Recitation: Functions

Discrete Mathematics TAs
(plaigarised primarily from the book)

May 11, 2022

1 Questions

1. $\boxed{\text{10 points}}$ Determine whether $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is onto if

   (a) $f(m, n) = 2m - n$

   Solution: This is clearly onto, since $f(0, -n) = n$ for every integer n.

   (b) $f(m, n) = m^2 - n^2$

   Solution: We know that $m^2 - n^2 = (m - n)(m + n)$. The resultant output is a factor of 2 numbers. We can see that the output 2 is not possible to generate in any case. We can justify this by contradiction. If $m^2 - n^2 = (m - n)(m + n) = 2$, then one of the factors must be 2 (the other being 1) or the factor must be -2 (the other being -1). In order to get 2 or -2, m and n must have same parity (both even or both odd) since the desired factor is even. In either case, both $m - n$ and $m + n$ are then even, so this expression is divisible by 4. This leads to a contradiction since 2 is not divisible by 4. $\perp$

   (c) $f(m, n) = m + n + 1$

   Solution: This is clearly onto, since $f(0, n - 1) = n$ for every integer n.

   (d) $f(m, n) = |m| - |n|$

   Solution: This is onto. To achieve negative values we set $m = 0$, and to achieve nonnegative values we set $n = 0$.

   (e) $f(m, n) = m^2 - 4$

   Solution: This is not onto, for the same reason as in part (b). In fact, the range here is clearly a subset of the range in that part.

2. $\boxed{\text{5 points}}$ if $f : A \to B$ is a one to one function show that there exists a function $g : B \to A$ such that g is onto.

Solution: A function is simply a relation. We can take the inverse of the relation , where an inverse is $R^{-1} = \{(b,a) \mid (a,b) \in f\}$, since for each a, there exists one unique b mapped to in f, we know that for each b that exists there is one a in R^{-1}, therefore the onto condition is met if R^{-1} is indeed a function. Some values in $b \in B$ might not have an associated pair (b,a) in R, let C be the set containing all these bs. Take any arbitrary value $a_0 \in A$ then we can construct a new relation g which is $R^{-1} \cup \{(b,a_0) \mid b \in C\}$. This relation is a function since each member from the domain B maps to one value in A. It is onto since each value in A is mapped to.

3. $\boxed{\text{5 points}}$ if $f : A \to B$ is a onto function show that there exists a function $g : B \to A$ such that g is one to one.

Solution: A function is simply a relation. We can take the inverse of the relation , where an inverse is $R^{-1} = \{(b,a) \mid (a,b) \in f\}$, Since f is onto, each b has some a that it is mapped to, and no two as are mapped by the same by b. R^{-1} might not be a function since one b might map to multiple a, in order to make it a function we can create a function g that is a subset of R^{-1} with only one element for each b to map to. The resulting is a function and is one-to-one since no two as are mapped by the same b.

4. $\boxed{\text{5 points}}$ Show that for every one-to-one and onto function $f : A \to B$ it has an inverse function g, such that $\forall a \in A, g(f(a)) = a$ and $\forall b \in A, f(g(b)) = b$

Solution: Proof. Let $f : A \to B$ be one-to-one and onto. We will define a function $g : B \to A$ as follows. Let $b \in B$. Since f is onto, there exists $a \in A$ such that $f(a) = b$. Let $g(b) = a$. Since f is injective, this a is unique, so g is well-defined.
Now we much check that g is the inverse of f. First we will show that $g(f(a)) = a$ for all a. Let $a \in A$. Let $b = f(a)$. Then, by definition, $f^{-1}(b) = a$. Then $g(f(a)) = g(f(a)) = g(b) = a$. Now we will show that $f(g(b))$ for all b. Let $b \in B$. Let $a = g(b)$. Then, by definition, $f(a) = b$. Then $f(g(b)) = f(g(b)) = f(a) = b$.

5. $\boxed{\text{7 points}}$
   1. Let I be the set of decimals of the form $0.d_1d_2d_3d_4...$ Construct a one to one function from I to $I \times I$

      Solution: $f(x) = (x,x)$ where $\forall x \in I$.

