

Strengthened stability analysis of discrete-time Lurie systems involving ReLU neural networks

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1. Motivation

Model-based control leverages a system model for control policy design with performance certificates. Learning-based methods, like RL, offer expressive policies but lack performance certificates, making them unsuitable for safety critical systems. To bridge this gap, we develop novel criteria to certificate the closed loop stability of systems using learning-based NN control policies, combining the strengths of both paradigms: performance certification from model-based methods and expressive policies from learning-based methods.

2. Problem setup

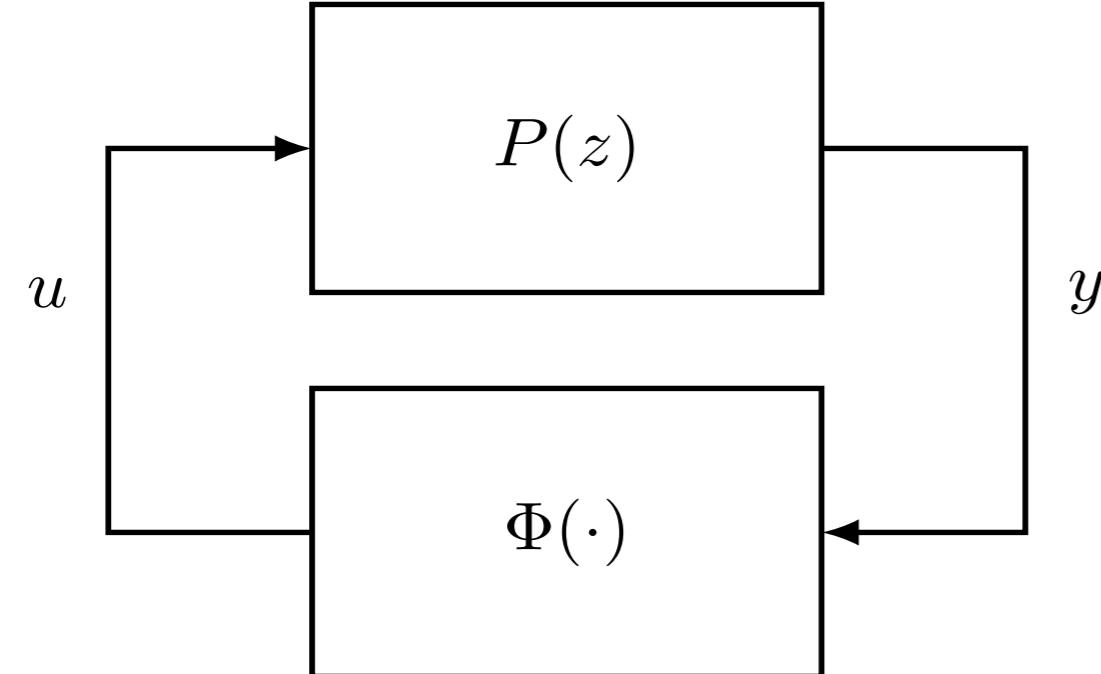


Figure 1. DT Lurie system with repeated ReLU nonlinearity

Consider the feedback interconnection in Fig. 1, where $P(z) \in \mathcal{RH}_\infty$ is a LTI system, with state space realisation (A, B, C, D) and the **repeated ReLU** $\Phi(\cdot)$ is just the ReLU function $\phi(\cdot)$ applied element-wise. Many systems involving a NN with ReLU activations are instances of such a setup. E.g. LTI system interconnected with an L -layer feed-forward NN.

