Methods of Mathematical Physics

—Lecture 2 Functions of a Complex Variable—

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Contents

Introduction

The concepts in ordinary calculus in the setting of $\mathbb{R}$, like convergence of sequences, or continuity and differentiability of functions, all rely on the notion of closeness of points in $\mathbb{R}$.

In order to do calculus with complex numbers, we need a notion of distance $d(z_1, z_2)$ between for pairs of complex numbers (z_1, z_2), and the first order of business is to explain what this notion is.

Metric on $\mathbb{C}$

A metric space is a pair (X, d), where X is a set and $d : X \times X \to \mathbb{R}$ is a function called a distance function or metric that satisfies the following conditions: for $x, y, z \in X$,

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ (symmetry);
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

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Example

Let $X = \mathbb{C}$, $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2) \in X$ and define
$d(z_1, z_2) = |z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$,
or $d(z_1, z_2) = |x_1 - x_2| + |y_1 - y_2|$,
or $d(z_1, z_2) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$.
Then $(\mathbb{C}, d)$ is a metric space.

Open discs, open sets, closed sets, compact sets, connected sets

- An open ball/disc $D(z_0, r)$ with center z_0 and radius $r > 0$ is defined by $D(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}$.
- A subset U of $\mathbb{C}$ is called open if for every $z \in U$, there exists an $r_z > 0$ such that $D(z, r_z) \subset U$. (z is an interior point)
- A set S is said to be closed when every limit point of S belongs to S. (A set $F \subset X$ is said to be closed if its complement, $X - F$, is open.)
- A subset S of $\mathbb{C}$ is called bounded if there exists a $M > 0$ such that for all $z \in S$, $|z| \leq M$. Thus S is contained in a big enough disc in the complex plane.
- A subset $K \subset \mathbb{C}$ is called compact if it is both closed and bounded.
- An open set is said to be connected if it cannot be represented as the union of two nonempty disjoint open sets. A nonempty open set in the complex plane is connected if and only if any two of its points can be joined by a polygonal arc[1] lying entirely in the set.

[1] By a polygonal arc we mean a continuous chain of a finite number of line segments.

Open and Closed Domain (or Region), Curves

- A nonempty **open connected** subset of the complex plane is called an open domain or an open region or, simply, a region.
- A curve or a continuous arc Γ in the complex plane is the set of points z in the complex plane determined by the equation

$$z = z(t) = x(t) + iy(t)$$

where $x(t)$ and $y(t)$ are real continuous functions of a real variable t defined on a real interval $\alpha \leq t \leq \beta$ where $\alpha \leq \beta$. We call $z(\alpha)$ and $z(\beta)$ the end points of Γ, $z(\alpha)$ being the initial point and $z(\beta)$ the terminal point of Γ. If $z(\alpha) = z(\beta), \Gamma$ is called a closed curve.
If the equation $z_0 = x(t) + iy(t)$ is satisfied by more than one value of t in the given range $I : \alpha \leq t \leq \beta$, then z_0 is said to be a multiple point. In particular, the multiple point is called a double point when the above equation is satisfied by two values of t in the given range I.

Jordan Arc and Simple Closed Jordan Curve

- A curve Γ is called a Jordan arc or a simple curve if it has no multiple points, i.e., if there exists some parametric representation

$$z = z(t) = x(t) + iy(t), \quad \alpha \leq t \leq \beta,$$

such that, if $t_1 \neq t_2$, then $z(t_1) \neq z(t_2)$, i.e., $z(t)$ is one-to-one. The simplest example of a Jordan arc is a straight line segment.

- If, in a Jordan arc, the initial and terminal points coincide, that is, if there is a double point corresponding to the end points (α and β) of the interval $I : \alpha \leq t \leq \beta$ and there is no other multiple point on it, then it is called a simple closed Jordan curve or simply a closed Jordan curve.

Convergence and continuity

A sequence $(z_n)_{n\in\mathbb{N}}$ is said to be convergent with limit L if for every $\epsilon > 0$, there exists an index $N \subset \mathbb{N}$ such that for every $n > N$, there holds that $|z_n - L| < \epsilon$. It follows from the triangle inequality that for a convergent sequence the limit is unique, and we write

$$\lim_{n\to\infty} z_n = L.$$

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Let S be a subset of $\mathbb{C}, z_0 \in S$ and $f : S \to \mathbb{C}$. Then f is said to be continuous at z_0 if for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $z \in S$ satisfies $|z - z_0| < \delta$, there holds that $|f(z) - f(z_0)| < \epsilon$.

f is said to be continuous if for every $z \in S$, f is continuous at z.

Definitions

Let D be an arbitrary non-empty point set of the complex plane. If z is allowed to denote any point of D, z is called a complex variable and D is called the domain of definition of z or simply the domain.

A complex variable w is said to be a function of the complex variable z if, to every value of z in a certain domain D, there corresponds **one or more values** of w. Thus, if w is a function of z, it is written as $w = f(z)$. We also say that f defines a mapping of D into the w-plane. The totality of values $f(z)$ corresponding to all z in D constitutes another set R of complex numbers, known as the range of the function f.

