

# Methods of Mathematical Physics

## —Lecture 2 Functions of a Complex Variable—

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## 1 Complex Integration

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# Introduction

In the theory of real variables, the integration is considered from two perspectives: the indefinite integration as an operation inverse to that of differentiation and the definite integration as the limit of a sum. The concept of the indefinite integral as the process of inverse differentiation in case a function of a real variable is extended to a function of a complex variable if the complex function  $f(z)$  is analytic. It means that, if  $f(z)$  is an analytic function of a complex variable  $z$  and

$$\int f(z)dz = F(z),$$

then the differential of  $F(z)$  is equal to  $f(z)$ , i.e.,  $F'(z) = f(z)$ .

However, the concept of the definite integral of a function of a real variable does not extend out, rightly to the domain of complex variables. For example, in the case of real variable, the path of integration of  $\int_a^b f(x)dx$  is always along the real axis from  $x = a$  to  $x = b$ . But, in the case of a complex function  $f(z)$ , the path of the definite integral

$$\int_a^b f(z)dz,$$

may be along any curve joining the points  $z = a$  and  $z = b$  and so its value depends upon the path (curve) of integration.

# Some Definitions

- Let  $[a, b]$  be a closed interval where  $a$  and  $b$  are real numbers. Subdivide the interval  $[a, b]$  into  $n$  sub-intervals:

$$[t_0, t_1], [t_1, t_2], [t_2, t_3], \dots, [t_{n-1}, t_n]$$

by inserting  $n - 1$  intermediate points  $t_1, t_2, \dots, t_{n-1}$  satisfying the inequalities:

$$a = t_0 < t_1 < t_2 < \dots < t_{n-1} < t_n = b.$$

Then the set

$$P = \{t_0, t_1, t_2, \dots, t_n\}$$

is called a partition of the interval  $[a, b]$  and the greatest of the numbers

$$t_1 - t_0, t_2 - t_1, \dots, t_n - t_{n-1}$$

is called the norm of the partition  $P$ , which is denoted by  $|P|$ .

- Suppose that a point  $z$  lies on an arc  $L$  is defined by

$$z = z(t) = x(t) + iy(t),$$

where  $t$  runs through the interval  $a \leq t \leq b$  and  $x(t), y(t)$  are continuous functions of  $t$ . Then the arc  $L$  is said to be a continuous arc.

# Some Definitions

- Arc  $L$  is said to be continuously differentiable or simply differentiable if  $z'(t)$  exists and is continuous. If, in addition to the existence of  $z'(t)$ , we also have  $z'(t) \neq 0$ , then we say that  $L$  is a regular arc (or a smooth arc). Thus a regular arc is characterized by the property that it has, at every point, a tangent whose direction is determined by  $\arg z'(t)$ . In fact, as  $t$  increases from  $a$  to  $b$ ,  $z$  continuously traces out the arc  $L$  and, at the same time,  $\arg z'(t)$  varies continuously since  $z'(t)$  changes continuously without vanishing.
- An arc  $L$  is said to be simple or a Jordan arc if  $z(t_1) = z(t_2)$  only when  $t_1 = t_2$ . If  $z(a) = z(b)$ , then the arc  $L$  is said to be a closed curve. If  $L$  is the arc defined by  $z = z(t) (a \leq t \leq b)$ , then the arc defined by

$$z = z(-t) \quad (-b \leq t \leq -a)$$

is said to be the opposite arc of  $L$  and is defined by  $-L$ .

# Complex Integrals

Let  $L$  be a Jordan arc defined by

$$z = z(t) = x(t) + iy(t) \quad (a \leq t \leq b)$$

and let  $f(z)$  be a function of a complex variable  $z$  which has a definite value at each point of a rectifiable arc  $L$ . Consider an arbitrary partition

$$P = \{a = t_0, t_1, t_2, \dots, t_{n-1}, t_n = b\}$$

of  $[a, b]$ . We divide the arc  $L$  into small arcs by means of the points  $z_0, z_1, z_2, \dots, z_{n-1}, z_n$ , which correspond to the values

$$a = t_0 < t_1 < t_2 < \dots < t_{n-1} < t_n = b$$

of the parameter  $t$ , and form the sum

$$\sum = \sum_{k=1}^n f(\zeta_k) (z_k - z_{k-1}) = \sum_{k=1}^n f(\zeta_k) \Delta z_k,$$

where  $z_k = z(t_k)$ ,  $\zeta_k = z(\alpha_k)$  and  $t_{k-1} \leq \alpha_k \leq t_k$ , is a point of  $L$  between  $z_{k-1}$  and  $z_k$ ,  $\Delta z_k = z_k - z_{k-1}$ .

