

Quantum statistical zero-knowledge

John Watrous
Department of Computer Science
University of Calgary
Calgary, Alberta, Canada
jwatrous@cpsc.ucalgary.ca

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Abstract

In this paper we propose a definition for (honest verifier) quantum statistical zero-knowledge interactive proof systems and study the resulting complexity class, which we denote QSZK. We prove several facts regarding this class:

- The following natural problem is a complete promise problem for QSZK: given instructions for preparing two mixed quantum states, are the states close together or far apart in the trace norm metric? By instructions for preparing a mixed quantum state we mean the description of a quantum circuit that produces the mixed state on some specified subset of its qubits, assuming all qubits are initially in the $|0\rangle$ state. This problem is a quantum generalization of the complete promise problem of Sahai and Vadhan [33] for (classical) statistical zero-knowledge.
- QSZK is closed under complement.
- $\text{QSZK} \subseteq \text{PSPACE}$. (At present it is not known if arbitrary quantum interactive proof systems can be simulated in PSPACE, even for one-round proof systems.)
- Any honest verifier quantum statistical zero-knowledge proof system can be parallelized to a two-message (i.e., one-round) honest verifier quantum statistical zero-knowledge proof system. (For arbitrary quantum interactive proof systems it is known how to parallelize to three messages, but not two.) Moreover, the one-round proof system can be taken to be such that the prover sends only one qubit to the verifier in order to achieve completeness and soundness error exponentially close to 0 and $1/2$, respectively.

These facts establish close connections between classical statistical zero-knowledge and our definition for quantum statistical zero-knowledge, and give some insight regarding the effect of this zero-knowledge restriction on quantum interactive proof systems.

1 Introduction

In recent years there has been an effort to better understand the potential advantages offered by computational models based on the laws of quantum physics as opposed to classical physics. Examples of such advantages include: polynomial time quantum algorithms for factoring, computing discrete logarithms, and several believed-to-be intractable group-theoretic and number-theoretic problems [10, 22, 23, 24, 28, 34, 38]; information-theoretically secure quantum key-distribution [5, 35]; and exponentially more efficient quantum than classical communication-complexity protocols [32]. Equally important for understanding the power of quantum models are upper bounds and impossibility proofs, such as the containment of BQP (bounded error quantum polynomial time) in PP [1, 14], the impossibility of quantum bit commitment [27], and the existence of oracles relative to which quantum computers have restricted power [4, 14].

In this paper we consider whether quantum variants of zero-knowledge proof systems offer any advantages over classical zero-knowledge proof systems. Zero-knowledge proof systems were first defined by Goldwasser, Micali, and Rackoff [20] in 1985, and have since been studied extensively in complexity theory and cryptography. Familiarity with the basics of zero-knowledge proof systems is assumed in this paper—readers not familiar with zero-knowledge proofs are referred to Goldreich [15, 16].

Several notions of zero-knowledge have been studied in the literature, but we will only consider *statistical* zero-knowledge in this paper. Moreover, we will focus on *honest verifier* statistical zero-knowledge, which means that it need only be possible for a polynomial-time simulator to approximate the view of a verifier that follows the specified protocol (as opposed to a verifier that may intentionally deviate from a given protocol in order to gain knowledge). In the classical case it has been proved that any honest verifier statistical zero-knowledge proof system can be transformed into a statistical zero-knowledge proof system against any verifier [18]. The class of languages having statistical zero-knowledge proof systems is denoted SZK; it is known that SZK is closed under complement [31], that $\text{SZK} \subseteq \text{AM}$ [3, 13], and that SZK has natural complete promise problems [19, 33]. Several interesting problems such as Graph Isomorphism and Quadratic Residuosity are known to be contained in SZK but are not known to be in BPP [17, 20]. For further information on statistical zero-knowledge we refer the reader to Okamoto [31], Sahai and Vadhan [33], and Vadhan [36].

To our knowledge, no formal definitions for *quantum* zero-knowledge proof systems have previously appeared in the literature. Despite this fact, the question of whether quantum models extend the class of problems having zero-knowledge proofs has been addressed by several researchers. For instance, the applicability of bit-commitment to zero-knowledge proof systems was one of the motivations behind investigating the possibility of quantum bit commitment [9]. The primary reason for the lack of formal definitions seems to be that difficulties arise when classical definitions for zero-knowledge are translated to the quantum setting in the most straightforward ways. More generally speaking, difficulties tend to arise in defining formal notions of security for quantum cryptographic models (to say nothing of proving security once a formal notion of security has been specified). For a discussion of some of these difficulties, including issues specific to quantum zero-knowledge, we refer the reader to van de Graaf [21].

We do not claim to resolve these difficulties in this paper, nor do we propose a definition for quantum zero-knowledge that we feel to be satisfying *from a cryptographic point of view*. Rather, our goal is to study the complexity-theoretic aspects of a very simple definition of quantum zero-

knowledge based on the notion of an honest verifier. Our primary motives for considering this definition are as follows.

1. Although we do not have satisfying definitions for quantum statistical zero-knowledge when the honest verifier assumption is absent, it is obvious that for any sensible definition that any quantum statistical zero-knowledge proof system would necessarily satisfy our honest verifier definition. Therefore, upper bounds on the power of honest verifier quantum zero-knowledge proof systems also hold for the arbitrary verifier case. (Our main results may be viewed as upper bound results.)
2. We hope that by investigating simple notions of quantum zero-knowledge we are taking steps toward the study and understanding of more cryptographically meaningful formal definitions of quantum zero-knowledge proof systems.
3. We are interested in the effect of zero-knowledge-type restrictions on the power of quantum interactive proof systems from a purely complexity-theoretic point of view. Indeed, we are able to prove some interesting facts about quantum statistical zero-knowledge proof systems that are not known to hold for arbitrary quantum interactive proofs, such as containment in PSPACE and parallelizability to two messages.

Our approach for studying a quantum variant of honest verifier statistical zero-knowledge parallels the approach of Sahai and Vadhan [33] for the classical case, which is based on the identification of a natural complete promise problem for the class SZK. We identify a complete promise problem for quantum statistical zero-knowledge that generalizes Sahai and Vadhan’s complete promise problem to the quantum setting. The problem, which we call the Quantum State Distinguishability problem, may be informally stated as follows: given instructions for preparing two mixed quantum states, are the states close together or far apart in the trace norm metric? The trace norm metric, which is discussed in more detail in the appendix, is an extension of the statistical difference metric to quantum states, and gives a natural way of measuring distances between quantum states. By instructions for preparing a mixed quantum state we mean the description of a quantum circuit that produces the mixed state on some specified subset of its qubits, assuming all qubits are initially in the $|0\rangle$ state. Naturally, the promise in this promise problem guarantees that the two mixed states given are indeed either close together or far apart.

Several facts about quantum statistical zero-knowledge proof systems and the resulting complexity class, which we denote QSZK, may be derived from the completeness of this problem. In particular, we prove that QSZK is closed under complement, that $\text{QSZK} \subseteq \text{PSPACE}$ (which is not known to hold for quantum interactive proof systems if the zero-knowledge condition is dropped, even in the case of one-round proof systems), and that any honest verifier quantum statistical zero-knowledge proof system can be parallelized to a one-round honest verifier quantum statistical zero-knowledge proof system in which the prover sends only one qubit to the verifier (in order to achieve completeness and soundness error exponentially close to 0 and $1/2$, respectively).

While our general approach follows the approach of Sahai and Vadhan, proofs of several of the key technical facts differ significantly from the classical case. For instance, the proofs of completeness and closure under complement rely heavily on properties of quantum states and thus have little resemblance to the proofs for the classical analogues of these facts.

Organization of the paper

Section 2 defines quantum interactive proof systems, the quantum statistical zero-knowledge property, and the Quantum State Distinguishability problem. Section 3 describes quantum zero-knowledge proof systems for the Quantum State Distinguishability problem and its complement. It is proved that the Quantum State Distinguishability problem is complete for QSZK in Section 4, and various corollaries of this fact as stated previously are stated more explicitly in this section. We conclude with Section 5, which mentions some open problems regarding quantum zero-knowledge. An overview of quantum circuits and some technical facts concerning the quantum formalism are contained in an appendix that follows the main part of the paper.

2 Preliminaries

In this section we define quantum interactive proof systems, the quantum statistical zero-knowledge property and the resulting class QSZK, and the Quantum State Distinguishability problem which is shown to be complete for QSZK in subsequent sections.

2.1 Quantum interactive proofs

Quantum interactive proofs were defined and studied in [26, 37]. As in the classical case, a quantum interactive proof system consists of two parties, a prover with unlimited computation power and a computationally bounded verifier. Quantum interactive proofs differ from classical interactive proofs in that the prover and verifier may send and process quantum information.

Formally, a quantum verifier is a polynomial-time computable mapping V where, for each input string x , $V(x)$ is interpreted as an encoding of a $k(|x|)$ -tuple $(V(x)_1, \dots, V(x)_{k(|x|)})$ of quantum circuits. These circuits represent the actions of the verifier at the different stages of the protocol, and are assumed to obey the properties of polynomial-time uniformly generated quantum circuits as discussed in the appendix. The qubits upon which each circuit $V(x)_j$ acts are divided into two sets: $q_V(|x|)$ qubits that are private to the verifier and $q_M(|x|)$ qubits that represent the communication channel between the prover and verifier. One of the verifier's private qubits is designated as the output qubit, which indicates whether the verifier accepts or rejects.

A quantum prover P is a function mapping each input x to an $l(|x|)$ -tuple $(P(x)_1, \dots, P(x)_{l(|x|)})$ of quantum circuits. Each of these circuits acts on $q_M(|x|) + q_P(|x|)$ qubits: $q_P(|x|)$ qubits that are private to the prover and $q_M(|x|)$ qubits representing the communication channel. Unlike the verifier, no restrictions are placed on the complexity of the mapping P , the gates from which each $P(x)_j$ is composed, or on the size of each $P(x)_j$, so in general we may simply view each $P(x)_j$ as an arbitrary unitary transformation.

