Assignment

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* ~~PCA~~

Q3. Consider a matrix $A$ of size $m \times n$, $m \leq n$ and $P = A^T A$ (size $n \times n$) and $Q = AA^T$ (size $m \times m$)

(a) Let $y$ be any $n \times 1$ size vector.

We have

$$y^T P y = y^T A^T A y$$
$$= (Ay)^T Ay$$
$$= \|Ay\|_2^2 \geq 0$$

$\because$ $Ay$ is a vector of size $m \times 1$ and $L_2$ norm square of it contains sum of square terms which will always be non-negative

$$\therefore \quad y^T P y \geq 0 \longrightarrow (1)$$

Similarly, we can prove $z^T Q z \geq 0$ where $z$ is any $m \times 1$ vector

$$z^T Q z = z^T A A^T z$$
$$= (A^T z)^T A^T z$$
$$= \|A^T z\|_2^2 \geq 0$$

$$\therefore \quad z^T Q z \geq 0 \longrightarrow (2)$$

Eqn (1) and (2) ~~are~~ ~~test~~ both resembles the test to check if a matrix is positive semi definite i.e a symmetric matrix with all non-negative eigen values.

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Lets see how equ$^n$ (1) is able to say that all the eigen values of P are non-negative

Consider a eigen value of P as $\lambda$ with $x$ as corresponding eigen vector.

$$\therefore \quad Px = \lambda x$$

$$x^T P x = \lambda x^T x \quad \cdots \text{premultiply by } x^T$$

but $\quad x^T P x \geq 0 \quad \cdots$ from equ$^n$ (1)

$$\therefore \quad \lambda x^T x \geq 0$$

$$\therefore \quad \lambda \|x\|_2^2 \geq 0$$

$$\therefore \quad \lambda \geq 0$$ as $2^{nd}$ term is always positive due to $x \neq \bar{0}$

As '$\lambda$' was any arbitrary eigen value, above is true for all eigen values of P.

If a symmetric matrix $X$ satisfies $y^T X y \geq 0$ for any real vector $y$ then its eigen values are non-negative & is thus positive semi-definite. $\longrightarrow$ (3)

$\therefore$ From equ$^n$ (1), (2) & (3) eigen values of P & Q are non-negative

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QS. 6.

Let 'u' be eigen vector of P with eigen value $\lambda$.

$\therefore \quad Pu = \lambda u$

$\therefore \quad A^T A u = \lambda u \quad \cdots\cdots \quad P = A^T A$

$\therefore \quad AA^T (Au) = \lambda (Au) \quad \cdots\cdots$ premultiply by A

$\therefore \quad Q (Au) = \lambda (Au)$

So we get $Au$ as eigen vector of Q with eigen value $\lambda$.

Similarly, let 'v' be eigen vector of Q with eigen value $\mu$

$\therefore \quad Qv = \mu v$

$\therefore \quad AA^T v = \mu v$

$\therefore \quad A^T A (A^T v) = \mu (A^T v) \quad \cdots$ premultiply by $A^T$

$\therefore \quad P (A^T v) = \mu (A^T v)$

So, here also, we get $A^T v$ as eigen vector of P with eigen value $\mu$.

$u \in R^n \quad \& \quad v \in R^m$

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Q3. c

Given: $v_i$ is eigen vector of $Q$

$$u_i = \frac{A^T v_i}{\|A^T v_i\|_2}$$

Consider

$$Au_i = A \frac{A^T v_i}{\|A^T v_i\|_2} \quad \text{---- substitute } u_i$$

$$= \frac{(AA^T) v_i}{\|A^T v_i\|_2}$$

$$= \frac{Q v_i}{\|A^T v_i\|_2}$$

$$= \frac{\mu_i v_i}{\|A^T v_i\|_2} \quad \text{---- } \mu_i \text{ is the eigen value having eigen vector } v_i$$

$$Au_i = \gamma_i v_i \quad \text{---- } \gamma_i = \frac{\mu_i}{\|A^T v_i\|_2}$$

---- $\mu_i$ is real and non-negative as its a eigen value of positive semi-definite matrix $Q$ (proved this in Q3. a) & denominator is positive

$$\therefore Au_i = \gamma_i v_i \quad \text{---- } \gamma_i \text{ is real \& non-negative}$$

Q3.d

As we have been following ; $u_i \cdots 1 \le i \le n$ are the eigen vectors of matrix $P = A^T A$ and $v_i \cdots 1 \le j \le m$ are that of matrix $Q = AA^T$

For symmetric matrices the eigen vectors are orthonormal.

$$\therefore \quad u_i^T u_j = \begin{cases} 1 & \cdots \; i = j \\ 0 & \; i \ne j \end{cases} \longrightarrow (1)$$

and

$$v_i^T v_j = \begin{cases} 1 & \cdots \; i = j \\ 0 & \; i \ne j \end{cases} \longrightarrow (2)$$

Consider,

$$U^T A V = \underset{m \times m}{\begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_m^T \end{bmatrix}} \underset{m \times n}{A} \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}_{n \times n}$$

$$= \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_m^T \end{bmatrix} \begin{bmatrix} Au_1 & Au_2 & \cdots & Au_n \end{bmatrix}$$

$$= \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_m^T \end{bmatrix} \begin{bmatrix} \sigma_1 v_1 & \sigma_2 v_2 & \cdots & \sigma_n v_n \end{bmatrix}$$

$\cdots \cdots \; Au_i = \sigma_i v_i$

proved in Q3-c

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$$= \begin{bmatrix} \sigma_1 & 0 & \cdots & \\ 0 & \sigma_2 & & \\ 0 & & \ddots & \\ & & & \sigma_n \end{bmatrix}_{m \times n}$$

$$= \Sigma$$

$\therefore$ using (2) $\cdots$ $\Sigma_{ij} = v_i^T \sigma_j v_j = \begin{cases} 0 & i \neq j \\ \sigma_j & i = j \end{cases}$

$\therefore \; U^T A V = \Sigma$

$\therefore \; U U^T A V = U \Sigma$

$\therefore \; A V V^T = U \Sigma V^T$

$\left.\right\}$ U and V are orthonormal

$\therefore \; \boxed{A = U \Sigma V^T}$ $\cdots$ proved.