Graph theory Problems

CS/MATH 113 team

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1. Let G be a Graph of Grith 4 in which every vertex has a degree k. prove that G will have atleast 2k vertices.

> **Solution:** Let G be a k-regular Graph, s.t. $grith(G) = 4$ and degree of each vertex in G is k.
> Let x, y be 2 vertices in G s.t. x and y are connected.
> As grith of G is 4, the length of the smallest cycle in G is 4.
> Therefore x and y cannot have any common neighbors as that would create a cycle of 3.
> The neighborhood of x and y are disjoint sets of size k, $N(x) = N(y) = k$
> Therefore there are atleaste $2k$ vertices in G
>
> □

2. Number of vertices of odd degree in a graph is always even.

> **Solution:** Let $G = (V, E)$ be a graph
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> $$|E| = \frac{1}{2}\sum_{v \in V} degree(v)$$
>
> Then $\sum_{v \in V} degree(v)$ must be even, so if an odd number of vertices have odd degree then $\sum_{v \in V} degree(v)$ would be odd.
> Therefore number of vertices of odd degree in a graph must be even.
>
> □

3. Prove that for any graph G and H, $G \cong H$ iff $\overline{G} \cong \overline{H}$.

> **Solution:** Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be 2 graphs.
> First we show that $G \cong H \implies \overline{G} \cong \overline{H}$
> Let $f : V(G) \to V(H)$ be an isomorphism from G to H.
> As f is an isomorphism $\forall u, v \in V(G)$, $uv \in E(G) \Leftrightarrow f(u)f(v) \in E(H)$.
> So for $\overline{G}$ we know $\forall u, v \in V(G)$, $uv \in E(\overline{G}) \Leftrightarrow uv \notin E(G)$
> Then as for $uv \in E(G) \Leftrightarrow f(u)f(v) \in E(H)$, then under f, $uv \notin E(G) \Leftrightarrow f(u)f(v) \notin E(H)$
> So we have that $uv \in E(\overline{G}) \Leftrightarrow f(u)f(v) \notin E(\overline{H})$
> So therefore f is an isomorphism from $\overline{G}$ to $\overline{H}$.
> The same argumentcan be used to show the converse.
>
> □

4. Show that isomorphism of simple graphs is an equivalence relation.

Solution: We show that isomorphism is reflexive, symmetric and transitive.
Reflexive: Let $G = (V, E)$, then we have bijection $f : V \to V$, where $\forall v \in V,\ v = f(v)$.
Therefore $G \cong H$
Symmetric: Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$,
Let $f : V(G) \to V(H)$ be an isomorphism from G to H, then inverse $f^{-1} : V(H) \to V(G)$ is an isomorphism from H to G
Therefore if $G \cong H$ then $H \cong G$
Transitive: Let $G = (V(G), E(G))$, $H = (V(H), E(H))$ and $K = (V(K), E(K))$,
Let $f : V(G) \to V(H)$ be an isomorphism from G to H, and $g : V(H) \to V(K)$ be an isomorphism from H to K, then $f \circ g$ is an isomorphism from G to K
Therefore if $G \cong H$ and $H \cong K$ then $G \cong K$

So isomorphism is an equivalence relation.

□

5. A graph with maximum degree at most k is $(k + 1)$-colorable

Solution: Proof. We use induction on the number of vertices in the graph, which we denote by n.
Base case: A 1-vertex graph has maximum degree 0 and is 1-colorable, so $P(1)$ is true.
Inductive Hypothesis: Assume that a n-vertex graph with maximum degree at most k is $(k+1)$-colorable.
Induction Step: Let G be an $(n + 1)$-vertex graph with maximum degree at most k.
Remove a vertex v, leaving an n-vertex graph G'.
The maximum degree of G' is at most k, and so G' is $(k+1)$-colorable by our Inductive Hypothesis.
Now add back vertex v. We can assign v a color different from all adjacent vertices, since v has degree at most k neighbors and $k + 1$ colors are available.
Therefore, G is $(k + 1)$-colorable.

□

6. Show that K_n has a Hamiltonian circuit whenever $n \geq 3$

Solution: We can form a Hamilton circuit in K_n beginning at any vertex. Such a circuit can be built by visiting vertices in any order we choose, as long as the path begins and ends at the same vertex and visits each other vertex exactly once. This is possible because there are edges in K_n between any two vertices.

□

7. Prove that in every set of 6 people there is a set if atleast 3 mutual aquantainces or 3 mutual strangers.

Solution: For any graph of 6 vertices if the size of independent set is less than 3 than then atleast 3 vertices in the connected set and vise versa.

□

8. What are the number of edges in k_n.

Solution: As every vertex in k_n has a degree $n-1$ then so the number of edges in k_n are:

$$\frac{n(n-1)}{2}$$

□

9. Show that every tree T as $\Delta(T) - 1$ leaves, where $\Delta(T)$ represented the maximum degree of a vertex in T.

Solution:

□

10. Prove or disprove that if every vertex of a graph G has a degree 2 then G is a cycle.

Solution: G can be 2 disconnected cycles, hence not true.

□

11. Prove that every n-vertex graph with atleast n edges contains a cycle.

Solution:

□

12. Prove that every graph G contains a path of length $\delta(G)$ and a cycle of length atleast $\delta(G) + 1$, where $\delta(G)$ represented the smallest degree of a vertex in G.

Solution:

□