Recitation: Induction

Discrete Mathematics TAs

May 7, 2022

1 Induction

The Induction Principle: let $P(n)$ be a statement which involves a natural number n, i.e., $n = 1, 2, 3...$, then $P(n)$ is true for all n if

1. $P(1)$ is true , and
2. $P(k) \to P(k+1)$ for all natural numbers k

2 Questions

1. $\boxed{\text{10 points}}$ Prove that the sum of adding the first n natural numbers is given by $\frac{n(n+1)}{2}$

Solution:

1. Step a) for $n = 1$, $\frac{1(2)}{2} = 1$
2. Step b) (the induction step): assume the result is true for $n = k$, i.e., assume

$$\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$$

The sum for $n = k+1$ may be written

$$\sum_{i=0}^{k+1} i = \sum_{i=0}^{k} i + (k+1)$$

Using the assumption this becomes

$$\sum_{i=0}^{k+1} i = \frac{k(k+1)}{2} + (k+1)$$

$$\frac{(k+1)(k+2)}{2}$$

which is the desired result for $n = k+1$

2. What is wrong with the following proof:
In the following proof we show that all horses are of the same color.
Base Case: The case with just one horse is trivial. If there is only one horse in the "group", then clearly all horses in that group have the same color.

Inductive step: Assume that n horses always are the same color. Consider a group consisting of $n+1$ horses. First, exclude one horse and look only at the other n horses; all these are the same color since n horses always are the same color. Likewise, exclude some other horse (not identical to the one first removed) and look only at the other n horses. By the same reasoning, these too, must also be of the same color. Therefore, the first horse that was excluded is of the same color as the non-excluded horses, who in turn are of the same color as the other excluded horse. Hence the first horse excluded, the non-excluded horses, and last horse excluded are all of the same color, and we have proven that: If n horses have the same color, then $n+1$ horses will also have the same color. And as we have already sown that the base case is true, therefore all horses are of the same color.

QED

Solution: The argument above makes the implicit assumption that the set of $n+1$ horses has the size at least 3, so that the two proper subsets of horses to which the induction assumption is applied would necessarily share a common element. This is not true at the first step of induction, i.e., when $n+1=2$ Let the two horses be horse A and horse B. When horse A is removed, it is true that the remaining horses in the set are the same color (only horse B remains). The same is true when horse B is removed. However the statement "the first horse in the group is of the same color as the horses in the middle" is meaningless, because there are no "horses in the middle" (common elements (horses) in the two sets). Therefore, the above proof has a logical link broken. The proof forms a falsidical paradox; it seems to show by valid reasoning something that is manifestly false, but in fact the reasoning is flawed.
source: `https://en.wikipedia.org/wiki/All_horses_are_the_same_color`

3. $\boxed{\text{10 points}}$ Prove by induction the sum of adding the first n odd numbers is given by $\sum_{j=1}^{n} 2j-1 = n^2$

Solution:

1. Step a) (the check): initial step of the proof, i.e., for $n=1, , \sum_{j=1}^{n}(2j-1) = 1 = 1^2$.
2. Step b) (the inductive step): we assume it is true for $n=k$ assume

$$\sum_{j=1}^{k}(2j-1) = k^2$$

and need to show that it follows that

$$\sum_{j=1}^{k+1}(2j-1) = (k+1)^2$$

Write this sum (over $k+1$ terms) as a sum over the first k terms plus the final term (where $j=k+1$)

$$\begin{aligned}\sum_{j=1}^{k+1}(2j-1) &= \sum_{j=1}^{k}(2j-1) + (2(k+1)-1) \\ &= k^2 + (2(k+1)-1) \\ &= k^2 + 2k + 1\end{aligned}$$

$$(k+1)^2$$

This completes step b) and by the Principle of Induction we have proven the result.

4. Show that $n! < n^n \ \forall n \in \mathbb{N} \setminus \{1\}$

Solution: Base case: $n = 2$

$$2 < 4$$

Induction hypothesis: $n = k$
Assume $k! < k^k$
Induction step: Assuming that the hypothesis is true for $n = k$ we need to show that it is true for $n = k + 1$.

$$(k+1)! = (k+1) \times k!$$

Since we know that $k! < k^k$ then

$$(k+1) \times k! < (k+1)k^k$$

As $k < k+1$ for $k \geq 1$ then $k^k < (k+1)^k$, this can be seen by dividing both sides by k^k, therefore we need to show that $1 < \frac{(k+1)^k}{k^k}$

$$1 < \left(\frac{k+1}{k}\right)^k$$

$\left(\frac{k+1}{k}\right)$ is bigger than 1, since $k < k+1$ and this coupled with the fact that $k > 1$ the statement is true.
Using this fact,

$$(k+1)! = (k+1) \times k! < (k+1)k^k < (k+1)(k+1)^k$$

$$(k+1) \times k! < (k+1)(k+1)^k$$

$$(k+1)! < (k+1)(k+1)^k$$

Therefore the inductive step is true and by the principle of induction, the statement is proved.

