Tarea 2

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1.1 Series de Fourier

Demostrar (con rigor matemático) los siguientes teoremas:

1. Si $f(t)$ es continua cuando $-T/2 \leq t \leq T/2$ con $f(-T/2) = f(T/2)$, y si la derivada $f'(t)$ es continua por tramos y diferenciable; entonces la serie de Fourier:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cos(n\omega_0 t) + b_n sin(n\omega_0 t)) \tag{1}$$

se puede diferenciar término por término para obtener:

$$f'(t) = \sum_{n=1}^{\infty} n\omega_0(-a_n sin(n\omega_0 t) + b_n cos(n\omega_0 t)) \tag{2}$$

Sea $f(t)$ continua por tramos en el intervalo $-T/2 \leq t \leq T/2$ y sea $f(t+T) = f(t)$. Demostrar que la serie de Fourier se puede integrar término por término para obtener:

$$\int_{t_1}^{t_2} f(t)dt = \frac{1}{2}a_0(t_2-t_1) + \sum_{n=1}^{\infty}\frac{1}{n\omega_0}[-b_n(cos(n\omega_0 t_2) - cos(n\omega_0 t_1)) + a_n(sen(n\omega_0 t_2) - sen(n\omega_0 t_1))] \tag{3}$$

Sabemos que si una función es continua por tramos en un intervalo $[-T/2, T/2]$, entonces el $\lim_{n\to\infty} \sum_{k=1}^{n} f(t_k)\Delta x_k$ existe en dicho tramo, y como $\int_{-T/2}^{T/2} f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(t_k)\Delta x_k$, se concluye que la integral de la función existe en $[-T/2, T/2]$.

$$\Rightarrow f(t) = a_0/2 + \sum_{n=1}^{\infty}(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t))$$

$$\Rightarrow \int_{t_1}^{t_2} f(t)\,dt = \int_{t_1}^{t_2} a_0/2\,dt + \sum_{n=1}^{\infty}\int_{t_1}^{t_2} a_n \cos(n\omega_0 t)\,dt + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)\,dt$$

$$= \frac{a_0(t_2-t_1)}{2} + \sum_{n=1}^{\infty}\frac{a_n}{n\omega_0}\sin(n\omega_0 t)\Big|_{t_1}^{t_2} - \sum_{n=1}^{\infty}\frac{b_n}{n\omega_0}\cos(n\omega_0 t)\Big|_{t_1}^{t_2}$$

$$= \frac{a_0(t_2-t_1)}{2} + \sum_{n=1}^{\infty}\frac{1}{n\omega_0}[-b_n(\cos(n\omega_0 t_2) - \cos(n\omega_0 t_1)) + a_n(\sin(n\omega_0 t_2) - \sin(n\omega_0 t_1))]$$

1.3 Función $\zeta(s)$ de Riemann

1. Integrar (analítcamente) la serie de Fourier de $f(t) = t^2$ en el intervalo $-\pi \le t \le \pi$ y $f(t+2\pi) = f(t)$.
2. Usando dicha integral y la identidad de Parseval, pensar en un programa para estimar numéricamente la función $\zeta(6)$ de Riemman:

$$\zeta(6) = \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}. \tag{5}$$

$$f(t) = t^2$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 = \frac{1}{\pi}\frac{t^3}{3}\Bigg|_{-\pi}^{\pi} = \frac{1}{3\pi}\left(\pi^3 - (-\pi)^3\right) = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \cos(\overset{n\omega_0 t}{nt})\,dt = \frac{1}{\pi}\left[\frac{t^2}{n}\sin(nt) + \frac{2t}{n^2}\cos(nt) - \frac{2}{n^3}\sin(nt)\right]_{-\pi}^{\pi}$$

$$= \frac{2}{\pi n^2}\left(\pi\cos(n\pi) + \pi\cos(n\pi)\right) = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\sin(nt)\,dt = 0 \text{ por ser impar.}$$

$$\Rightarrow f(t) = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos(nt)$$

$$\int f(t)\,dt = \int \frac{2\pi^2}{3}dt + \sum_{n=1}^{\infty}\int \frac{4(-1)^n}{n^2}\cos(nt)\,dt$$

$$\frac{t^3}{3} = \frac{2\pi^2}{3}t + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin(nt)$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin(nt) = \frac{1}{12}\left(t^3 - 2\pi^2 t\right)$$

