Strengthened stability analysis of discrete-time Lurie systems involving ReLU neural networks

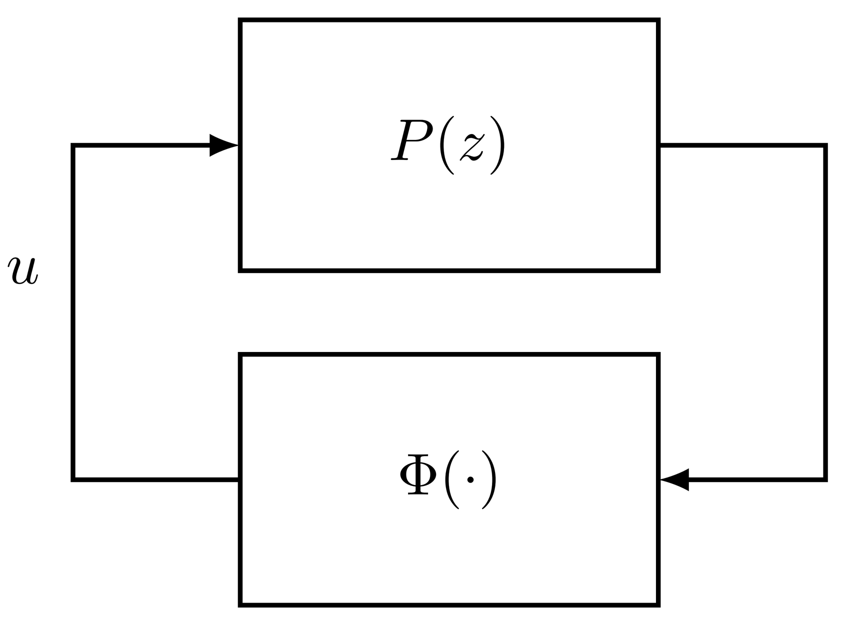
Carl R. Richardson[1], Matthew C. Turner[1], Steve R. Gunn[1], Ross Drummond[2]

University of Southampton [1] University of Sheffield [2]

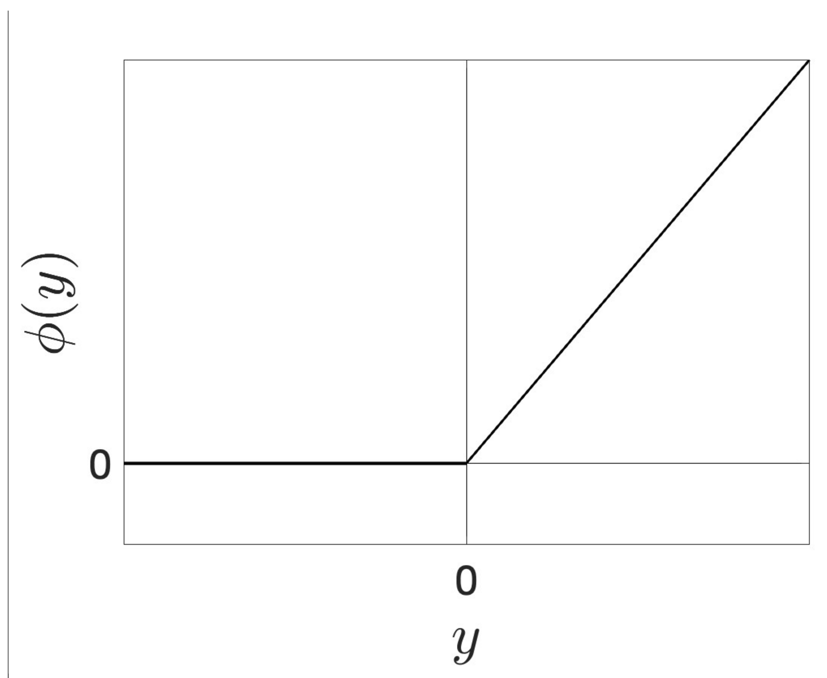
1. Motivation

Model-based control leverages a system model for control policy design with performance certificates. *Learning-based* methods, like RL, offer expressive policies/models but lack performance certificates, making them unsuitable for safety critical systems. To bridge this gap, we develop novel criteria to certificate the closed loop stability of systems involving learning-based NN policies/models, combining the strengths of both paradigms: performance certification from model-based methods and expressive policies/models from learning-based methods.