5. Given a finite set A, show that $|P(A)| = 2^{|A|}$

Solution: `https://proofwiki.org/wiki/Cardinality_of_Power_Set_of_Finite_Set`

6. Prove that 2 divides $n^2 + n$ whenever n is a positive integer.

Solution: Base case: $n = 1$
$1^2 + 1 = 2$ and 2 divides 2
Induction hypothesis: $n = k$
Assume 2 divides $k^2 + k$
Induction Step: $n = k + 1$

$$(k+1)^2 + (k+1) = k^2 + 2k + 1 + k + 1 = k^2 + k + (2k+2) = k^2 + k + 2(k+1)$$

2 divides $2(k+1)$ and 2 divides k^2+k from inductive hypothesis

7. Given a set of natural numbers from 1 to n, it possible to arrange the numbers in an order where the average of 2 numbers doesn't lie between them.

Solution: A very detailed proof:
We would first define average in our context:
let $AVG : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$

$$AVG(x,y) = \frac{x+y}{2} \quad where \quad x, y \in \mathbb{N}$$

let $A = \{x | x \in \mathbb{N} \wedge 1 \leq x \leq n, n \in \mathbb{N}\}$
We will be using strong induction to to prove our statement.

Base case:

Our AVG function requires 2 inputs so our smallest possible case is of 2 numbers.
$n = 2$
$\Rightarrow A = \{1, 2\}$
This case is trivial as $AVG(1,2) = 1.5$, and $1.5 \notin A$
As for strong induction we will be exploring further cases too.
$n = 3$
$\Rightarrow A = \{1, 2, 3\}$
We can have (2,1,3) as one such arrangement, as $AVG(1,2) = 1.5$ and $1.5 \notin A$, $AVG(1,3) = 2$ and 2 is not between 1 and 3, $AVG(2,3) = 2.5$ and $2.5 \notin A$.
$n = 4$
$\Rightarrow A = \{1, 2, 3, 4\}$
We can have (2,4,1,3) as one such arrangement, as $AVG(2,4) = 3$ and 3 is not between 2 and 4, $AVG(1,2) = 1.5$ and $1.5 \notin A$, $AVG(2,3) = 2.5$ and $2.5 \notin A$, $AVG(1,4) = 2.5$ and $2.5 \notin A$, $AVG(3,4) = 3.5$ and $3.5 \notin A$, $AVG(1,3) = 2$ and 2 is not between 1 and 3.

Induction Hypothesis:

Our statement holds true for all $n < k$

Induction Step:

Now we shall explore 2 possible cases.
Case 1 : k is even
$\Rightarrow k = 2m$ where $m \in \mathbb{N} \Rightarrow m < k$
Thus as from our induction hypothesis, there exists and ordering that satisfies our condition for $n = m$.
Let $O_m = (x_1, x_2, x_3, ...x_m)$ be one such ordering, where $\{x_1, x_2, x_3, ...x_m\} = \{1, 2, 3...m\}$.
The idea is to split the even and order numbers.
As for $a, b \in \mathbb{N}$ if a is odd and b is even then $a + b$ is odd.
All even natural numbers can be written in the form $2n$ where $n \in \mathbb{N}$, and all odd natural numbers can be written as $2n - 1$ where $n \in \mathbb{N}$.