A verifier V and a prover P are compatible if for all inputs x we have (i) each $V(x)_i$ and $P(x)_j$ agree on the number $q_M(|x|)$ of message qubits upon which they act, and (ii) $k(|x|) = \lfloor m(|x|)/2+1 \rfloor$ and $l(|x|) = \lfloor m(|x|)/2 + 1/2 \rfloor$ for some $m(|x|)$ (representing the number of messages exchanged). We say that V is an m -message verifier and P is an m -message prover in this case. Whenever we discuss an interaction between a prover and verifier, we naturally assume they are compatible.

Given a verifier V , a prover P , and an input x , we define a quantum circuit $(V(x), P(x))$ acting on $q(|x|) = q_V(|x|) + q_M(|x|) + q_P(|x|)$ qubits as follows. If $m(|x|)$ is even, circuits

$$V(x)_1, P(x)_1, \dots, P(x)_{m(|x|)/2}, V(x)_{m(|x|)/2+1}$$

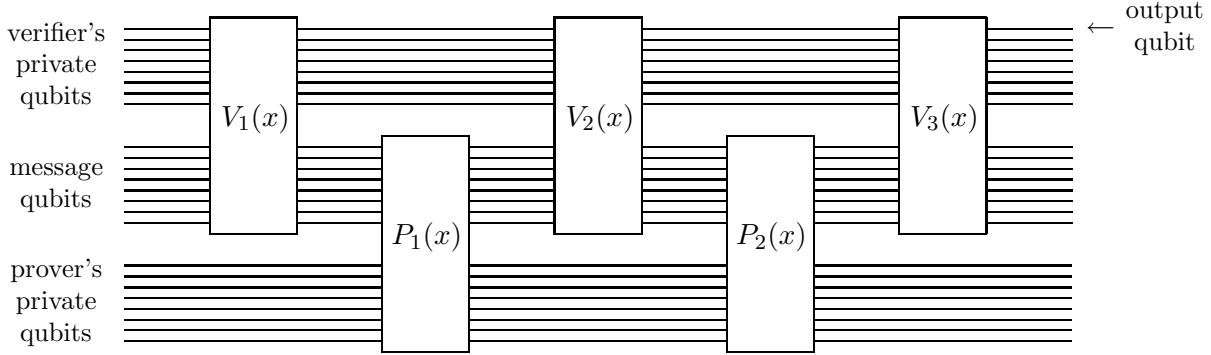


Figure 1: Quantum circuit for a 4-message quantum interactive proof system

are applied in sequence, each to the $q_V(|x|) + q_M(|x|)$ verifier/message qubits or to the $q_M(|x|) + q_P(|x|)$ message/prover qubits accordingly. This situation is illustrated in Figure 1 for the case $m(|x|) = 4$. If $m(|x|)$ is odd the situation is similar, except that the prover applies the first circuit, so circuits

$$P(x)_1, V(x)_1, \dots, P(x)_{(m(|x|)+1)/2}, V(x)_{(m(|x|)+1)/2}$$

are applied in sequence. Thus, it is assumed that the prover always sends the last message (since there would be no point for the verifier to send a message without a response).

Now, for a given input x , the probability that the pair (V, P) accepts x is defined to be the probability that an observation of the verifier's output qubit (in the $\{|0\rangle, |1\rangle\}$ basis) yields the value 1, after the circuit $(V(x), P(x))$ is applied to a collection of $q(|x|)$ qubits each initially in the $|0\rangle$ state. We define a function $\text{max_accept}(V(x))$ (the maximum acceptance probability of $V(x)$) to be the probability that (V, P) accepts x maximized over all possible m -message provers P .

A language A is said to have an m -message quantum interactive proof system with completeness error ε_c and soundness error ε_s , where ε_c and ε_s may be functions of the input length, if the exists an m -message verifier V such that

- (i) if $x \in A$ then $\text{max_accept}(V(x)) \geq 1 - \varepsilon_c(|x|)$, and
- (ii) if $x \notin A$ then $\text{max_accept}(V(x)) \leq \varepsilon_s(|x|)$.

We also say that (V, P) is a quantum interactive proof system for A with completeness error ε_c and soundness error ε_s if V satisfies these properties and P is a prover that succeeds in convincing V to accept with probability at least $1 - \varepsilon_c(|x|)$ when $x \in A$.

The following conventions will be used when discussing quantum interactive proof systems. Assume we have a prover P , a verifier V , and an input x . For readability we generally drop the arguments x and $|x|$ in the various functions above when it is understood (e.g., we write V_j and P_j to denote $V(x)_j$ and $P(x)_j$ for each j , and we write m to denote $m(|x|)$). We let \mathcal{V} , \mathcal{M} , and \mathcal{P} denote the Hilbert spaces corresponding to the verifier's qubits, the message qubits, and the prover's qubits, respectively. At a given instant, the state of the qubits in the circuit (V, P) is thus a unit vector in the space $\mathcal{V} \otimes \mathcal{M} \otimes \mathcal{P}$. Throughout this paper, we assume that operators acting

on subsystems of a given system are extended to the entire system by tensoring with the identity. For instance, for a 4-message proof system as illustrated in Figure 1, the state of the system after all circuits have been applied is $V_3 P_2 V_2 P_1 V_1 |0^q\rangle$.

2.2 (Honest verifier) quantum statistical zero-knowledge

Now we discuss the zero-knowledge property for quantum interactive proofs. A short discussion of our definition follows in subsection 2.3.

In the classical case, the zero-knowledge property concerns the distribution of possible conversations between the prover and verifier from the verifier's point of view. In the quantum case, we cannot consider the verifier's view of the *entire* interaction in terms of a single quantum state in any physically meaningful way (this issue is discussed in subsection 2.3 below), so instead we consider the mixed quantum state of the verifier's private qubits together with the message qubits at various times during the protocol. This gives a reasonably natural way of characterizing the verifier's view of the interaction.

It will be sufficient to consider the verifier's view after each message is sent (since the verifier's views at all other times are easily obtained from the views after each message is sent by running the verifier's circuits). The zero-knowledge property will be that the mixed states representing the verifier's view after each message is sent should be approximable to within negligible trace distance by a polynomial-size (uniformly generated) quantum circuit on accepted inputs. We formalize this notion presently.

First, given a collection $\{\rho_y\}$ of mixed states, let us say that the collection is *polynomial-time preparable* if there exists a polynomial-time uniformly generated family $\{Q_y\}$ of quantum circuits, each having a specified collection of output qubits, such that the following holds. For each y , the state ρ_y is the mixed state obtained by running Q_y with all input qubits initialized to the $|0\rangle$ state and then tracing out all non-output qubits.

Next, given a verifier V and a prover P , we define a function $\text{view}_{V,P}(x, j)$ to be the mixed state of the verifier and message qubits after j messages have been sent during an execution of the proof system on input x . For example, if j and m (the total number of messages) are both even, then

$$\text{view}_{V,P}(x, j) = \text{tr}_P P(x)_{j/2} V(x)_{j/2} \cdots P(x)_1 V(x)_1 |0^q\rangle \langle 0^q| V(x)_1^\dagger P(x)_1^\dagger \cdots V(x)_{j/2}^\dagger P(x)_{j/2}^\dagger.$$

The other three cases are defined similarly.

Finally, given a verifier V and a prover P , we say that the pair (V, P) is an *honest verifier quantum statistical zero-knowledge proof system* for a language A if

1. (V, P) is an interactive proof system for A , and
2. there exists a polynomial-time preparable set $\{\sigma_{x,i}\}$ such that

$$x \in A \Rightarrow \|\sigma_{x,i} - \text{view}_{V,P}(x, i)\|_{\text{tr}} \leq \delta(|x|)$$

for some negligible function δ (i.e., $\delta(n) < 1/p(n)$ for sufficiently large n for all polynomials p).

The polynomial-time preparable set $\{\sigma_{x,i}\}$ corresponds to the output of a polynomial-time simulator. The completeness and soundness error of an honest verifier quantum statistical zero-knowledge proof system are determined by the underlying proof system.

Finally we define QSZK (honest verifier quantum statistical zero-knowledge) to be the class of languages having honest verifier quantum statistical zero-knowledge proof systems with completeness and soundness error at most $1/3$. We note that sequential repetition of honest verifier quantum statistical zero-knowledge proof systems reduces completeness and soundness error exponentially while preserving the zero-knowledge property. Thus, we may equivalently define QSZK to be the class of languages having honest verifier quantum statistical zero-knowledge proof systems with completeness and soundness error at most $2^{-p(n)}$ for any chosen polynomial p , or with completeness and soundness error satisfying $(1 - \varepsilon_c(n)) \geq \varepsilon_s(n) + 1/p(n)$ for some polynomial p (assuming that $\varepsilon_c(n)$ and $\varepsilon_s(n)$ are computable in time polynomial in n).

2.3 Notes on the definition

A few notes regarding our definition are in order. First, aside from the obvious difference of quantum vs. classical information, our definition differs from the standard definition for classical honest-verifier statistical zero-knowledge in the following sense. In the classical case, the simulator randomly outputs a transcript representing the *entire interaction* between the prover and verifier, while our definition requires only that the view of the verifier *at each instant* can approximated by a simulator. The main reason for this difference is that the notion of a transcript of a quantum interaction is counter to the nature of quantum information—in general, there is no physically meaningful way to define a transcript of a quantum interaction. For instance, if a verifier were to copy down everything it sees during an interaction in order to produce such a transcript, this would be tantamount to the verifier measuring everything it sees, which could spoil the properties of the protocol.

This suggests the following question about *classical* honest verifier statistical zero-knowledge: is the standard definition equivalent to a definition that is analogous to ours (i.e., requiring only that a simulator exists that takes as input any time t and outputs something that is statistically close to the verifier’s view at time t). We will not attempt to answer this question in this paper.