3. Quadratic constraints satisfied by the repeated ReLU

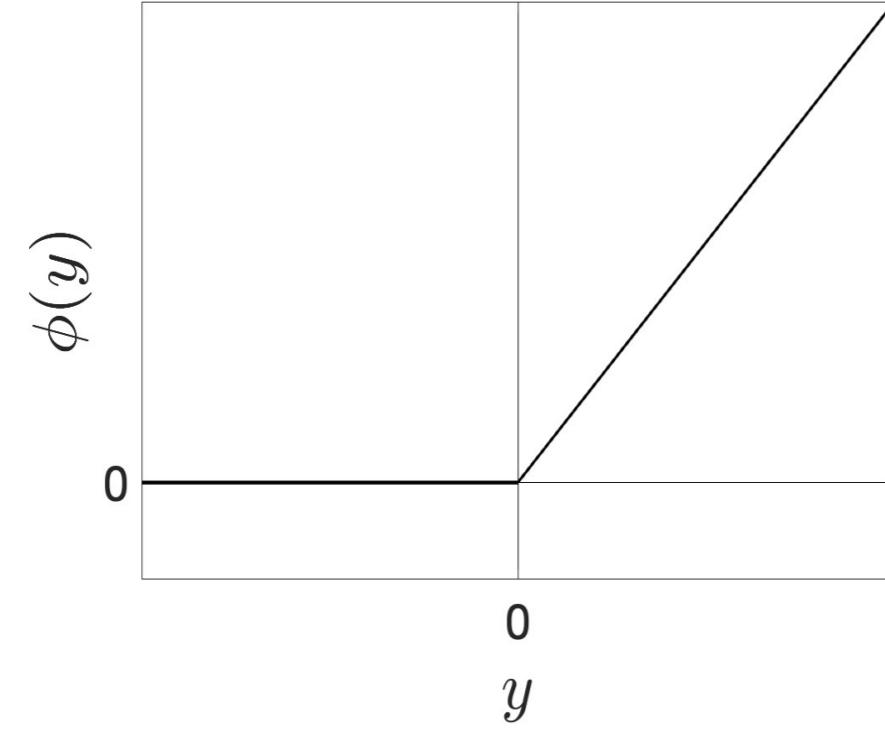


Figure 2. ReLU function

Let $\Phi(\cdot)$ be the repeated ReLU and $\Psi(\tilde{y}, y) := \Phi(\tilde{y}) - \Phi(y)$. If $\mathbf{Q}_{11} \in \mathbb{R}_{\geq 0}^{m \times m}$, $\mathbf{V} \in \mathcal{Z}^m$, $\mathbf{W} \in \mathcal{D}_+^m$ then the following quadratic inequalities hold:

$$\Phi(y)' \mathbf{Q}_{11} \Phi(\tilde{y}) \geq 0 \quad \forall y, \tilde{y} \in \mathbb{R}^m \quad \text{Positivity QC} \quad (1)$$

$$\Phi(y)' \mathbf{V} [\tilde{y} - \Phi(\tilde{y})] \geq 0 \quad \forall y \in \mathbb{R}^m \quad \text{Sector-like QC} \quad (2)$$

$$\Psi(\tilde{y}, y)' \mathbf{W} [\tilde{y} - y - \Psi(\tilde{y}, y)] \geq 0 \quad \forall \tilde{y}, y \in \mathbb{R}^m \quad \text{Slope-restricted QC} \quad (3)$$

4. DT Circle-like Criterion

Consider the DT Lurie system in Fig. 1. If there exists $\mathbf{P} \in \mathcal{S}_+^n$, $\mathbf{V} \in \mathcal{Z}^m$, $\mathbf{Q}_{11} \in \mathbb{R}_{\geq 0}^{m \times m}$ such that:

$$\begin{bmatrix} \mathbf{A}' \mathbf{P} \mathbf{A} - \mathbf{P} & \mathbf{A}' \mathbf{P} \mathbf{B} + \mathbf{C}' \mathbf{V}' \\ * & \mathbf{B}' \mathbf{P} \mathbf{B} + \text{He}(\mathbf{Q}_{11} - \mathbf{V}(I - D)) \end{bmatrix} \prec 0$$

then the origin of the DT Lurie system is **globally asymptotically stable (GAS)**.

The proof follows standard Lyapunov arguments starting from $V_c(x) = x' \mathbf{P} x$ and appending (1), (2) to $\Delta V_c(x)$ to setup the **LMI**.

5. DT Popov-like Criterion

Consider the DT Lurie system in Fig. 1 with $D = 0$. If there exists $\mathbf{P} \in \mathcal{S}_+^n$; $\mathbf{\Lambda}, \mathbf{W} \in \mathcal{D}_+^m$; $\mathbf{V} \in \mathcal{Z}^m$; $\mathbf{Q}_{11}, \tilde{\mathbf{Q}}_{11} \in \mathbb{R}_{\geq 0}^{m \times m}$ such that:

$$\begin{bmatrix} \mathbf{X}_{11} & \mathbf{X}_{12} & (\mathbf{A} - \mathbf{I})' \mathbf{C}' \mathbf{\Lambda} \\ * & \mathbf{X}_{22} & \mathbf{B}' \mathbf{C}' \mathbf{\Lambda} + \tilde{\mathbf{Q}}_{11} \\ * & * & -2\mathbf{W} \end{bmatrix} \prec 0$$

