The Clifford Group Forms a 3-design

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This is a simplified version of [Web15], explaining why the Clifford group forms a 3-design with quite primary mathematical tools. However if you are not familiar with unitary design and Clifford group you might still feel hard to follow. In this case, I prepare some good references on unitary design and Clifford group for you, so at least you would know more about this two interesting topics.

1 Preliminary

1.1 Pauli basis and Clifford group

The 4 Pauli matrices $\{I, X, Y, Z\}$ acting on 1 qubit can be extended to n-qubit system by taking n-fold tensor product. Denote $P_n = \{I, X, Y, Z\}^{\otimes n}$ the n-qubit Pauli matrices, then P_n forms an orthogonal basis of $L(\mathbb{C}^{2^n})$, i. e., $\forall p, q \in P_n$, $\mathrm{Tr}(pq) = 2^n \delta_{pq}$.

Therefore, we can expand any matrix $A \in L(\mathbb{C}^{2^n})$ in the Pauli basis. If we write

$$A = \sum_p \alpha_p p, \tag{1}$$

then we can compute the coefficients α_q by considering $\mathrm{Tr}Aq$:

$$\mathrm{Tr}(Aq) = \sum_p \mathrm{Tr}(\alpha_p pq) = 2^n \alpha_q \Rightarrow \alpha_q = \frac{1}{2^n}\mathrm{Tr}(Aq) \tag{2}$$

A useful fact (but not useful here) is that for all Hermitian matrix H, the coefficients are real:

$$\begin{aligned}\mathrm{Tr}(Hp) = \sum_{i,j} H_{i,j} p_{j,i} =& \frac{1}{2}\sum_{i,j}(H_{i,j}p_{j,i} + H_{j,i}p_{i,j}) \\ =& \frac{1}{2}\sum_{i,j}(H_{i,j}p_{j,i} + H^*_{i,j}p^*_{j,i}) = \sum_{i,j}\mathrm{Re}(H_{i,j}p_{j,i})\end{aligned} \tag{3}$$

To define the Clifford group, it is not necessary to turn the Pauli matrices into a group. We can start by observing the behavior of Clifford gates. Suppose a unitary gate c maps a Pauli matrix p to a Pauli matrix q (p and q can be equal), up to some global phase $e^{\mathrm{i}\theta}$:

$$cpc^\dagger = e^{\mathrm{i}\theta} q, \ p, q \in P_n$$

then it should not be hard to have the following observations:

- all possible c form a group
- c and $e^{\mathrm{i}\psi}c$ have the same action on p
- $e^{\mathrm{i}\theta} = \pm 1$, $\forall p \in P_n$ (because p has only ± 1 eigenvalues)

- if $p_1 \neq p_2$, then $cp_1c^\dagger \neq cp_2c^\dagger$, i. e., c permutes P_n
- $p = I$ iff $q = I$
- if p_1 and p_2 (anti-)commute, then $cp_1c^\dagger$ and $cp_2c^\dagger$ (anti-)commute

Interestingly, the above observations are enough to characterize the Clifford group. The above observations give us hints to define the Clifford group in a proper manner so that we can evaluate the size of Clifford groups. We first give some extended definitions of P_n. Since we do not care conjugation on the identity operator, we define $\hat{P}_n = P_n \setminus \{I_{2^n}\}$. We also need to introduce the ± 1 coefficient $\overline{P}_n = \{\pm p,\ p \in \hat{P}_n\}$. According to the definitions we have $|P_n| = 4^n$, $|\hat{P}_n| = 4^n - 1$ and $|\overline{P}_n| = 2(4^n - 1)$.

The Clifford group is then defined as follows:

$$C_n = \{c \in U(2^n) : \forall p \in \hat{P}_n, cpc^\dagger \in \overline{P}_n\}/U(1)$$

The size of the Clifford groups can be evaluated in a comprehensive way by this definition. We give out the result directly below and the evaluating method can be found in this document (you can get many useful knowledge about the Clifford group in this document).

$$|C_n| = \Pi_{j=1}^n 2(4^j - 1)4^j$$

Another useful fact (but not useful here) is that the Clifford group can be generated by $\{H, S, \mathrm{CNOT}\}$.