Thus, we cannot claim that our definition is a direct quantum analogue of the standard classical definition. However, rather than trying to give a direct quantum analogue of the classical definition, or aim has been to provide a definition that (i) is clearly weaker than any reasonable definition for (not necessarily honest verifier) quantum statistical zero-knowledge in order to prove upper bounds on the resulting complexity class, but strong enough to allow interesting bounds to be proved, (ii) satisfies the intuitive notion of honest verifier statistical zero-knowledge, and (iii) is as simple as possible. We certainly do not suggest that our definition is the only natural definition for honest-verifier quantum statistical zero-knowledge. However, our results suggest that our definition yields a complexity class that is a natural quantum variant of classical statistical zero-knowledge, given the similarity of the complete promise problems.

2.4 The quantum state distinguishability problem

A promise problem consists of two disjoint sets $A_{\text{yes}}, A_{\text{no}} \subseteq \Sigma^*$. The computational task associated with a promise problem is as follows: we are given some $x \in A_{\text{yes}} \cup A_{\text{no}}$, and the goal is to accept if $x \in A_{\text{yes}}$ and to reject if $x \in A_{\text{no}}$. Thus, the input is *promised* to be an element of $A_{\text{yes}} \cup A_{\text{no}}$, with no requirement made in case the input string is not in $A_{\text{yes}} \cup A_{\text{no}}$. Ordinary decision problems are a special case of promise problem where $A_{\text{yes}} \cup A_{\text{no}} = \Sigma^*$. See Even, Selman, and Yacobi [12]

for further information on promise problems. Our above definition for QSZK is stated in terms of decision problems, but may be extended to promise problems in the straightforward way.

In this paper we will focus on the following promise problem, which is parameterized by constants α and β satisfying $0 \leq \alpha < \beta \leq 1$. (We will focus on a restricted version of this problem where $\alpha < \beta^2$.)

(α, β) -Quantum State Distinguishability ((α, β)-QSD)

Input: Quantum circuits Q_0 and Q_1 , each acting on m qubits and having k specified output qubits.

Promise: Letting ρ_i denote the mixed state obtained by running Q_i on state $|0^m\rangle$ and discarding (tracing out) the non-output qubits, for $i = 0, 1$, we have either

$$\|\rho_0 - \rho_1\|_{\text{tr}} \leq \alpha \quad \text{or} \quad \|\rho_0 - \rho_1\|_{\text{tr}} \geq \beta.$$

Output: Accept if $\|\rho_0 - \rho_1\|_{\text{tr}} \geq \beta$, reject if $\|\rho_0 - \rho_1\|_{\text{tr}} \leq \alpha$.

3 Quantum SZK proofs for state distinguishability

In this section we discuss constructions for manipulating trace distances of outputs of quantum circuits, then present quantum zero-knowledge protocols for the (α, β) -QSD problem and its complement that are based on these constructions. The conclusion will be that (α, β) -QSD and its complement are in QSZK for any constants α and β satisfying $\alpha < \beta^2$.

3.1 Manipulating trace distance

Sahai and Vadhan [33] give constructions for manipulating the statistical distance between given polynomial-time sampleable distributions. These constructions generalize to the trace distance between polynomial-time preparable mixed quantum states with essentially no changes. The following theorem describes the main consequence of the constructions.

Theorem 1 *Fix constants α and β satisfying $0 \leq \alpha < \beta^2 \leq 1$. There is a (deterministic) polynomial-time procedure that, on input $(Q_0, Q_1, 1^n)$ where Q_0 and Q_1 are descriptions of quantum circuits specifying mixed states ρ_0 and ρ_1 , outputs descriptions of quantum circuits (R_0, R_1) (each having size polynomial in n and in the size of Q_0 and Q_1) specifying mixed states ξ_0 and ξ_1 satisfying the following.*

$$\begin{aligned} \|\rho_0 - \rho_1\|_{\text{tr}} < \alpha &\Rightarrow \|\xi_0 - \xi_1\|_{\text{tr}} < 2^{-n}, \\ \|\rho_0 - \rho_1\|_{\text{tr}} > \beta &\Rightarrow \|\xi_0 - \xi_1\|_{\text{tr}} > 1 - 2^{-n}. \end{aligned}$$

The remainder of this subsection contains a proof of this theorem. The proof relies on the following two lemmas.

Lemma 2 *There is a (deterministic) polynomial-time procedure that, on input $(Q_0, Q_1, 1^r)$ where Q_0 and Q_1 are descriptions of quantum circuits each having k specified output qubits, outputs (R_0, R_1) , where R_0 and R_1 are descriptions of quantum circuits each having rk specified output*

qubits and satisfying the following. Letting ρ_0, ρ_1, ξ_0 , and ξ_1 denote the mixed states obtained by running Q_0, Q_1, R_0 , and R_1 with all inputs in the $|0\rangle$ state and tracing out the output qubits, we have

$$\|\xi_0 - \xi_1\|_{\text{tr}} = \|\rho_0 - \rho_1\|_{\text{tr}}^r.$$

Proof. The circuit R_0 operates as follows: choose $b_1, \dots, b_{r-1} \in \{0, 1\}$ independently and uniformly, set $b_r = b_1 \oplus \dots \oplus b_{r-1}$, and output the state $\rho_{b_1} \otimes \dots \otimes \rho_{b_r}$ (by running Q_{b_1}, \dots, Q_{b_r} on r separate collections of k qubits). The circuit R_1 operates similarly, except b_r is flipped: randomly choose $b_1, \dots, b_{r-1} \in \{0, 1\}$ uniformly, set $b_r = 1 \oplus b_1 \oplus \dots \oplus b_{r-1}$, and output the state $\rho_{b_1} \otimes \dots \otimes \rho_{b_r}$. In both cases, the random choices are easily implemented using the Hadamard transform, and the construction of the circuits is straightforward. The required inequality $\|\xi_0 - \xi_1\|_{\text{tr}} = \|\rho_0 - \rho_1\|_{\text{tr}}^r$ follows from Proposition 15 (in the appendix) along with a simple proof by induction. ■

Lemma 3 *There is a (deterministic) polynomial-time procedure that, on input $(Q_0, Q_1, 1^r)$ where Q_0 and Q_1 are descriptions of quantum circuits each having k specified output qubits, outputs (R_0, R_1) , where R_0 and R_1 are descriptions of quantum circuits each having rk specified output qubits and satisfying the following. Letting ρ_0, ρ_1, ξ_0 , and ξ_1 denote the mixed states obtained by running Q_0, Q_1, R_0 , and R_1 with all inputs in the $|0\rangle$ state and tracing out the output qubits, we have*

$$1 - \exp\left(-\frac{r}{2}\|\rho_0 - \rho_1\|_{\text{tr}}^2\right) \leq \|\xi_0 - \xi_1\|_{\text{tr}} \leq r\|\rho_0 - \rho_1\|_{\text{tr}}.$$

Proof. R_0 and R_1 are each simply obtained by running r independent copies of Q_0 and Q_1 , respectively. Thus $\xi_i = \rho_i^{\otimes r}$ for $i = 0, 1$. The bounds on $\|\xi_0 - \xi_1\|_{\text{tr}}$ follow from Lemma 20 (in the appendix). ■

Proof of Theorem 1. We assume Q_0 and Q_1 each act on m qubits and have k specified output qubits for some choice of m and k .

Apply the construction in Lemma 2 to $(Q_0, Q_1, 1^r)$, where $r = \lceil \log(8n)/\log(\beta^2/\alpha) \rceil$. The result is circuits Q'_0 and Q'_1 that produce states ρ'_0 and ρ'_1 satisfying

$$\begin{aligned} \|\rho_0 - \rho_1\|_{\text{tr}} < \alpha &\Rightarrow \|\rho'_0 - \rho'_1\|_{\text{tr}} < \alpha^r \\ \|\rho_0 - \rho_1\|_{\text{tr}} > \beta &\Rightarrow \|\rho'_0 - \rho'_1\|_{\text{tr}} > \beta^r. \end{aligned}$$

Now apply the construction from Lemma 3 to $(Q'_0, Q'_1, 1^s)$, where $s = \lfloor \alpha^{-r}/2 \rfloor$. This results in circuits Q''_0 and Q''_1 that produce ρ''_0 and ρ''_1 such that

$$\begin{aligned} \|\rho_0 - \rho_1\|_{\text{tr}} < \alpha &\Rightarrow \|\rho''_0 - \rho''_1\|_{\text{tr}} < \alpha^r \alpha^{-r}/2 = 1/2, \\ \|\rho_0 - \rho_1\|_{\text{tr}} > \beta &\Rightarrow \|\rho''_0 - \rho''_1\|_{\text{tr}} > 1 - \exp\left(-\frac{s}{2}\beta^{2r}\right) \geq 1 - e^{-2n+1}. \end{aligned}$$

Finally, again apply the construction from Lemma 2, this time to $(Q''_0, Q''_1, 1^n)$. This results in circuits R_0 and R_1 that produce states ξ_0 and ξ_1 satisfying

$$\begin{aligned} \|\rho_0 - \rho_1\|_{\text{tr}} < \alpha &\Rightarrow \|\xi_0 - \xi_1\|_{\text{tr}} < 2^{-n}, \\ \|\rho_0 - \rho_1\|_{\text{tr}} > \beta &\Rightarrow \|\xi_0 - \xi_1\|_{\text{tr}} > (1 - e^{-2n+1})^n > 1 - 2^{-n}. \end{aligned}$$

The circuits R_0 and R_1 have size polynomial in n and the size of Q_0 and Q_1 as required. ■

3.2 Distance test

Here we describe a quantum statistical zero-knowledge protocol for Quantum State Distinguishability. The protocol is identical in principle to several classical zero-knowledge protocols, including the well-known Graph Non-isomorphism protocol of Goldreich, Micali, and Wigderson [17] and Quadratic Non-residuosity protocol of Goldwasser, Micali, and Rackoff [20].

