, \begin{pmatrix} k(x,x)+\sigma^2 & k(x,x^*) \\ k(x^*,x) & k(x^*,x^*) \end{pmatrix}\right]$$

condition possibility ⇓

$P(f(x^*) \mid Y, X, X^*) \xrightarrow{\text{formula}}$

formula:
Given $X \sim (\mu, \Sigma)$

$X = \begin{pmatrix} x_a \\ x_b \end{pmatrix}$ $\mu = \begin{pmatrix} \mu_a \\ \mu_b \end{pmatrix}$ $\Sigma = \begin{pmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{pmatrix}$

$x_b \mid x_a \sim N(\mu_{b|a}, \Sigma_{b|a})$

$\mu_{b|a} = \Sigma_{ba}\Sigma_{aa}^{-1}(x_a - \mu_a) + \mu_b$

$\Sigma_{b|a} = \Sigma_{bb} - \Sigma_{ba}\Sigma_{aa}^{-1}\Sigma_{ab}$

We can tackle the Gaussian process motion planning problem with this pattern.

LTV-SDE:

$$\dot{\boldsymbol{\theta}}(t) = \mathbf{A}(t)\theta(t) + \mathbf{u}(t) + \mathbf{F}(t)\mathbf{w}(t)$$

Gaussian process:

$$\mathbf{w}(t) \sim \mathcal{GP}(\mathbf{0}, \mathbf{Q}_C\delta(t - t'))$$

Answer for LTV-SDE:

$$\boldsymbol{\theta}(t) = \boldsymbol{\Phi}(t, t_0)\boldsymbol{\theta}_0 + \int_{t_0}^{t} \boldsymbol{\Phi}(t, s)(\mathbf{u}(s) + \mathbf{F}(s)\mathbf{w}(s))\mathrm{d}s$$

The mean and coveriance of Gaussian process can be got from LTV-SDE answer.

$$\widetilde{\mu}(t) = \boldsymbol{\Phi}(t, t_0)\boldsymbol{\mu}_0 + \int_{t_0}^{t} \boldsymbol{\Phi}(t, s)\mathbf{u}(s)\ \mathrm{d}s \qquad ($$

$$\widetilde{\mathcal{K}}(t, t') = \boldsymbol{\Phi}(t, t_0)\mathcal{K}_0\boldsymbol{\Phi}(t', t_0)^{\mathrm{T}} + \int_{t_0}^{\min(t, t')} \boldsymbol{\Phi}(t, s)\mathbf{F}(s)\mathbf{Q}_C\mathbf{F}(s)^{\mathrm{T}}\boldsymbol{\Phi}(t', s)^{\mathrm{T}}\mathrm{d}s$$