$$X_{11} = \mathbf{A}' \mathbf{P} \mathbf{A} - \mathbf{P} + \text{He}((\mathbf{A} - \mathbf{I})' \mathbf{C}' \mathbf{\Lambda} (\mathbf{A} - \mathbf{I}))$$

$$X_{12} = \mathbf{A}' \mathbf{P} \mathbf{B} + \mathbf{C}' \mathbf{V}' + (\mathbf{A} - \mathbf{I})' \mathbf{C}' \mathbf{\Lambda} + 2(\mathbf{A} - \mathbf{I})' \mathbf{C}' \mathbf{\Lambda} \mathbf{C} \mathbf{B}$$

$$X_{22} = \mathbf{B}' \mathbf{P} \mathbf{B} + \text{He}(\mathbf{Q}_{11} - \mathbf{V} + \mathbf{\Lambda} \mathbf{C} \mathbf{B} + \mathbf{B}' \mathbf{C}' \mathbf{\Lambda} \mathbf{C} \mathbf{B} + \tilde{\mathbf{Q}}_{11})$$

then the origin of the DT Lurie system is **GAS**.

The proof follows Lyapunov arguments starting from $V_p(x) = x' \mathbf{P} x + 2 \int_0^y \mathbf{\Lambda} \Phi(\sigma) \cdot d\sigma$ and appending (1)-(3) to $\Delta V_p(x)$.

6. Local stability analysis

The DT Lurie system in Fig. 1 has a unique equilibrium point at the origin if all matrices in the set \mathcal{D} are full rank.

$$\mathcal{D} = \left\{ \mathbf{A} - \mathbf{I} + \mathbf{B} \mathbf{U} (\mathbf{I} - \mathbf{D} \mathbf{U})^{-1} \mathbf{C} \mid \mathbf{U} \in \mathcal{U} \right\}$$

The 2^m possible permutations of \mathbf{U} are captured by the set:

$$\mathcal{U} = \{ \text{diag}(u_1, \dots, u_m) \mid u_i \in \{0, 1\} \text{ and } i \in 1, 2, \dots, m \}.$$

If the origin of the DT Lurie system in Fig. 1 is a unique equilibrium point and a ball of radius r_x can be established as a region of attraction; then, the origin is actually **GAS**.

7. Numerical examples

Ex	m	Max. series gain (left) and # of decision variables (right)					
		Circle	Circle-like	Popov	Popov-like	Park	
1	3	20.8	48	39.4	63	411	51
3	4	0.52	10	0.68	38	0.52	14
6	4	0.19	25	0.39	53	0.19	29
				3310	66	450	579
				0.68	42	0.52	207
				0.50	57	0.22	402

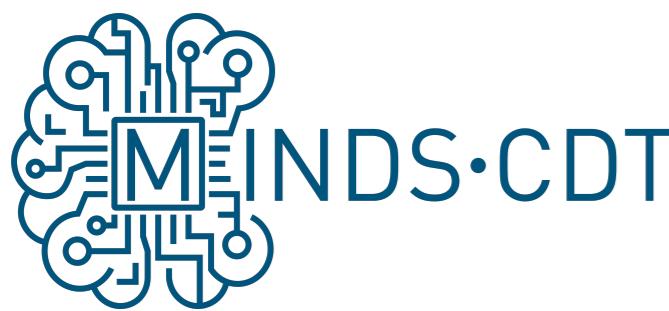
Table 1. Maximum series gain α for which GAS can be verified using various criteria

8. Conclusion

- The strengthened criteria are specialised for the repeated ReLU.
- New criteria strike an appealing balance of **conservatism and complexity** compared to existing criteria.
- Section 6 shows that if global stability of the DT Lurie system in Fig. 1 cannot be established, then it is **futile to attempt to establish local stability**. This is counterintuitive since it is typical, in absolute stability analysis, to attempt a local stability analysis if one cannot establish global stability.

9. Notation

\mathcal{RH}_∞	real rational transfer function matrices
\mathcal{Z}^m	$m \times m$ matrices with non-positive off-diagonal elements
$\mathbb{R}_{\geq 0}^{m \times m}$	$m \times m$ matrices with non-negative elements
\mathcal{S}_+^m	$m \times m$ symmetric positive definite matrices
\mathcal{D}_+^m	diagonal subset of \mathcal{S}_+^m
$\text{He}(\cdot)$	$\text{He}(A) := A + A'$
m	number of NN activation functions



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