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MATH 3503: Tashfeen's Discrete Mathematics
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Homework 3

Question 1. Please read chapters 3 and 4 of Chartrand et al. and write a couple sentences about a topic/example/concept that you found difficult or interesting and why?

Question 2. Consider the following quantified statement: For every real number x, there exists a positive real number y such that $y < x^2$.

(a) Express this quantified statement in symbols.

$$\forall x \in \mathbb{R}. \exists y \in \mathbb{R}. (y > 0 \wedge y < x^2)$$

(b) Express the negation of this quantified statement in symbols.

$$\exists x \in \mathbb{R}. \forall y \in \mathbb{R}. (y \leq 0 \vee y \geq x^2)$$

(c) Express the negation of this quantified statement in words.
For a real number x, there does not exist a positive real number y such that $y < x^2$

Question 3. Prove that if r and s are rational numbers, then $r - s$ is a rational number.

Proof. if $(r \in \mathbb{R} \wedge s \in \mathbb{R})$ then $(r - s) \in \mathbb{R}$

1) Assume $r = \frac{x}{y} \wedge s = \frac{j}{k}$
2) Rational numbers are numbers that can be expressed as a fraction
3) Then, $(r - s) = (\frac{x}{y} - \frac{j}{k}) = \frac{xk - yj}{yk}$
4) since (r - s) can be written as a function then (r - s) is a rational number
5) Therefore, $(r - s) \in \mathbb{Q}$

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Question 4. Let x and y be integers. Prove that if $x + y \geq 9$, then either $x \geq 5$ or $y \geq 5$.

Proof by Contrapositive. $x \in \mathbb{Z} \wedge y \in \mathbb{Z}. x + y \geq 9. x \geq 5 \vee y \geq 5$

1) Assume $(x \in \mathbb{Z} \wedge y \in \mathbb{Z}) \wedge (x < 5 \wedge y < 5)$
2) Then $(x \leq 4 \wedge y \leq 4)$
3) Therefore, $(x + y) \leq (4 + 4) < 9$

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Question 5. Let m and n be two integers. Prove that mn and $m + n$ are both even if and only if m and n are both even.

Proof. $(m \in \mathbb{Z} \wedge n \in \mathbb{Z})$. mn is even $\wedge$ m + n is even if m is even $\wedge$ n is even
Assume $m = 2x \wedge n = 2y$
Then $mn = 4xy \wedge (m + n) = (2x + 2y)$
Therefore, $mn \wedge (m + n)$ are even because they can be written in terms of 2

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Therefore, $m \wedge n$ are even becuase they can be written in terms of 2

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Question 6. Disprove: Let A, B and C be sets. If $A \cup B = A \cup C$, then $B = C$.

1) Assume $A = \{1, 2\}, B = \{2, 3\}, C = \{3\}$
2) Then $A \cup B = A \cup C$, but $B \neq C$
3) Therefore If $A \cup B = A \cup C$, then $B = C$ is not true

Question 7. Prove that if a and b are positive real numbers, then $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$.

Proof. 1) Assume $a = x^2 \wedge b = x^2$
2) Then $\sqrt{a} + \sqrt{b} = \sqrt{x^2} + \sqrt{x^2} = x + x = 2x$
$\sqrt{a+b} = \sqrt{2x^2} = x\sqrt{2}$
3) Therefore $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$ can be written as $2x \neq x\sqrt{2}$ which is true

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Question 8. Let $r \geq 2$ be an integer. Prove that $1 + r + r^2 + \cdots + r^n = \frac{r^{n+1}-1}{r-1}$ for every positive integer n.

$$r \in \mathbb{Z}, \geq 2 \quad n \in \mathbb{Z}, > 0 \quad 1 + r + r^2 + \cdots + r^n = \frac{r^{n+1}-1}{r-1}$$

Proof by Induction. **Basis:** Take $n \in \{...\}$ then we have,

$$\frac{r^{1+1}-1}{r-1} = \frac{r^2-1}{r-1} = \frac{(r-1)(r+1)}{r-1} = r+1 \quad \frac{r^{0+1}-1}{r-1} = 1$$

