Recitation: Relation Proofs

Fix your relations

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Questions

1. $\boxed{\text{5 points}}$ Given Theorem 1 and Definition 1 below, prove Theorem 2.

   Theorem 1. *Let R be an equivalence relation on a set A. The following statements for elements a and b of A are equivalent.*

   $$(i) aRb \quad (ii) [a] = [b] \quad (iii) [a] \cap [b] \neq \emptyset$$

   Definition 1. *A partition of a set, A, is a set of non-empty subsets, A_i, of A, such that every element a in A is in exactly one of these subsets (i.e. A is a disjoint union of the subsets). [Wikipedia]*

   Theorem 2. *Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S.*

   Solution: Following Definition 1, we have to show that:

   1. each equivalence class is non-empty, and
   2. every element of S is in exactly one equivalence class.

   1. each equivalence class is non-empty:
   Consider an equivalence class, C, with a representative element, $a \in S$. By definition,

   $$C = [a] = \{b \in S \mid (a, b) \in R\}.$$

   As R is reflexive, $(a, a) \in R$. Therefore $a \in C$ and the equivalence class is non-empty.

   2. every element of S is in exactly one equivalence class:
   As R is reflexive, an element $a \in S$, belongs to $[a]$. That is, **every element belongs to some equivalence class**. To show that this class is unique, i.e. a does not belong to any equivalence class other than $[a]$, let us assume the opposite, i.e. a belongs to another equivalence class, say $[b]$ for some $b \in S$ where $[a] \neq [b]$.

   Since $a \in [b]$, we know that bRa and, by symmetry, aRb. Then, from Theorem 1, $[a] = [b]$. This contradicts our initial assumption that $[a] \neq [b]$.

   Therefore a belongs to $[a]$ only. $\square$

2. $\boxed{\text{5 points}}$ Prove the following theorem.

Theorem 3. *Let R be an equivalence relation on a set A. The following statements for elements a and b of A are equivalent.*

$$(i) aRb \quad (ii)[a] = [b] \quad (iii)[a] \cap [b] \neq \emptyset$$

Solution: Proof: We first show that (i) implies (ii). Assume that aRb. We will prove that $[a] = [b]$ by showing $[a] \subseteq [b]$ and $[b] \subseteq [a]$. Suppose $c \in [a]$. Then aRc. Because aRb and R is symmetric, we know that bRa. Furthermore, because R is transitive and bRa and aRc, it follows that bRc. Hence, $c \in [b]$. This shows that $[a] \subseteq [b]$. The proof that $[b] \subseteq [a]$ is similar; it is left as an exercise for the reader.

Second, we will show that (ii) implies (iii). Assume that $[a] = [b]$. It follows that $[a] \cap [b] \neq \emptyset$ because $[a]$ is nonempty (because $a \in [a]$ because R is reflexive).

Next, we will show that (iii) implies (i). Suppose that $[a] \cap [b] \neq \emptyset$. Then there is an element c with $c \in [a]$ and $c \in [b]$. In other words, aRc and bRc. By the symmetric property, cRb. Then by transitivity, because aRc and cRb, we have aRb.

Because (i) implies (ii), (ii) implies (iii), and (iii) implies (i), the three statements, $(i), (ii)$, and (iii), are equivalent. □

3. $\boxed{\text{5 points}}$ Given non empty set S and a partition A_i of S. Prove that $R = \bigcup_{A_i}(A_i \times A_i)$ is an equivalence relation on S. Moreover the equivalence classes of R are exactly A_i.

Solution: Let $x \in S$. Since the union of the sets in the partition A_i give us S, x must belong to a set in A_i. Since $x \in A \implies \exists i(x \in A_i)$, (x, x) would exist in $A_i \times A_i$ and hence R, therefore the relation is reflexive.
The relation is symmetric. Suppose $(x, y) \in R$ for some x, y, then $\exists i(x \in A_i \wedge y \in A_i)$ therefore $A_i \times A_i$ would also contain (y, x) since $y \in A_i$ and $x \in A_i$, therefore the relation is symmetric.
Suppose $(x, y) \in R$ and $(y, z) \in R$. Then $\exists i, (x \in A_i \wedge y \in A_i)$ and $\exists j, (y \in A_j \wedge z \in A_i)$. Since sets in a partition are disjoint, since $y \in A_i \wedge y \in A_j$ therefore $A_i = A_j$ since $A_i \cap A_j \neq \emptyset$. Hence $x \in A_i \wedge z \in A_i$, therefore (x, z) would be in $A_i \times A_i$ and hence in R, therefore the relation is transitive.
Hence R is an equivalence relation.
The moreover part is already proved in Question 1.