# Complex Integrals

If this sum  $\sum$  tends to a unique limit  $I$  as  $n \rightarrow \infty$  and the norm of  $P$ , i.e.,  $|P|$  tends to zero, then we say that  $f(z)$  is integrable from  $a$  to  $b$  along the arc  $L$  and we write

$$I = \int_L f(z) dz.$$

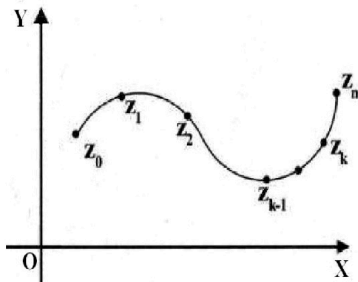
We also call  $\int_L f(z) dz$  the complex line integral or, simply, the line integral of  $f(z)$  along the arc  $L$  or the definite integral of  $f(z)$  from  $a$  to  $b$  along  $L$ . The sense of direction of integration is from  $a$  to  $b$ , since the points  $x(t) + iy(t)$ , for increasing values of  $t$ , are oriented in the very sense on the arc  $L$ . In fact, the value of  $t$  depends not only on the end points of arc  $L$ , but also on the actual form. Thus we have

$$\int_L f(z) dz = \lim_{|P| \rightarrow 0, n \rightarrow \infty} \sum_{k=1}^n f(\zeta_k) (z_k - z_{k-1}).$$

Note that an integral of this type exists under pretty general conditions. However, we may do without the assumption that  $x'(t)$  and  $y'(t)$  exist at each point of  $L$ . In fact, the continuity of  $f(z)$  on  $L$  is a sufficient condition.



# Complex Integrals



**Figure:** The complex line integral of  $f(z)$  along the arc  $L$ .

# Evaluation of Integrals by the Direct Definition

## Example

- 1  $\int_L dz$ ;
- 2  $\int_L |dz|$ ;
- 3  $\int_L z dz$ , where  $L$  is any rectifiable<sup>a</sup> joining the points  $z = \alpha$  and  $z = \beta$ .

<sup>a</sup>The length of the polygonal arc, obtained by joining successively  $z_0$  and  $z_1$ ,  $z_1$  and  $z_2$ ,  $\dots$ ,  $z_{n-1}$  and  $z_n$ , by straight line segments is given by

$$\Sigma = \sum_{k=1}^n L_k = \sum_{k=1}^n |z_k - z_{k-1}|$$

where  $L_k = \text{Arc } z_{k-1}z_k$  ( $k = 1, 2, \dots, n$ ) and  $z_l = z(t_l)$  ( $l = 0, 1, \dots, n$ ). If this sum  $\Sigma$  tends to a unique limit  $l$ , say, as  $n \rightarrow \infty$  and the norm of the partition  $P$  tends to zero, then we say that the arc  $L$  defined by  $z = x(t) + iy(t)$  ( $a \leq t \leq b$ ) is rectifiable and its length is  $l$ . Rectifiable Jordan arcs with continuously turning tangents are called regular arcs.

# Properties of Complex Integrals

Some elementary properties of complex integrals are as follows:

- ①  $\int_L [f(z) + g(z)] dz = \int_L f(z) dz + \int_L g(z) dz.$
- ②  $\int_L f(z) dz = - \int_{-L} f(z) dz$ , where by  $-L$  we mean the curve  $L$  traversed in the opposite direction.
- ③  $\int_{L_1+L_2} f(z) dz = \int_{L_1} f(z) dz + \int_{L_2} f(z) dz$ , where the terminal point of  $L_1$  coincides with the initial point of  $L_2$ .
- ④  $\int_L c f(z) dz = c \int_L f(z) dz$ , where  $c$  is any complex constant.
- ⑤ We have

$$\begin{aligned} & \int_L [c_1 f_1(z) + c_2 f_2(z) + \cdots + c_n f_n(z)] dz \\ &= c_1 \int_L f_1(z) dz + c_2 \int_L f_2(z) dz + \cdots + c_n \int_L f_n(z) dz. \end{aligned}$$

- ⑥  $|\int_L f(z) dz| \leq \int_L |f(z)| |dz|.$

# Integrations along Regular Arcs

## Theorem

Let  $f(z)$  be continuous on the regular arc  $L$  which is defined by

$$z = z(t) = x(t) + iy(t) \quad (a \leq t \leq b).$$

Then  $f(z)$  is integrable along  $L$  and

$$\int_L f(z) dz = \int_a^b F(t) \{ \dot{x}(t) + i\dot{y}(t) \} dt,$$

where  $F(t)$  denotes the value of  $f(z)$  at the point  $z = x(t) + iy(t)$  of  $L$  corresponding to the parameter value  $t$ .

