Practice Problems: Relations
CS 113 Discrete Mathematics

Overworked TAs <3

Habib University – Spring 2022

Definitions

Let A and B be sets. A **binary relation** from A to B is a subset of $A \times B$
Relation on a set is a relation from A to A
A relation R on a set A is called **reflexive** if $(a, a) \in R$ for every element $a \in A$
Reflexive in predicate logic: $\forall a((a, a) \in R)$
A relation R on a set A is called **symmetric** if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$.
Symmetric in predicate logic: $\forall a, \forall b, ((a, b) \in R \implies (b, a) \in R)$
A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$ then $a = b$ is called anti-symmetric.
Anti-symmetric in predicate logic: $\forall a, \forall b, (((a, b) \in R \wedge (b, a) \in R) \implies (a = b)$
A relation R on a set A is called **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for all $a, b, c \in A$.
Transitive in predicate logic: $\forall a, \forall b, \forall c(((a, b) \in R \wedge (b, c) \in R) \implies (a, c) \in R$

Advice from the book

This section gives the basic terminology, especially the important notions of reflexivity, symmetry, anti-symmetry, and transitivity. If we are given a relation as a set of ordered pairs, then reflexivity is easy to check for: we make sure that each element is related to itself. Symmetry is also fairly easy to test for: we make sure that no pair (a, b) is in the relation without its opposite (b, a) being present as well. To check for anti-symmetry we make sure that no pair (a, b) with $a \neq b$ and its opposite are both in the relation. In other words, at most one of (a, b) and (b, a) is in the relation if $a \neq b$. Transitivity is much harder to verify, since there are many triples of elements to check. A common mistake to try to avoid is forgetting that a transitive relation that has pairs (a, b) and (b, a) must also include (a, a) and (b, b).

Questions

1. Those that don't break their morals, are true to themselves. Only those with morals can break them. Those that are true to themselves can be trusted. Therefore the person who has no

morals can be trusted. Write and justify this argument in predicate logic. (Hint: use vacuous proof)

Solution:

$$\begin{aligned} M(x,y) &: \text{Person } x \text{ has moral } y \\ B(x,y) &: \text{Person } x \text{ breaks moral } y \\ T(x) &: \text{Person } x \text{ is true to themselves} \\ Tr(x) &: \text{Person } x \text{ can be trusted} \end{aligned}$$

We know that

$$\forall x(\forall y(B(x,y) \implies M(x,y))) \tag{1}$$
$$\forall x(\forall y(\neg B(x,y)) \implies T(x)) \tag{2}$$
$$\forall x(T(x) \implies Tr(x)) \tag{3}$$

and for our person c who has no morals

$$\forall y(\neg M(c,y))$$

We can instantiate c therefore we know that

$$\forall y B(c,y) \implies M(c,y) \tag{4}$$
$$\forall y(\neg B(c,y)) \implies T(c) \tag{5}$$
$$T(c) \implies Tr(c) \tag{6}$$

and by initializing a moral d we know

$$B(c,d) \implies M(c,d) \tag{7}$$

Since we know that c has no morals, then for an arbitrary moral d we know that he can't break them by Modus Tollens. Therefore by universal generalisation, we know that for c,$\forall y(\neg B(x,y))$ therefore by modus ponens we know that they are true to themselves ($T(c)$) and since they are true to themselves, by Modus Ponens, we can conclude they can be trusted.

2. Prove that $x^2 + y^2 \geq 2xy$ for all real numbers x and y using backward reasoning.

Solution: Proof: ASSUME that x and y are real numbers. SHOW that $x^2 + y^2 \geq 2xy$.
But
$x^2 + y^2 \geq 2xy$ . . . is TRUE
iff

$x^2 + y^2 - 2xy \geq 0$ . . . is TRUE
iff
$x^2 - 2xy + y^2 \geq 0$ . . . is TRUE
iff
$(x - y)^2 \geq 0$ . . . is TRUE .

Since the last statement is TRUE, all of the equivalent statements are TRUE. In particular, $x^2 + y^2 \geq 2xy$.

3. (Recognize whether several important relations are reflexive or not over their respective domain.) Which of the following statements are True and which are False?

   1. $\forall x \in \mathbb{R}(x = x)$
   2. $\forall x \in \mathbb{R}(x \neq x)$.
   3. $\forall x \in \mathbb{R}(x < x)$.
   4. $\forall x \in \mathbb{R}(x \geq x)$.
   5. $\forall a \in \mathbb{N}(a \mid a)$
   6. $\forall X \in 2^U(X \subseteq X)$ where U is the universal set.