In the present case the goal of the prover is to prove that two mixed quantum states are far apart in the trace norm metric. A proof system for this problem is that the verifier simply prepares one of the two states, chosen at random, and sends it to the prover, and the prover is challenged to identify which of the two states the verifier sent. If the states are indeed far apart, the prover can determine which state was sent by performing an appropriate measurement, while if the states are close together, the prover cannot reliably tell the difference between the states because there does not exist a measurement that distinguishes them. By requiring that the verifier first apply the construction from the previous section, an exponentially small error is achieved, which makes it very easy to prove that the zero-knowledge property holds. A more precise description of the protocol is as follows:

- Verifier: Apply the construction of Theorem 1 to $(Q_0, Q_1, 1^n)$ for n exceeding the length of the input (Q_0, Q_1) . Let R_0 and R_1 denote the constructed circuits, and ξ_0 and ξ_1 the associated mixed states. Choose $b \in \{0, 1\}$ uniformly and send ξ_b to the prover.
- Honest prover: Perform the optimal measurement for distinguishing ξ_0 and ξ_1 . Let \tilde{b} be 0 if the measurement indicates the state is ξ_0 , and 1 if the measurement indicates the state is ξ_1 . Send \tilde{b} to the verifier.
- Verifier: *Accept* if $b = \tilde{b}$ and *reject* otherwise.

Based on this protocol, we have the following theorem.

Theorem 4 *Let α and β be constants satisfying $0 \leq \alpha < \beta^2 \leq 1$. Then (α, β) -QSD \in QSZK.*

Proof. First we discuss the completeness and soundness of the proof system, then prove that the zero-knowledge property holds.

For the completeness property of the protocol, we assume that the prover receives one of ξ_0 and ξ_1 such that $\|\xi_0 - \xi_1\|_{\text{tr}} > 1 - 2^{-n}$, and thus can distinguish the two cases with probability of error bounded by 2^{-n} by performing an appropriate measurement. Specifically, the prover can apply the measurement described by orthogonal projections $\{\Pi_0, \Pi_1\}$ where Π_0 maximizes $\text{tr} \Pi_0 (\xi_0 - \xi_1)$ and $\Pi_1 = I - \Pi_0$. This gives an outcome of 0 with probability at least $1 - 2^{-n}$ in case the verifier sent ξ_0 and gives an outcome of 1 with probability at least $1 - 2^{-n}$ in case the verifier sent ξ_1 . This will cause the verifier to accept with probability at least $1 - 2^{-n}$.

For the soundness condition, we assume the prover receives either ξ_0 or ξ_1 where $\|\xi_0 - \xi_1\|_{\text{tr}} < 2^{-n}$, and then the prover returns a single bit to the verifier. There is no loss of generality in assuming that the bit sent by the prover is measured immediately upon being received by the verifier, since this would not change the verifier's decision to accept or reject. Thus, we may treat this bit as being the outcome of a measurement of whichever state ξ_0 or ξ_1 was initially sent by the verifier. Since the trace distance between these two states is at most 2^{-n} , no measurement can

distinguish the states with bias exceeding 2^{-n} . Consequently the prover has probability at most $1/2 + 2^{-n}$ of correctly answering $\tilde{b} = b$.

Finally, the zero-knowledge property is straightforward—the state of the verifier and message qubits after the first message is obtained by applying V_1 (the verifier's first transformation), and the state of the verifier and message qubits after the prover's response is approximated by applying V_1 , tracing out the message qubits, then setting \tilde{b} to b . Since the completeness error is exponentially small, this gives a negligible error for the simulator. ■

3.3 Closeness test

Now we consider a protocol for the complement of (α, β) -QSD. Unlike the previous protocol this protocol seems to have no classical analogue, relying heavily on non-classical properties of quantum states.

We begin with a description of the protocol, which is as follows:

- Verifier: Apply the construction of Theorem 1 to $(Q_0, Q_1, 1^{n+1})$ for n exceeding the length of the input (Q_0, Q_1) . Let R_0 and R_1 denote the constructed circuits, and ξ_0 and ξ_1 the associated mixed states. Let t be the number of qubits on which R_0 and R_1 act. Apply R_0 to $|0^t\rangle$ and send the prover **only** the non-output qubits (that is, the qubits that would be traced-out to yield ξ_0).
- Honest prover: Apply unitary transformation U (described below) to the qubits sent by the verifier, then send these qubits back to the verifier.
- Verifier: Apply R_1^\dagger to the output qubits of R_0 (which were not sent to the prover in the first message) together with the qubits received from the prover. Measure the resulting qubits: *accept* if the result is 0^t , and *reject* otherwise.

The correctness of the protocol is closely related to the Schmidt decomposition of bipartite quantum states, which states the following. If $|\phi\rangle \in \mathcal{H} \otimes \mathcal{K}$ is a pure, bipartite quantum state, then it is possible to write

$$|\phi\rangle = \sum_{i=1}^n \sqrt{p_i} |\psi_i\rangle |\nu_i\rangle$$

for positive real numbers p_1, \dots, p_n and orthonormal sets $\{|\psi_1\rangle, \dots, |\psi_n\rangle\}$ and $\{|\nu_1\rangle, \dots, |\nu_n\rangle\}$. Such sets may be obtained by letting $\{|\psi_1\rangle, \dots, |\psi_n\rangle\}$ be an orthonormal collection of eigenvectors of $\rho = \text{tr}_{\mathcal{K}} |\phi\rangle\langle\phi|$ having nonzero eigenvalues and taking p_1, \dots, p_n to be the corresponding nonzero eigenvalues, which are therefore positive since ρ is positive semidefinite. At this point $|\nu_1\rangle, \dots, |\nu_n\rangle$ are determined, and can be shown to be orthonormal. Consequently, if we have two bipartite states $|\phi\rangle, |\phi'\rangle \in \mathcal{H} \otimes \mathcal{K}$ that give the same mixed state when the second system is traced-out, i.e., $\text{tr}_{\mathcal{K}} |\phi\rangle\langle\phi| = \text{tr}_{\mathcal{K}} |\phi'\rangle\langle\phi'| = \rho$, then there must exist a unitary operator U acting on \mathcal{K} such that $(I \otimes U)|\phi\rangle = |\phi'\rangle$. The operator U is simply a change of basis taking $|\nu_i\rangle$ to $|\nu'_i\rangle$ for each i , where the vectors $|\nu'_1\rangle, \dots, |\nu'_n\rangle$ are given by

$$|\phi'\rangle = \sum_{i=1}^n \sqrt{p_i} |\psi_i\rangle |\nu'_i\rangle.$$

In case $\rho = \text{tr}_{\mathcal{K}} |\phi\rangle\langle\phi|$ and $\rho' = \text{tr}_{\mathcal{K}} |\phi'\rangle\langle\phi'|$ are not identical, but are close together in the trace norm metric, an approximate version of this fact holds: there exists a unitary operator U acting on \mathcal{K} such that $(I \otimes U)|\phi\rangle$ and $|\phi'\rangle$ are close in Euclidean norm. For the above protocol, the states $|\phi\rangle$ and $|\phi'\rangle$ are the states produced by R_0 and R_1 , \mathcal{K} is the space corresponding to the qubits sent to the prover, and U corresponds to the action of the prover.

We formalize this argument in the proof of the following theorem.

Theorem 5 *Let α and β satisfy $0 \leq \alpha < \beta^2 \leq 1$. Then (α, β) -QSD $\in \text{co-QSZK}$.*

Proof. First let us consider the completeness condition. If $(Q_0, Q_1) \notin (\alpha, \beta)$ -QSD then we have $\|\xi_0 - \xi_1\|_{\text{tr}} < 2^{-(n+1)}$ and thus $F(\xi_0, \xi_1) > 1 - 2^{-(n+1)}$ (where $F(\xi_0, \xi_1)$ denotes the fidelity of ξ_0 and ξ_1). The states $R_0|0^t\rangle$ and $R_1|0^t\rangle$ are purifications of ξ_0 and ξ_1 , respectively, so by Lemma 21 (in the appendix) there exists a unitary transformation U acting only on the non-output qubits of $R_0|0^t\rangle$ (i.e., the qubits sent to the prover) such that $\|(I \otimes U)R_0|0^t\rangle - R_1|0^t\rangle\| \leq 2^{-n/2}$. This is the transformation U performed by the honest prover. The verifier accepts with probability

$$|\langle 0^t | R_1^\dagger (I \otimes U) R_0 | 0^t \rangle|^2 \geq \left(1 - \frac{1}{2} \|R_1|0^t\rangle - (I \otimes U)R_0|0^t\rangle\|^2\right)^2 > 1 - 2^{-n}.$$

The soundness of the proof system may be proved as follows. Assume $(Q_0, Q_1) \in (\alpha, \beta)$ -QSD, so that $\|\xi_0 - \xi_1\|_{\text{tr}} > 1 - 2^{-(n+1)}$, and thus $F(\xi_0, \xi_1) < 2^{-n/2}$. The verifier prepares $R_0|0^t\rangle$ and sends the non-output qubits to the prover. The most general action of the prover is to apply some arbitrary unitary transformation to the qubits sent by the verifier along with any number of its own private qubits, and then return some number of these qubits to the verifier. Let σ denote the mixed state of the verifier's private qubits and the message qubits immediately after the prover has sent its message. As usual we let \mathcal{V} denote the space corresponding to the verifier's private qubits and \mathcal{M} the space corresponding to the message qubits, so that $\sigma \in \mathbf{D}(\mathcal{V} \otimes \mathcal{M})$ and $\text{tr}_{\mathcal{M}} \sigma = \xi_0$. (The fact that $\text{tr}_{\mathcal{M}} \sigma = \xi_0$ follows from the fact that the prover has not touched the verifier's private qubits, so that they must still be in state ξ_0 .) The verifier applies R_1^\dagger and measures, which results in *accept* with probability $\langle 0^t | R_1^\dagger \sigma R_1 | 0^t \rangle$. Since $R_1|0^t\rangle$ is a purification of ξ_1 , we have that $\langle 0^t | R_1^\dagger \sigma R_1 | 0^t \rangle \leq F(\xi_0, \xi_1)^2 < 2^{-n}$ by Lemma 19 (in the appendix). Thus the verifier accepts with exponentially small probability.