1.2 Permutation operator

For each permutation $\pi \in S_t$, where S_t is the symmetric group (all different permutations) on t elements, there is a permutation operator W_π permuting the subsystems according to π:

$$W_\pi : \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_t \mapsto \rho_{\pi(1)} \otimes \rho_{\pi(2)} \otimes \cdots \otimes \rho_{\pi(t)}$$

where each $\rho_i, i \in [1, t]$ is a n-qubit quantum state.

Of course we can write down the matrix expression of any permutation operator by considering its action on computational bases. But this expression is useless in our proof. Instead, we consider expanding permutation operators in the Pauli base. We give examples in the $t = 3$ case where we have interest. Note that each subsystem is now a n-qubit system. Then a Pauli operator on the *three* n-qubit subsystems is

$$p_1 \otimes p_2 \otimes p_3,\ \ p_1, p_2, p_3 \in P_n$$

Let's now try to compute the coefficients of Pauli basis of $W_{(123)} : \rho_1 \otimes \rho_2 \otimes \rho_3 \mapsto \rho_2 \otimes \rho_3 \otimes \rho_1$:

$$\begin{aligned}
\alpha_{p_1p_2p_3} =& \frac{1}{2^{3n}} \mathrm{Tr}(W_{123}(p_1 \otimes p_2 \otimes p_3)) \\
=& \frac{1}{2^{3n}} \sum_{i_1,i_2,i_3} \langle i_1i_2i_3|W_{123}(p_1 \otimes p_2 \otimes p_3)|i_1i_2i_3\rangle \\
=& \frac{1}{2^{3n}} \sum_{i_1,i_2,i_3} \langle i_1i_2i_3|(p_3|i_3\rangle \otimes p_1|i_1\rangle \otimes p_2|i_2\rangle) \\
=& \frac{1}{2^{3n}} \sum_{i_1,i_2,i_3} \langle i_1|p_3|i_3\rangle\langle i_2|p_1|i_1\rangle\langle i_3|p_2|i_2\rangle \\
=& \frac{1}{2^{3n}} \sum_{i_1,i_2,i_3} \langle i_3|p_2|i_2\rangle\langle i_2|p_1|i_1\rangle\langle i_1|p_3|i_3\rangle \\
=& \frac{1}{2^{3n}} \mathrm{Tr}(p_2p_1p_3)
\end{aligned}$$

Note that p_1, p_2, p_3 are Pauli basis on n-qubit system. So

$$\alpha_{p_1,p_2,p_3} = \frac{1}{2^{3n}}\mathrm{Tr}(p_2 p_1 p_3) = \begin{cases} \frac{1}{2^{2n}} e^{-\mathrm{i}\psi}, & \text{if } p_3 = e^{\mathrm{i}\psi} p_2 p_1 \\ 0, & \text{elsewise} \end{cases}$$

Then we can expand W_{123} in the Pauli Base:

$$W_{123} = \sum_{p_1,p_2,p_3} \alpha_{p_1,p_2,p_3} p_1 \otimes p_2 \otimes p_3 = \frac{1}{2^{2n}} \sum_{p_1,p_2} p_1 \otimes p_2 \otimes p_2 p_1$$

In a similar manner we can expand all permutation operators for $t = 3$:

$$W_{12} = \frac{1}{2^n}\sum_p p \otimes p \otimes I_{2^n},\ W_{13} = \frac{1}{2^n}\sum_p p \otimes I_{2^n} \otimes p,\ W_{23} = \frac{1}{2^n}\sum_p I_{2^n} \otimes p \otimes p$$

$$W_{123} = \frac{1}{2^{2n}} \sum_{p_1,p_2} p_1 \otimes p_2 \otimes p_2 p_1,\ W_{321} = \frac{1}{2^{2n}} \sum_{p_1,p_2} p_1 \otimes p_2 \otimes p_1 p_2$$

Another useful fact is that W_π commute with $V^{\otimes t}$ for any V, since $V^{\otimes t}$ acts equivalently on each subsystem.

1.3 Unitary t-design

There are several (equivalent) definitions of unitary t-design. In this note we refer to the definition given below:

For Haar random unitary U we define $T_t(X)$ by

$$T_t(X) = \int_{\mu(U)} d\mu(U) U^{\otimes t} X U^{\dagger\otimes t}$$

For a ensemble of unitaries $\mathcal{E} = \{(\alpha_U, U)\}$ we define

$$\Psi_{\mathcal{E},t}(X) = \sum_{(\alpha_U,U)\in\mathcal{E}} \alpha_U U^{\otimes t} X U^{\dagger\otimes t}$$

And $\mathcal{E}$ forms a (weighted) t-design iff $\Psi_{\mathcal{E},t}(X) = T_t(X)$ for all linear operator X.

