Recitation: Graphs

The Final Recitation

May 7, 2022

What is isomorphism?

The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a one-to-one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such a function f is called an isomorphism. Two simple graphs that are not isomorphic are called nonisomorphic.

Questions

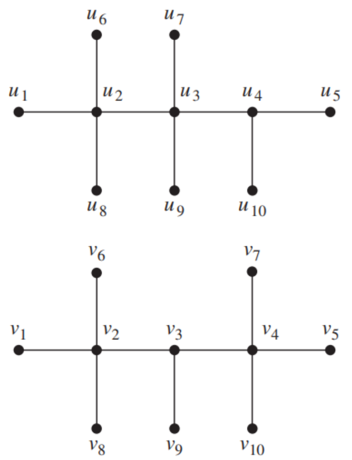
1. Show that isomorphism of simple graphs is an equivalence relation.

> **Solution:** We must show that being isomorphic is reflexive, symmetric, and transitive.
> It is reflexive since the identity function from a graph to itself provides the isomorphism (the one-to-one correspondence)-certainly the identity function preserves adjacency and nonadjacency.
> It is symmetric, since if f is a one-to-one **correspondence** that makes G_1 isomorphic to G_2, then f^{-1} is a one-to-one **correspondence** that makes G_2 isomorphic to G_1; that is, f^{-1} is a one-to-one and onto function from V_2 to V_1 such that c and d are adjacent in G_2 if and only if $f^{-1}(c)$ and $f^{-1}(d)$ are adjacent in G_1.
> It is transitive, since if f is a one-to-one correspondence that makes G_1 isomorphic to G_2, and g is a one-to-one correspondence that makes G_2 isomorphic to G_3, then $g \circ f$ is a one-to-one correspondence that makes G_1 isomorphic to G_3.

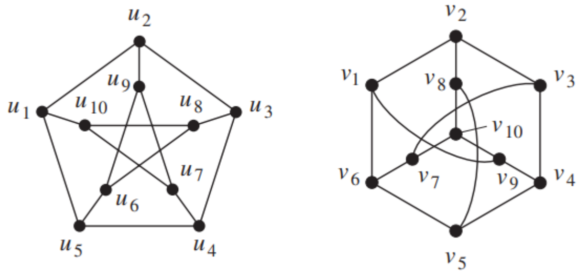
2. Suppose that G and H are isomorphic simple graphs. Show that their complementary graphs $\overline{G}$ and $\overline{H}$ are also isomorphic.

> **Solution:** This is immediate from the definition, since an edge is in $\overline{G}$ if and only if it is not in G, if and only if the corresponding edge is not in H, if and only if the corresponding edge is in $\overline{H}$.
>
> Edge in $\overline{G}$ = Edge not in G = Mapped Edge (from G) not in H = Mapped Edge (from G) in $\overline{H}$
>
> We can use the same map from vertices in G to H for vertices in $\overline{G}$ to vertices in $\overline{H}$. Maintaining adjacency and nonadjacency information.

3. Determine whether the given pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.



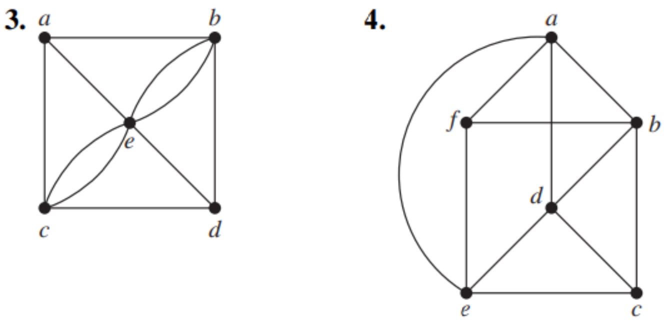
(a) Part a



(b) Part b

Solution: a) These graphs are not isomorphic. In the first graph the vertices of degree 4 are adjacent. This is not true of the second graph.
b) These are isomorphic. We can see that each vertex has 3 degrees. We can assign one node arbitrarily and assign the rest of the values based on that. One isomorphism is $f(u_1) = v_1$, $f(u_2) = v_9, f(u_3) = v_4, f(u_4) = v_3, f(u_5) = v_2, f(u_6) = v_8, f(u_7) = v_7, f(u_8) = v_5$, $f(u_9) = v_{10}$ and $f(u_{10}) = v_6$

4. Determine whether each given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists. exists.
Definition: An Euler circuit in a graph G is a simple circuit containing every edge of G. An Euler path in G is a simple path containing every edge of G.
Theorem 1: A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree
Theorem 2: A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.