Finally, the zero-knowledge property is again straightforward. We define a simulator that outputs $R_0|0^t\rangle$ for the verifier's view as the first message is being sent and $R_1|0^t\rangle$ for the verifier's view after the second message. The simulator is perfect for the first message, and has trace distance at most 2^{-n} from the actual view of the verifier interacting with the prover defined above for the second message. ■

4 Completeness of quantum state distinguishability for QSZK

The notion of a promise problem being complete for a given class is defined in the most straightforward way; in the case of QSZK we say that a promise problem $B = (B_{\text{yes}}, B_{\text{no}})$ is complete for QSZK if (i) $B \in \text{QSZK}$, and (ii) for every promise problem $A = (A_{\text{yes}}, A_{\text{no}}) \in \text{QSZK}$ there is a deterministic polynomial-time computable function f such that for all x we have $x \in A_{\text{yes}} \Rightarrow f(x) \in B_{\text{yes}}$ and $x \in A_{\text{no}} \Rightarrow f(x) \in B_{\text{no}}$. In this section we prove that (α, β) -QSD is complete for QSZK whenever α and β are constants satisfying $0 < \alpha < \beta^2 < 1$.

Theorem 6 Let α and β satisfy $0 < \alpha < \beta^2 < 1$. Then (α, β) -QSD is complete for QSZK.

By Theorems 4 and 5 we have that (α, β) -QSD is in $\text{QSZK} \cap \text{co-QSZK}$ provided $\alpha < \beta^2$. In order to prove Theorem 6 it will therefore suffice to show that, for any promise problem $A \in \text{QSZK}$, A reduces to the complement of (α, β) -QSD. After describing the reduction f , the main facts to be proved will therefore be

- (i) $x \in A_{\text{yes}} \Rightarrow f(x) \in (\alpha, \beta)\text{-QSD}_{\text{no}}$, and
- (ii) $x \in A_{\text{no}} \Rightarrow f(x) \in (\alpha, \beta)\text{-QSD}_{\text{yes}}$.

The following technical lemma will be useful in the proof.

Lemma 7 Let V be an m -message verifier and x an input such that $m = m(|x|)$ is even and $\max_{\text{accept}}(V(x)) \leq \varepsilon$. Let $k = m/2 + 1$, so that $V(x) = (V_1, \dots, V_k)$. Let $\rho_0, \dots, \rho_{k-1} \in \mathbf{D}(\mathcal{V} \otimes \mathcal{M})$, let $\xi_i = V_i \rho_{i-1} V_i^\dagger$ for $i = 1, \dots, k$, and assume that $\rho_0 = |0^{q_V+q_M}\rangle\langle 0^{q_V+q_M}|$ (i.e., ρ_0 denotes the initial state of the qubits) and $\text{tr}(\Pi_{\text{acc}} \xi_k) = 1$ for Π_{acc} denoting the projection onto states for which the output qubit has value 1 (i.e., ξ_k is a state where the verifier accepts with certainty). Then

$$\|\text{tr}_{\mathcal{M}} \xi_1 \otimes \dots \otimes \text{tr}_{\mathcal{M}} \xi_{k-1} - \text{tr}_{\mathcal{M}} \rho_1 \otimes \dots \otimes \text{tr}_{\mathcal{M}} \rho_{k-1}\|_{\text{tr}} \geq \frac{(1 - \sqrt{\varepsilon})^2}{3(k-1)}.$$

Proof. Let $|\phi_0\rangle = |0^q\rangle$, which is a purification of ρ_0 , let $|\phi_1\rangle, \dots, |\phi_{k-1}\rangle \in \mathcal{V} \otimes \mathcal{M} \otimes \mathcal{P}$ be purifications of $\rho_1, \dots, \rho_{k-1}$, and set $|\psi_i\rangle = V_i |\phi_{i-1}\rangle$ for $i = 1, \dots, k$. (As usual, we extend each V_i to a unitary operator on $\mathcal{V} \otimes \mathcal{M} \otimes \mathcal{P}$ by tensoring with the identity on \mathcal{P}). Note that $|\psi_1\rangle, \dots, |\psi_k\rangle$ are necessarily purifications of ξ_1, \dots, ξ_k .

Define

$$\delta_i = 1 - F(\text{tr}_{\mathcal{M}} \xi_i, \text{tr}_{\mathcal{M}} \rho_i)$$

for $i = 1, \dots, k-1$. By Lemma 21 (in the appendix) there exists a unitary operator $P_i \in \mathbf{U}(\mathcal{M} \otimes \mathcal{P})$ such that

$$\|P_i |\psi_i\rangle - |\phi_i\rangle\| \leq \sqrt{2\delta_i}.$$

Now, for each $i = 2, \dots, k$, we have

$$\begin{aligned} & \|V_i P_{i-1} V_{i-1} \cdots P_1 V_1 |\phi_0\rangle - |\psi_i\rangle\| \\ &= \|P_{i-1} V_{i-1} \cdots P_1 V_1 |\phi_0\rangle - |\phi_{i-1}\rangle\| \\ &\leq \|P_{i-1} V_{i-1} \cdots P_1 V_1 |\phi_0\rangle - P_{i-1} |\psi_{i-1}\rangle\| + \|P_{i-1} |\psi_{i-1}\rangle - |\phi_{i-1}\rangle\| \\ &\leq \|V_{i-1} \cdots P_1 V_1 |\phi_0\rangle - |\psi_{i-1}\rangle\| + \sqrt{2\delta_{i-1}}, \end{aligned}$$

so that

$$\|V_k P_{k-1} V_{k-1} \cdots P_1 V_1 |\phi_0\rangle - |\psi_k\rangle\| \leq \sum_{i=1}^{k-1} \sqrt{2\delta_i}.$$

Consequently, since $\|\Pi_{\text{acc}} |\psi_k\rangle\| = 1$, we must have

$$\|\Pi_{\text{acc}} V_k P_{k-1} V_{k-1} \cdots P_1 V_1 |\phi_0\rangle\| \geq 1 - \sum_{i=1}^{k-1} \sqrt{2\delta_i}.$$

Since $\max\text{-accept}(V(x)) \leq \varepsilon$ and $|\phi_0\rangle$ is the initial state of $(V(x), P(x))$, this implies

$$\sum_{i=1}^{k-1} \sqrt{2\delta_i} \geq 1 - \sqrt{\varepsilon}. \quad (1)$$

Now, by Proposition 13, we have

$$F(\text{tr}_{\mathcal{M}} \xi_1 \otimes \cdots \otimes \text{tr}_{\mathcal{M}} \xi_{k-1}, \text{tr}_{\mathcal{M}} \rho_1 \otimes \cdots \otimes \text{tr}_{\mathcal{M}} \rho_{k-1}) = \prod_{i=1}^{k-1} F(\text{tr}_{\mathcal{M}} \xi_i, \text{tr}_{\mathcal{M}} \rho_i) \leq \prod_{i=1}^{k-1} (1 - \delta_i).$$

Subject to the constraint in Eq. 1, we have

$$\prod_{i=1}^{k-1} (1 - \delta_i) \leq \left(1 - \frac{(1 - \sqrt{\varepsilon})^2}{2(k-1)^2}\right)^{k-1} \leq \exp\left(-\frac{(1 - \sqrt{\varepsilon})^2}{2(k-1)}\right) \leq 1 - \frac{(1 - \sqrt{\varepsilon})^2}{3(k-1)}.$$

Thus,

$$\|\text{tr}_{\mathcal{M}} \xi_1 \otimes \cdots \otimes \text{tr}_{\mathcal{M}} \xi_{k-1} - \text{tr}_{\mathcal{M}} \rho_1 \otimes \cdots \otimes \text{tr}_{\mathcal{M}} \rho_{k-1}\|_{\text{tr}} \geq \frac{(1 - \sqrt{\varepsilon})^2}{3(k-1)}$$

as required. ■

Proof of Theorem 6. Let $A \in \text{QSZK}$, and let (V, P) be an honest verifier quantum statistical zero-knowledge proof system for A with completeness and soundness error smaller than 2^{-n} for inputs of length n . Such a proof system exists, since sequential repetition reduces completeness and soundness errors exponentially while preserving the zero-knowledge property of honest verifier quantum statistical zero-knowledge proof systems. Let $m = m(|x|)$ be the number of messages exchanged by P and V . For simplicity we assume that the number of messages m is even for all x (adding an initial move where the verifier sends some arbitrary state if necessary). Thus, the verifier will apply transformations V_1, \dots, V_k for $k = m/2 + 1$, and will send the first message in the protocol. We let $\{\sigma_{x,j}\}$ correspond to the mixed states output by the simulator for (V, P) as discussed in Section 2. The quantum circuits that produce the states $\{\sigma_{x,j}\}$ are used implicitly in the reduction.

First, we describe, for any fixed input x , the following quantum states:

1. Let ρ_0 be the state in which all verifier and message qubits are in state $|0\rangle$.
2. Let ξ_k denote the state obtained by applying V_k to $\sigma_{x,m}$, discarding the output qubit, and replacing it with a qubit in state $|1\rangle$.
3. Let $\rho_i = \sigma_{x,2i}$ for $i = 1, \dots, k-2$ and let $\rho_{k-1} = V_k^\dagger \xi_k V_k$.
4. Let $\xi_i = V_i \rho_{i-1} V_i^\dagger$ for $i = 1, \dots, k-1$.