As a is even $a = 2c$ where $c \in \mathbb{N}$ and as b is odd $b = 2d - 1$ where $d \in \mathbb{N}$.
Then $a+b = 2c+2d-1 = 2(c+d)-1$, as natural numbers are closed under addition $c+d \in \mathbb{N} \Rightarrow a+b$ is in the form $2n - 1 \Rightarrow a + b$ is odd.
As an odd number is not a multiple of 2, then for $b \in \mathbb{N}$ if b is odd $\frac{b}{2} \notin \mathbb{N}$.

$$As\ \ b = 2d - 1 \Rightarrow \frac{b}{2} = \frac{2d-1}{2} = d - \frac{1}{2} \wedge \frac{1}{2} \notin \mathbb{N} \Rightarrow d - \frac{1}{2} \notin \mathbb{N}$$

Let $A_m = \{x | x \in \mathbb{N} \wedge 1 \leq x \leq m\}$ and $A_k = \{x | x \in \mathbb{N} \wedge 1 \leq x \leq k\}$

$$k = 2m \Rightarrow |A_k| = 2|A_m|$$

$$A_m = \{1, 2, ..., m\}$$

$$A_k = \{1, 2, 3, 4, ..., k-3, k-2, k-1, k\}$$

$$A_k = \{2(1)-1, 2(1), 2(2)-1, 2(2), ..., 2(m-1)-1, 2(m-1), 2m-1, 2m\}$$

We can split A_k into two sets A_o and A_e:

$$A_o = \{2(1)-1, 2(2)-1, ..., 2(m-1)-1, 2m-1\}$$

$$A_e = \{2(1), 2(2), ..., 2(m-1), 2m\}$$

As $AVG(2x, 2y) = \frac{2x+2y}{2} = 2\left(\frac{x+y}{2}\right) = 2AVG(x, y)$
As $AVG(2x-1, 2y-1) = \frac{2x-1+2y-1}{2} = 2\left(\frac{x+y}{2}\right) - 1 = 2AVG(x, y) - 1$ And we know an ordering that satisfy our condition exist for $n = m$ from out induction hypothesis.
We can construct an ordering from A_e:

$$A_e = \{y_i | i \in \mathbb{N} \wedge 1 \leq i \leq m\} = \{2(1), 2(2), ..., 2(m-1), 2m\}$$

We can order A_e the same way as we ordered A_m,
where $\forall y_i \in A_e \exists x_i \in A_m : y_i = 2x_i$ where $1 \leq i \leq m$ and x_i is the element at ith index in the ordering of $n = m$.
Let O_e be the ordering of A_e

$$O_e = (y_1, y_2, y_3, ...y_m)$$

We know this ordering would satisfy our condition as $AVG(2a, 2b) = 2AVG(a, b)$ as $AVG(x_i, x_j)$ doesn't lie between x_i and x_j $\forall x_i, x_j \in O_m$, $AVG(y_i, y_j)$ won't lie between y_i and y_j as $y_i = 2x_i$ and $y_j = 2x_j$.

We can construct a similar ordering from A_o:

$$A_o = \{z_i | i \in \mathbb{N} \wedge 1 \leq i \leq m\} = \{2(1)-1, 2(2)-1, ..., 2(m-1)-1, 2m-1\}$$

We can order A_o the same way as we ordered A_m,
where $\forall z_i \in A_o \exists x_i \in A_m : z_i = 2x_i - 1$ where $1 \leq i \leq m$ and x_i is the element at ith index in the ordering of $n = m$.

Let O_o be the ordering of A_o

$$O_o = (z_1, z_2, z_3, ...z_m)$$

We know this ordering would satisfy our condition as
$AVG(2a-1, 2b-1) = 2AVG(a,b) - 1$
as $AVG(x_i, x_j)$ doesn't lie between x_i and x_j $\forall x_i, x_j \in O_m$,
$AVG(z_i, z_j)$ won't lie between z_i and z_j as $z_i = 2x_i - 1$ and $z_j = 2x_j - 1$.

The elements in A_e are even, the elements of A_o are odd.
As if $a, b \in \mathbb{N}$ and a is even and b is odd, $\Rightarrow a + b$ is odd $\Rightarrow \frac{a+b}{2} \notin \mathbb{N}$

$$A_k = A_e \cup A_o \wedge \forall x \in A_k, x \in \mathbb{N}$$

$$\forall y \in A_e \forall z \in A_o, \frac{y+z}{2} \notin \mathbb{N} \Rightarrow AVG(y,z) \notin A_k$$

The ordering O_k of A_k can be constructed as:

$$O_k = (y_1, y_2, y_3, ..., y_m, z_1, z_2, z_3, ..., z_m)$$

$$As\ \ A_k = \{y_1, y_2, y_3, ..., y_m, z_1, z_2, z_3, ..., z_m\}$$

Thus we can have an ordering that satisfies our condition $n = k$ if k is even.
Case 2 : k is odd
Then $k = 2m - 1$
Then we know $k+1$ is even as $k + 1 = 2m - 1 + 1 = 2m$
We know there exists an ordering for $n = 2m$ where $m < k$.
As for our case $m < k$
Let $A_{k+1} = \{x | x \in \mathbb{N} \wedge 1 \leq x \leq k+1\}$
Then $\exists O_{k+1}$ where O_{k+1} is an ordering on A_{k+1} that satisfies our condition.

