Recitation: Relations

The world is a hierarchy

May 7, 2022

Definitions/Important Concepts

Partition and Equivalence Classes

Equivalent: If R is an equivalence relation, a is equivalent to b if aRb.

Partition: A partition of a set S is a collection of disjoint nonempty subsets of S that have S as their union

Equivalence Class of a w.r.t R ($[a]_R$): The set of all elements of A that are equivalent to a.

Ordering

Poset: A set S together with a partial ordering R is called a partially ordered set, or poset, and is denoted by (S, R). Members of S are called elements of the poset.

Comparable: The elements a and b of a poset $(S, \preceq)$ are called comparable if either $a \preceq b$ or $b \preceq a$.

Total Order: If $(S, \preceq)$ is a poset and every two elements of S are comparable, S is called a totally ordered or linearly ordered set, and $\preceq$ is called a total order or a linear order.

Maximal Element: An element of a poset that is not less than any other element of the poset.

<u>Alternate</u>: a is maximal in the poset $(S,\preceq)$ if $\nexists b \in S : a \prec b$.

Minimal Element: An element of a poset that is not greater than any other element of the poset.

Alternate: a is minimal in the poset $(S,\preceq)$ if $\nexists b \in S : b \prec a$.

Greatest Element: An element of a poset greater than all other elements in this set.

<u>Alternate:</u> a is the greatest element if $\forall b \in A : b \preceq a$

Least Element: An element of a poset less than all other elements in this set.

<u>Alternate</u>:a is the least element if $\forall b \in A : a \preceq b$

Upper Bound of a set: An element in a poset greater than all other elements in the set.

<u>Alternate:</u> Assume $A \subseteq (S, \preceq).u \in S$ is an upper bound of A if $\forall a \in A : a \preceq u$

Lower Bound of a set: An element in a poset less than all other elements in the set.

<u>Alternate:</u> Assume $A \subseteq (S, \preceq).l \in S$ is a lower bound of A if $\forall a \in A : l \preceq u$

Least Upper Bound of a set: An upper bound of the set that is less than all other upper bounds.

Greatest Lower Bound of a set: A lower bound of the set that is greater than all other lower bounds.

Questions

1. Justify that for an equivalence relation R for an equivalence class $[\mathbf{c}]$ $\forall a, b \in [\mathbf{c}](a, b) \in R$, or every element in the equivalence class has relation with every element in the equivalence class

> **Solution:** The equivalence class $[\mathbf{c}]$ has all pairs (c, a) where $a \in [\mathbf{c}]$, therefore for any arbitrary $a, b \in [\mathbf{c}]$, the pairs (c, a) and (c, b) exist in R respectively. Since R is an equivalence relation, due to symmetry (a, c) also exists and using transitivity, since (a, c) and (c, b) exist, so does the the link (a, b).

2. Determine the number of different equivalence relations on a set with three elements a, b, c by listing them.

> **Solution:** An equivalence relation divides the underlying set into equivalence classes. The equivalence classes determine the relation, and the relation determines the equivalence classes. It will probably be easier to count in how many ways we can divide our set into equivalence classes.
>
> 1. $\{\{a\}, \{b\}, \{c\}\}$
> 2. $\{\{a, b\}, \{c\}\}$
> 3. $\{\{a, c\}, \{b\}\}$
> 4. $\{\{a\}, \{b, c\}\}$
> 5. $\{\{a, b, c\}\}$

3. Let us assume that F is a relation on the set $\mathbb{R}$ real numbers defined by xFy if and only if $x - y$ is an integer. Prove that F is an equivalence relation on $\mathbb{R}$.

> **Solution:** Reflexive: Suppose $x \in \mathbb{R}$. Then $x - x = 0$, which is an integer. Thus, xFx.
> Symmetric: Suppose $x, y \in \mathbb{R}$ and xFy. Then $x - y$ is an integer. Since $y - x = -(x - y), y - x$ is also an integer. Thus, yFx.
> Suppose $x, y \in \mathbb{R}, xFy$ and yFz. Then $x - y$ and $y - z$ are integers. Thus, the sum $(x - y) + (y - z) = x - z$ is also an integer, and so xFz.
> Thus, F is an equivalence relation on $\mathbb{R}$.

   (a) What is the equivalence class of 1 for this equivalence relation?

   > **Solution:** The equivalence class of 1 is the set of all real numbers that differ from 1 by an integer. Obviously this is the set of all integers.

   (b) What is the equivalence class of 1/2 for this equivalence relation?