2. Problem setup



(a) DT Lurie system with repeated ReLU nonlinearity



(b) ReLU function

Consider the feedback interconnection in Fig. 1a, where $P(z) \in \mathcal{RH}_\infty$ is a LTI system, with state space realisation (A, B, C, D) and the **repeated ReLU** $\Phi(\cdot)$ is just the ReLU function $\phi(\cdot)$ applied element-wise. Many systems involving a NN with ReLU activations are instances of such a setup. E.g. LTI system interconnected with an L-layer feed-forward NN (without biases).

3. Properties satisfied by the ReLU and Repeated ReLU functions

$\phi(v) \geq 0$	$\forall v \in \Re$	Positive
$\phi(v) - v \geq 0$	$\forall v \in \Re$	Positive complement
$\phi(v)(v - \phi(v)) = 0$	$\forall v \in \Re$	Complementarity
$\phi(\alpha v) = \alpha\phi(v)$	$\forall v \in \Re, \alpha \in \Re_{\geq 0}$	Positive homogeneity
$0 \leq \frac{\phi(\tilde{v})-\phi(v)}{\tilde{v}-v} \leq 1$	$\forall \tilde{v}, v \neq \tilde{v} \in \Re$	Slope-restricted

Table 1. Properties of the ReLU function

Let $\Phi(\cdot)$ be the repeated ReLU and $\Psi(\tilde{y}, y) := \Phi(\tilde{y}) - \Phi(y)$. If $\mathbf{Q}_{11} \in \Re^{m\times m}_{\geq 0}$, $\mathbf{V} \in \mathcal{Z}^m$, $\mathbf{W} \in \mathcal{D}^m_+$ then the following quadratic inequalities hold:

$$\Phi(y)'\mathbf{Q}_{11}\Phi(\tilde{y}) \geq 0 \quad \forall y, \tilde{y} \in \Re^m \qquad \text{Positivity QC} \qquad (1)$$

$$\Phi(y)'\mathbf{V}[y - \Phi(y)] \geq 0 \quad \forall y \in \Re^m \qquad \text{Sector-like QC} \qquad (2)$$

$$\Psi(\tilde{y}, y)'\mathbf{W}[\tilde{y} - y - \Psi(\tilde{y}, y)] \geq 0 \quad \forall \tilde{y}, y \in \Re^m \qquad \text{Slope-restricted QC} \qquad (3)$$

4. DT Circle-like Criterion

Consider the DT Lurie system in Fig. 1a. If there exists $\mathbf{P} \in \mathcal{S}^n_+$, $\mathbf{V} \in \mathcal{Z}^m$, $\mathbf{Q}_{11} \in \Re^{m\times m}_{\geq 0}$ such that:

$$\begin{bmatrix} A'\mathbf{P}A - \mathbf{P} & A'\mathbf{P}B + C'\mathbf{V}' \\ \star & B'\mathbf{P}B + He(\mathbf{Q}_{11} - \mathbf{V}(I - D)) \end{bmatrix} \prec 0$$

then the origin of the DT Lurie system is **globally asymptotically stable (GAS)**.

The proof follows standard Lyapunov arguments starting from $V_c(x) = x'Px$ and appending (1), (2) to $\Delta V_c(x)$ to setup the **LMI**.

