Compressible Pressure-Based Solver

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Compressibility Effects

- Compressible form of the continuity equation introduces density into the system

$$\frac{\partial \rho}{\partial t} + \nabla\bullet(\rho \mathbf{u}) = 0$$

- In the analysis, we shall attempt to derive the equation set in general terms. For external aerodynamics, it is typical to use the ideal gas law as the constitutive relation connecting pressure p and density ρ:

$$\rho = \frac{P}{RT} = \psi P$$

where ψ is compressibility:

$$\psi = \frac{1}{RT}$$

Compressibility Effects

- The principle is the same for more general expressions. In this case, presence of density also couples in the energy equation because temperature T appears in the constitutive relation

$$\frac{\partial(\rho e)}{\partial t} + \nabla_\bullet(\rho e \mathbf{u}) - \nabla_\bullet(\lambda \nabla T) = \rho \mathbf{g}_\bullet \mathbf{u} - \nabla_\bullet(P\,\mathbf{u})$$
$$- \nabla_\bullet\left(\frac{2}{3}\,\mu(\nabla_\bullet \mathbf{u})\,\mathbf{u}\right) + \nabla_\bullet\left[\mu\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right)_\bullet \mathbf{u}\right] + \rho Q,$$

- Momentum equation is in a form very simular to before: note the presence of (non-constant) density in all terms. Also, unlike the incompressible form, we shall now deal with dynamic pressure and viscosity in the lace of their kinematic equivalents

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla_\bullet(\rho \mathbf{u}\mathbf{u}) - \nabla_\bullet\left[\mu\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right)\right] = \rho \mathbf{g} - \nabla\left(P + \frac{2}{3}\,\mu \nabla_\bullet \mathbf{u}\right)$$

Compressibility Effects

- In the incompressible form, the $\nabla\bullet\left[\mu(\nabla\mathbf{u})^T\right]$ term was dropped due to $\nabla\bullet\mathbf{u} = 0$:

$$\nabla\bullet\left[\mu(\nabla\mathbf{u})^T\right] = \nabla\mathbf{u}\bullet\nabla\mu + \mu\nabla(\nabla\bullet\mathbf{u})$$

  where the first term disappears for $\mu = const.$ and the second for $\nabla\bullet\mathbf{u} = 0$. In compressible flows, this is not the case and the term remains

Compressible Pressure Equation

- The basic idea in the derivation is identical to the incompressible formulation: we shall use the semi-discretised form of the momentum equation

$$a_P^{\mathbf{u}}\mathbf{u}_P = \mathbf{H}(\mathbf{u}) - \nabla P$$

and

$$\mathbf{u}_P = (a_P^{\mathbf{u}})^{-1}(\mathbf{H}(\mathbf{u}) - \nabla P)$$

- Substituting this into the continuity equation will not yield the pressure equation directly: we need to handle the density-pressure relation

Compressible Pressure Equation

- The first step is the transformation of the rate-of-change term. Using the chain rule on $\rho = \rho(p, \ldots)$, it follows:

$$\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial P}\frac{\partial P}{\partial t}$$

  From the ideal gas law, it follows

$$\frac{\partial \rho}{\partial P} = \psi$$

- Looking at the divergence term, we will substitute the expression for $\mathbf{u}$ and try to present ρ in terms of P as appropriate

$$\nabla\bullet(\rho\mathbf{u}) = \nabla\bullet\left[\rho(a_P^{\mathbf{u}})^{-1}\mathbf{H}(\mathbf{u})\right] - \nabla\bullet\left[\rho\,(a_P^{\mathbf{u}})^{-1}\nabla P\right]$$

- The first term is under divergence and we will attempt to convert it into a convection term. Using $\rho = \psi\, P$, it follows:

$$\nabla\bullet\left[\rho(a_P^{\mathbf{u}})^{-1}\mathbf{H}(\mathbf{u})\right] = \nabla\bullet\left[\psi\, P(a_P^{\mathbf{u}})^{-1}\mathbf{H}(\mathbf{u})\right] = \nabla\bullet(F_p\, P)$$

Compressible Pressure Equation

- Here F_p is the flux featuring in the convective effects in the pressure.

$$F_p = \psi \, (a_P^{\mathbf{u}})^{-1} \mathbf{H}(\mathbf{u})$$

- The second term produces a Laplace operator similar to the incompressible form and needs to be preserved. The working variable is pressure and we will leave the term in the current form. Note the additional ρ pre-factor, which will remain untouched; otherwise the term would be a non-linear function of P
- Combining the above, we reach the compressible form of the pressure equation:

$$\frac{\partial(\psi \, P)}{\partial t} + \nabla\bullet\left[\psi (a_P^{\mathbf{u}})^{-1} \mathbf{H}(\mathbf{u}) \, P\right] - \nabla\bullet\left[\rho \, (a_P^{\mathbf{u}})^{-1} \nabla P\right] = 0$$

- A pleasant surprise is that the pressure equation is in standard form: it consists of a rate of change, convection and diffusion terms. However, flux F_p is not a volume/mass flux as was the case before. This is good news: discretisation of a standard form can be handle in a stable, accurate and bounded manner

Pressure-Velocity-Energy Coupling

- Discretisation of the momentum equation is performed in standard way. Pressure gradient term is left in a differential form:

$$a_P^{\mathbf{u}} \mathbf{u}_P = \mathbf{H}(\mathbf{u}) - \nabla P$$

- Using the elements of the momentum equation, a sonic flux is assembled as:

$$F_p = \psi \, (a_P^{\mathbf{u}})^{-1} \mathbf{H}(\mathbf{u})$$

- Pressure equation is derived by substituting the expression for $\mathbf{u}$ and expressing density in terms of pressure

$$\frac{\partial(\psi \, P)}{\partial t} + \nabla \bullet (F_p \, P) - \nabla \bullet \left[\rho \, (a_P^{\mathbf{u}})^{-1} \nabla P\right] = 0$$