$$O_{k+1} = (x_1, x_2, ..., x_k, x_k + 1)$$

If ordering O on some A satisfies our condition, then the ordering O_r obtained from removing an element r from O satisfies the condition for the set $A/\{r\}$.
As we have an ordering on A_{k+1} that satisfies our condition, we can remove $k+1$ from O_{k+1}, the ordering O_k obtained from this satisfies our condition for A_k.
As $O_k = (x_1, x_2, ..., x_k) \Rightarrow \{x_1, x_2, ..., x_k\} = A_k = \{1, 2, ...k\}$. Thus we can have an ordering that satisfies our condition $n = k$ if k is odd.
As we have exhausted all the cases for $n = k$

$$\forall n \in \mathbb{N}, n < k\ \ being\ \ true\ \ \Rightarrow\ \ n = k\ \ is\ \ true$$

Where being true means that we can construct an ordering that satisfies our condition.
With the principal of strong induction we can conclude that our statement holds true for all $n \in \mathbb{N}$.
Given a set of natural numbers from 1 to n, it possible to arrange the numbers in an order where the average of 2 numbers doesn't lie between them.

QED

8. There is a midnight party in the jungle and you are out to track the elusive unicorn. Because all the guests are in costume, it is difficult to spot the unicorn visually. Instead you are relying on the fact that the unicorn smells like a rainbow. Your own sense of smell is overpowered by all the scents from the guests, the jungle, and the smoke from the party machine. So you are going to question the animals at the party who possess a keener sense of smell. Animals do not lie.

   Due to differences in olfactory systems among the guests, the set of animals that smell like the rainbow varies for each guest. The unicorn is special in that it smells like the rainbow to every animal and does not perceive a rainbow smell from any other animal.

   Your Animalese is weak–the only question you can ask a guest is whether another guest smells like the rainbow to them.

   Use mathematical induction to show that if there are n animals at the party, then you can find the unicorn, if it is attending, with a total of at most $3(n-1)$ questions.

Solution: The premise can be proved in multiple ways, we will be going over 2 proofs for it.

Proof 1:
As the question requires a proof by induction we would be first using induction to prove the premise. We will be using strong induction to prove the premise.

Base case:
The premise is true for $n = 2$, as if there are 2 guests we can ask a maximum of 3 questions to find the unicorn.
Lets say the two guests are a and b, we can ask a to smell b, is b smells like rainbow to a then that means a is not the unicorn. Then we ask b to smell a is a doesn't smell like rainbow to b then b is the unicorn, if a does smell like rainbow to b that means neither of them are the unicorn.
Thus we can find the unicorn or that its not there with 2 questions $2 < 3(2-1)$

Inductive hypothesis:
We assume that we can find the unicorn if it it exists in maximum of $3(n-1)$ questions for any n number of guests.

Inductive Step:
Our inductive hypothesis should imply our that the premise holds for $n+1$ guests too. That is to say that for $n+1$ guest the unicorn can be found in at maximum of $3(n)$ questions

If $n + 1$ guests are attending, lets say we take a random guest a, we ask a to smell a random guest b, either b smells like rainbow to a, then a is not the unicorn, or b does not smell like rainbow to a then b is not the unicorn.
As we find either a or b is not the unicorn with 1 question, let x be the guest that we found that is not the unicorn.
As we know x is not the unicorn, we can remove x from the set of guests leaving n guest.
From our inductive hypothesis we assumed that we can find the unicorn if its attending from n guest with $3(n-1)$ questions. As without x we have n guests left, from those n guests we can find the unicorn, if it exists with $3(n-1)$ questions, we have used $3(n-1)+1$ questions till now. If no unicorn was found from n guests then we end at $3(n-1)+1$ questions.
If a unicorn was found from n guest we need to check if it is indeed the unicorn.

Let y be the guest we found from n remaining guest that can be the unicorn, we ask y to smell x and x to smell y, if y smells like rainbow to x and x doesn't smell like rainbow to y then y is the unicorn, if not then the unicorn isn't attending.
This adds 2 more questions, making $3(n-1)+3$ questions

$$3(n-1)+3 = 3n$$

Thus the unicorn can be found from any n guests and if it exists within $3(n-1)$ questions.