This definition implies that the Clifford group may form unitary designs if we notice that we can expand X in the Pauli basis.

(If you are curious about the equivalent definitions, as well as the approximate version of t-design, Ref. [Low10] will be very much helpful.)

2 Propositions

We now give some propositions about t-design and Clifford group, which help clarify the proof sketch and may also be useful somewhere else.

Proposition 1. For any X and any ensemble $\mathcal{E}$, $T_t(X) = T_t(\Psi_{\mathcal{E},t}(X))$.

Proof. The unitary invariance of the Haar measure directly implies that

$$\int_{\mu(U)} d\mu(U) U^{\otimes t} X U^{\dagger\otimes t} = \int_{\mu(U)} d\mu(U) (UV)^{\otimes t} X (UV)^{\dagger\otimes t} = \int_{\mu(U)} d\mu(U) U^{\otimes t} (V^{\otimes t} X V^{\dagger\otimes t}) U^{\dagger\otimes t}$$

Proposition 2. For any permutation operator W_π, $T_t(W_\pi) = W_\pi$.

Proof. Recall that W_π and $U^{\otimes t}$ commute. Then

$$\int_{\mu(U)} d\mu(U) U^{\otimes t} W_\pi U^{\dagger\otimes t} = W_\pi \int_{\mu(U)} d\mu(U) U^{\otimes t} U^{\dagger\otimes t} = W_\pi$$

Apparently $\Psi_{\mathcal{E},t}(W_\pi) = W_\pi$ for the same reason. It is also true if we replace W_π by linear combinations of W_π, due to the linearity of $T_t(\cdot)$.

3 Sketch of the proof

We can prove a ensemble forms 3-design by using proposition 1 and 2 jointly. As long as we can prove that $\Psi_{\mathcal{E},3}(X)$ can be written as the linear combination of permutation operators:

$$\Psi_{\mathcal{E},3}(X) = \sum_{\pi} \alpha_\pi(X) W_\pi,$$

then we have

$$T_3(X) \xlongequal{\text{Prop.1}} T_3(\Psi_{\mathcal{E},3}(X)) \xlongequal{\text{to be proved}} T_3(\sum_{\pi} \alpha_\pi(X) W_\pi) \xlongequal{\text{Prop.2}} \sum_{\pi} \alpha_\pi(X) W_\pi = \Psi_{\mathcal{E},3}(X),$$

which complete the proof of 3-design.

We do not prove directly that $\Psi_{\text{Clifford},3}(X)$ can be written as the linear combination of permutation operators. Instead, we introduce the some properties of the Clifford group, namely Pauli invariant, Pauli mixing and Pauli 2-mixing, which characterize how *uniformly* the Clifford gates act on P_n. In next section we will show that the Clifford group is Pauli 2-mixing. Then we prove that for any Pauli 2-mixing ensemble, $\Psi_{\text{Pauli 2-mixing},3}(X)$ can be written as the linear combination of permutation operators, thus any Pauli 2-mixing ensemble forms a 3-design. Thus the Clifford group forms a 3-design. This technique allows us to find smaller ensembles that forms a 3-design, e. g., some subsets of Clifford group is sufficient for Pauli 2-mixing, therefore forms a 3-design.

4 Detail of the proof

4.1 Pauli-variant, Pauli mixing and Pauli 2-mixing

In this subsection we prove that the Clifford group maps uniformly on any Pauli matrix and any pair of Pauli matrices.

Definition 1. (Pauli-invariant) An ensemble $\mathcal{E}$ is (right) *Pauli-invariant* if it is unchanged by a Pauli transformation: $\forall p \in P_n$,

$$(\alpha_U, U) \in \mathcal{E} \;\Rightarrow\; (\alpha_U, e^{\mathrm{i}\theta_U} U p) \in \mathcal{E}$$

If every elements in $\mathcal{E}$ is equally weighted, i. e., $\alpha_i = \frac{1}{|\mathcal{E}|}$, then the action of any Pauli matrix on a Pauli-invariant ensemble can be regarded as a permutation of the ensemble. It should not be hard to realize that the (equally weighted) Clifford group obeys this:

Lemma 1. The Clifford group is Pauli-invariant.

proof. First if $c \in C_n$ then $cp \in C_n$. Then for any pair of distinct Clifford gate $c_1 \neq c_2$, $c_1 p \neq c_2 p$.