Inductive Hypothesis: Assume for some positive integer n,

$$n = k \quad \frac{r^{k+1}-1}{r-1}$$

Inductive Step: We show the bellow by induction on n,

$$(1 + r + r^2 + \cdots + r^k) + r^{k+1} =$$
$$\frac{r^{k+1}-1}{r-1} + r^{k+1} =$$
$$\frac{r^{k+1}-1}{r-1} + \frac{r^{k+1}(r-1)}{r-1} =$$
$$\frac{r^{k+1}-1+(r^{k+1}r-r^{k+1})}{r-1} =$$
$$\frac{r^{k+1}-1+(r^{k+1}r-r^{k+1})}{r-1} =$$
$$\frac{r^{k+1}-1+(r^{k+2}-r^{k+1})}{r-1} =$$
$$\frac{r^{k+2}-1}{r-1} =$$
$$\frac{r^{(k+1)+1}-1}{r-1}$$

Then by induction on n we showed that

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1}-1}{r-1}$$

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Question 9. Prove that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n+1}$ for every integer $n \geq 3$.

Proof. Base Case

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} > \sqrt{4}$$

Hypothesis

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} > \sqrt{k+1}$$

Inductive Step

$$(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}}) + \frac{1}{\sqrt{k+1}} > \sqrt{k+1} + \frac{1}{\sqrt{k+1}}$$

$$\sqrt{k+1} + \frac{1}{\sqrt{k+1}} = \frac{\sqrt{k+1}\sqrt{k+1}}{\sqrt{k+1}} + \frac{1}{\sqrt{k+1}} = \frac{k+2}{\sqrt{k+1}}$$

$$\frac{k+2}{\sqrt{k+1}} > \sqrt{(k+1)+1} =$$

$$(\frac{k+2}{\sqrt{k+1}})^2 > (\sqrt{(k+1)+1})^2 =$$

$$\frac{(k+2)^2}{k+1} > k+2 =$$

$$(k+2)^2 > (k+2)(k+1)$$

$$k+2 > k+1$$

By proof induction on n we show that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n+1}$$

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Question 10. A sequence $a_1, a_2, a_3, \cdots$ is defined recursively by $a_1 = 3$ and $a_n = 2a_{n-1} + 1$ for $n \geq 2$.

(a) Determine a_2, a_3, a_4, and a_5.

$$a_2 = 7$$
$$a_3 = 15$$
$$a_4 = 31$$
$$a_5 = 63$$

(b) Based on the values obtained in (a), make a guess for a formula for a_n for every positive integer n and use induction to verify that your guess is correct.

Proof. formula

$$a_0 = 1 \quad \text{and} \quad a_n = 2a_{n-1} + 1$$

Base

$$a_1 = 3$$

hypothesis

$$a_k = 2a_{k-1} + 1$$

inductive step

$$a_{k+1} = 2(2a_{k-1} + 1) + 1$$
$$a_{k+1} = 4a_{k-1} + 3$$
$$a_5 = 4a_3 + 3$$
$$63 = 4(15) + 3$$
$$63 = 63$$

Therefore

$$a_k = 2a_{k-1} + 1$$

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Question 11. In Example 4.36, we saw that n^{th} Fibonacci number $F_n \leq 2^n$. Prove that $F_n \leq (\frac{5}{3})^n$ for every positive integer n.

Proof. Base

$$F_1 \leq \frac{5}{3}^1 = 1 \leq \frac{5}{3}$$

Hypothesis

$$F_k \leq \frac{5}{3}^k$$

Inductive step

$$F_{k+1} \leq \frac{5}{3}^{k+1}$$
$$F_{k+1} \leq \frac{5}{3}F_k \leq \frac{5}{3}^{k+1}$$

Therefore

$$F_{k+1} \leq \frac{5}{3}^{k+1}$$

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Question 12. A sequence $\{a_n\}$ is defined recursively by $a_1 = 5$, $a_2 = 7$ and $a_n = 3a_{n-1} - 2a_{n-2} - 2$ for $n \geq 3$. Prove that $a_n = 2n + 3$ for every positive integer n.

Proof. Base

$$a_3 = 9$$

Hypothesis

$$a_k = 2k + 3$$

Inductive step

$$a_{k+1} = 2(k+1) + 3 = 2k + 5$$
$$2n + 5 = 3a_{n-1} - 2a_{n-2} - 2$$
$$2(3) + 5 =$$

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