Strengthened Circle and Popov Criteria and the Analysis of ReLU Neural Networks

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Motivation

Neural network (NN) based control has become of interest due to recent successes in RL and IL. Advantages include:

- Ability to learn expressive policies from data
- Accurate model of the system is not necessary
- Single controller for systems with interacting sub-systems

Problem: need some form of safety guarantees before deploying on safety critical systems e.g., closed loop stability.

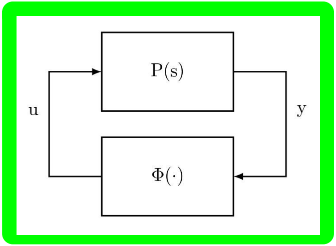
Neural Network Analysis

Several systems involving NNs can be modelled by a *Lurie system*: an LTI component interconnected with a nonlinearity.

The nonlinearity, $\Phi(\cdot)$, is comprised of all the NN activations and the NN weights are part of the LTI component. For example:

- LTI system in feedback with an L-layer feedforward NN
- Rate-based recurrent neural network (RNN)

NN analysis: the stability analysis of a Lurie system representing a system involving a NN.



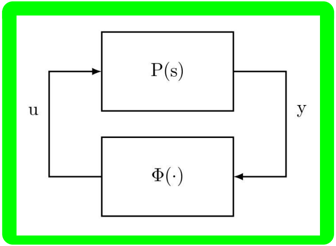
Absolute Stability

Lurie systems have been well-studied in the context of the *Absolute Stability* Problem.

That is, $\Phi(\cdot)$ is assumed to satisfy some condition (E.g., Sector-bounded, Slope-restricted, ...) and the problem is to verify stability for the set of nonlinearities which satisfy that condition.

Existing Absolute Stability Criteria include: Circle, Popov, Park, Zames-Falb multipliers, ...

$$\begin{aligned} \dot{x} &= Ax + Bu & x &\in \Re^n \\ y &= Cx + Du & y &\in \Re^m \\ u &= \Phi(y) & u &\in \Re^m \end{aligned} \quad \Longleftrightarrow$$



Existing Absolute Stability Criteria

Typically follow Lyapunov style proofs where positive terms, in the form of quadratic constraints (QCs), are appended to the time derivative such that the final condition can be posed as an LMI.

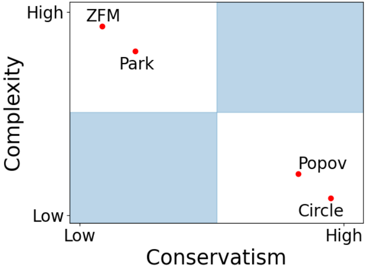
What differentiates the existing criteria?

1. Class of nonlinearities 2. Conservatism 3. Complexity

<u>Problem</u>: In NN analysis - m is typically large.

- Higher complexity criteria scale poorly with m
- Lower complexity criteria become too conservative

$$
\begin{aligned}
\dot{x} &= Ax + Bu & x &\in \Re^n \\
y &= Cx + Du & y &\in \Re^m \\
u &= \Phi(y) & u &\in \Re^m
\end{aligned}
$$



Contribution

Address the trade-off between complexity and conservatism in absolute stability problems encountered in NN analysis.

- Focus on the case when $\Phi(\cdot)$ is the repeated ReLU, as the ReLU function, $\phi(\cdot)$, is a popular activation function in deep learning.
- By reducing the size of the class of nonlinearities, can conservatism and complexity of the stability criteria be reduced?

<u>Problem 1</u>: Find tractable Lyapunov-based conditions which ensure the origin of the Lurie system (1) is globally asymptotically stable (GAS) when $\Phi(\cdot)$ is the repeated ReLU.

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \\ u &= \Phi(y) \end{aligned} \tag{1}$$

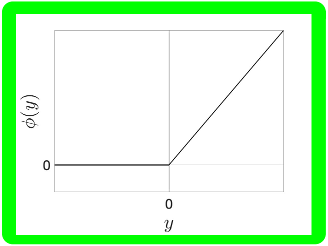
Properties of the ReLU function

Definition 1 (ReLU function)

$$\phi(y) = \begin{cases} y & y \geq 0 \\ 0 & y < 0 \end{cases}$$

Definition 2 (Repeated ReLU)

If $\phi(\cdot)$ is the ReLU, the repeated ReLU is: $\Phi(\cdot) = \left[\phi(\cdot) \ldots \phi(\cdot)\right]'$