Since $z = x + iy$, $f(z)$ will be of the form $u + iv$, where u and v are functions of two real variables x and y. We may then write

$$w = f(z) = u(x, y) + iv(x, y).$$

Single-valued and multiple-valued Functions

A function $f(z)$ of the complex variable z with domain of definition D and range R is said to be single-valued or one-valued if w takes only one value in R for each value of z in D.

If there correspond two or more values of $f(z)$ in R for some or all values of z in D, then $f(z)$ is called a multiple-valued or many-valued function of z.

Limits of Functions

Let $f(z)$ be a function of z defined in some neighborhood of a point z_0. The function $f(z)$ is said to have the limit ℓ as z tends to z_0 if, to each positive arbitrary number ϵ, there exists a positive number δ depending upon ϵ with the property that

$$|f(z) - \ell| < \epsilon$$

for all z such that $0 < |z - z_0| < \delta$ and $z \neq z_0$. In other words, there exists a deleted neighborhood of the point $z = z_0$ in which $|f(z) - \ell|$ can be made as small as we please. Symbolically, we write $\lim_{z \to z_0} f(z) = \ell$.

Continuity

Let G be an open set in $\mathbb{C}$ and let $f\colon G \to \mathbb{C}$. Then f is said to be continuous at a point z_0 in G if, given any positive number ϵ, we can find a member $\delta > 0$ depending in general on ϵ and z_0 such that

$$|f(z) - f(z_0)| < \epsilon$$

for all $z \in G$ in the neighborhood $|z - z_0| < \delta$ of z_0.

It follows from the above definition and the definition of limit that f is continuous at $z = z_0$ if

$$\lim_{z \to z_0} f(z) = f(z_0)\,.$$

If a function is continuous at every point of G, it is said to be continuous in G.

Continuity in terms of Real & Imaginary Parts of $f(z)$

If $f(z) = u(z, y) + iv(x, y)$, then it can be easily shown that f is a continuous function of z if and only if $u(x, y)$ and $v(x, y)$ are separately continuous functions of x and y.

Let f and g be continuous functions from X into $\mathbb{C}$ and let $a, b \in \mathbb{C}$. Then $af + bg$ and fg are both continuous. Also, f/g is continuous provided $g(x) \neq 0$ for every x in X.

A continuous function of a continuous function is a continuous function; that is, if $f : X \to Y$ and $g : Y \to Z$ are continuous functions, then $g \circ f$ where $(g \circ f)(x) = g(f(x))$ is a continuous function from X into Z.

Complex differentiability

In this section we will learn three main things:

1. The definition of complex differentiability.
2. The Cauchy-Riemann equations.
3. The geometric meaning of the complex derivative $f'(z_0)$.

The central result in this section is the necessity and (under mild conditions) sufficiency of the Cauchy-Riemann equations for the complex differentiability of a function in an open set.

Differentiability

If G is an open set in $\mathbb{C}$ and $f\colon G \to \mathbb{C}$ is a function, then f is said to be differentiable at a point z_0 in G if, for any positive number ϵ, we can find a positive number δ depending on ϵ and possibly on z_0 such that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon$$

for all $z \in G$ in the neighborhood of z_0 defined by $|z - z_0| < \delta$.

If f is differentiable at each point of G, then we say that f is differentiable on G.

An example

Example

If $f(z) = \frac{x^3 y(y-ix)}{x^6+y^2}$ $(z \neq 0), f(0) = 0$, prove that $\frac{f(z)-f(0)}{z-0} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in any manner.

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Proof.

Let $z \to 0$ along $y = mx$ (radius vector). Then we have

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{z\to 0} \frac{x^3 y(y-ix)}{(x^6+y^2)(x+iy)} = \lim_{x\to 0} \frac{x^3 mx(mx-ix)}{(x^6+m^2x^2))(x+imx)}$$
$$= \lim_{x\to 0} \frac{m(m-i)\cdot x^2}{(m^2+x^4)(1+im)} = 0.$$

Now, let $z \to 0$ along the path $y = x^3$. Then, for $x \neq 0$

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{x\to 0} \frac{x^6\left(x^3-ix\right)}{(x^6+x^6)(x+ix^3)} = \lim_{x\to 0} \frac{\left(x^2-i\right)}{2\left(1+ix^2\right)} = -\frac{i}{2}.$$

Differentiability

Theorem

If $f : G \to C$ is differentiable at a point z_0 in G, then f is continuous at z_0.

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Proof.

Consider the following identity:

$$\begin{aligned}\lim_{z\to z_0}|f(z)-f(z_0)| &= \left[\lim_{z\to z_0}\frac{|f(z)-f(z_0)|}{|z-z_0|}\right]\cdot\left[\lim_{z\to z_0}|z-z_0|\right]\\ &= f'(z_0)\cdot 0\\ &= 0\end{aligned}$$

that is, $\lim_{z\to z_0} f(z) = f(z_0)$. Thus it follows that $f(z)$ is continuous at z_0. This completes the proof. $\square$

Differentiability

The converse of the above theorem is not necessarily true.