4. $\boxed{\text{5 points}}$ Show that a finite nonempty poset has a maximal element.

Solution: Proof: Choose an element a_0 of S. If a_0 is not maximal, then there is an element a_1 with $a_1 \succ a_0$. If a_1 is not maximal, there is an element a_2 with $a_2 \succ a_1$. Continue this process, so that if a_n is not maximal, there is an element a_{n+1} with $a_{n+1} \succ a_n$. Because there are only a finite number of elements in the poset, this process must end with a maximal element a_n.

5. $\boxed{\text{5 points}}$ The following is a summary of someone's attempt to prove that exists only one unique God. Find the error in the proof.

Proof: Assume there is more than one unique God.
Insert convincing argument to show that this leads to a contradiction.
Since having more than one unique God results in a contradiction, there exists only one unique God.
□

Solution: The proof shows that there is not more than one unique God, this would mean that the number of unique Gods is not > 1 hence is less than equals to 1. This means that there is either 0 Gods or 1 God which is not the same as saying there exists one unique God.

6. $\boxed{\text{5 points}}$ "Prove the least element is unique when it exists.". State the error in the following proofs.

Proof 1: Assume that Least element exists and it is not unique. Let a be a least element and g be another least element and $a \neq g$. Then by the definition of least element, $g \prec a$. But we assumed that a is the least element. Therefore, this is a contradiction and hence we can say that, The least element is unique when it exists.
Proof 2: Let's assume that there is more than one unique least element, with a and b being two distinct least elements. For the relation R to be antisymmetric, a must be equal to be ($a = b$), given that $(a, b) \in R$ and $(b, a) \in R$. Since a and b are equal to each other, we can conclude that there is only one unique least element.

Solution: Proof 1: The definition of least element states that the element a is the least element if $\forall b \in A : a \preceq b$. We have to justify that this is the same as $\forall b \in A \setminus \{a\} : a \prec b$. There is no justification of this given. Emphasis needs to be put on why this is a contradiction.
Proof 2: The phrasing is off, but the idea makes sense. The correct phrasing should be that $(a, b) \in R$, since a is least element and $(b, a) \in R$ since b is the least element. This can only happen when $a = b$ due to antisymmetry which leads us to a contradiction as we assumed $a \neq b$, meaning there is only one least element if such an element exists.

7. $\boxed{\text{5 points}}$ The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$

Solution: The proof can be found in the book on Pg 581.

8. $\boxed{\text{5 points}}$ Fix the problem in the proof. "Show that for a reflexive relation R, $R^{-1} \subseteq R \circ R^{-1}$"

Proof: By reflexivity, $(a, a) \in R$, therefore $(b, a) \in R \circ R^{-1}$ $\square$

Solution: The phrasing is an issue. What is a, what is b?
Since the relationship is reflexive, we know that $\forall a, (a, a) \in R$. By definition of R^{-1}, we know that it contains all pairs (b, a) such that $(a, b) \in R$. We need to show that every element in R^{-1} is a member of $R \circ R^{-1}$. We can take any arbitrary pair $(b, a) \in R^{-1}$, and take the element $(a, a) \in R$ (We can do this since relation is reflexive), therefore the arbitrary element (b, a) would lie in the composite of R^{-1} and R: $R \circ R^{-1}$. Therefore our statement is true.

9. $\boxed{\text{5 points}}$ Show that a subset of an antisymmetric relation is also antisymmetric.

Solution: Suppose that $R1 \subseteq R2$ and that $R2$ is antisymmetric. We must show that $R1$ is also antisymmetric. Let $(a, b) \in R1$ and $(b, a) \in R1$. Since these two pairs are also both in $R2$,we been knew that $a = b$, as desired.

10. $\boxed{\text{5 points}}$ A relation R is called circular if aRb and bRc imply that cRa. Show that R is reflexive and circular if and only if it is an equivalence relation.

Solution: First suppose that R is reflexive and circular. We need to show that R is symmetric and transitive. Let $(a, b) \in R$. Since also $(b, b) \in R$, it follows by circularity that $(b, a) \in R$; this proves symmetry. Now if $(a, b) \in R$ and $(b, c) \in R$, then by circularity $(c, a) \in R$ and so by symmetry $(a, c) \in R$; thus R is transitive. Conversely, transitivity and symmetry immediately imply circularity, so every equivalence relation is reflexive and circular.