Strengthened Circle and Popov Criteria for the stability analysis of feedback systems with ReLU neural networks

Carl R. Richardson, Matthew C. Turner, Steve R. Gunn

University of Southampton

1. Motivation

Neural network (NN) based control has become of interest due to recent successes in RL. To extend its range of application to safety critical systems, performance certificates, such as closed loop stability, must be verified.

Problem 1: Can properties of the ReLU function be leveraged to find non-conservative, convex stability criteria for Lurie systems in Fig. 1?

2. Problem setup

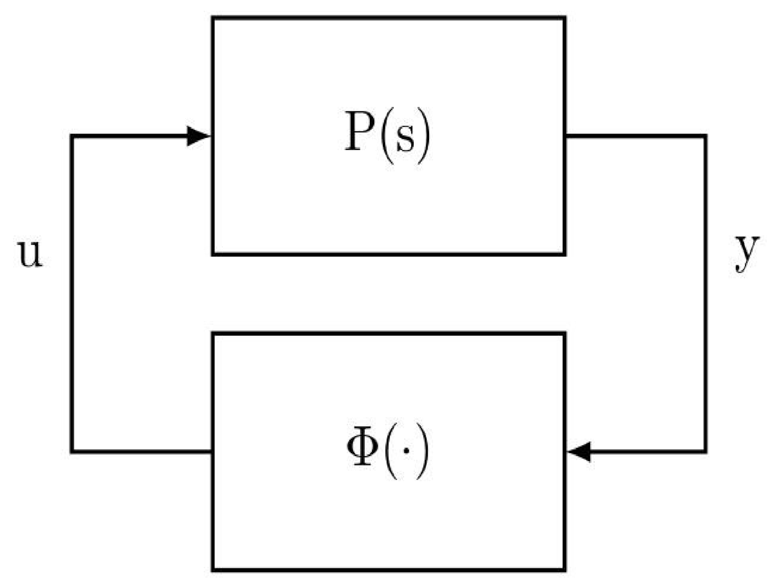

Figure 1. Lurie system with repeated ReLU nonlinearity

Consider the feedback interconnection in Fig. 1, where $P(s) \in \mathcal{RH}_\infty$ is a LTI system, with state space realisation (A, B, C, D) and the **repeated ReLU** $\Phi(\cdot)$ is just the ReLU function $\phi(\cdot)$ applied element-wise. Many systems involving a NN with ReLU activations are instances of such a setup. E.g. continuous-time RNN and a LTI system interconnected with a feed-forward NN.

3. Properties of the ReLU function

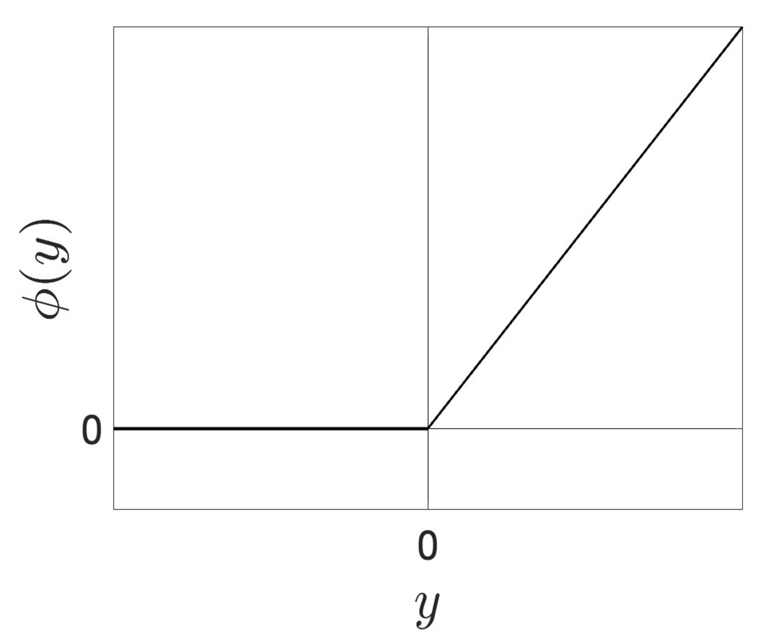

Figure 2. ReLU function

$\phi(y) \geq 0$	$\forall y \in \Re$	Positivity
$\phi(\beta y) = \beta\phi(y)$	$\forall y \in \Re, \beta \in \Re_{\geq 0}$	Positive homogeneity
$\phi(y) - y \geq 0$	$\forall y \in \Re$	Positive complement
$\phi(y)(y - \phi(y)) = 0$	$\forall y \in \Re$	Complementarity
$0 \leq \frac{\phi(y)}{y} \leq 1$	$\forall y \in \Re$	Sector-boundedness
$0 \leq \frac{\phi(\tilde{y})-\phi(y)}{\tilde{y}-y} \leq 1$	$\forall \tilde{y}, y \neq \tilde{y} \in \Re$	Slope-restriction

