Recitation: Relations

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Definitions

Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$

Let R be a relation on the set A. The powers R^n, $n = 1, 2, 3, \ldots$, are defined recursively as follows:

$$R^1 = R \text{ and } R^{n+1} = R^n \circ R$$

Questions

1. Find the error in the "proof" of the following "theorem".
   Theorem: Let R be a relation on a set A that is symmetric and transitive. Then R is reflexive.
   Proof: Let $a \in A$. Take an element $b \in A$ such that $(a, b) \in R$. Because R is symmetric, we also have $(b, a) \in R$. Now using the transitive property, we can conclude that $(a, a) \in R$ because $(a, b) \in R$ and $(b, a) \in R$.

   Solution: The second sentence of the proof asks us to "take an element $b \in A$ such that $(a, b) \in R$." There is no guarantee that such an element exists for the taking. This is the only mistake in the proof. If one could be guaranteed that each element in A is related to at least one element, then symmetry and transitivity would indeed imply reflexivity. Without this assumption, however, the proof and the proposition are wrong. As a simple example, take the relation $\varnothing$ on any nonempty set. This relation is vacuously symmetric and transitive, but not reflexive. Here is another counterexample: the relation $\{(1, 1), (1, 2), (2, 1), (2, 2)\}$ on the set $\{1, 2, 3\}$.

2. Let
   - $R_1 = \{(a, b) \in \mathbb{R}^2 \mid a > b\}$, the "greater than" relation
   - $R_2 = \{(a, b) \in \mathbb{R}^2 \mid a \geq b\}$, the "greater than or equal to" relation
   - $R_3 = \{(a, b) \in \mathbb{R}^2 \mid a < b\}$, the "less than" relation

   Find

   (a) $R_1 \circ R_1$

   Solution: Answer: R_1
   Let (a, b) and (b, c) in R_1. For (a, c) to be in $R_1 \circ R_1$, we must find an element b such that $(a, b) \in R_1$ and $(b, c) \in R_1$. This means that $a > b$ and $b > c$. Clearly this can be done if and only if $a > c$ to begin with. But that is precisely the statement that $(a, c) \in R_1$. Therefore we have $R_1 \circ R_1 = R_1$.

(b) $R_1 \circ R_2$

Solution: **Answer:** R_1
For (a, c) to be in $R_1 \circ R_2$, we must find an element b such that $(a, b) \in R_2$ and $(b, c) \in R_1$. This means that $a \geq b$ and $b > c$. Clearly this can be done if and only if $a > c$ to begin with. But that is precisely the statement that $(a, c) \in R_1$. Therefore we have $R_1 \circ R_2 = R_1$.

(c) $R_1 \circ R_3$

Solution: **Answer:** $\mathbb{R}^2$ For (a, c) to be in $R_1 \circ R_3$, we must find an element b such that $(a, b) \in R_3$ and $(b, c) \in R_1$. This means that $a < b$ and $b > c$. Clearly this can always be done simply by choosing b to be large enough. Therefore we have $R_1 \circ R_3 = \mathbb{R}^2$, the relation that always holds.

3. The relation R on a set A is transitive if and only if $R \circ R \subseteq R$

Solution: In order to prove this we need to prove two things.
If $R \circ R \subseteq R$ then R on set A is transitive.
Proof: Assume that $R \circ R \subseteq R$, and suppose that $(x, y) \in R$ and $(y, z) \in R$. By definition of $R \circ R$, it follows that $(x, z) \in R \circ R$, but since $R \circ R \subseteq R$ we have $(x, z) \subseteq R$. We have proved that if $(x, y) \in R$ and $(y, z) \subseteq R$ then $(x, z) \subseteq R$: hence, R is transitive.

If R on set A is transitive then $R \circ R \subseteq R$.
Proof: If R is transitive then for any arbitrary $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$. The set $R \circ R$ contains elements (x, z) where $(x, y) \in R$ and $(y, z) \in R$. Due to transitivity, we know that any arbitrary (x, z) where $(x, y) \in R$ and $(y, z) \in R$ will be in R. Then since all elements $(x, z) \in R \circ R$ belong to R, therefore $R \circ R \subseteq R$

4. Let R be the relation on the set $\{1, 2, 3, 4, 5\}$ containing the ordered pairs $(1, 1), (1, 2), (1, 3), (2, 3), (2, 4), (3, 1), (3, 4), (3, 5), (4, 2), (4, 5), (5, 1), (5, 2)$, and $(5, 4)$. Find R^2, R^3, R^4, R^5

Solution: R^2 :
R^2 will be $M_R \times M_R$

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$M_R \odot M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The non zero entries in the matrix $M_R \odot M_R$ tells the elements related in R^2. So, R^2 contains all pairs belonging to $\{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4, 5\}$ except $(2, 3)$ and $(4, 5)$.