   2. Find either an onto function from I to $I \times I$ or a one to one function from for $I \times I$ to I

      Solution: Simply use cantor's diagnolization proof to map $I \times I$ to I

   3. Do I **and** $I \times I$ have the same cardinality?

      Solution: Yes as there is a bijection between them

6. $\boxed{\text{5 points}}$ Let $f : A \to B$ and $g : B \to C$
If $C_0 \subseteq C$, show that $(g \circ f)^{-1}(C_0) = f^{-1}(g^{-1}(C_0))$

Solution: If $C_0 \subset C$, show that $(g \circ f)^{-1}(C_0) = f^{-1}\left(g^{-1}(C_0)\right)$. For any element $a \in A$, we have

$$\begin{aligned} a \in (g \circ f)^{-1}(C_0) &\Leftrightarrow (g \circ f)(a) \in C_0 \\ &\Leftrightarrow g(f(a)) \in C_0 \\ &\Leftrightarrow f(a) \in g^{-1}(C_0) \\ &\Leftrightarrow a \in f^{-1}\left(g^{-1}(C_0)\right) \end{aligned}$$

Thus, the two sets are equal.

7. $\boxed{\text{5 points}}$ Let $f : A \to B$ and $g : B \to C$
If f and g are injective, show that $g \circ f$ is injective.

Solution: Suppose $g \circ f(x) = g \circ f(y)$, we show that this implies $x = y$

$$g(f(x)) = g(f(y))$$

As g is injective, then $f(x) = f(y)$
As f is injective, then $x = y$
$g \circ f$ is injective

8. $\boxed{\text{5 points}}$ Let $f : A \to B$ and $g : B \to C$
If f and g are surjective, show that $g \circ f$ is surjective.

Solution: Let $z \in C$
As g is surjective $\exists y \in B$, s.t. $g(y) = z$
As f is surjective $\exists x \in A$, s.t. $f(x) = y$
Therefore $z = g(f(x)) = g \circ f(x)$, thus $z \in range(g \circ f)$.
$g \circ f$ is surjective

9. $\boxed{\text{5 points}}$ In mathematics and computer science, the floor function is the function that takes as input a real number x, and gives as output the greatest integer less than or equal to x.
Let g be a function from the set A to the set B. Let S be a subset of B. We define the inverse image of S to be the subset of A whose elements are precisely all pre-images of all elements of S. We denote the inverse image of S by $g^{-1}(S)$, so $g^{-1}(S) = \{a \in A \mid g(a) \in S\}$. (Beware: The notation g^{-1} is used in two different ways. Do not confuse the notation introduced here with the notation $g^{-1}(y)$ for the value at y of the inverse of the invertible function g. Notice also that $g^{-1}(S)$, the inverse image of the set S, makes sense for all functions g , not just invertible functions.)
Let $g(x) = \lfloor x \rfloor$. Find

(a) $g^{-1}(\{0\})$.

Solution: Answer: $\{x \mid 0 \leq x < 1\}$

(b) $g^{-1}(\{-1, 0, 1\})$

Solution: Answer: $\{x \mid -1 \leq x < 2\}$

(c) $g^{-1}(\{x \mid 0 < x < 1\})$.

Solution: There is no integer in range $0 < x < 1$,hence no number will give this output.
Answer: $\emptyset$

10. $\boxed{\text{5 points}}$ Let f be a function from the set A to the set B. Let S be a subset of B. We define the inverse image of S to be the subset of A whose elements are precisely all pre-images of all elements of S. We denote the inverse image of S by $f^{-1}(S)$, so $f^{-1}(S) = \{a \in A \mid f(a) \in S\}$. (Beware: The notation f^{-1} is used in two different ways. Do not confuse the notation introduced here with the notation $f^{-1}(y)$ for the value at y of the inverse of the invertible function f. Notice also that $f^{-1}(S)$, the inverse image of the set S, makes sense for all functions f , not just invertible functions.)
Let f be a function from A to B. Let S and T be subsets of B. Show that
$f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$.

Solution: We need to prove two things. First suppose $x \in f^{-1}(S \cap T)$. This means that $f(x) \in S \cap T$. Therefore $f(x) \in S$ and $f(x) \in T$. That means $x \in f^{-1}(S)$, and $x \in f^{-1}(T)$. Thus we have shown that $f^{-1}(S \cap T) \subseteq f^{-1}(S) \cap f^{-1}(T)$.
Conversely, suppose that $x \in f^{-1}(S) \cap f^{-1}(T)$. Then $x \in f^{-1}(S)$ and $x \in f^{-1}(T)$, so $f(x) \in S$ and $f(x) \in T$. Thus we know that $f(x) \in S \cap T$, so by definition $x \in f^{-1}(S \cap T)$. This shows that $f^{-1}(S) \cap f^{-1}(T) \subseteq f^{-1}(S \cap T)$, as desired.