Solution:

1. This is True. Equality is a reflexive relation since an element is equal to itself.
2. This is False. Not equal is not a reflexive relation.
3. This is False. Less than is not a reflexive relation since an element can't be less than itself.
4. This is True. Since an element $a = a$, Greater than or equals is a reflexive relation
5. This is True. Divides is a reflexive relation since an element is a factor of itself.
6. This is True. Since a set is a subset of itself, this is a reflexive relation.

4. For each of these relations on the set $\{1, 2, 3, 4\}$, decide whether it is reflexive, whether it is symmetric, whether it is anti-symmetric and whether it is transitive.

   1. $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$
   2. $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$
   3. $\{(2, 4), (4, 2)\}$
   4. $\{(1, 2), (2, 3), (3, 4)\}$
   5. $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$

6. $\{(1,3),(1,4),(2,3),(2,4),(3,1),(3,4)\}$

Solution:

	Reflexive	Symmetric	Anti-symmetric	Transitive
$\{(2,2),(2,3),(2,4),(3,2),(3,3),(3,4)\}$	No	No	No	Yes
$\{(1,1),(1,2),(2,1),(2,2),(3,3),(4,4)\}$	Yes	Yes	No	Yes
$\{(2,4),(4,2)\}$	No	Yes	No	No
$\{(1,2),(2,3),(3,4)\}$	No	No	Yes	No
$\{(1,1),(2,2),(3,3),(4,4)\}$	Yes	Yes	Yes	Yes
$\{(1,3),(1,4),(2,3),(2,4),(3,1),(3,4)\}$	No	No	No	No

a) This relation is not reflexive, since it does not include, for instance $(1,1)$. It is not symmetric, since it includes, for instance, $(2,4)$ but not $(4,2)$. It is not anti-symmetric since it includes both $(2,3)$ and $(3,2)$, but $2 \neq 3$. It is transitive. To see this we have to check that whenever it includes (a,b) and (b,c), then it also includes (a,c). We can ignore the element 1 since it never appears. If (a,b) is in this relation, then by inspection we see that a must be either 2 or 3 . But $(2,c)$ and $(3,c)$ are in the relation for all $c \neq 1$; thus (a,c) has to be in this relation whenever (a,b) and (b,c) are. This proves that the relation is transitive. Note that it is very tedious to prove transitivity for an arbitrary list of ordered pairs.
b) This relation is reflexive, since all the pairs $(1,1),(2,2),(3,3)$, and $(4,4)$ are in it. It is clearly symmetric, the only nontrivial case to note being that both $(1,2)$ and $(2,1)$ are in the relation. It is not anti-symmetric because both $(1,2)$ and $(2,1)$ are in the relation. It is transitive; the only nontrivial cases to note are that since both $(1,2)$ and $(2,1)$ are in the relation, we need to have (and do have) both $(1,1)$ and $(2,2)$ included as well.
c) This relation clearly is not reflexive and clearly is symmetric. It is not anti-symmetric since both $(2,4)$ and $(4,2)$ are in the relation. It is not transitive, since although $(2,4)$ and $(4,2)$ are in the relation, (2,2) is not.
d) This relation is clearly not reflexive. It is not symmetric, since, for instance, (1,2) is included but $(2,1)$ is not. It is anti-symmetric, since there are no cases of (a,b) and (b,a) both being in the relation. It is not transitive, since although $(1,2)$ and $(2,3)$ are in the relation, $(1,3)$ is not.
e) This relation is clearly reflexive and symmetric. It is trivially anti-symmetric since there are no pairs (a,b) in the relation with $a \neq b$. It is trivially transitive, since the only time the hypothesis $(a,b) \in R \wedge (b,c) \in R$ is met is when $a = b = c$.
f) This relation is clearly not reflexive. The presence of $(1,4)$ and absence of $(4,1)$ shows that it is not symmetric. The presence of both $(1,3)$ and $(3,1)$ shows that it is not anti-symmetric. It is not transitive; both $(2,3)$ and $(3,1)$ are in the relation, but $(2,1)$ is not, for instance.

5. (Be able to describe the relations determined by Boolean operations.) Describe these relations over the set of Boolean variables.

