Multivariable Calculus

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1 Understanding Hessian Matrix

Let's think a function $f(x_1, x_2, x_3, ...x_n)$ and the hessian matrix of this function will be :

$$H = \begin{pmatrix} \frac{\delta^2 f}{\delta x_1 x_1} & \frac{\delta^2 f}{\delta x_1 x_2} & \frac{\delta^2 f}{\delta x_1 x_3} & . & . & . & \frac{\delta^2 f}{\delta x_1 x_n} \\ \frac{\delta^2 f}{\delta x_2 x_1} & \frac{\delta^2 f}{\delta x_2 x_2} & \frac{\delta^2 f}{\delta x_2 x_3} & . & . & . & \frac{\delta^2 f}{\delta x_2 x_n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \frac{\delta^2 f}{\delta x_n x_1} & \frac{\delta^2 f}{\delta x_n x_2} & \frac{\delta^2 f}{\delta x_n x_3} & . & . & . & \frac{\delta^2 f}{\delta x_n x_n} \end{pmatrix}$$

But we know that the jacobian of function $f(x_1, x_2, x_3, ...x_n)$ can be written as : $J = [\frac{\delta f}{\delta x_1}, \frac{\delta f}{\delta x_2}, \frac{\delta f}{\delta x_3}, ..., \frac{\delta f}{\delta x_n}]$, than think that $J_i = \frac{\delta f}{\delta x_i}$, again function f can be written as :$J(f) = [J_1, J_2, J_3, ...J_n]$,So finally the hessian can be found in the form jacobian which is :

$$H = \begin{pmatrix} \frac{\delta^2 J_1}{\delta x_1} & \frac{\delta J_1}{\delta x_2} & \frac{\delta J_1}{\delta x_3} & . & . & . & \frac{\delta^2 J_1}{\delta x_n} \\ \frac{\delta J_2}{\delta x_1} & \frac{\delta J_2}{\delta x_2} & \frac{\delta J_2}{\delta x_3} & . & . & . & \frac{\delta J_2}{\delta x_n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \frac{\delta J_n}{\delta x_1} & \frac{\delta J_n}{\delta x_2} & \frac{\delta J_n}{\delta x_3} & . & . & . & \frac{\delta J_n}{\delta x_n} \end{pmatrix}$$

Now, a function $f(x, y, z) = x^2 \cdot y \cdot z$, the Jacobian of f will be $J = [\frac{\delta f}{\delta x}, \frac{\delta f}{\delta y}, \frac{\delta f}{\delta z}]$ So the jacobian of function f(x,y,z) is, $J = [2 \cdot x \cdot y \cdot z, x^2 \cdot z, x^2 \cdot y]$ and let's think $J = [J_1, J_2, J_3]$ and so $J_1 = 2xyz$,$J_2 = x^2 z$,$J_3 = x^2 y$.

Then from our past discussion we know that the Hessian of $f(x, y, z)$ will be looks like :

$$H(f) = \begin{pmatrix} \frac{\delta^2 f}{\delta x^2} & \frac{\delta^2 f}{\delta xy} & \frac{\delta^2 f}{\delta xz} \\ \frac{\delta^2 f}{\delta yx} & \frac{\delta^2 f}{\delta y^2} & \frac{\delta^2 f}{\delta yz} \\ \frac{\delta^2 f}{\delta zx} & \frac{\delta^2 f}{\delta zy} & \frac{\delta^2 f}{\delta z^2} \end{pmatrix} = \begin{pmatrix} 2yz & 2xz & 2xy \\ 2xz & 0 & x^2 \\ 2xy & x^2 & 0 \end{pmatrix} = \begin{pmatrix} \frac{\delta J_1}{\delta x} & \frac{\delta J_1}{\delta y} & \frac{\delta J_1}{\delta z} \\ \frac{\delta J_2}{\delta x} & \frac{\delta J_2}{\delta y} & \frac{\delta J_2}{\delta z} \\ \frac{\delta J_3}{\delta x} & \frac{\delta J_3}{\delta y} & \frac{\delta J_3}{\delta z} \end{pmatrix}$$