1: Sums and Products

(a) Compute the following sums.

1. $$\sum_{i=0}^{N} 1 =$$

2. $$\sum_{k=1}^{K} \sum_{t=1}^{T} 1 =$$

3. $$\sum_{k=1}^{K} \sum_{t=1}^{T} 0.5^k =$$

4. $$\sum_{k=1}^{\infty} \sum_{t=1}^{T} 0.5^k =$$

(b) The notation

$$\prod_{i=1}^{N} p_i$$

denotes the product with N factors:

$$\prod_{i=1}^{N} p_i = p_1 p_2 \cdots p_N.$$

Compute the following products.

1. $$\prod_{i=1}^{M} \frac{1}{\theta} =$$

2. $$\prod_{k=1}^{K} \frac{k}{k+1} =$$

3. $$\ln\left(\prod_{k=1}^{K} e^k\right) =$$

2: Asymptotics and Trends

For each of the following functions $f(x)$ below:

- Find its limits $\lim_{x\to\pm\infty} f(x)$ as x approaches $\pm\infty$.
- Choose the values of x where $f(x)$ is differentiable, i.e. $f'(x)$ exists.
- Choose the values of x where $f(x)$ is also strictly increasing, i.e. $f'(x) > 0$.

1. For $f(x) = \max(0, x)$:
   - $$\lim_{x\to-\infty} f(x) =$$
   - $$\lim_{x\to+\infty} f(x) =$$
   - Interval on which $f(x)$ is differentiable.
   - Interval on which $f'(x)$ is differentiable.

2. For
   $$f(x) = \frac{1}{1+e^{-x}}:$$
   - $$\lim_{x\to-\infty} f(x) =$$
   - $$\lim_{x\to+\infty} f(x) =$$
   - Interval on which $f(x)$ is differentiable.
   - Interval on which $f'(x)$ is differentiable.

3: Points and Vectors

A list of n numbers can be thought of as a point or a vector in n-dimensional space. In this course, we will think of n-dimensional vectors $[x_1, x_2, \ldots, x_n]$ flexibly as points and as vectors.

1. Dot Products and Norm
   Recall the dot product of a pair of vectors a and b:
   $$a \cdot b = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$
   where $a = [a_1, a_2, \ldots, a_n]$ and $b = [b_1, b_2, \ldots, b_n]$. When thinking about a and b as vectors in n-dimensional space, we can also express the dot product as
   $$a \cdot b = ||a|| ||b|| \cos \theta,$$
   where θ is the angle formed between the vectors a and b in n-dimensional Euclidean space. Here, $||a||$ refers to the length, also known as norm, of a:
   $$||a|| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}.$$
   - What is the length of the vector $[0.4, 0.3]$?
   - What is the length of the vector $[-0.15, 0.2]$?
   - What is the angle between $[0.4, 0.3]$ and $[-0.15, 0.2]$? Choose the answer that lies between 0 and π.

2. Dot Products and Orthogonality
   Given 3-dimensional vectors $x^{(1)} = [a_1, a_2, a_3]$ and $x^{(2)} = [a_1, -a_2, a_3]$, when is $x^{(1)}$ orthogonal to $x^{(2)}$, i.e. the angle between them is $\pi/2$?

3. Unit Vectors
   A unit vector is a vector with length 1. The length of a vector is also called its norm. Given any vector x, write down the unit vector pointing in the same direction as x?

4. Projections
   Recall from linear algebra the definition of the projection of one vector onto another. As before, we have 3-dimensional vectors $x^{(1)} = [a_1, a_2, a_3]$ and $x^{(2)} = [a_1, -a_2, a_3]$. Which of these vectors is in the same direction as the projection of $x^{(1)}$ onto $x^{(2)}$?
   - $x^{(1)}$
   - $x^{(2)}$
   - $x^{(1)} + x^{(2)}$

5. What is the signed magnitude c of the projection of $x^{(1)}$ onto $x^{(2)}$? More precisely, let u be the unit vector in the direction of the correct choice in previous part, find a number c such that the projection is cu.

4: Planes

A hyperplane in n dimensions is a $n-1$ dimensional subspace. For instance, a hyperplane in 2-dimensional space can be any line in that space and a hyperplane in 3-dimensional space can be any plane in that space. A hyperplane separates a space into two sides.
In general, a hyperplane in n-dimensional space be written as

$$\theta_0 + \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n = 0.$$

For example, a hyperplane in two dimensions, which is a line, can be expressed as
$Ax_1 + Bx_2 + C = 0$.
Using this representation of a plane, we can define a plane given an n-dimensional vector $\theta = [\theta_1, \theta_2+, \ldots, \theta_n]$ and offset θ_0. This vector and offset combination would define the plane $\theta_0 + \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n = 0$. One feature of this representation is that the vector θ is normal to the plane.

1. Number of Representations. Given a d-dimensional vector θ and a scalar offset θ_0 which describe a hyperplane $P : \theta \cdot x + \theta_0 = 0$. How many alternative descriptions θ' and θ'_0 are there for this plane P?

2. To check if a vector x is orthogonal to a plane P characterized by θ and θ_0, what do we check?

3. Perpendicular Distance to Plane. Given a point x in n-dimensional space and a hyperplane described by θ and θ_0, find the signed distance between the hyperplane and x. This is equal to the perpendicular distance between the hyperplane and x, and is positive when x is on the same side of the plane as θ points and negative when x is on the opposite side.

4. Orthogonal Projection onto Plane. Find an expression for the orthogonal projection of a point v onto a plane P that is characterized by θ and θ_0. Write your answer in terms of v, θ, and θ_0.

5: 1D Optimization

1. Let $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 10$ defined on the interval $[-4, 4]$. Let x_1 and x_2 be the critical points of f, and let's impose that $x_1 < x_2$. What are x_1 and x_2? What is $f''(x_1)$ and $f''(x_2)$?
2. Using the function and critical points x_1 and x_2 from the previous part, according to the second derivative test, what can you say about x_1?
3. What can you say about x_2?
4. At what value of x is the (global) minimum value of $f(x)$ attained on the interval $[-4, 4]$?
5. At what value of x is the (globbal) maximum value of $f(x)$ attained on the interval $[-4, 4]$?
6. Strict Concavity. Which of the following functions are strictly concave? (Choose all that apply). (Recall that a twice-differential function $f : I \to \mathbb{R}$, where I is a subset of $\mathbb{R}$, is strictly concave if $f''(x) < 0$ for all $x \in I$.)
   - $f_1(x) = x$ on $\mathbb{R}$
   - $f_2(x) = -e^{-x}$ on $\mathbb{R}$
   - $f_3(x) = x^{0.99}$ on the interval $(0, \infty)$
   - $f_4(x) = x^2$ on $\mathbb{R}$