Pressure-Velocity-Energy Coupling

- Face flux expression is assembled in a similar way as before

$$F = \mathbf{s}_f \bullet \left[\psi \, (a_P^\mathbf{u})^{-1} \mathbf{H}(\mathbf{u})\right]_f P_f - \rho \, (a_P^\mathbf{u})^{-1} \mathbf{s}_f \bullet \nabla P$$

  and is evaluated from the pressure solution

- Density can be evaluated either from the constitutive relation:

$$\rho = \frac{P}{RT} = \psi P$$

  or from the continuity equation. Note that at this stage the face flux (= velocity field) is known and the equation can be explicitly evaluated for ρ

- Depending on the kind of physics and the level of coupling, the energy equation may or may not be added to the above. It is in standard form but contains source and sink terms which need to be considered with care

Coupling Algorithm

- The pressure-velocity coupling issue in compressible flows is identical to its incompressible equivalent: in order to solve the momentum equation, we need to know the pressure, whose role is to impose the continuity constraint on the velocity
- In the limit of zero Ma number, the pressure equation reduces to its incompressible form
- With this in mind, we can re-use the incompressible coupling algorithms: SIMPLE and PISO
- In cases of rapidly changing temperature distribution (because of the changes in source/sink terms in the energy equation), changing temperature will considerably change the compressibility ψ. For correct results, coupling between pressure and temperature needs to be preserved and the energy equation is added into the loop

Boundary Conditions

- We have shown that for incompressible flows boundary conditions on pressure and velocity are not independent: two equations are coupled and badly posed set of boundary conditions may result in an ill-defined system
- In compressible flows, we need to account for 3 variables (ρ, $\mathbf{u}$, e) handled together. The issue is the same: number of prescribed values at the boundary depends on the number of characteristics pointing into the domain:
  - Supersonic inlet: 3 variables are specified
  - Subsonic inlet: 2 variables
  - Subsonic outlet 1 variable
  - Supersonic outlet: no variables
- Inappropriate specification of boundary conditions or location of boundaries may result in an ill-defined problem: numerical garbage

Additional Coupled Equations

Coupling to Other Equations

- Compared with the importance and strength of pressure-velocity (or pressure-velocity-energy) coupling, other equations that appear in the system are coupled more loosely
- We shall consider two typical sets of equations: turbulence and chemical reactions

Additional Coupled Equations

Turbulence

- Simple turbulence models are based on the Boussinesq approximation, where μ_t acts as turbulent viscosity. Coupling of turbulence to the momentum equation is relatively benign: the Laplace operator will handle it without trouble
- In all cases, momentum to turbulence coupling will thus be handled in a segregated manner
- In 2-equation models, the coupling between two equations may be strong (depending on the model formulation). Thus, turbulence equations may be solved together – keep in mind that only linear coupling may be made implicit
- A special case is Reynolds stress transport model: the momentum equation is formally saddle-point with respect to R; R is governed by its own equation. In most cases, it is sufficient to handle RSTM models as an explicit extension of the reduced 2-equation model (note that $k = tr(\mathbf{R})$). From time to time, the model will blow up, but careful discretisation usually handles is sufficiently well

Additional Coupled Equations

Chemistry and Species

- Chemical species equations are coupled to pressure and temperature, but more strongly coupled to each other. Coupling to the rest of the system is through material properties (which depend on the chemical composition of the fluid) and temperature.
- Only in rare cases it is possible to solve chemistry in a segregated manner: a coupled chemistry solver is preferred
- The second option is a 2-step strategy. Local equilibrium solution is sought for chemical reactions using an ordinary differential equation (ODE) solver, which is followed be a segregated transport step

Pressure- vs Density-Based Solver

Density-Based Solver

- Coupled equations are solved together: flux formulation enforces the coupling and entropy condition
- The solver is explicit and non-linear in nature: propagating waves. Extension to implicit solver is approximate and done through linearisation
- Limitation on Courant number are handled specially: multigrid is a favoured acceleration technique
- Problem exist at the incompressibility limit: formulation breaks down

Pressure- vs Density-Based Solver

Pressure–Based Solver

- Equation set is decoupled and each equation is solved in turn: segregated solver approach
- Equation coupling is handled by evaluating the coupling terms from the available solution and updating equations in an iteration loop
- Density equation is reformulated as an equation for the pressure. In the incompressible limit, it reduces to a the pressure-velocity system described above: incompressible flows are handled naturally
- Equation segregation implies that matrices are created and inverted one at a time, re-using the storage released the storage from the previous equation. This results is a considerably lower overall storage requirement
- Flux calculation is performed one equation at a time, consistent with the segregated approach. As a consequence, the entropy condition is regularly violated (!)

Pressure- vs Density-Based Solver

Variable Density or Transonic Formulation

- To follow the discussion, note that the cost of solving an elliptic equation (characterised by a symmetric matrix) is half of the equivalent cost for the assymetric solver
- For low Mach number or variable compressibility flows, it is known in advance that the pressure equation is dominated by the Laplace operator. Discretised version of it creates a symmetric matrix
- In subsonic high-Ma or transonic flows, importance of convection becomes more important. However, changed nature of the equation (transport is local) makes it easier to solve
- **Variable compressibility formulation** handles the convection explicitly: the matrix remains symmetric but total cost is reduced with minimal impact on accuracy