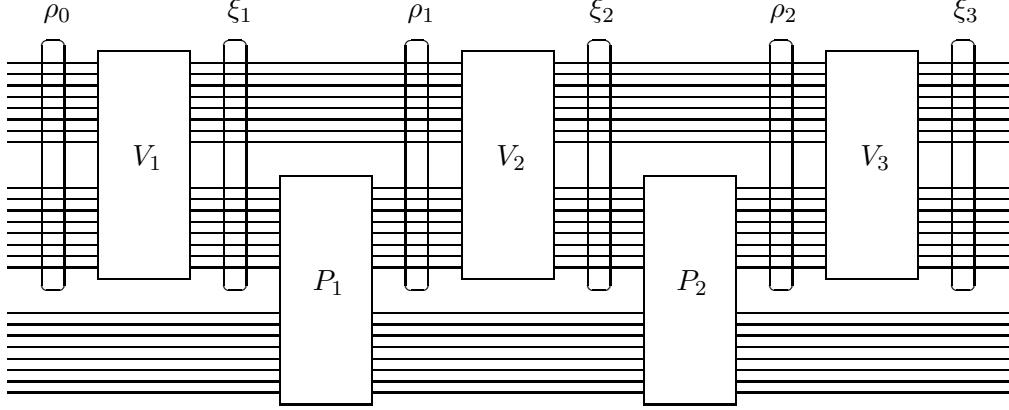


Figure 2: States $\rho_0, \dots, \rho_{k-1}$ and ξ_1, \dots, ξ_k for $m = 4, k = 3$.

These states are illustrated in Figure 2 for the case $m = 4$ (meaning that these states will be close approximations to the illustrated states given an input $x \in A_{\text{yes}}$). Let $Q_0 = Q_0(x)$ and $Q_1 = Q_1(x)$ be quantum circuits that output $\gamma_0 = \text{tr}_{\mathcal{M}}(\rho_1) \otimes \dots \otimes \text{tr}_{\mathcal{M}}(\rho_{k-1})$ and $\gamma_1 = \text{tr}_{\mathcal{M}}(\xi_1) \otimes \dots \otimes \text{tr}_{\mathcal{M}}(\xi_{k-1})$, respectively (assuming the circuits are applied to the state $|0^m\rangle$ for appropriate m and non-output qubits are traced out in the usual way). These circuits are easily constructed based on V and on the simulator for (V, P) .

We claim that the following implications hold:

$$x \in A_{\text{yes}} \Rightarrow \|\gamma_0 - \gamma_1\|_{\text{tr}} < \delta(|x|) \quad \text{and} \quad x \in A_{\text{no}} \Rightarrow \|\gamma_0 - \gamma_1\|_{\text{tr}} > c/k$$

where $\delta(|x|)$ is a negligible function (determined by the accuracy of the simulator for (V, P)) and $c > 0$ is constant. The second implication follows immediately from Lemma 7. To prove the first implication, consider states $\rho'_0, \dots, \rho'_{k-1}$ and ξ'_1, \dots, ξ'_k obtained precisely as in the description of Q_0 and Q_1 , except replacing $\sigma_{x,j}$ with $\text{view}_{V,P}(x, j)$, the actual view of the verifier V while interacting with P , for each x and j . We necessarily have $\text{tr}_{\mathcal{M}} \xi'_i = \text{tr}_{\mathcal{M}} \rho'_i$ for $i = 1, \dots, k-2$. Since measuring the output qubit of $V_k \text{view}_{V,P}(x, m) V_k^\dagger$ gives 1 with probability at least $1 - 2^{-|x|}$, replacing the output qubit with a qubit in state $|1\rangle$ has little effect on this state. Specifically, we deduce that $\|\text{tr}_{\mathcal{M}} \xi'_{k-1} - \text{tr}_{\mathcal{M}} \rho'_{k-1}\|_{\text{tr}} < 2^{-|x|/2}$. Thus, the quantity

$$\|\text{tr}_{\mathcal{M}}(\rho'_1) \otimes \dots \otimes \text{tr}_{\mathcal{M}}(\rho'_{k-1}) - \text{tr}_{\mathcal{M}}(\xi'_1) \otimes \dots \otimes \text{tr}_{\mathcal{M}}(\xi'_{k-1})\|_{\text{tr}}$$

is negligible. Now, since the simulator deviates from $\text{view}_{V,P}$ by a negligible quantity on each input, the inequality

$$\|\text{tr}_{\mathcal{M}}(\rho_1) \otimes \dots \otimes \text{tr}_{\mathcal{M}}(\rho_{k-1}) - \text{tr}_{\mathcal{M}}(\xi_1) \otimes \dots \otimes \text{tr}_{\mathcal{M}}(\xi_{k-1})\|_{\text{tr}} < \delta(|x|)$$

for some negligible $\delta(|x|)$ follows from the triangle inequality.

Finally, by applying the constructions from Lemmas 2 and 3 to (Q_0, Q_1) appropriately results in circuits R_0 and R_1 that specify mixed states γ_0 and γ_1 , respectively, such that

- (i) $x \in A_{\text{yes}} \Rightarrow \|\gamma_0 - \gamma_1\|_{\text{tr}} < \alpha$, and

(ii) $x \in A_{\text{no}} \Rightarrow \|\gamma_0 - \gamma_1\|_{\text{tr}} > \beta$

for any chosen constants $\alpha, \beta \in (0, 1)$.

Thus, $x \in A_{\text{yes}}$ implies $(R_0, R_1) \in (\alpha, \beta)\text{-QSD}_{\text{no}}$ and $x \in A_{\text{no}}$ implies $(R_0, R_1) \in (\alpha, \beta)\text{-QSD}_{\text{yes}}$ as required. \blacksquare

Corollary 8 *QSZK is closed under complement.*

Corollary 9 *For any language $L \in \text{QSZK}$ there is a 2-message honest verifier quantum statistical zero-knowledge proof system with exponentially small completeness error and soundness error exponentially close to $1/2$ in which the prover's message to the verifier consists of a single bit.*

Corollary 8 follows from Theorem 6 together with Theorem 5, and Corollary 9 follows from Theorem 6 and the proof of Theorem 4.

Corollary 10 *$\text{QSZK} \subseteq \text{PSPACE}$.*

In order to prove this Corollary, let us consider the following problem.

Trace Norm Approximation (TNA)

Input: An $n \times n$ matrix X (with entries having rational real and imaginary parts) and an accuracy parameter 1^k .

Output: A nonnegative rational number r satisfying $|r - \|X\|_{\text{tr}}| < 2^{-k}$.

Proposition 11 *$\text{TNA} \in \text{NC}$.*

Proof. [Sketch] Consider the following algorithm.

1. Compute $Y = XX^\dagger$.
2. Compute the characteristic polynomial of Y (the coefficients will be real since Y is necessarily Hermitian).
3. Calculate the n roots $\lambda_1, \dots, \lambda_n$ of the characteristic polynomial of Y to $O(k + \log n)$ bits of precision.
4. Compute $r = \frac{1}{2} \sum_{j=1}^n \sqrt{\lambda_j}$, where each square root is approximated to $O(k + \log n)$ bits of precision, and output r .

The output r is an approximation to one-half the trace of $\sqrt{XX^\dagger}$, which is $\|X\|_{\text{tr}}$. The approximation is correct to $O(k)$ bits of precision as required. Each step can be performed in NC; simple arithmetic operations and multiplication of matrices are well-known to be in NC, the fact that the characteristic polynomial can be computed in NC was shown by Csanky [11], and polynomial root approximation was shown to be in NC by Neff [29]. \blacksquare

Proof of Corollary 10. [Sketch] By Theorem 6 it suffices to show that $(\alpha, \beta)\text{-QSD}$ is in PSPACE. Recall that for any function $s(n) \geq \log n$, $\text{NC}(2^s)$ denotes the class of languages computable by space $O(s)$ -uniform boolean circuits having size $2^{O(s)}$ and depth $s^{O(1)}$ [8]. The class $\text{NC}(2^s)$ is contained in $\text{DSPACE}(s^{O(1)})$ [7]. Thus, it will suffice to prove that $(\alpha, \beta)\text{-QSD}$ is contained in $\text{NC}(2^n)$.

Let (Q_0, Q_1) be an input pair of quantum circuits specifying density matrices (ρ_0, ρ_1) on k qubits, and let n be the length of the description of the pair (Q_0, Q_1) . Obviously we may assume $k \leq n$, the number of qubits m on which Q_0 and Q_1 act satisfies $m \leq n$, and each of Q_0 and Q_1 contains at most n gates. We assume Q_0 and Q_1 are composed of gates that can be described by unitary matrices having entries with rational real and imaginary parts (see Section A). Thus, ρ_0 and ρ_1 correspond to $N \times N$ matrices where $N \leq 2^n$, and for each entry of ρ_0 and ρ_1 the numerators and denominators of the real and imaginary parts are $O(n)$ -bit integers.

For each $i = 0, 1$ it is possible to compute $|\psi_i\rangle = Q_i|0^m\rangle$ (expressed as a 2^m -dimensional vector with rational real and imaginary parts) in $\text{NC}(2^n)$, simply by computing the product of the matrices corresponding to each individual gate. (In fact, there are better ways to do this from a complexity-theoretic standpoint [14], but this method is sufficient for our needs.) Once these vectors are computed, it is possible to compute $\rho_0 - \rho_1$ in $\text{NC}(2^n)$ by constructing $|\psi_0\rangle\langle\psi_0|$ and $|\psi_1\rangle\langle\psi_1|$, performing the partial trace on the non-output qubits for each matrix (which involves computing a sum of at most 2^n matrices, each of which is obtained by multiplying $|\psi_i\rangle\langle\psi_i|$ on the left and on the right by a $2^k \times 2^m$ or $2^m \times 2^k$ matrix, respectively, as in the definition of the partial trace), and then computing the difference of the resulting matrices. Once we have $\rho_0 - \rho_1$, we may use the method described in Proposition 11 to compute $\|\rho_0 - \rho_1\|_{\text{tr}}$ in $\text{NC}(2^n)$ (which is NC with respect to the size of $\rho_0 - \rho_1$). Since it is only required that the cases $\|\rho_0 - \rho_1\|_{\text{tr}} \leq \alpha$ and $\|\rho_0 - \rho_1\|_{\text{tr}} \geq \beta$ be discriminated, $\|\rho_0 - \rho_1\|_{\text{tr}}$ need in fact only be computed to $O(1)$ bits of precision. This completes the proof. ■

5 Conclusion

We have given a simple definition for honest verifier quantum statistical zero-knowledge and proved several facts about the resulting complexity class. Many questions regarding quantum statistical zero-knowledge, and quantum zero-knowledge more generally, are left open. For instance:

- What are other natural definitions for quantum statistical zero-knowledge, and how do they compare to our definition? In particular, how does our definition for honest verifier quantum statistical zero-knowledge compare to possible definitions for (not necessarily honest verifier) quantum statistical zero-knowledge? Are there quantum protocols that satisfy intuitive notions of statistical zero-knowledge that do not satisfy our definition?
- What is the most reasonable definition for computational quantum zero-knowledge, and what can be said about this class?
- What further relations among QSZK and other complexity classes can be shown? Is there a better upper bound than PSPACE? Is it possible that $\text{NP} \subseteq \text{QSZK}$, or do unexpected consequences result from such an assumption?
- The Quantum State Distinguishability problem is natural from the perspective of quantum computation and quantum information theory, but is rather unnatural outside of this scope. Are there more natural problems that are candidates for problems in QSZK but not in SZK?

