Newton Solver Reference

François Faure and Matthieu Nesme

January 30, 2015

Abstract

Detailing the non-linear time-stepping scheme implemented in the CompliantNLImplicitSolver component.

1 Notations

- x: positions
- v: velocities
- $f(x, v, t)$: forces for given positions and velocities at time t
- h: time step
- $(.)^-$, $(.)^+$: a state at, respectively, the beginning and the end of the time step
- $\Delta x = x^+ - x^-$: variation of position during the time step
- $\Delta v = v^+ - v^-$: variation of velocity during the time step
- α, β: blending parameters such as $f^* = \alpha f^+ + (1-\alpha) f^-$ and $v^* = \beta v^+ + (1-\beta) v^-$. Corresponding Data are called implicitVelocity and implicitPosition.
- M: mass

2 Euler Integration

$$\left\{ \begin{array}{rcl} \Delta x & = & h v^* \\ M \Delta v & = & h f^* \end{array} \right.$$

from explicit $\alpha = \beta = 0$ to implicit $\alpha = \beta = 1$.

3 Non-linear Solver

The method computes the next velocity v^+, such that $e \equiv M\Delta v - hf^*$ is satisfied. (Note that other time discretizations are implemented to rather compute the new acceleration or Δv, similar development can be done being careful of the time step scaling.)

Based on the Newton's method, an approximate solution is iteratively improved by solving a linear equation system based on the Jacobian of the residual of the equation to satisfy. A first guess is computed with the regular, linearized system (cf the linear time-stepping scheme in the Compliant plugin doc).

Stating that

$$e \equiv M(v^+ - v^-) - h(\alpha f^+ + (1-\alpha) f^-)$$

we obtain the jacobian

$$\frac{\partial e}{\partial v^+} = M - h\alpha \frac{\partial f^+}{\partial v^+}$$

A first order approximation of the Taylor serie of $f^+ = f(x^+, v^+, t+h)$ gives

$$\begin{aligned} f^+ &= f(x^-, v^-, t) + \tfrac{\partial f}{\partial x}\Delta x + \tfrac{\partial f}{\partial v}\Delta v \\ &= f^- + K\Delta x + B\Delta v \\ &= f^- + K(h(\beta v^+ + (1-\beta)v^-)) + B(v^+ - v^-) \end{aligned}$$

with $K = \frac{\partial f}{\partial x}$ the stiffness matrix and $B = \frac{\partial f}{\partial v}$ the damping matrix.

So

$$\frac{\partial f^+}{\partial v^+} = h\beta K + B$$

and

$$\frac{\partial e}{\partial v^+} = M - h\alpha B - h^2 \alpha\beta K$$

4 Constraints

Bilateral, holonomic constraint $\phi(x) = 0$, combined with the ODE leads to $Jv = 0$ with $J = \frac{\partial \phi}{\partial x}$, the constraint forces are $-J^T\lambda^+$ with λ the Lagrange multipliers.

The error becomes

$$e \equiv M(v^+ - v^-) - h(\alpha(f^+ - J^T\lambda^+) + (1-\alpha) f^-)$$

For compliant constraints $C\lambda = -\phi$,

$$e \equiv M(v^+ - v^-) - h(\alpha(f^+ - J^T\lambda^+) + (1-\alpha) f^-) - \sum_{i \in iterations} (Ch\lambda_i^+ - \phi_i^+)$$

Note the tricky correction accumulation all along the iterations for compliant constraints.

Unilateral constraints $\phi(x) >= 0$ are handled the same way, expect they participate to the error only when they are violated (i.e. when then generate a force λ).

5 Newton Step Length

Two different strategies are implemented:

- naïve sub-step approach: a predefinied portion (Data 0<newtonStepLength<1) of the correction is applied successively while the error is decreasing.
- Backtracking algorithm (Data newtonStepLength=1): try to find the maximum amount of correction to apply that decreased "sufficiently" the error. The line search described in *Numerical Recipies* (chapter Globally Convergent Methods for Nonlinear Systems of Equations) is employed.