$\phi(y) \geq 0$	$\forall y \in \Re$	Positivity
$\phi(y) - y \geq 0$	$\forall y \in \Re$	Positive complement
$\phi(y)(y - \phi(y)) = 0$	$\forall y \in \Re$	Complementarity
$0 \leq \frac{\phi(y)}{y} \leq 1$	$\forall y \in \Re$	Sector-bounded
$0 \leq \frac{\phi(y)-\phi(\tilde{y})}{y-\tilde{y}} \leq 1$	$\forall y, \tilde{y} \neq y \in \Re$	Slope-restricted

Standard QCs Satisfied by the Repeated ReLU

Fact 3 (Sector-bounded QC)

Let $\Phi(\cdot) : \Re^m \to \Re^m_{\geq 0}$ *be the repeated ReLU. If* $\mathbf{V} \in \mathcal{D}^m_+$ *then the following QC holds:*

$$\Phi(y)'\mathbf{V}\Big(y - \Phi(y)\Big) \geq 0 \quad \forall\, y \in \Re^m$$

Fact 4 (Slope-restricted QC)

Let $\Phi(\cdot) : \Re^m \to \Re^m_{\geq 0}$ *be the repeated ReLU and* $\Psi(\tilde{y}, y) := \Phi(\tilde{y}) - \Phi(y)$. *If* $\mathbf{W} \in \mathcal{D}^m_+$ *then the following QC is satisfied:*

$$\Psi(\tilde{y}, y)'\mathbf{W}\Big(\tilde{y} - y - \Psi(\tilde{y}, y)\Big) \geq 0 \quad \forall\, \tilde{y}, y \in \Re^m$$

Positive QC

Fact 5 (Positivity QC)

Let $\Phi(\cdot) : \Re^m \to \Re^m_{\geq 0}$ be the repeated ReLU. If $\mathbf{Q} \in \Re^{m \times m}_{\geq 0}$ then the following QC holds:

$$\Phi(y)' \mathbf{Q} \Phi(\tilde{y}) \geq 0 \quad \forall\, y, \tilde{y} \in \Re^m$$

Note that this also holds for the special case $\tilde{y} = y$.

Proof:

$\phi(y) \geq 0 \;\; \forall y \in \Re$	Positivity

$\sum_{i=1}^m \sum_{j=1}^m \phi(y_i) q_{ij} \phi(\tilde{y}_j) \geq 0$ where $q_{ij} \geq 0$

$\Re^{m \times m}_{\geq 0}$: $\mathbf{Q} = [q_{ij}]$ where $q_{ij} \geq 0$

Sector-like QC

Fact 6 (Sector-like QC)

Let $\Phi(\cdot) : \Re^m \to \Re^m_{\geq 0}$ be the repeated ReLU. If $\mathbf{V} \in \mathcal{Z}^m$ then the following QC holds:

$$\Phi(y)'\mathbf{V}\Big(y - \Phi(y)\Big) \geq 0 \;\; \forall \; y \in \Re^m$$

Proof:

$\phi(y) \geq 0$	$\forall y \in \Re$	Positivity
$\phi(y) - y \geq 0$	$\forall y \in \Re$	Positive complement
$\phi(y)(y - \phi(y)) = 0$	$\forall y \in \Re$	Complementarity

$$\textstyle\sum_{i=1}^m \sum_{j=1, j\neq i}^m \phi(y_i) v_{ij}\big(y_j - \phi(y_j)\big) \geq 0 \qquad \text{where } v_{ij} \leq 0$$

if $i = j$: $$\textstyle\sum_{i=1}^m \phi(y_i) v_{ii}\big(y_i - \phi(y_i)\big) \geq 0 \qquad \text{where } v_{ii} \in \Re$$

$\mathcal{Z}^m : \mathbf{V} = [v_{ij}]$ where $v_{ii} \in \Re$ and $v_{ij} \leq 0 \;\; \forall i \neq j$

Circle and Popov Criteria

Consider the Lurie system with $\Phi(\cdot)$ as the Repeated ReLU. Assume the system is well-posed. Then the proof of global asymptotic stability (GAS) is outlined as follows:

$$\begin{aligned} \dot{x} &= Ax + B\Phi(y) \quad x \in \Re^n \\ y &= Cx + D\Phi(y) \quad y \in \Re^m \end{aligned}$$

Circle Criterion:

1. $V_c(x) = x'\mathbf{P}x > 0 \;\forall\, x \neq 0$ where $\mathbf{P} \in \mathcal{S}_+^n$.
2. Show $\dot{V}_c(x) + \Phi(y)'\mathbf{V}\big(y - \Phi(y)\big) < 0 \;\forall\, x \neq 0$ for some $\mathbf{V} \in \mathcal{D}_+^m$.