1. $p \wedge q = \text{True}$

2. $p \wedge q =$ False
3. $p \implies q =$ True
4. $p \implies q =$ False
5. $p \equiv q =$ True

Solution:

1. $\{(1,1)\}$
2. $\{(0,0),(0,1),(1,0)\}$
3. $\{(0,0),(0,1),(1,1)\}$
4. $\{(1,0)\}$
5. $\{(0,0),(1,1)\}$

6. Is the "divides" relation on the set of positive integers symmetric? Is it antisymmetric?

Solution: This relation is not symmetric because $1 \mid 2$, but 2 does not divide 1. It is anti-symmetric, for if a and b are positive integers with $a \mid b$ and $b \mid a$, then $a = b$.

7. How many reflexive relations are there on a set with n elements?

Solution: A relation R on a set A is a subset of $A \times A$. Consequently, a relation is determined by specifying whether each of the n^2 ordered pairs in $A \times A$ is in R. However, if R is reflexive, each of the n ordered pairs (a, a) for $a \in A$ must be in R. Each of the other $n(n-1)$ ordered pairs of the form (a, b), where $a \neq b$, may or may not be in R. Hence, by the product rule for counting, there are $2^{n(n-1)}$ reflexive relations [this is the number of ways to choose whether each element (a, b), with $a \neq b$, belongs to R

8. Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) | a \text{ divides } b\}$?

Solution: Because (a, b) is in R if and only if a and b are positive integers not exceeding 4 such that a divides b, we see that $R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}$.

9. Show that if a, b, and c are real numbers and $a \neq 0$, then there is a unique solution of the equation $ax + b = c$.

Solution: Assume there exist 2 solutions x_1 and x_2 of the equations such that

$$x_1 \neq x_2$$

$$ax_1 + b = c \text{ and } ax_2 + b = c$$

Now we reach a contradiction and show that $x_1 = x_2$

$$ax_1 + b = ax_2 + b = c$$

$$ax_1 + b = ax_2 + b$$

$$ax_1 + b - b = ax_2 + b - b$$

$$ax_1 = ax_2$$

$$\frac{1}{a} \times ax_1 = \frac{1}{a} \times ax_2$$

$$x_1 = x_2$$

10. Prove that between every two rational numbers there is an irrational number.
Hint: use the fact that $\frac{1}{\sqrt{2}}$ is irrational and $0 < \frac{1}{\sqrt{2}} < 1$, and an irrational number added or multiplied with a rational number is still an irrational number.

Solution: Take a and b as two consecutive rational numbers, s.t. $b > a$.
We know $\frac{1}{\sqrt{2}}$ is irrational and

$$0 < \frac{1}{\sqrt{2}} < 1$$

As $b > a$, $b - a > 0$
Multiply each side by $b - a$

$$0 < \frac{b-a}{\sqrt{2}} < b - a$$

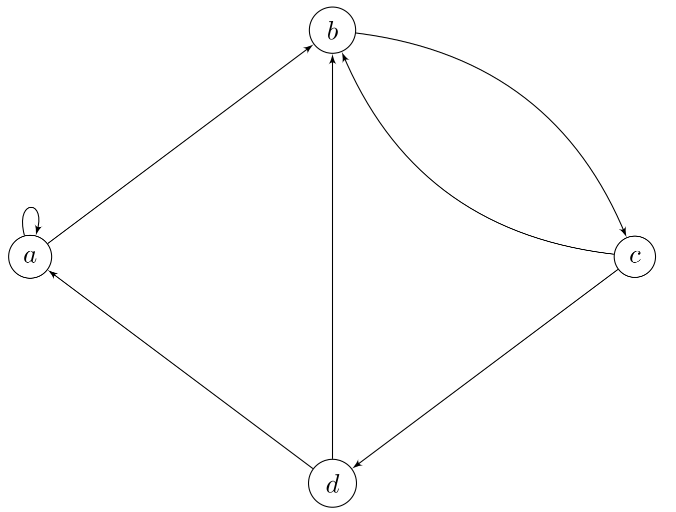
$\frac{b-a}{\sqrt{2}}$ is still irrational

Now we add a to each side

$$a < \frac{b-a}{\sqrt{2}} + a < b$$

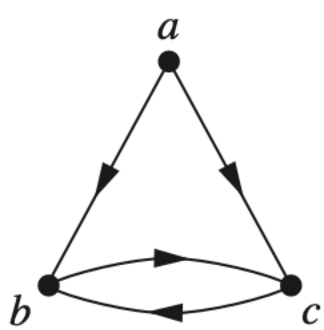
$\frac{b-a}{\sqrt{2}} + a$ is irrational and exists between a and b