   > **Solution:** The equivalence class of 1/2 is the set of all real numbers that differ from 1/2 by an integer, namely $1/2, 3/2, 5/2$, etc., and $-1/2, -3/2$, etc. These are often called half-integers. We could write this set as $\{(2n + 1)/2 \mid n \in \mathbb{Z}\}$

4. Can you draw the Hasse diagram for the relation $\{(a, b) \mid a > b\}$ on $\{1, 2, 3, 4, 6, 8, 12\}$? If not, then why?

Solution: The relation given is not a partial ordering since it is not a reflexive relation. For example, $(1, 1) \notin R$

5. Which of these relations(R) on $\{0, 1, 2, 3\}$ are partial orderings? Determine the properties of a partial ordering that the others lack.

1. $\{(0, 0), (2, 2), (3, 3)\}$

Solution: This relation is not reflexive because 1 is not related to itself. Therefore R is not a partial ordering. The relation is antisymmetric, because the only way for a to be related to b is for a to equal b . Similarly, the relation is transitive, because if a is related to b , and b is related to c, then necessarily $a = b = c \neq 1$ so a is related to c.

b) $\{(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 3)\}$

Solution: This is a partial ordering, because it is reflexive and the pairs $(2, 0)$ and $(2, 3)$ will not introduce any violations of antisymmetry or transitivity.

c) $\{(0, 0), (1, 1), (1, 2), (2, 2), (3, 1), (3, 3)\}$

Solution: This is not a partial ordering, because it is not transitive: $3R1$ and $1R2$, but 3 is not related to 2 . It is reflexive and the pairs $(1, 2)$ and $(3, 1)$ will not introduce any violations of antisymmetry.

d) $\{(0, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (2, 3), (3, 0), (3, 3)\}$

Solution: This is not a partial ordering, because it is not transitive: $1R2$ and $2R0$, but 1 is not related to 0 . It is reflexive and the non reflexive pairs will not introduce any violations of antisymmetry.

e) $\{(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (3, 3)\}$

Solution: The relation is clearly reflexive, but it is not antisymmetric ($0R1$ and $1R0$, but $0 \neq 1$) and not transitive ($2R0$ and $0R1$, but 2 is not related to 1).

6. Is (S, R) a poset if S is the set of all people in the world and $(a, b) \in R$, where a and b are people, if

a) a is no shorter than b ?

b) a weighs more than b ?

c) $a = b$ or a is a descendant of b ?

d) a and b do not have a common friend?

Solution:

a) Since there surely are unequal people of the same height (to whatever degree of precision heights are measured), this relation is not antisymmetric, so (S, R) cannot be a poset.

b) Since nobody weighs more than herself, this relation is not reflexive, so (S, R) cannot be a poset.

c) This is a poset. The equality clause in the definition of R guarantees that R is reflexive. To check antisymmetry and transitivity it suffices to consider unequal elements (these rules hold for equal elements trivially). If a is a descendant of b , then b cannot be a descendant of a (for one thing, a descendant needs to be born after any ancestor), so the relation is vacuously antisymmetric. If a is a descendant of b , and b is a descendant of c, then by the way "descendant" is defined, we know that a is a descendant of c; thus R is transitive.

d) This relation is not reflexive, because anyone and himself have a common friend.

7. Which of these are posets?

   1. $(\mathbf{Z}, =)$

   Solution: The equality relation on any set satisfies all three conditions and is therefore a partial partial ordering. (It is the smallest partial partial ordering; reflexivity insures that every partial order contains at least all the pairs (a, a).)

   2. $(\mathbf{Z}, \nmid)$

   Solution: This is not a poset. The relation is not reflexive, since it is not true, for instance, that $2 \nmid 2$. (It also is not antisymmetric and not transitive.)

8. Determine whether the relations represented by these zero-one matrices are partial orders.

   a.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

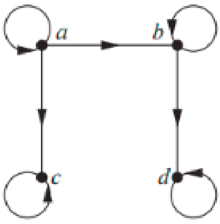
   Solution: The relation is reflexive((1,1),(2,2),(3,3) exists). It is antisymmetric((3,1) exists but (1,3) doesn't hence vacously true). We see that this relation is a partial order, since the pair $(3, 1)$ can cause no problem with transitivity($(3, 1)$ and $(1, 1)$ implies $(3, 1)$).

   b.

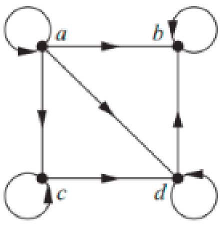
$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

> **Solution:** The relation is not transitive ((1, 3) and (3, 4) are present, but not (1, 4)) and therefore not a partial order.

9. Determine whether relations with the following directed graph is a partial order.

   (a) 

   > **Solution:** This relation is not transitive (there are arrows from a to b and from b to d, but there is no arrow from a to d) , so it is not a partial order.