For R^3:
$R^3 = R^2 \circ R$

$$M_R{}^2 \odot M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

R^3 contains all pairs belonging to $\{1,2,3,4,5\} \times \{1,2,3,4,5\}$
For R^4:
$R^4 = R^3 \circ R$
$M_{R^3} \odot M_R$ will also be filled with 1's. Hence, will contain all the ordered pairs belonging to $\{1,2,3,4,5\} \times \{1,2,3,4,5\}$
For R^5:
$R^5 = R^4 \circ R$
$M_{R^4} \times M_R$ will also be filled with 1's. Hence, will contain all the ordered pairs belonging to $\{1,2,3,4,5\} \times \{1,2,3,4,5\}$

5. Show that for a reflexive relation R, $R^{-1} \subseteq R \circ R^{-1}$

Solution: In order to show this statement, we need to show that every element in R^{-1} is a subset of $R \circ R^{-1}$. We can take any arbitrary pair $(b,a) \in R^{-1}$, and take the element $(a,a) \in R$ (We can do this since relation is reflexive), therefore the element (b,a) would lie in the composite of R^{-1} and R: $R \circ R^{-1}$. Therefore our statement is true.

6. Represent each of these relations on $\{1,2,3\}$ with a matrix

- $\{(1,1),(1,2),(1,3)\}$

Solution:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- $\{(1,2),(2,1),(2,2),(3,3)\}$

Solution:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

7. List the ordered pairs in the relations on $\{1,2,3\}$ corresponding to these matrices

1.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Solution: $\{(1,1),(1,3),(2,2),(3,1),(3,3)\}$

2.

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Solution: $\{(1,2),(2,2),(3,2)\}$

8. How can the matrix representing a relation R on a set A be used to determine whether the relation is irreflexive?

Solution: An irreflexive relation is one in which no element is related to itself. In the matrix, this means that there are no 1's on the main diagonal (position m_{ii} for some i). Equivalently, the relation is irreflexive if and only if every entry on the main diagonal of the matrix is 0.

9. How many nonzero entries does the matrix representing the relation R on $A = \{1, 2, 3, ..., 100\}$ consisting of the first 100 positive integers have if R is

   1. $\{(a,b) \mid a > b\}$
   2. $\{(a,b) \mid a = 1\}$
   3. $\{(a,b) \mid a \neq b\}$

Solution: Note that the total number of entries in the matrix is $100^2 = 10,000$.

1. There is a 1 in the matrix for each pair of distinct positive integers not exceeding 100 , namely in position (a,b) where $a > b$. Thus the answer is the number of subsets of size 2 from a set of 100 elements, i.e., $C(100,2) = 4950$.
2. The entire first row of this matrix corresponds to $a = 1$. Therefore the matrix has 100 nonzero entries.
3. There is a 1 in the matrix at each position except the 100 positions on the main diagonal. Therefore the answer is $100^2 - 100 = 9900$.

10. How can the directed graph of a relation R on a finite set A be used to determine whether a relation is asymmetric?

Solution: A relation R is asymmetric if there are never two edges in opposite direction between distinct nodes.

11. Let R be a relation on a set A. Explain how to use the directed graph representing R to obtain the directed graph representing the inverse relation R^{-1}. Note that $R^{-1} = \{(b,a) \mid (a,b) \in R\}$

Solution: Since the inverse relation consists of all pairs (b, a) for which (a, b) is in the original relation, we just have to take the digraph for R and reverse the direction on every edge.

12. Which of these relations on the set of all people are equivalence relations? Determine the properties of an equivalence relation that the others lack.

    1. $\{(a, b) \mid a$ and b are the same age$\}$
    2. $\{(a, b) \mid a$ and b have the same parents$\}$
    3. $\{(a, b) \mid a$ and b share a common parent$\}$
    4. $\{(a, b) \mid a$ and b have met$\}$
    5. $\{(a, b) \mid a$ and b speak a common language$\}$

Solution:

1. Is an equivalence relation since it is reflexive (A person has their own age), symmetric (if a has the same age as b, b has the same age as a) and transitive (if a, b have the same age x and b, c have the same age y then $x = y$ since b has one age and hence a and c have the same age).