## Proof.

Consider the sum

$$\sum = \sum_{k=1}^n f(\zeta_k) (z_k - z_{k-1})$$

where  $\zeta_k$  is a point of  $L$  between  $z_{k-1}$  and  $z_k$  and  $\zeta_k = z(\tau_k)$ ,  $t_{k-1} \leq \tau_k \leq t_k$ . Write

$$F(t) = \phi(t) + i\psi(t),$$

# Integrations along Regular Arcs

where  $\phi(t)$  and  $\psi(t)$  are real-valued functions of parameter  $t$ . Then we have

$$\begin{aligned}\sum &= \sum_{k=1}^n \phi(\tau_k)(x_k - x_{k-1}) + i \sum_{k=1}^n \psi(\tau_k)(x_k - x_{k-1}) + i \sum_{k=1}^n \phi(\tau_k)(y_k - y_{k-1}) - \sum_{k=1}^n \psi(\tau_k)(y_k - y_{k-1}) \\ &= \Sigma_1 + i\Sigma_2 + i\Sigma_3 - \Sigma_4, \text{ say.}\end{aligned}$$

By the mean value theorem of differential calculus, the first sum is

$$\Sigma_1 = \sum_{k=1}^n \phi(\tau_k) \dot{x}(\tau'_k)(t_k - t_{k-1}),$$

where  $t_{k-1} \leq \tau'_k \leq t_k$ . Let  $\Sigma'_1 = \sum_{k=1}^n \phi(t_k) \dot{x}(t_k)(t_k - t_{k-1})$ , by making the norm of  $P$ , i.e.,  $|P|$  sufficiently small, we show that

$$|\Sigma_1 - \Sigma'_1| < M(\epsilon)\epsilon.$$

Now, by the hypothesis,  $\phi(t)$  and  $\dot{x}(t)$  are continuous and, since every continuous function is bounded, there exists a positive number  $M$  such that the inequalities:

$$|\phi(t)| \leq M, \quad |\dot{x}(t)| \leq M$$

hold for  $a \leq t \leq b$ .

# Integrations along Regular Arcs

Again, since a continuous function is necessarily uniformly continuous, we can preassign an arbitrary positive number  $\epsilon$  and then we can choose a positive number  $\delta = \delta(\epsilon)$  such that

$$|\phi(t) - \phi(t')| < \epsilon, \quad |\dot{\phi}(t) - \dot{\phi}(t')| < \epsilon,$$

where  $|t - t'| < \delta$ . Hence, if  $|P| < \delta$ , then we have

$$\begin{aligned} |\phi(\tau_k) \dot{\phi}(\tau'_k) - \phi(t_k) \dot{\phi}(t_k)| &= |\phi(\tau_k) \{\dot{\phi}(\tau'_k) - \dot{\phi}(t_k)\} + \dot{\phi}(t_k) \{\phi(\tau_k) - \phi(t_k)\}| \\ &\leq |\phi(\tau_k)| |\dot{\phi}(\tau'_k) - \dot{\phi}(t_k)| + |\dot{\phi}(t_k)| |\phi(\tau_k) - \phi(t_k)| < 2M\epsilon \end{aligned}$$

and so it follows that  $|\Sigma_1 - \Sigma'_1| < 2M\epsilon(b-a)$ .

By the definition of the integral of functions of a real variable,  $\Sigma'_1$  tends to the limit

$$\int_a^b \phi(t) \dot{\phi}(t) dt = \lim_{n \rightarrow \infty, |P| \rightarrow 0} \Sigma'_1.$$

The remaining  $\Sigma'$ 's tend to corresponding limits in the same manner. Then  $\Sigma$  tends to the limit

$$\int_a^b \{\phi(t) \dot{\psi}(t) - \psi(t) \dot{\phi}(t)\} dt + i \int_a^b \{\psi(t) \dot{\phi}(t) + \phi(t) \dot{\psi}(t)\} dt = \int_a^b F(t) \{\dot{\phi}(t) + i \dot{\psi}(t)\} dt.$$

# Examples

## Example

- 1 Evaluate  $\int_C \frac{dz}{z}$ , where  $C$  is the circle with center at the origin and radius  $r$ .
- 2 Evaluate  $\int_C \frac{dz}{z-\alpha}$ , where  $C$  represents a circle  $|z-\alpha| = r$ .
- 3 Evaluate  $\int_C f(z)dz$ , if  $f(z) \equiv 1$ , and  $C$  is any smooth curve.

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- 3 Evaluate  $\int_C f(z)dz$ , if  $f(z) \equiv 1$ , and  $C$  is any smooth curve.

