Strengthened stability analysis of discrete-time Lurie systems involving ReLU neural networks

Carl R. Richardson, Matthew C. Turner, Steve R. Gunn

1. Motivation

Model-based control leverages a system model and interpretable control policies accompanied with performance certificates. On the other hand, learning-based methods, like RL, offer expressive policies and models but lack performance certificates. To bridge this gap, we develop novel criteria to certificate the closed loop stability of systems involving NNs, thus combining the strengths of both paradigms: expressive policies/models accompanied by performance certification.

2. Problem setup

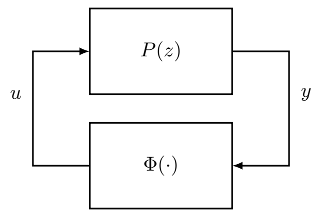

Figure 1. DT Lurie system with repeated ReLU nonlinearity

Consider the feedback interconnection in Fig. 1, where $P(z) \in \mathcal{RH}_\infty$ is a LTI system, with state space realisation (A, B, C, D) and the repeated ReLU is just the ReLU function applied element-wise. Many systems involving a NN with ReLU activations are instances of such a setup. E.g. LTI system interconnected with a feed-forward NN.

3. Quadratic constraints satisfied by the repeated ReLU

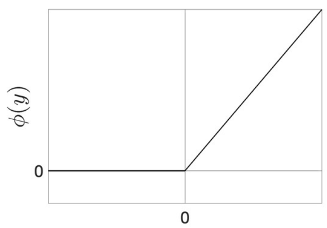

Figure 2. ReLU function

Let $\Phi(\cdot)$ be the repeated ReLU and $\Psi(\tilde{y}, y) := \Phi(\tilde{y}) - \Phi(y)$. If $V \in \mathcal{Z}^m, Q_{11} \in \mathbb{R}^{m\times m}_{\geq 0}, W \in \mathcal{D}^m_+$ then the following quadratic inequalities hold:

$$\Phi(y)'\mathbf{Q}_{11}\Phi(\tilde{y}) \geq 0 \quad \forall y, \tilde{y} \in \Re^m \qquad \text{Positivity QC} \qquad (1)$$

$$\Phi(y)'\mathbf{V}[y - \Phi(y)] \geq 0 \quad \forall y \in \Re^m \qquad \text{Sector-like QC} \qquad (2)$$

$$\Psi(\tilde{y}, y)'\mathbf{W}[\tilde{y} - y - \Psi(\tilde{y}, y)] \geq 0 \quad \forall \tilde{y}, y \in \Re^m \qquad \text{Slope-restricted QC} \qquad (3)$$

4. DT Circle-like Criterion

Consider the Lurie system in Fig. 1. If there exists $P \in \mathcal{S}^n_+, V \in \mathcal{Z}^m, Q_{11} \in \mathbb{R}^{m\times m}_{\geq 0}$ such that:

$$\begin{bmatrix} A'\mathbf{P}A - \mathbf{P} & A'\mathbf{P}B + C'\mathbf{V}' \\ \star & B'\mathbf{P}B + He(\mathbf{Q}_{11} - \mathbf{V}(I - D)) \end{bmatrix} \prec 0$$

then the origin of the DT Lurie system is **globally asymptotically stable (GAS)**.
The proof follows standard Lyapunov arguments starting from $V_c(x) = x'Px$ and appending (1), (2) to $\Delta V_c(x)$ to setup the **LMI.**

5. DT Popov-like Criterion

Consider the Lurie system in Fig. 1 with $D = 0$. If there exists $P \in \mathcal{S}^n_+$; $\Lambda \in \mathcal{D}^m_+$; $V \in \mathcal{Z}^m$; $Q_{11}, \tilde{Q}_{11} \in \mathbb{R}^{m\times m}_{\geq 0}$ such that:

$$\begin{bmatrix} X_{11} & X_{12} \\ \star & X_{22} \end{bmatrix} \prec 0$$

$$X_{11} = A'\mathbf{P}A - \mathbf{P} + He\left((A - I)'C'\mathbf{\Lambda}C(A - I)\right)$$

$$X_{12} = A'\mathbf{P}B + C'\mathbf{V}' + (A - I)'C'\mathbf{\Lambda} + 2(A - I)'C'\mathbf{\Lambda}CB$$

$$X_{22} = B'\mathbf{P}B + He\left(\mathbf{Q}_{11} - \mathbf{V} + \mathbf{\Lambda}CB + B'C'\mathbf{\Lambda}CB + \tilde{\mathbf{Q}}_{11}\right)$$

then the origin of the DT Lurie system is **GAS**.
The proof follows non-standard Lyapunov arguments starting from $V_p(x) = x'Px + 2\int_0^{Hy} \Lambda\Phi(\sigma)\cdot d\sigma$ and appending (1)-(3) to $\Delta V_p(x)$.

6. Local stability analysis

The DT Lurie system in Fig. 1 has a unique equilibrium point at the origin if all matrices in the set D are full rank.

$$\mathcal{D} = \left\{A - I + BU(I - DU)^{-1}C \mid U \in \mathcal{U}\right\}$$

The 2^m possible permutations of U are captured by the set:

$$\mathcal{U} = \{\text{diag}(u_1, \ldots, u_m) \mid \; u_i \in 0, 1 \text{ and } i \in 1, 2, \ldots, m\}.$$

If the origin of the DT Lurie system is a unique equilibrium point and a ball of radius r_x can be established as a region of attraction; then, the origin is actually **GAS**.

7. Numerical examples

Ex	n	m	Max. series gain (left) and # of decision variables (right)										Nyquist
			Circle		Circle-like		Popov		Popov-like		Park		gain
1	9	3	20.8	48	39.4	63	411	51	**3310**	66	450	579	6666
3	3	4	0.52	10	**0.68**	38	0.52	14	**0.68**	42	0.52	207	0.69
6	6	4	0.19	25	0.39	53	0.19	29	**0.50**	57	0.22	402	0.82

Table 1. Comparison of maximum series gain α for which GAS can be maintained using various criteria

8. Conclusion

- The strengthened criteria are specialised for the repeated ReLU.
- New criteria strike an appealing balance of **conservatism and complexity** compared to existing criteria.
- Section 6 shows that if GAS of the DT Lurie system in Fig. 1 cannot be established, then it is **futile to attempt to establish local stability**. This is counterintuitive since it is typical, in absolute stability analysis, to attempt a local stability analysis if one cannot establish GAS.

9. Notation

$\mathcal{RH}_\infty$	real rational transfer function matrices
$\mathcal{Z}^m$	$m \times m$ matrices with non-positive off-diagonal elements
$\Re^{m\times m}_{\geq 0}$	$m \times m$ matrices with non-negative elements
$\mathcal{S}^m_+$	$m \times m$ symmetric positive definite matrices
$\mathcal{D}^m_+$	diagonal subset of $\mathcal{S}^m_+$
$He(\cdot)$	$He(A) := A + A'$
m	number of NN activation functions

https://github.com/CR-Richardson/DT-Max-Series-Gain