Solution: a) Since there are two vertices of odd degree (a and d), this graph has no Euler circuit, but it does have an Euler path starting at a and ending at d. We can find such a path by inspection,

or by using the splicing idea explained in the book (10.5) One such path is $a, e, c, e, b, e, d, b, a, c, d$.
b) This graph has no Euler circuit, since the degree of vertex c (for one) is odd. There is an Euler path between the two vertices of odd degree. One such path is $f, a, b, c, d, e, f, b, d, a, e, c$.
Link that might be helpful in finding Euler paths and circuits.

5. A graph with maximum degree at most k is $(k+1)$-colorable

Solution: Proof. We use induction on the number of vertices in the graph, which we denote by n. Let $P(n)$ be the proposition that an n-vertex graph with maximum degree at most k is $(k+1)$-colorable. A 1-vertex graph has maximum degree 0 and is 1-colorable, so $P(1)$ is true.

Now assume that $P(n)$ is true, and let G be an $(n+1)$-vertex graph with maximum degree at most k. Remove a vertex v, leaving an n-vertex graph G'. The maximum degree of G' is at most k, and so G' is $(k+1)$-colorable by our assumption $P(n)$. Now add back vertex v. We can assign v a color different from all adjacent vertices, since v has degree at most k neighbors and $k+1$ colors are available. Therefore, G is $(k+1)$-colorable. The theorem follows by induction.

6. Show that K_n has a Hamiltonian circuit whenever $n \geq 3$

Solution: We can form a Hamilton circuit in K_n beginning at any vertex. Such a circuit can be built by visiting vertices in any order we choose, as long as the path begins and ends at the same vertex and visits each other vertex exactly once. This is possible because there are edges in K_n between any two vertices.

7. **Iftar Party:** Since the semester has ended Khubaib decides to host an iftar party for every DM TA other than himself (Haania, Ifrah, Mujtaba and Affan).
Dishes: Ramen, Qeema samosas, Aloo ke pakorey, hash browns, Fish Fingers, Pizza, Pork
Haania likes Aloo ke pakorey and Qeema samosas
Ifrah likes Ramen, fish fingers
Mujtaba likes Qeema samosas, Hash browns and fish fingers
Affan likes Aloo ke pakorey
Khubaib wants that every TA has one dish to themselves (that they share with no one) that they like. Is it possible? Show using the Hall Marriage Theorem. If there is, mention what is that matching.

Solution: This problem is essentially that of a complete matching. Therefore we need to look at all possible subsets.