We are often interested in subsets of the Clifford group, of which the elements have same action on a given Pauli matrix. For an Clifford ensemble $\mathcal{E}_{\text{Clifford}} = \{(\alpha_c, c) : c \in C_n\}$, we denote the sub-ensemble in which the elements takes $p \in \hat{P}_n$ to $q \in \overline{P}_n$ by $\mathcal{E}_{p \to q}$:

$$\mathcal{E}_{p \to q} = \{(\alpha_c, c) \in \mathcal{E}_{\text{Clifford}} : cpc^\dagger = q\}$$

Similarly we can define the sub-ensemble in which the elements takes $p_1, p_2 \in \hat{P}_n$ to $q_1, q_2 \in \overline{P}_n$ by $\mathcal{E}_{p_1, p_2 \to q_1, q_2}$:

$$\mathcal{E}_{p_1, p_2 \to q_1, q_2} = \{(\alpha_c, c) \in \mathcal{E}_{\text{Clifford}} : cp_1c^\dagger = q_1, cp_2c^\dagger = q_2\}$$

Now we can define Pauli-mixing explicitly:

Definition 2. (Pauli-mixing) An ensemble $\mathcal{E}$ is *Pauli-mixing* if it maps any Pauli matrix uniformly: $\forall p \in \hat{P}_n$ and $q \in \overline{P}_n$,

$$\sum_{(\alpha_c, c) \in \mathcal{E}_{p \to q}} \alpha_c = \frac{1}{|\overline{P}_n|} = \frac{1}{2(4^n - 1)}$$

Lemma 2. The Clifford group is Pauli-mixing.

proof. Lemma 2 can be proved by the technique used in the proof of Lemma 3.

If we try to define a uniform map of any pair of Pauli matrices, we should be careful since the map must preserve the commutation relation of the pair. Considering pairs of $q \in \overline{P}_n$, we define

$$H_0 = \{(q_1, q_2) \in \overline{P}_n \times \overline{P}_n : q_1 \neq \pm q_2, q_1 q_2 = q_2 q_1\}$$

$$H_1 = \{(q_1, q_2) \in \overline{P}_n \times \overline{P}_n : q_1 \neq \pm q_2, q_1 q_2 = -q_2 q_1\}$$

Then we have $(cp_1c^\dagger, cp_2c^\dagger) \in H_0$ for commute p_1, p_2, and $(cp_1c^\dagger, cp_2c^\dagger) \in H_1$ for anti-commute p_1, p_2, where $p_1 \neq p_2$. It should not be hard to evaluate that $|H_0| = 2(4^n - 1)(4^n - 4)$ and $|H_1| = 2(4^n - 1)4^n$. Now, we want an ensemble *uniformly* maps any Pauli pair, we may mean that, if p_1, p_2 commute, then the ensemble maps (p_1, p_2) uniformly to H_0, and if p_1, p_2 anti-commute, then the ensemble maps (p_1, p_2) uniformly to H_1. Then Pauli 2-mixing is defined as follows:

Definition 3. (Pauli 2-mixing) An ensemble $\mathcal{E}$ is *Pauli 2-mixing* if it maps any Pauli matrix uniformly meanwhile preserves the commutation relation: $\forall p \in \hat{P}_n$ and $q \in \overline{P}_n$,

$$\sum_{(\alpha_c, c) \in \mathcal{E}_{p_1, p_2 \to q_1, q_2}} \alpha_c = \frac{1}{|H|} = \begin{cases} \frac{1}{2(4^n-1)(4^n-4)}, & \text{if } p_1 p_2 = p_2 p_1 \\ \frac{1}{2(4^n-1)4^n}, & \text{if } p_1 p_2 = -p_2 p_1 \end{cases}$$

Lemma 3. The Clifford group is Pauli 2-mixing.