Solutions:

1

$$\int_C \frac{dz}{z} = \int_0^{2\pi} \frac{1}{re^{it}} rie^{it} dt = i \int_0^{2\pi} dt = 2\pi i.$$

2

$$\int_C \frac{dz}{z-\alpha} = \int_0^{2\pi} \frac{1}{re^{it}} rie^{it} dt = i \int_0^{2\pi} dt = 2\pi i.$$

3

$$\int_C f(z)dz = \int_a^b \dot{z}(t)dt = z(b) - z(a).$$



# An important integral

## Theorem

Let  $C$  be a circular path with center  $z_0$  and radius  $r > 0$  traversed in the anticlockwise direction. Then

$$\int_C (z - z_0)^n dz = \begin{cases} 2\pi i & \text{if } n = -1, \\ 0 & \text{if } n \neq -1. \end{cases}$$

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## Proof.

We have  $C(t) = z_0 + r \exp(it)$ ,  $t \in [0, 2\pi]$ , and so  $C'(t) = i r \exp(it)$ ,  $t \in [0, 2\pi]$ .

1 : When  $n = -1$ , we have

$$\int_C (z - z_0)^n dz = \int_C (z - z_0)^{-1} dz = \int_0^{2\pi} \frac{1}{r \exp(it)} \cdot i r \exp(it) dt = \int_0^{2\pi} i dt = 2\pi i.$$

2 : When  $n \neq -1$ , we have

$$\begin{aligned} \int_C (z - z_0)^n dz &= \int_0^{2\pi} r^n \exp(nit) \cdot i r \exp(it) dt = \int_0^{2\pi} i r^{n+1} \exp(i(n+1)t) dt \\ &= -r^{n+1} \int_0^{2\pi} \sin((n+1)t) dt + i r^{n+1} \int_0^{2\pi} \cos((n+1)t) dt = 0 + 0 = 0. \end{aligned}$$

# Integrations along Regular Arcs

## Theorem

Let  $C$  be the given curve. Then

$$\int_{-C} f(z) dz = - \int_C f(z) dz.$$

## Proof.

We have

$$\int_{-C} f(z) dz = - \int_a^b f(z(b+a-t)) \dot{z}(b+a-t) dt$$

Now expanding the integral into real and imaginary parts and applying the change of variable theorem to each real integral, we obtain

$$\int_{-C} f(z) dz = \int_b^a f(z(t)) \dot{z}(t) dt = - \int_C f(z) dz.$$



# Complex Integrals as Sum of Two Real Line Integrals

## Example

- 1 Prove that the value of the integral of  $\frac{1}{z}$  along a semi-circular arc  $|z| = a$  from  $-a$  to  $+a$  is  $-\pi i$  or  $\pi i$  if the arc lies above or below the real axis.
- 2 Find the value of the integral  $\int_0^{1+i} (x - y + ix^2) dz$ 
  - along the straight line from  $z = 0$  as  $z = 1 + i$ ;  $\left(\frac{-1+i}{3}\right)$
  - along the real axis from  $z = 0$  to  $z = 1$  and then along a line parallel to the imaginary axis from  $z = 1$  to  $z = 1 + i$ .  $\left(\frac{-1}{2} + \frac{5i}{6}\right)$
- 3 Evaluate the integral  $\int_0^{1+i} z^2 dz$ .
- 4 Evaluate the integral  $\int_{-2+i}^{5+3i} z^3 dz$ .

# The Absolute Value of Complex Integrals

## Theorem

Let  $f(z)$  be continuous on a contour  $L$  of length  $l$  and let  $|f(z)| \leq M$  for every point  $z$  on  $L$ . Then we have

$$\left| \int_L f(z) dz \right| \leq Ml.$$

# The Absolute Value of Complex Integrals

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$$\left| \int_L f(z) dz \right| \leq Ml.$$

## Proof.

Without loss of generality, we may assume that  $L$  is a regular arc. Now, we have

$$\Sigma = \sum_{k=1}^n f(z_k) (z_k - z_{k-1}).$$

Since the modulus of the sum is less than or equal to the sum of the moduli, we have

$$|\Sigma| = \left| \sum_{k=1}^n f(z_k) (z_k - z_{k-1}) \right| \leq \sum_{k=1}^n |f(z_k) (z_k - z_{k-1})| = \sum_{k=1}^n |f(z_k)| |z_k - z_{k-1}| \leq M \sum_{k=1}^n |z_k - z_{k-1}|.$$

Therefore, we have

$$\lim_{n \rightarrow \infty} \left| \sum_{k=1}^n f(z_k) (z_k - z_{k-1}) \right| \leq M \lim_{n \rightarrow \infty} \sum_{k=1}^n |z_k - z_{k-1}| \Rightarrow \left| \int_L f(z) dz \right| \leq M \int_L |dz| = Ml.$$