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\begin{aligned}
N(\{\}) &= \{\} \\
N(\{\text{Haania}\}) &= \{\text{Aloo ke pakorey, Qeema samosas}\} \\
N(\{\text{Ifrah}\}) &= \{\text{Ramen, fish fingers}\} \\
N(\{\text{Mujtaba}\}) &= \{\text{Qeema samosas, Hash browns, fish fingers}\} \\
N(\{\text{Affan}\}) &= \{\text{Aloo ke pakorey}\} \\
N(\{\text{Haania,Ifrah}\}) &= \{\text{Aloo ke pakorey, Qeema samosas,Ramen, fish fingers}\} \\
N(\{\text{Haania,Mujtaba}\}) &= \{\text{Aloo ke pakorey, Qeema samosas, Hash browns and fish fingers}\} \\
N(\{\text{Haania,Affan}\}) &= \{\text{Aloo ke pakorey,Qeema samosas}\} \\
N(\{\text{Ifrah,Mujtaba}\}) &= \{\text{Ramen, fish fingers,Qeema samosas, Hash browns}\} \\
N(\{\text{Ifrah,Affan}\}) &= \{\text{Ramen, fish fingers,Aloo ke pakorey}\} \\
N(\{\text{Mujtaba,Affan}\}) &= \{\text{Qeema samosas, Hash browns,fish fingers,Aloo ke pakorey}\} \\
N(\{\text{Haania, Mujtaba,Ifrah}\}) &= \{\text{Aloo ke pakorey, Qeema samosas, Hash browns, fish fingers, Ramen}\} \\
N(\{\text{Haania, Mujtaba,Affan}\}) &= \{\text{Aloo ke pakorey, Qeema samosas, Hash browns, fish fingers}\} \\
N(\{\text{Affan, Mujtaba,Ifrah}\}) &= \{\text{Qeema samosas, Hash browns,fish fingers,Aloo ke pakorey, Ramen}\} \\
N(\{\text{Affan,Haania,Ifrah}\}) &= \{\text{Aloo ke pakorey, Qeema samosas,Ramen, fish fingers}\} \\
N(\{\text{Haania, Mujtaba,Ifrah,Affan}\}) &= \{\text{Aloo ke pakorey, Qeema samosas, Hash browns, fish fingers, Ramen}\}
\end{aligned}
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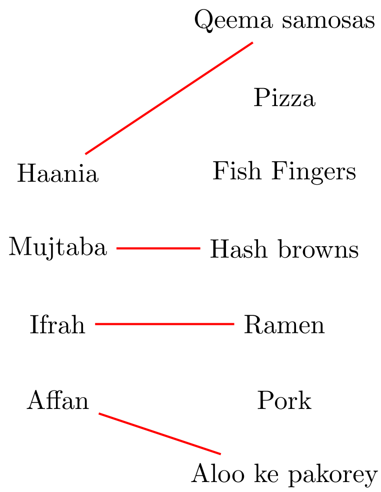
We can see that each subset A of TAs, the neighborhood is the same size or bigger therefore a solution does exist and a solution is as follows



1 Message from the TAs

From Ifrah (Codename: Ifrah-senpai): The course was very diverse in terms of the people who took this course either voluntarily or just to complete the part of the grid. People from different batches and different majors all trying to learn something that makes the real basis of Computer Science. Many of you must have struggled a lot. Whether be the constant change of mode of classes or jumping from one hard abstract concept to another. But not all struggles are bad. You guys struggled and yet tried your very best; Handling back to back exams, visiting us in our hours (and even after-hours at 1am to which we sent you links on why sleep is important XD), asking relevant questions during recitations. Sometimes hard work shows an immediate result as good grades, but sometimes it does not. But it should not discourage you. That hard work is the evidence that you dedicated yourself to something and regardless of the result you should be proud of yourself. This is the first time I was TA-ing for a course and I am really glad this was the first one. I really hope I helped you out with concepts, even if it was something small. I hope to see you guys around on campus and feel free to ask me for help in anything. Will miss the time spent together.

From Khubaib: I will keep things simple. Make lots of mistakes over your 4 years but make sure to fix them as well. Making mistakes is half the story. I do know Zuckerburg said move fast and break things, but someone ultimately has to pick them up. So move fast but at times reflect and pick those broken pieces. Buddy system is really important to me. Don't hate your buddy (most of you don't) and I hope you remain good friends with them. Forgive me if I am rash or abrupt, I do have your best interest at heart, although sometimes it might not come across that way. You have all become better as a result of your hard work, remember that. Also remember that if you do good, you are standing on effort by other people, your instructors, buddies, TAs, parents etc. This is my last TAship at Habib and it was enjoyable. I hope some of you become TAs for this course and tell me how you like it.

From Affan: Dear people who have taken discrete mathematics this semester. I was asked to share some departing words and advice with you all. In which regard I must firstly congratulate all of you for putting in relentless effort that this course required. I hope that apart from technical skills like latex, logic and proofs you were also able to take away soft skills like communication, team work and time management. I would request all of you to value and work on your these skills throughout your 4 years at Habib. My advice for people has always been specific to their circumstances and now that I have to give general advice the job is much tougher. Perhaps the best advice I can give is be a good person, actively self reflect on your values and actions and try to make change wherever necessary. Look after the people around you, the plants, the animals and be empathetic. Always assume people have good intentions and make respectable decisions that don't compromise your values. Above all try to do the things that bring you joy.