proof. Suppose that p_1, p_2 commute. Arbitrarily take (q_1, q_2) and (r_1, r_2) from H_0. There must exists some Clifford gate c_0 such that $c_0 q_1 c_0^\dagger = r_1$ and $c_0 q_2 c_0^\dagger = r_2$. Then $\forall c \in \mathcal{E}_{p_1, p_2 \to q_1, q_2}$, $c_0 c \in \mathcal{E}_{p_1, p_2 \to r_1, r_2}$. So $|\mathcal{E}_{p_1, p_2 \to q_1, q_2}| \leq |\mathcal{E}_{p_1, p_2 \to r_1, r_2}|$. As the (q_1, q_2) and (r_1, r_2) are arbitrarily taken, $|\mathcal{E}_{p_1, p_2 \to q_1, q_2}| = |\mathcal{E}_{p_1, p_2 \to r_1, r_2}|$ for all $(q_1, q_2) \in H_0$ and $(r_1, r_2) \in H_0$. Similar analysis works if p_1, p_2 anti-commute. This finishes the proof of Lemma 3.

4.2 Pauli 2-mixing implies 3-design

Recall that we need only to show that $\Psi_{\text{Pauli 2-mixing},3}(X)$ can be written as the linear combination of permutation operators:

$$\Psi_{\text{Pauli 2-mixing},3}(X) = \sum_{\pi} \alpha_\pi(X) W_\pi,$$

for all linear operator X. In fact, we can only consider the case where X is Pauli matrix $X \in P_{3n}$ since any linear operator can be expanded as the linear combination of Pauli matrices.

We do this proof by case analysis. Suppose $X = p_1 \otimes p_2 \otimes p_3$. where $p_1, p_2, p_3 \in P_n$. There are 3 cases according to the relationship between p_1, p_2, p_3:

case 1. Suppose that at least one of the 3 Pauli matrices is I_{2^n}. In this case, the whole problem reduces to a 2-design or 1-design problem. If we assume that we've already know the Clifford group forms a 2-design then we finish this case. If you are not satisfied about this assumption, the techniques in the proof of *case 3.* will let you know how to prove that the Clifford group forms a 2-design.

case 2. Now p_1, p_2, p_3 are non-identity. if we in this case suppose that $p_3 \not\propto p_1 p_2$, i. e., $p_1 p_2 p_3 \not\propto I_{2^n}$, then there exist a non-identity Pauli matrix r that anti-commute with $p_1 p_2 p_3$. This implies r anti-commute with either 1 or 3 of p_1, p_2, p_3, which gives

$$\begin{aligned}
\Psi_{\mathcal{E},3}(X) = &\sum_{(\alpha_U, U)\in\mathcal{E}} \alpha_U U p_1 U^\dagger \otimes U p_2 U^\dagger \otimes U p_3 U^\dagger \\
=&\frac{1}{2}\sum_{(\alpha_U, U)\in\mathcal{E}} \alpha_U (U p_1 U^\dagger \otimes U p_2 U^\dagger \otimes U p_3 U^\dagger + U r p_1 r U^\dagger \otimes U r p_2 r U^\dagger \otimes U r p_3 r U^\dagger) \\
=&\frac{1}{2}\sum_{(\alpha_U, U)\in\mathcal{E}} \alpha_U U p_1 U^\dagger \otimes U p_2 U^\dagger \otimes U p_3 U^\dagger (1 + (-1)^{F(r,p_1)+F(r,p_2)+F(r,p_3)}) = 0
\end{aligned}$$

where $F(p,q) = 0$ if p, q commute and 1 if p, q anti-commute. Therefore $T_3(X) = \Psi_{\mathcal{E},3}(X) = 0$, which finishes this case.

case 3. The only case left is that p_1, p_2, p_3 are non-identity and $p_3 \propto p_1 p_2$. We can let $X = p_1 \otimes p_2 \otimes p_1 p_2$ (this will take X out of the Pauli bases but will not effect the correctness of our proof). Note that now $p_1 \neq p_2$, implying $(q_1, q_2) = (U p_1 U^\dagger, U p_2 U^\dagger) \in H_{F(p_1,p_2)}$. Assume that p_1 and p_2 commute, we can write