Popov Criterion:

1. $V_p(x) = x'\mathbf{P}x + 2\int_0^y \mathbf{\Lambda}\Phi(\sigma) \cdot d\sigma > 0 \;\forall\, x \neq 0$ where $\mathbf{P} \in \mathcal{S}_+^n$ and $\mathbf{\Lambda} \in \mathcal{D}_+^m$.
2. Show $\dot{V}_p(x) + \Phi(y)'\mathbf{V}\big(y - \Phi(y)\big) < 0 \;\forall\, x \neq 0$ for some $\mathbf{V} \in \mathcal{D}_+^m$.

Main Results: Circle-like Criterion

Theorem 7 (Circle-like Criterion)

Consider the Lurie system (1) with $\Phi(\cdot)$ the repeated ReLU. Assume the system is well-posed. If there exists $\mathbf{P} \in \mathcal{S}_+^n$, $\mathbf{V} \in \mathcal{Z}^m$, $\mathbf{Q} \in \Re_{\geq 0}^{m\times m}$ such that:

$$\begin{bmatrix} He(A'\mathbf{P}) & \mathbf{P}B + C'\mathbf{V}' \\ \star & He\Big(\mathbf{Q} - \mathbf{V}(I - D)\Big) \end{bmatrix} \prec 0 \qquad (2)$$

then the origin of (1) is GAS.

- Solution space of the Circle Criterion is a subset of (2) when $\mathbf{V}$ and $\mathbf{Q}$ are reduced to $\mathbf{V} \in \mathcal{D}_+^m$ and $\mathbf{Q} = 0$. Hence, Theorem 7 must be less conservative than the Circle Criterion.

[0] $He(U) := U + U'$

Main Results: Popov-like Criterion

Theorem 8 (Popov-like Criterion)

Consider the Lurie system (1) with $\Phi(\cdot)$ the repeated ReLU and let $D = 0$. If there exists $\mathbf{P} \in \mathcal{S}_+^n$; $\mathbf{\Lambda}, \mathbf{W} \in \mathcal{D}_+^m$; $\mathbf{V} \in \mathcal{Z}^m$; $\mathbf{Q}, \tilde{\mathbf{Q}} \in \Re_{\geq 0}^{m \times m}$ such that:

$$\begin{bmatrix} He(A'\mathbf{P}) & \mathbf{P}B + C'\mathbf{V}' + A'C'\mathbf{\Lambda} & A'C'\mathbf{\Lambda} \\ \star & He(\tilde{\mathbf{Q}} + \mathbf{Q} - \mathbf{V} + \mathbf{\Lambda}CB) & B'C'\mathbf{\Lambda} + \tilde{\mathbf{Q}} \\ \star & \star & -2\mathbf{W} \end{bmatrix} \prec 0 \qquad (3)$$

then the origin of (1) is GAS.

- Solution space of the Popov Criterion is a subset of (3). Hence, Theorem 8 must be less conservative than the Popov Criterion.

[0] $He(U) := U + U'$

Numerical Examples

			Max. series gain (left) and # of decision variables (right)											
Ex	n	m	Circle		Th. 7		Popov		Th. 8[1]		Park		Zames-Falb	
1	9	3	20.87	48	39.52	63	434.3	51	9990	66	448.7	87	435.83	252
2	3	3	89.90	9	89.90	24	89.90	12	89.91	27	89.90	30	89.90	45
3	3	4	0.52	10	0.68	38	0.52	14	0.68	42	0.52	40	0.55	48
4	6	4	0.19	25	0.39	53	0.19	29	0.50	57	0.22	67	0.27	129
5	8	4	0.09	40	0.12	68	0.09	44	0.14	72	0.10	90	0.10	208
6	5	5	2.02	20	2.02	65	2.02	25	2.02	70	2.02	70	2.02	100

- Theorem 7 and Theorem 8 were of similar complexity, slightly higher than Popov and, in general, slightly less than Park.
- Theorem 8 is of equal or less conservatism in all of the examples. It strikes an appealing balance of reduced conservatism and complexity.
- Theorem 7 is of equal or less conservatism than all *existing criteria* in 83% of examples. It also strikes an appealing balance between reduced conservatism and complexity.

[1]The convex relaxation $\mathbf{H} = I$ was used.

Conclusion

- Proposed strengthened Circle and Popov Criteria for the analysis of Lurie systems with the Repeated ReLU nonlinearity.
- Criteria were built upon novel QCs for the Repeated ReLU.
- New criteria demonstrated lower levels of conservatism whilst retaining the computational appeal of Circle and Popov. *This addresses the large m problem in NN analysis.*