Identdad de Parseval

$$\frac{1}{L}\int_{-L}^{L}[f(x)]^2\,dx = \frac{(a_0)^2}{2} + \sum_{n=1}^{\infty}\left[(a_n)^2 + (b_n)^2\right]$$

$$\Rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{1}{12}\left(t^3 - 2\pi t\right)\right]^2 dt = \sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^6} = \zeta(6)$$

Series de Fourier

1.1

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos(nw_0t)+b_n\sin(nw_0t))$$

$$\int_{t_1}^{t_2}f(t)dt=\int_{t_1}^{t_2}\frac{a_0}{2}dt+\int_{t_1}^{t_2}\sum_{n=1}^{\infty}(a_n\cos(nw_0t)+b_n\sin(nw_0t))dt$$

$$=\frac{a_0}{2}(t_2-t_1)+\sum_{n=1}^{\infty}\frac{1}{nw_0}\left(a_n\sin(nw_0t)-b_n\cos(nw_0t)\right)\Bigg|_{t_1}^{t_2}$$

$$=\frac{a_0}{2}(t_2-t_1)+\sum_{n=1}^{\infty}\frac{1}{nw_0}\left(a_n[\sin(nw_0t_2)-\sin(nw_0t_1)]-b_n[\cos(nw_0t_2)-\cos(nw_0t_1)\right.$$

1.2

$$f(t)=t\rightarrow a_0=\int_{-t}^{t}f(t)dt\xrightarrow[(-\pi,\pi)]{\text{Intervalo}}a_0=\int_{-\pi}^{\pi}t\,dt=\frac{t^2}{2}\Bigg|_{-\pi}^{\pi}=0$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}t\cos(nw_0t)dt=\frac{1}{n^2w_0^2}\cos(nw_0t)\Bigg|_{-\pi}^{\pi}=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}t\sin(nw_0t)dt=\frac{-t}{\pi nw_0}\cos(nw_0t)\Bigg|_{-\pi}^{\pi}=-\frac{2\cancel{\pi}}{\cancel{\pi}nw_0}\cos(nw_0\pi)$$

$$=\frac{-2\cos(nw_0\pi)}{nw_0}=-\frac{2(-1)^n}{n}=\frac{2(-1)^{n-1}}{n}$$

Excluyendo w_0

$$f(t)=\sum_{n=1}^{\infty}\frac{2(-1)^n}{n}\sin(nt)$$

$$f(t+2\pi)+f(t)=2t+2\pi$$

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}(2t+2\pi)dt=\frac{(t^2+2\pi t)}{2\pi}\Bigg|_{-\pi}^{\pi}=\frac{4\pi^2}{2\pi}=2\pi$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}(2t+2\pi)\cos(nt)dt=0$$

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$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(2t+2\pi)\sin(nt)\,dt = \frac{-(2t+2\pi)}{\pi n}\cos(nt)\Big|_{-\pi}^{\pi}$$

$$= \frac{-4}{n}\cos(n\pi) = \frac{4}{n}(-1)^{n-1}$$

$$f(t+2\pi) + f(t) = \sum_{n=1}^{\infty} \frac{4}{n}(-1)^{n-1}\sin(nt) \quad + \pi$$

1.3 $f(t) = t^2 \qquad (-\pi, \pi)$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2\,dt = \frac{t^3}{6\pi}\Big|_{-\pi}^{\pi} = \frac{\cancel{2}\pi^{\cancel{3}\,2}}{\cancel{6\pi}} = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\cos(nt)\,dt = \frac{2\cancel{\pi}\,\cancel{n}\cos(\pi n)}{\cancel{\pi}\, n^3} = 4\,\frac{\cos(\pi n)}{n^2}$$

$$a_n = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\sin(nt)\,dt = 0$$

$$f(t) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}$$