$$\begin{aligned}
\Psi_{\mathcal{E},3}(X) &= \sum_{(\alpha_U, U)\in\mathcal{E}} \alpha_U U p_1 U^\dagger \otimes U p_2 U^\dagger \otimes U p_1 p_2 U^\dagger \\
&= \sum_{(\alpha_U, U)\in\mathcal{E}} \alpha_U U p_1 U^\dagger \otimes U p_2 U^\dagger \otimes U p_1 U^\dagger U p_2 U^\dagger \\
&= \sum_{(q_1,q_2)\in H_0} \sum_{(\alpha_U, U)\in\mathcal{E}_{p_1,p_2\to q_1,q_2}} \alpha_U U p_1 U^\dagger \otimes U p_2 U^\dagger \otimes U p_1 U^\dagger U p_2 U^\dagger \\
&= \sum_{(q_1,q_2)\in H_0} \frac{1}{|H_0|} q_1 \otimes q_2 \otimes q_1 q_2 \\
&= \sum_{p_1\neq p_2\in\hat{P}_n, F(p_1,p_2)=0} \frac{4}{|H_0|} p_1 \otimes p_2 \otimes p_1 p_2
\end{aligned}$$

Recall that

$$2^{2n} W_{321} = \sum_{p_1,p_2\in P_n} p_1 \otimes p_2 \otimes p_1 p_2 = \sum_{p_1,p_2\in P_n, F(p_1,p_2)=0} p_1 \otimes p_2 \otimes p_1 p_2 + \sum_{p_1,p_2\in P_n, F(p_1,p_2)=1} p_1 \otimes p_2 \otimes p_1 p_2$$

$$2^{2n} W_{123} = \sum_{p_1,p_2\in P_n} p_1 \otimes p_2 \otimes p_2 p_1 = \sum_{p_1,p_2\in P_n, F(p_1,p_2)=0} p_1 \otimes p_2 \otimes p_1 p_2 - \sum_{p_1,p_2\in P_n, F(p_1,p_2)=1} p_1 \otimes p_2 \otimes p_1 p_2$$

$$W_{12} = \frac{1}{2^n}\sum_p p \otimes p \otimes I_{2^n}, \;\; W_{13} = \frac{1}{2^n}\sum_p p \otimes I_{2^n} \otimes p, \;\; W_{23} = \frac{1}{2^n}\sum_p I_{2^n} \otimes p \otimes p$$

We have

$$\Psi_{\mathcal{E},3}(X) = \frac{4}{|H_0|}(2^{2n-1}(W_{123} + W_{321}) - 2^n (W_{12} + W_{23} + W_{13}) + 2 I_{2^n} \otimes I_{2^n} \otimes I_{2^n})$$

With similar analysis we can know that if p_1 and p_2 anti-commute,

$$\Psi_{\mathcal{E},3}(X) = \frac{4}{|H_1|} 2^{2n-1} (-W_{123} + W_{321})$$

This finished the whole proof that the Clifford group forms a 3-design.

5 If you are still interest...

We roughly argue that the Clifford group does not form a 4-design, and the generated (d-dimensional for $d \neq 2^n$) Clifford group does not forms a 3-design.

The sketch of proof in Section 3 is not enough to prove that any ensemble does not form a t-design, as "$\Psi_{\mathcal{E},t}(X) = \sum_\pi \alpha_\pi(X) W_\pi \Rightarrow \mathcal{E}$ is t-design" does not ensure that "$\Psi_{\mathcal{E},t}(X) \neq \sum_\pi \alpha_\pi(X) W_\pi \Rightarrow \mathcal{E}$ is not t-design". Luckily, we have the following proposition ensuring the latter statement:

Proposition 3. For any X, $T_t(X) = \sum_\pi \alpha_\pi(X) W_\pi$.

(Unluckily, the proof of **Proposition 3.** uses Von Neumann's double commutant theorem or representation theory, which requires heavy mathematical machinery.)

With **Proposition 3.** we can say that if $\Psi_{\mathcal{E},t}(X)$ can not be written as a linear combination of permutation operators, then $\mathcal{E}$ is not a t-design. In Ref. [Web15] the author proves that in the t=4 case and the $d \neq 2^n$ case the coefficients will not match if we assume that group forms a t-design. So the Clifford group does not form a 4-design, and the generated (d-dimensional for $d \neq 2^n$) Clifford group does not forms a 3-design.

References

[Low10] Richard A. Low. Pseudo-randomness and learning in quantum computation, 2010.

[Web15] Zak Webb. The clifford group forms a unitary 3-design, 2015.