On the method of Chabauty and Coleman

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Intro: history

   

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Conjecture (Mordell[1], 1922)

A curve over $\mathbb{Q}$ of genus $g \geq 2$ has only finitely many rational points.

[1] Louis Mordell. "On the rational solutions of the indeterminate equations of the third and fourth degrees". In: *Proc. Cambridge Phil. Soc.* 21 (1922), pp. 179–192.

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Theorem (Chabauty[2], 1941)

The Mordell conjecture holds for curves for which the rank of their Jacobian is strictly less than their genus.

[2] Claude Chabauty. "Sur les points rationnels des courbes algébriques de genre supérieur à l'unité". (French). In: *C. R. Acad. Sci. Paris* 212 (1941), pp. 882–885.

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Theorem (Faltings[3], 1983)

The Mordell conjecture is true: Any curve over $\mathbb{Q}$ of genus $g \geq 2$ has only finitely many rational points.

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Theorem (Coleman[4], 1985)

Chabauty's argument can be used to give an effective bound on $\#C(\mathbb{Q})$, for curves C for which the rank of their Jacobian is strictly less than their genus.

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[4] Robert F. Coleman. "Effective Chabauty". In: *Duke Math. J.* 52.3 (1985), pp. 765–770.

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For the rest of this presentation we assume that $r < g$.

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Then we have the *Abel–Jacobi embedding*

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Enter Coleman

Now, let's see what Coleman did with this.

Interlude: Coleman integration

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Lemma

If $r < g$ then there exists an annihilating differential ω.

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This gives an effective bound for $\#C(\mathbb{Q})$!

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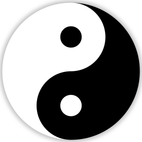

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Stoll[5] described a variation on the method called Selmer–Chabauty, which tries to circumvent this problem using only the 2-Selmer group of J.

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Stoll[5] described a variation on the method called Selmer–Chabauty, which tries to circumvent this problem using only the 2-Selmer group of J. One drawback is that this method may sometimes fail.

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Questions?