11. Draw the directed graph that represents the relation $\{(a, a), (a, b), (b, c), (c, b), (c, d), (d, a), (d, b)\}$

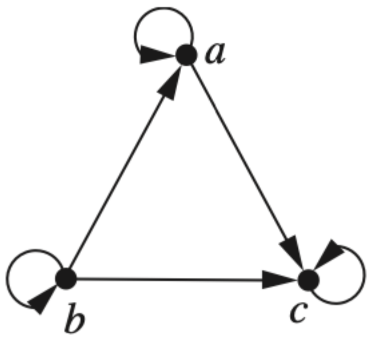


Solution:

12. For each of the following digraphs, list down the ordered pairs and determine whether the relations represented by the digraphs are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.

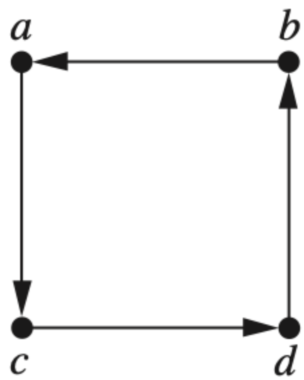
(a) 

Solution: Not reflexive since (a, a) doesn't exist.
Irreflexive since $(a, a), (b, b), (c, c)$ do not exist.
Not symmetric since (a, b) exists but not (b, a).
Not anti-symmetric since $(b, c), (c, b)$ both in R but $b \neq c$.
Not transitive since $(b, c), (c, b)$ in R but not (b, b)



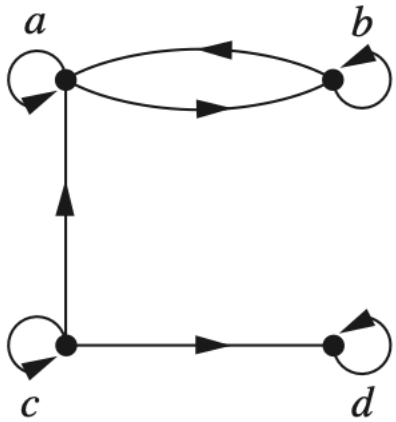
(b)

Solution: Reflexive since all nodes have self loops.
Since it is reflexive, it isn't irreflexive.
Not symmetric since (a, c) exists but not (c, a).
Anti symmetric since if $a \neq b$, then $(a, b), (b, a)$ do not exist in R at the same time.
Is transitive. Follows the definition.



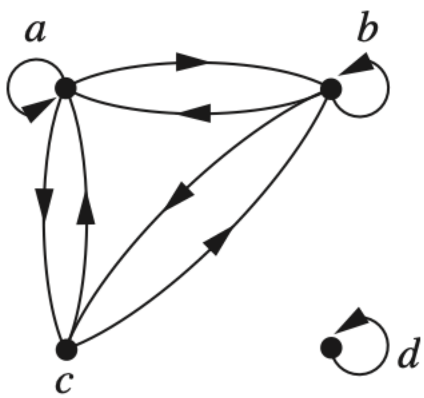
(c)

Solution: Is not reflexive since $(a, a) \notin R$
Is irreflexive since no self loops.
Is not symmetric since edge from b to a but not a to b.
Is Antisymmetric since between two distinct nodes, there aren't two edges in the opposite direction.
Not transitive since edge from b to a and a to c but not b to c.



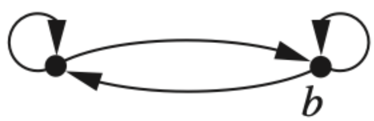
(d)

Solution: Since each node has self loops, this is reflexive.
Since it is reflexive, it isn't irreflexive.
Not symmetric since edge from c to d but not d to c.
Not transitive since edge from c to a and a to b but not c to b.



(e)

Solution: Not reflexive since c doesn't have a self loop.
Not irreflexive since a has self loop.
Symmetric since if there is an edge from a node to another, there is an edge in the opposite direction.
Not antisymmetric since edge from a to b and vice versa when $b \neq a$.



(f)

Solution: Reflexive since all nodes have self loops.
Not irreflexive since it is reflexive.
Symmetric since if there is an edge from a node to another, there is an edge in the opposite direction
Not antisymmetric since edge from c to d and vice versa when $c \neq d$.