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Appendix

A Quantum circuits and the quantum formalism

We assume that the reader is familiar with the basics of quantum computation, including the notion of (pure) quantum states, unitary operators, and projective (or von Neumann) measurements. We also assume familiarity with the quantum circuit model. For further background information we refer the reader to Nielsen and Chuang [30], Berthiaume [6], and Kitaev [25]. In this paper we will rely heavily on the so-called *density matrix formalism*, which we briefly discuss below. This formalism is discussed in detail by Nielsen and Chuang.

We use the following notion of a uniform family of quantum circuits. A family $\{Q_x\}$ of quantum circuits is said to be *polynomial-time uniformly generated* if there exists a deterministic procedure that, on input x , outputs a description of Q_x and runs in time polynomial in x . It is assumed that the number of gates in any circuit is not more than the length of that circuit’s description (i.e., no compact descriptions of large circuits are allowed), so that Q_x must have size polynomial in $|x|$. We also assume that quantum circuits are composed of gates from some reasonable, universal, finite set of (unitary) gates. By “reasonable” we mean, for instance, that gates cannot be defined by matrices with non-computable, or difficult to compute, entries. In fact, it will be helpful later to use the fact that any quantum circuit composed of gates from any reasonable set of basis gates can be efficiently simulated by a quantum circuit consisting only of gates from a finite collection whose corresponding matrices have only entries with rational real and imaginary parts. See, for instance, Section 4.5.3 in Nielsen and Chuang for further discussion. It should be noted that our notion of uniformity is somewhat nonstandard, since we allow an input x to be given to the procedure generating the circuits rather than just $|x|$ written in unary (with x given as input to the circuit itself). This does not change the computational power for the resulting class of quantum circuits, however, and we find that it is more convenient to describe quantum interactive proof systems using this notion.

Now we briefly discuss the density matrix formalism. Among other things, this formalism provides a way to describe subsystems of quantum systems, which is helpful when considering quantum interactive proof systems and crucial for extending the notion of zero-knowledge to the quantum setting.

Recall that a *pure (quantum) state* (or superposition) of an n -qubit quantum system is a unit vector in the Hilbert space¹ $\mathcal{H} = \ell_2(\{0, 1\}^n)$, and corresponding to each pure state $|\psi\rangle \in \mathcal{H}$ is a linear functional $\langle\psi|$ that maps each vector $|\phi\rangle$ to the inner product $\langle\psi|\phi\rangle$ (conjugate-linear in the first argument). A *mixed state* of a quantum system is a state that may be described by a distribution on (not necessarily orthogonal) pure states. A collection $\{(p_k, |\psi_k\rangle)\}$ such that $0 \leq p_k$, $\sum_k p_k = 1$, and each $|\psi_k\rangle$ is a pure state is called a *mixture*: for each k , the system is in state $|\psi_k\rangle$ with probability p_k . For a given mixture $\{(p_k, |\psi_k\rangle)\}$, we associate a *density matrix* ρ having operator representation $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$. Necessary and sufficient conditions for a given matrix ρ to be a density matrix (i.e., to represent some mixed state) are (i) ρ must be positive semidefinite, and (ii) ρ must have unit trace. Two mixtures can be distinguished (in a statistical sense) if and only if they yield different density matrices, and for this reason we interpret a given density matrix ρ as being a canonical representation of a given mixed state. Unitary transformations and

¹All Hilbert spaces referred to in this paper are assumed to be finite dimensional.

measurements work as follows on density matrices. Applying a unitary operator U to ρ yields $U\rho U^\dagger$, and measuring a mixed state ρ according to a (projective) measurement described by some complete, orthogonal set of projections $\{\Pi_1, \dots, \Pi_l\}$ yields result j with probability $\text{tr} \Pi_j \rho$.

The quantum circuit model has been extended to the density matrix formalism by Aharonov, Kitaev, and Nisan [2], who show that the resulting model (which allows more general types of gates than the usual model, such as “measurement gates”) is equivalent in power to the usual model in which only unitary gates are allowed. As stated above, we assume all quantum circuits in our model consist of only unitary gates, which causes no loss of generality following from this equivalence.

In order to describe the density matrix formalism further, it will be helpful at this point to introduce some notation. For a given Hilbert space \mathcal{H} , let $\mathbf{L}(\mathcal{H})$ denote the set of linear operators on \mathcal{H} , let $\mathbf{D}(\mathcal{H})$ denote the set of positive semidefinite operators on \mathcal{H} having unit trace (so that $\mathbf{D}(\mathcal{H})$ may be identified with the set of mixed states of a given system), let $\mathbf{U}(\mathcal{H})$ denote the set of unitary operators on \mathcal{H} , and let $\mathbf{P}(\mathcal{H})$ denote the set of projection operators on \mathcal{H} .

Given Hilbert spaces \mathcal{H} and \mathcal{K} , we define a mapping $\text{tr}_{\mathcal{K}} : \mathbf{D}(\mathcal{H} \otimes \mathcal{K}) \rightarrow \mathbf{D}(\mathcal{H})$ as follows:

$$\text{tr}_{\mathcal{K}} \rho = \sum_{j=1}^n (I \otimes \langle e_j |) \rho (I \otimes |e_j \rangle),$$

where $\{|e_1\rangle, \dots, |e_n\rangle\}$ is any orthonormal basis of \mathcal{K} . This mapping is known as the *partial trace*, and has the following intuitive meaning: given a mixed state $\rho \in \mathbf{D}(\mathcal{H} \otimes \mathcal{K})$ of a bipartite system (meaning that the first part of the system corresponds to \mathcal{H} and the second part to \mathcal{K}), $\text{tr}_{\mathcal{K}} \rho$ is the mixed state of the first part of the system obtained by discarding or not considering the second part of the system. To say that a particular part of a quantum system is *traced out* means that the partial trace is performed, removing this part of the system from consideration.

A *purification* of a given mixed state $\rho \in \mathbf{D}(\mathcal{H})$ is any pure state $|\psi\rangle$ of a larger quantum system that gives ρ when part of the system is traced out. In other words, we have $|\psi\rangle \in \mathcal{H} \otimes \mathcal{K}$ for some Hilbert space \mathcal{K} such that $\text{tr}_{\mathcal{K}} |\psi\rangle \langle \psi| = \rho$.

For $X \in \mathbf{L}(\mathcal{H})$ define

$$\|X\|_{\text{tr}} = \frac{1}{2} \text{tr} \sqrt{X^\dagger X}.$$

(Recall that for any positive semidefinite matrix A there is a unique positive semidefinite matrix denoted \sqrt{A} that satisfies $(\sqrt{A})^2 = A$.) The function $\|\cdot\|_{\text{tr}}$ is a norm called the *trace norm*, and generalizes the norm induced by the statistical difference or total variation distance (i.e., one-half the ℓ_1 norm). For any normal matrix X , the trace norm is simply one-half the sum of the absolute values of the eigenvalues of X . For any $X \in \mathbf{L}(\mathcal{H})$ we have $\|X\|_{\text{tr}} = \max_A |\text{tr} AX|$, where the maximum is over all positive semidefinite $A \in \mathbf{L}(\mathcal{H})$ with $\|A\| \leq 1$. Alternately we may take the maximum to be over all projections $A \in \mathbf{P}(\mathcal{H})$, which does not change the maximum value.

Given two mixed states $\rho, \xi \in \mathbf{D}(\mathcal{H})$, define the *fidelity* of ρ and ξ by

$$F(\rho, \xi) = \text{tr} \sqrt{\rho^{1/2} \xi \rho^{1/2}}.$$

For all $\rho, \xi \in \mathbf{D}(\mathcal{H})$ we have $1 - F(\rho, \xi) \leq \|\rho - \xi\|_{\text{tr}} \leq \sqrt{1 - F(\rho, \xi)^2}$. This and several other facts about the trace norm and the fidelity are discussed in the next section.

B Basic properties of fidelity and the trace distance

In this section of the appendix we give proofs or references for basic facts about trace distance and fidelity that are used elsewhere in the paper.

Proposition 12 *For all $\rho, \xi \in \mathbf{D}(\mathcal{H})$ we have*

$$1 - F(\rho, \xi) \leq \|\rho - \xi\|_{\text{tr}} \leq \sqrt{1 - F(\rho, \xi)^2}.$$

See Section 9.2.3 of Nielsen and Chuang [30] for a proof.