2. Is an equivalence relation since it is reflexive (A person has the same parents as himself/herself/ pronoun of your choice), symmetric (if a has the same parents as b, b must have the same parents as a) and transitive (if a, b have the same parents and b, c have the same parents then since b has the same parents with a and c therefore a and c have the same parents).

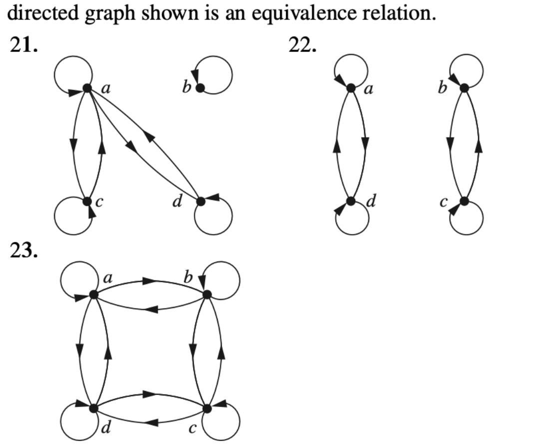
3. This is not an equivalence relation, since it need not be transitive.(We assume that biological parentage is at issue here, so it is possible for A to be the child of W and X, B to be the child of X and Y, and C to be the child of Y and Z. Then A is related to B, and B is related to C, but A is not related to C.)

4. Not transitive. a might have met b and not c but b could have met c.

5. This is not transitive. For example, let a speak just Urdu, b speak Urdu and English and c speak just English. (a, b) and (b, c) exist in the relation however (a, c) doesn't as they share no common language.

13. Justify whether the given relations are equivalence relations or not.

In Exercises 21–23 determine whether the relation with the directed graph shown is an equivalence relation.



Solution: In order for a relation to be an equivalence relation, it must be reflexive, symmetric and transitive.

1. 21 is reflexive and symmetric but it isn't transitive since (c, d) $((c, a)$ and $(a, d))$ and (d, c) $((d, a)$ and $(d, c))$ aren't there. Therefore it isn't an equivalence relation.
2. Since 22 is reflexive, symmetric and also transitive, it is an equivalence relation.
3. 23 is both reflexive and symmetric but not transitive since some edges like (a, c) are missing $((a, b)$ and $(b, c))$. Therefore this isn't an equivalence relation.

14. Justify that for an equivalence relation R for an equivalence class $[\mathbf{c}]$ $\forall a, b \in [\mathbf{c}], (a, b)$, or every element in the equivalence class has relation with every element in the equivalence class

Solution: The equivalence class $[\mathbf{c}]$ has all pairs (c, a) where $a \in [\mathbf{c}]$, therefore for any arbitrary $a, b \in [\mathbf{c}]$, the pairs (c, a) and (c, b) exist in R respectively. Since R is an equivalence relation, due to symmetry (a, c) also exists and using transitivity, since (a, c) and (c, b) exist, so does the the link (a, b).

15. Let R be a relation that is reflexive and transitive. Prove that $R^2 = R$.

Solution: By Q3, we know that $R^2 \subseteq R$, therefore we just need to show that $R \subseteq R^2$. We need to show that all elements of form $(a, b) \in R$ are also in R^2. We can take any arbitrary element $(a, b) \in R$, we know that $(a, a) \in R$ as well since it is transitive. Since (a, a) and $(a, b) \in R$, we know that $(a, b) \in R \circ R = R^2$. Therefore we have shown that all elements in R are in R^2 hence $R \subseteq R^2$. Since $R \subseteq R^2$ and $R^2 \subseteq R$, $R^2 = R$

16. Show that $R \circ R = R$ for equivalence relations

Solution: Since Equivalence relations are reflexive and transitive, using the above statement, we can conclude that the statement is true.

17. Let R be the relation $\{(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)\}$, and let S be the relation $\{(2, 1), (3, 1), (3, 2), (4, 2)\}$. Find $S \circ R$

Solution: Since $(1, 2) \in R$ and $(2, 1) \in S$, we have $(1, 1) \in S \circ R$. We use similar reasoning to form the rest of the pairs in the composition, giving us the answer $\{(1, 1), (1, 2), (2, 1), (2, 2)\}$.