5. DT Popov-like Criterion

Consider the DT Lurie system in Fig. 1a with $D = 0$. If there exists $\mathbf{P} \in \mathcal{S}^n_+$; $\mathbf{\Lambda}, \mathbf{W} \in \mathcal{D}^m_+$; $\mathbf{V} \in \mathcal{Z}^m$; $\mathbf{Q}_{11}, \tilde{\mathbf{Q}}_{11} \in \Re^{m\times m}_{\geq 0}$ such that:

$$\begin{bmatrix} X_{11} & X_{12} & (A - I)'C'\mathbf{\Lambda} \\ \star & X_{22} & B'C'\mathbf{\Lambda} + \tilde{\mathbf{Q}}_{11} \\ \star & \star & -2\mathbf{W} \end{bmatrix} \prec 0$$

$$X_{11} = A'\mathbf{P}A - \mathbf{P} + He\left((A - I)'C'\mathbf{\Lambda}C(A - I)\right)$$

$$X_{12} = A'\mathbf{P}B + C'\mathbf{V}' + (A - I)'C'\mathbf{\Lambda} + 2(A - I)'C'\mathbf{\Lambda}CB$$

$$X_{22} = B'\mathbf{P}B + He\left(\mathbf{Q}_{11} - \mathbf{V} + \mathbf{\Lambda}CB + B'C'\mathbf{\Lambda}CB + \tilde{\mathbf{Q}}_{11}\right)$$

then the origin of the DT Lurie system is **GAS**.

The proof follows Lyapunov arguments starting from $V_p(x) = x'\mathbf{P}x + 2\int_0^y \mathbf{\Lambda}\Phi(\sigma) \cdot d\sigma$ and appending (1)-(3) to $\Delta V_p(x)$.

6. Local ≡ Global stability analysis

The DT Lurie system in Fig. 1a has a unique equilibrium point at the origin if all matrices in the set $\mathcal{D}$ are full rank.

$$\mathcal{D} = \left\{A - I + BU(I - DU)^{-1}C \,|\, U \in \mathcal{U}\right\}$$

The 2^m possible permutations of U are captured by the set:

$$\mathcal{U} = \{\mathrm{diag}(u_1, \ldots, u_m) \,|\, u_i \in 0, 1 \text{ and } i \in 1, 2, \ldots, m\}.$$

If the origin of the DT Lurie system is a unique equilibrium point and a ball of any radius r_x can be established as a region of attraction, then, the origin is in fact GAS.

The proof relies on the positive homogeneity property of the ReLU function (Table 1). This allows any positively scaled DT Lurie system, with a nonlinearity satisfying the positive homogeneity property, to be expressed in the same form as the unscaled system.

7. Numerical examples

			Max. series gain (left) and # of decision variables (right)										Nyquist
Ex	n	m	Circle		Circle-like		Popov		Popov-like		Park		gain
1	9	3	20.8	48	39.4	63	411	51	**3310**	66	450	579	6666
3	3	4	0.52	10	**0.68**	38	0.52	14	**0.68**	42	0.52	207	0.69
6	6	4	0.19	25	0.39	53	0.19	29	**0.50**	57	0.22	402	0.82

Table 2. Maximum series gain α for which GAS can be verified using various criteria

8. Conclusion

- The strengthened criteria are specialised for the repeated ReLU.
- New criteria strike an appealing balance of **conservatism and complexity** compared to existing criteria.
- Section 6 shows that if GAS of the DT Lurie system in Fig. 1a cannot be established, then it is **futile to attempt to establish local stability**. This is counter intuitive since it is typical, in absolute stability analysis, to attempt a local stability analysis if one cannot establish GAS.

9. Notation

$\mathcal{RH}_\infty$	real rational transfer function matrices
$\mathcal{Z}^m$	$m \times m$ matrices with non-positive off-diagonal elements
$\Re^{m\times m}_{\geq 0}$	$m \times m$ matrices with non-negative elements
$\mathcal{S}^m_+$	$m \times m$ symmetric positive definite matrices
$\mathcal{D}^m_+$	diagonal subset of $\mathcal{S}^m_+$
$He(\cdot)$	$He(A) := A + A'$
m	number of NN activation functions

https://github.com/CR-Richardson/DT-Max-Series-Gain cr2g16@soton.ac.uk