Proposition 13 *For any $\rho_1, \xi_1 \in \mathbf{D}(\mathcal{H})$ and $\rho_2, \xi_2 \in \mathbf{D}(\mathcal{K})$ we have*

$$F(\rho_1 \otimes \rho_2, \xi_1 \otimes \xi_2) = F(\rho_1, \xi_1) F(\rho_2, \xi_2).$$

Proof. For given positive semidefinite matrices A and B we have $\sqrt{A \otimes B} = \sqrt{A} \otimes \sqrt{B}$ and $\text{tr } A \otimes B = (\text{tr } A)(\text{tr } B)$. Thus,

$$\begin{aligned} F(\rho_1 \otimes \rho_2, \xi_1 \otimes \xi_2) &= \text{tr} \sqrt{(\rho_1 \otimes \rho_2)^{1/2} (\xi_1 \otimes \xi_2) (\rho_1 \otimes \rho_2)^{1/2}} \\ &= \text{tr} \left(\sqrt{\rho_1^{1/2} \xi_1 \rho_1^{1/2}} \otimes \sqrt{\rho_2^{1/2} \xi_2 \rho_2^{1/2}} \right) \\ &= F(\rho_1, \xi_1) F(\rho_2, \xi_2) \end{aligned}$$

as required. ■

Proposition 14 *Let $A \in \mathbf{L}(\mathcal{H})$ and $B \in \mathbf{L}(\mathcal{K})$. Then $\|A \otimes B\|_{\text{tr}} = 2 \|A\|_{\text{tr}} \|B\|_{\text{tr}}$.*

Proof. We have

$$\begin{aligned} \|A \otimes B\|_{\text{tr}} &= \frac{1}{2} \text{tr} \sqrt{A^\dagger A \otimes B^\dagger B} \\ &= \frac{1}{2} \text{tr} \sqrt{A^\dagger A} \otimes \sqrt{B^\dagger B} \\ &= \frac{1}{2} \left(\text{tr} \sqrt{A^\dagger A} \right) \left(\text{tr} \sqrt{B^\dagger B} \right) \\ &= 2 \|A\|_{\text{tr}} \|B\|_{\text{tr}} \end{aligned}$$

as required. ■

Proposition 15 *Let $\rho_0, \rho_1 \in \mathbf{D}(\mathcal{H})$ and $\xi_0, \xi_1 \in \mathbf{D}(\mathcal{K})$. Define*

$$\begin{aligned} \gamma_0 &= \frac{1}{2}(\rho_0 \otimes \xi_0) + \frac{1}{2}(\rho_1 \otimes \xi_1), \\ \gamma_1 &= \frac{1}{2}(\rho_0 \otimes \xi_1) + \frac{1}{2}(\rho_1 \otimes \xi_0). \end{aligned}$$

Then $\|\gamma_0 - \gamma_1\|_{\text{tr}} = \|\rho_0 - \rho_1\|_{\text{tr}} \|\xi_0 - \xi_1\|_{\text{tr}}$.

Proof. We have

$$\begin{aligned}
\|\gamma_0 - \gamma_1\|_{\text{tr}} &= \left\| \frac{1}{2}(\rho_0 \otimes \xi_0) + \frac{1}{2}(\rho_1 \otimes \xi_1) - \frac{1}{2}(\rho_0 \otimes \xi_1) - \frac{1}{2}(\rho_1 \otimes \xi_0) \right\|_{\text{tr}} \\
&= \left\| \frac{1}{2}(\rho_0 - \rho_1) \otimes (\xi_0 - \xi_1) \right\|_{\text{tr}} \\
&= \|\rho_0 - \rho_1\|_{\text{tr}} \cdot \|\xi_0 - \xi_1\|_{\text{tr}}
\end{aligned}$$

as required. \blacksquare

Proposition 16 Let $\rho_0, \rho_1 \in \mathbf{D}(\mathcal{H})$ and $\xi_0, \xi_1 \in \mathbf{D}(\mathcal{K})$. Then

$$\|\rho_0 \otimes \xi_0 - \rho_1 \otimes \xi_1\|_{\text{tr}} \leq \|\rho_0 - \rho_1\|_{\text{tr}} + \|\xi_0 - \xi_1\|_{\text{tr}}.$$

Proof. We have

$$\begin{aligned}
\|\rho_0 \otimes \xi_0 - \rho_1 \otimes \xi_1\|_{\text{tr}} &\leq \|\rho_0 \otimes \xi_0 - \rho_1 \otimes \xi_0\|_{\text{tr}} + \|\rho_1 \otimes \xi_0 - \rho_1 \otimes \xi_1\|_{\text{tr}} \\
&= \|(\rho_0 - \rho_1) \otimes \xi_0\|_{\text{tr}} + \|\rho_1 \otimes (\xi_0 - \xi_1)\|_{\text{tr}} \\
&= \|\rho_0 - \rho_1\|_{\text{tr}} + \|\xi_0 - \xi_1\|_{\text{tr}}.
\end{aligned}$$

as required. \blacksquare

Theorem 17 Let $|\phi\rangle, |\psi\rangle \in \mathcal{H} \otimes \mathcal{K}$ satisfy $\text{tr}_{\mathcal{K}} |\phi\rangle\langle\phi| = \text{tr}_{\mathcal{K}} |\psi\rangle\langle\psi| = \rho$ for some $\rho \in \mathbf{D}(\mathcal{H})$. Then there exists $U \in \mathbf{U}(\mathcal{K})$ such that $(I \otimes U)|\phi\rangle = |\psi\rangle$.

See Section 2.5 of Nielsen and Chuang [30] for a proof.

Theorem 18 Let $\rho, \xi \in \mathbf{D}(\mathcal{H})$, and let \mathcal{K} be such that there exist purifications $|\phi_0\rangle, |\psi_0\rangle \in \mathcal{H} \otimes \mathcal{K}$ of ρ and ξ , respectively (i.e., $\text{tr}_{\mathcal{K}} |\phi_0\rangle\langle\phi_0| = \rho$ and $\text{tr}_{\mathcal{K}} |\psi_0\rangle\langle\psi_0| = \xi$). Then

$$F(\rho, \xi) = \max_{|\phi\rangle, |\psi\rangle} |\langle\phi|\psi\rangle|,$$

where the maximum is over all purifications $|\phi\rangle, |\psi\rangle \in \mathcal{H} \otimes \mathcal{K}$ of ρ and ξ , respectively.

See Section 9.2.2 of Nielsen and Chuang [30] for a proof.

Lemma 19 Let $\rho, \xi \in \mathbf{D}(\mathcal{H})$ and let $\sigma \in \mathbf{D}(\mathcal{H} \otimes \mathcal{K})$ satisfy $\text{tr}_{\mathcal{K}} \sigma = \rho$. Let $|\psi\rangle \in \mathcal{H} \otimes \mathcal{K}$ be a purification of ξ , i.e., $\text{tr}_{\mathcal{K}} |\psi\rangle\langle\psi| = \xi$. Then $\langle\psi|\rho|\psi\rangle \leq F(\rho, \xi)^2$.

Proof. We have

$$\sqrt{\langle\psi|\rho|\psi\rangle} = F(|\psi\rangle\langle\psi|, \rho) = \max_{|\phi_0\rangle, |\psi_0\rangle} |\langle\phi_0|\psi_0\rangle| \leq F(\rho, \xi).$$

Here the maximum is over purifications of ρ and $|\psi\rangle\langle\psi|$. The inequality follows from the fact that any purification of $|\psi\rangle\langle\psi|$ is also a purification of ξ . \blacksquare

Lemma 20 Let $\rho, \xi \in \mathbf{D}(\mathcal{H})$ satisfy $\|\rho - \xi\|_{\text{tr}} = \varepsilon$. Then

$$1 - e^{-k\varepsilon^2/2} < \|\rho^{\otimes k} - \xi^{\otimes k}\|_{\text{tr}} \leq k\varepsilon.$$

Proof. The second inequality follows immediately from Proposition 16. Let us prove the first inequality. We have

$$\begin{aligned} \|\rho^{\otimes k} - \xi^{\otimes k}\|_{\text{tr}} &\geq 1 - F(\rho^{\otimes k}, \xi^{\otimes k}) = 1 - F(\rho, \xi)^k \geq 1 - \left(\sqrt{1 - \|\rho - \xi\|_{\text{tr}}^2}\right)^k \\ &= 1 - (1 - \varepsilon^2)^{\frac{k}{2}} = 1 - (1 - \varepsilon^2)^{\frac{1}{\varepsilon^2} \cdot \frac{k\varepsilon^2}{2}} > 1 - e^{-\frac{k\varepsilon^2}{2}} \end{aligned}$$

as required. \blacksquare

Lemma 21 Let $\rho, \xi \in \mathbf{D}(\mathcal{H})$ satisfy $F(\rho, \xi) \geq 1 - \varepsilon$ and let $|\phi\rangle, |\psi\rangle \in \mathcal{H} \otimes \mathcal{K}$ be purifications of ρ and ξ , respectively, i.e., $\text{tr}_{\mathcal{K}} |\phi\rangle\langle\phi| = \rho$ and $\text{tr}_{\mathcal{K}} |\psi\rangle\langle\psi| = \xi$. Then there exists $U \in \mathbf{U}(\mathcal{K})$ such that

$$\|(I \otimes U)|\phi\rangle - |\psi\rangle\| \leq \sqrt{2\varepsilon}.$$

Proof. By Theorem 18 we have

$$F(\rho, \xi) = \max_{|\phi_0\rangle, |\psi_0\rangle} |\langle\phi_0|\psi_0\rangle|,$$

where the maximum is over all purifications $|\phi_0\rangle, |\psi_0\rangle \in \mathcal{K}$ of ρ and ξ , respectively. Let $|\phi_0\rangle$ and $|\psi_0\rangle$ be pure states achieving this maximum, and assume without loss of generality that $\langle\phi_0|\psi_0\rangle$ is a nonnegative real number.

Since $|\phi\rangle$ and $|\phi_0\rangle$ are both purifications of ρ , we have by Theorem 17 that there exists $V \in \mathbf{U}(\mathcal{K})$ such that $|\phi_0\rangle = (I \otimes V)|\phi\rangle$. Similarly, there exists $W \in \mathbf{U}(\mathcal{K})$ such that $|\psi_0\rangle = (I \otimes W)|\psi\rangle$.

Define $U = V^\dagger W$. Then

$$\|(I \otimes U)|\phi\rangle - |\psi\rangle\| = \|(I \otimes W)|\phi\rangle - (I \otimes V)|\psi\rangle\| = \||\phi_0\rangle - |\psi_0\rangle\| = \sqrt{2 - 2\langle\phi_0|\psi_0\rangle} \leq \sqrt{2\varepsilon}$$

as required. \blacksquare