QED

Proof 2:
One algorithmic way of finding the unicorn if its attending in $3(n-1)$ questions would be:

Start of from a random guest a, a is a possible candidate for the unicorn. We ask a to smell some random guest b, if b smells like rainbow to a, then that means that a cannot be the unicorn, and if b is a possible candidate to be the unicorn. Now we ask b to smell some guest c if c does smell like rainbow to b then c cannot be the unicorn, now we ask b to smell some guest d, if d smells like rainbow to b that means b is not the unicorn while d is a possible candidate for the unicorn. No we can remove a, b, c from the set and continue the smelling process from d, lets say for each guess that d smelled, no one smelled like rainbow to d. That means in the unicorn is attending then it must be d.
This would take a maximum of $n-1$ questions in case the unicorn is found, cause we don't ask any questions to the person who doesn't smell like rainbow. And in case everyone smells like rainbow we don't ask any questions from the last person we smell. In case everyone no one smells like rainbow, then a is the possible candidate to be the unicorn, again in which case we ask $n-1$ questions from a
Let the person we found to be the unicorn be x Now we check if x is the unicorn or not. We ask x to smell all the other guests that adds $n-1$ questions, if no one smells like rainbow to x, then we ask all the other guests to smell x, that adds $n-1$ more questions.If x smells like rainbow to everyone that makes the total of maximum questions $3(n-1)$.

Thus the unicorn can be found from any n guests and if it exists within $3(n-1)$ questions.

QED

9. Given a $2^n \times 2^n$ board where $\forall n \in \mathbb{Z}, n \geq 0$, if we remove a single square from the board, the remaining board can be tiled entirely by L-shaped trominoes shown in Figure 1. Prove this using induction.

(a) T_1 (b) T_2 (c) T_3 (d) T_4

Figure 1: All Possible Trominoes

Solution:

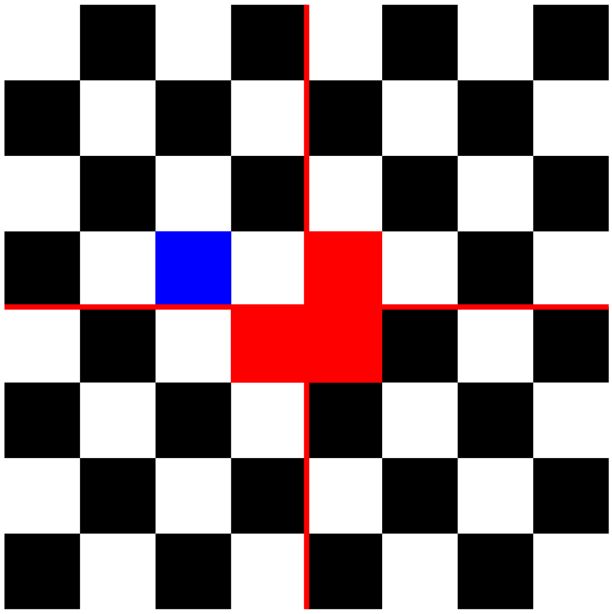

Figure 2: Blue: Tile removed. Red: Tile placed

Proof:

The proof is by induction.

Base Case: When $n = 0$, the statement is true, since there is nothing left to cover.

Assuming $n = k$ is true, if that implies $n = k + 1$ is true, then since we know the base case is true, then by induction the statement is true, $\forall n \in \mathbb{Z}, n \geq 0$

Induction Hypothesis: Assume the statement holds true for $n = k$ case.

Induction Step: Given a board of size $2^{k+1} \times 2^{k+1}$, where $k \geq 0$, the board can be broken into 4 quadrants of size $2^k \times 2^k$ each.

Let i be the quadrant that contains the removed square (1 being Top Left, 2 being Top Right, 3 being Bottom Left and 4 being Bottom Right). By placing T_i in the center of the $2^k \times 2^k$ grid (Figure 2 shows placing of T_1), each quadrant now has one square "removed". This makes each quadrant into a subproblem of size $2^k \times 2^k$, these subproblems can be solved for each quadrant, therefore boards of size $2^{k+1} \times 2^{k+1}$ can be solved and the induction step is true. Hence the statement is true, $\forall n \geq 0$.