Table 1. Properties of the ReLU function

4. Quadratic constraints (QC)

Let $\Phi(\cdot)$ be the repeated ReLU and $\Psi(\tilde{y}, y) := \Phi(\tilde{y}) - \Phi(y)$. If $\mathbf{V} \in \mathcal{Z}^m$, $\mathbf{Q}_{11} \in \Re^{m\times m}_{\geq 0}$, $\mathbf{W} \in \mathcal{D}^m_+$ then the following quadratic inequalities hold:

$$\Phi(y)'\mathbf{Q}_{11}\Phi(\tilde{y}) \geq 0 \quad \forall y, \tilde{y} \in \Re^m \quad \text{Positivity QC} \quad (1)$$

$$\Phi(y)'\mathbf{V}[y - \Phi(y)] \geq 0 \quad \forall y \in \Re^m \quad \text{Sector-like QC} \quad (2)$$

$$\Psi(\tilde{y}, y)'\mathbf{W}[\tilde{y} - y - \Psi(\tilde{y}, y)] \geq 0 \quad \forall \tilde{y}, y \in \Re^m \quad \text{Slope-restricted QC} \quad (3)$$

These QC were derived by leveraging properties of the ReLU function.

5. Circle-like Criterion

Consider the Lurie system in Fig. 1. If there exists $\mathbf{P} \in \mathcal{S}^m_+$, $\mathbf{V} \in \mathcal{Z}^m$, $\mathbf{Q}_{11} \in \Re^{m\times m}_{\geq 0}$ such that:

$$\begin{bmatrix} He(A'\mathbf{P}) & \mathbf{P}B + C'\mathbf{V}' \\ \star & He(\mathbf{Q}_{11} - \mathbf{V}(I - D)) \end{bmatrix} \prec 0$$

then the origin of the Lurie system is **globally asymptotically stable**.

The proof follows standard Lyapunov arguments starting from $V_c(x) = x'Px$ and appending (1), (2) to $\dot{V}_c(x)$ to setup the **LMI**.

6. Popov-like Criterion

Consider the Lurie system in Fig. 1 with $D = 0$. If there exists $\mathbf{P} \in \mathcal{S}^m_+$; $\mathbf{H} \in \Re^{m\times m}$; $\mathbf{\Lambda}, \mathbf{W} \in \mathcal{D}^m_+$; $\mathbf{V} \in \mathcal{Z}^m$; $\mathbf{Q}_{11}, \tilde{\mathbf{Q}}_{11} \in \Re^{m\times m}_{\geq 0}$ such that:

$$\begin{bmatrix} He(A'\mathbf{P}) & \mathbf{P}B + C'\mathbf{V}' + A'C'\mathbf{H}'\mathbf{\Lambda} & A'C'\mathbf{H}'\mathbf{\Lambda} + C'(\mathbf{H}' - I)\mathbf{W} \\ \star & He(\mathbf{\Lambda}\mathbf{H}CB - \mathbf{V} + \tilde{\mathbf{Q}}_{11} + \mathbf{Q}_{11}) & B'C'\mathbf{H}'\mathbf{\Lambda} + \tilde{\mathbf{Q}}_{11} \\ \star & \star & -2\mathbf{W} \end{bmatrix} \prec 0$$

then the origin of the Lurie system is **globally asymptotically stable**.

The proof follows non-standard Lyapunov arguments starting from $V_p(x) = x'\mathbf{P}x + 2\int_0^{\mathbf{H}y} \mathbf{\Lambda}\Phi(\sigma) \cdot d\sigma$ and appending (1)-(3) to $\dot{V}_p(x)$. **Convex relaxations** were required to transform this **BMI** into an **LMI** to satisfy **Problem 1**.

7. Numerical examples

		Maximum value of α						
Ex.	m	Circle	**Circle-like**	Popov	**Popov-like 1**	**Popov-like 2**	Park	ZF
1	3	20	39	434	**52430**	9990	448	435
3	4	0.52	**0.68**	0.52	**0.68**	**0.68**	0.52	0.55
6	4	0.19	**0.39**	0.19	0.41	**0.50**	0.22	0.27
7	4	0.09	**0.12**	0.09	0.12	**0.14**	0.10	0.10

Table 2. Comparison of maximum series gain α for which global asymptotic stability can be maintained using various criteria

8. Conclusion

- The strengthened criteria are specialised for the repeated ReLU.
- The new criteria have, potentially, **much lower levels of conservatism** than the standard Circle and Popov Criteria whilst retaining much of their computational appeal.
- **Popov-like 2** in particular, provides low levels of conservatism without the high computational overhead of approaches such as Zames-Falb analysis.
- The highlight numerical result is the **Circle-like Criterion being less conservative than Park and Zames-Falb** (Ex. 3, 6, 7). This gives one confidence that the strengthened criteria should perform well when used in NN analysis, where m may be orders of magnitude higher.

9. Notation

$\mathcal{RH}_\infty$	real rational transfer function matrices
$\mathcal{Z}^m$	$m \times m$ matrices with non-positive off-diagonal elements
$\Re^{m\times m}_{\geq 0}$	$m \times m$ matrices with non-negative elements
$\mathcal{S}^m_+$	$m \times m$ symmetric positive definite matrices
$\mathcal{D}^m_+$	diagonal subset of $\mathcal{S}^m_+$
$He(\cdot)$	$He(A) := A + A'$
m	number of NN activation functions

https://github.com/CR-Richardson/Max-Series-Gain

cr2g16@soton.ac.uk