Combinatorics Problems

CS/MATH 113 team

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1. Prove that at a party where there are at least two people, there are two people who know the same number of other people there. Given that everyone atleast knows the host.

> **Solution:** Let there be n number of people at the party where $n \in \mathbb{N}$ and $n \geq 2$
> Now the maximum number of people a person can know is $n-1$
> The minimum number of people someone can know is 1 (the host).
> Now we have n pigeons and $n-1$ holes, so with the pigeon hole principle atleast 1 hole will have more than 1 pigeon.
> So assigning number of people that know that amount of people from 1 to $n-1$, there would be atleast 1 number k where 2 people are assigned.
>
> $\square$

2. How many 4-permutations of the positive integers not exceeding 100 contain exactly three consecutive integers k, $k+1$, $k+2$, in the correct order where these consecutive integers can be separated by other integers in the permutation?
Some example of such 4 permuations would be $(1, 2, 5, 3), (69, 42, 43, 44), (50, 51, 52, 54)$, etc.

> **Solution:** As we have a 4-permutation for positive interger less than of equal to 100 where k, $k+1$, and $k+2$ occurs then $k \leq 98$
> We need to choose 4 integers k, $k+1$, $k+2$, and x.
> The number of ways we can choose k is ${}^{98}C_1 = 98$.
> Number of ways x can be chosen from the remaining 97 intgers is ${}^{97}C_1 = 97$.
> The number of arrangements we can have where integers are in correct order are 4:
>
> $$(k, k+1, k+2, x)$$
> $$(k, k+1, x, k+2)$$
> $$(k, x, k+1, k+2)$$
> $$(x, k, k+1, k+2)$$
>
> Number of ways 4 integers can be selected such that all 4 are in the correct order is ${}^{97}C_1 = 97$. So in total we have $(98 \times 97 \times 4) - 97 = 37927$
>
> $\square$

3. Show that for any connected graph atleast 2 vertices have the same degree.

Solution: Let $G = (V, E)$ be a connected graph with n vertices.
Then for every vertex $v \in V$, $1 \leq degree(v) \leq n - 1$.
There are n vertices and $n - 1$ possible values for degree.
With pigeonhole principle we know that atleast 2 vertices will have the same degree.

$\square$

4.