   (b) 

   > **Solution:** This relation is not transitive (there is no arrow from c to b), so it is not a partial order.

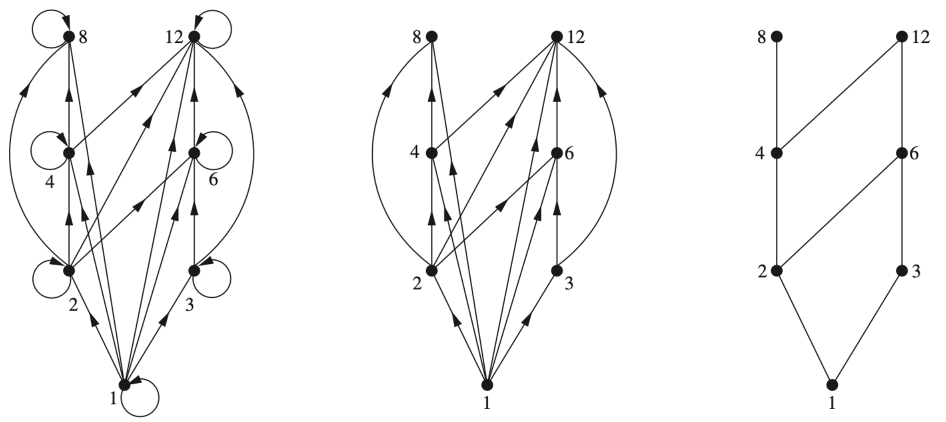
   (c) 

   > **Solution:** This relation is a partial order, since it has all three properties-it is reflexive (there is an arrow at each point), antisymmetric (there are no pairs of arrows going in opposite directions between two different points and it's vacously true(aRb exists but bRa doesn't, hence the premise aRb $\wedge$ bRa in definition of antisymmetric is false)), and transitive (there is no missing arrow from some a to b when there is an arrow from a to b and b to b).

   (d) 

   > **Solution:** Similar to c.

10. **EXAMPLE:** Draw the Hasse diagram representing the partial ordering $\{(a, b) \mid \text{a divides b}\}$ on $\{1, 2, 3, 4, 6, 8, 12\}$.



Solution:

11. Prove that the greatest element is unique when it exists.

Solution: Assume that more than one unique greatest element. Let a, b be two distinct elements that are the greatest elements. Therefore we know that $(a, b) \in R$, and $(b, a) \in R$, due to antisymmetry, this can only happen when $a = b$, this leads us to a contradiction, therefore there is only one greatest element.

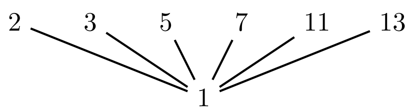
12. Prove the least element is unique when it exists.

Solution: Assume that more than one unique greatest element. Let a, b be two distinct elements that are the least elements. Therefore we know that $(b, a) \in R$, and $(a, b) \in R$, due to antisymmetry, this can only happen when $a = b$, this leads us to a contradiction, therefore there is only one least element.

13. Draw a Hasse Diagram for the following cases.

(a) Divisibility on the set $\{1, 2, 3, 5, 7, 11, 13\}$

Solution:



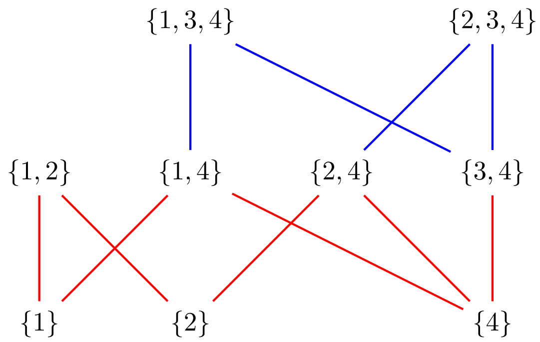
(b) less than or equal to relation on $\{0, 2, 5, 10, 11, 15\}$.

Solution:

14. Answer these questions for the poset $(\{\{1\},\{2\},\{4\},\{1,2\},\{1,4\},\{2,4\},\{3,4\},\{1,3,4\},\{2,3,4\}\},\subseteq)$.

(a) Draw a Hasse Diagram representing this poset.

Solution:



(b) Find the maximal elements.

Solution: The maximal elements are the ones without any elements lying above them in the Hasse diagram, namely $\{1,2\}, \{1,3,4\}$, and $\{2,3,4\}$.

(c) Find the minimal elements.

Solution: The minimal elements are the ones without any elements lying below them in the Hasse diagram, namely $\{1\}, \{2\}$, and $\{4\}$.

(d) Is there a greatest element?

Solution: There is no greatest element, since there is more than one maximal element, none of which is greater than the others.

(e) Is there a least element?

Solution: There is no least element, since there is more than one minimal element, none of which is less than the others.

(f) Find all upper bounds of $\{\{2\},\{4\}\}$.

Solution: The upper bounds are the sets containing both $\{2\}$ and $\{4\}$ as subsets, i.e., the sets containing both 2 and 4 as elements. Pictorially, these are the elements lying above both $\{2\}$ and $\{4\}$ (in the sense of there being a path in the diagram), namely $\{2, 4\}$ and $\{2, 3, 4\}$.

(g) Find the least upper bound of $\{\{2\}, \{4\}\}$, if it exists.

Solution: The least upper bound is an upper bound that is less than every other upper bound. We found the upper bounds in part (e), and since $\{2, 4\}$ is less than (i.e., a subset of) $\{2, 3, 4\}$, we conclude that $\{2, 4\}$ is the least upper bound.

(h) Find all lower bounds of $\{\{1, 3, 4\}, \{2, 3, 4\}\}$.

Solution: To be a lower bound of both $\{1, 3, 4\}$ and $\{2, 3, 4\}$, a set must be a subset of each, and so must be a subset of their intersection, $\{3, 4\}$. There are only two such subsets in our poset, namely $\{3, 4\}$ and $\{4\}$. In the diagram, these are the points which lie below both $\{1, 3, 4\}$ and $\{2, 3, 4\}$.

(i) Find the greatest lower bound of $\{\{1, 3, 4\}, \{2, 3, 4\}\}$, if it exists

Solution: The greatest lower bound is a lower bound that is greater than every other lower bound. We found the lower bounds in part (g), and since $\{3, 4\}$ is greater than (i.e., a superset of) $\{4\}$, we conclude that $\{3, 4\}$ is the greatest lower bound

15. Let **R** be the relation on the set of all colorings of the 2 × 2 checkerboard where each of the four squares is colored either red or blue so that (C1 , C2), where C1 and C2 are 2 × 2 checkerboards with each of their four squares colored blue or red, belongs to R if and only if C2 can be obtained from C1 either by rotating the checkerboard or by rotating it and then reflecting it.

(a) Show that **R** is an equivalence relation.

Solution: There are 4 possible rotations rotate 0°, rotate 90°, rotate 180°, rotate 270°, that we shall call $R_0, R_{90}, R_{180}, R_{270}$.
Similiarly we have 4 possible reflections reflect horizontal, reflect vertical, reflect right diagonal, reflect left diagonal, that we shall call H, V, D, D'.
For **R** to be an equivalence relation **R** needs to be reflexive, symmetric and transitive.
Let $C1$ be a board, $C1$ can be obtained by applying R_0 on $C1$ thus the relation is reflexive.
The actions are closed in composition (can be seen b y constructing a Cayley table) or argued intuitively by cases like $R_{90} + H = V$ and so forth (too lazy to type all of them out)
Thus every $A_1 \circ A_2 \circ .. \circ A_n = A_m$ where all $A_1, A_2, .., , A_n, A_m$ are either a rotation of reflection thus applying actions in composition gives some other actions this thus the relation is transitive.
For every action A there exist an inverse s.t $A \circ A^{-1} = R_0$ which means the board returns to its orignal state. Thus if $(C1, C2) \in \mathbf{R}$ then $(C2, C1) \in \mathbf{R}$ as is there exist rotation and reflection that takes C1 to C2 then there exist inverse reflection and rotation that takes C2 to C1.

(b) What are the equivalence classes of **R**?

Solution: There are 16 possible checkerboards.
You can go by drawing all 16 possible boards. classes are

- 4 reds
- 4 blues
- 3 reds 1 blue
- 3 blues 1 red
- 2 red 2 blue where 2 blues are adjacent to each other
- 2 red 2 blue where 2 blues are diagonal to each other

16. Show that the partition of the set of bit strings of length 16 formed by equivalence classes of bit strings that agree on the last eight bits is a refinement of the partition formed from the equivalence classes of bit strings that agree on the last four bits.
Definition: A partition P1 is called a refinement of the partition P2 if every set in P1 is a subset of one of the sets in P2.

Solution: By the definition, we need to show that every set in the first partition is a subset of some set in the second partition. Let A be a set in the first partition. So A is the set of all bit strings of length 16 that agree on their last eight bits. Pick a particular element x of A, and suppose that the last four bits of x are *abcd*. Then the set of all bit strings of length 16 whose last four bits are *abcd* is one of the sets in the second partition, and clearly every string in A is in that set, since every string in A agrees with x on the last eight bits, and therefore perforce agrees on the last four bits.