For example, take the function $|z|^2$ which is continuous in all finite regions of the z-plane. It has, however, a derivative only at the origin, since, when $z \neq z_0$ and $z_0 \neq 0$, we have, for $f(z) = |z|^2$,

$$
\begin{aligned}
\frac{f(z) - f(z_0)}{z - z_0} &= \frac{|z|^2 - |z_0|^2}{z - z_0} = \frac{z\bar{z} - z_0\bar{z}_0}{z - z_0} \\
&= \frac{z\bar{z} - z_0\bar{z} + z_0\bar{z} - z_0\bar{z}_0}{z - z_0} = \bar{z} + z_0\frac{\bar{z} - \bar{z}_0}{z - z_0} \\
&= \bar{z} + z_0\frac{\rho(\cos\theta - i\sin\theta)}{\rho(\cos\theta + i\sin\theta)} = \bar{z} + z_0(\cos 2\theta - i\sin 2\theta),
\end{aligned}
$$

where $\rho = |z - z_0|$ and $\theta = \arg(z - z_0)$. Clearly, $\lim\limits_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ does not exist since the limit depends upon $\arg(z - z_0)$. However, when $z_0 = 0$, the expression reduces to $\bar{z}$ which tends to 0 with z tends to 0 .

Definition

Definition

- The function f is analytic at z_0 if $f(z)$ is differentiable in some neighborhood of z_0 (open region including z_0);
- The function f is analytic in a region if it is analytic at all points in that region;
- The function f is holomorphic if it is analytic. The terms are synonyms.
- An analytic function is entire if its region of analyticity includes all points in $\mathbb{C}$, the finite complex plane, excluding infinity.

If we describe a function as analytic, without specifying any point or region, that means there is some region within which it is analytic.

Rules of Differentiation

Theorem

If f and g are analytic on G, where $g(z) \neq 0$, then

1. $(f \pm g)'(z) = f'(z) \pm g'(z)$.
2. $(cf)'(z) = cf'(z)$, *where c is a complex constant.*
3. $(f \cdot g)'(z) = f(z) \cdot g'(z) + g(z) \cdot f'(z)$.
4. $\left(\frac{f}{g}\right)'(z) = \frac{g(z)f'(z) - f(z)g'(z)}{[g(z)]^2}$ *where $g(z) \neq 0$.*

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4. $\left(\frac{f}{g}\right)'(z) = \frac{g(z)f'(z) - f(z)g'(z)}{[g(z)]^2}$ *where $g(z) \neq 0$.*

Theorem (Chain Rule)

If f and g are analytic on G and Ω, respectively, and suppose $f(G) \subset \Omega$, then $g \circ f$ is analytic on G and for all z in G,

$$(g \circ f)'(z) = g'(f(z))f'(z).$$

The chain rule shows that an analytic function of an analytic function is analytic.

Examples

Example

Show that the function $f(z) = z^n$ where n is a positive integer is an analytic function. Furthermore, polynomials and rational Functions are analytic functions.

Cauchy-Riemann Equations

Theorem

A necessary condition for a function $f(z) = u(x, y) + iv(x, y)$ to be analytic at any point $z = x + iy$ of the domain D of f is that the four partial derivatives u_x, u_y, u_y and v_x should exist and satisfy the equation

$$u_x = v_y, \quad u_y = -v_x. \tag{1}$$

The equations given in (1) are known as the Cauchy–Riemann equations.

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Example

Show that the function $f(z) = u + iv$, where

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}\,(z \neq 0), \quad f(0) = 0,$$

is continuous and the Cauchy-Riemann equations are satisfied at the origin, but $f'(0)$ does not exist.

Sufficient conditions

Theorem

The one-valued function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D if the four partial derivatives u_x, v_x, u_y and v_y exist, are continuous and satisfy the Cauchy-Riemann equations at each point D.

$$f'(z) = u_x + iv_x = v_y - iu_y.$$

Conjugate Functions

Definition

If a function $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ is analytic in a domain D, then the functions u and v of two variables x and y are called conjugate functions.

Harmonic Functions

Definition

A real-valued function $u(x, y)$ is said to be harmonic in a domain D if, for all $x, y \in D$, all second-order partial derivatives exist and are continuous and satisfies Laplace's equation, that is,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Theorem

If the harmonic functions u and v satisfy the Cauchy-Riemann equations, then $u + iv$ is an analytic function.

Examples

Example

Show that the functions $u(x, y) = e^x \cos y$ is harmonic. Determine its harmonic conjugate $v(x, y)$ and the analytic function $f(z) = u + iv$.

Example

If $u = x^2 - y^2$ and $v = -\frac{y}{x^2+y^2}$, then show that both u and v satisfy Laplace's equation, but $u + iv$ is not an analytic function of z.

Example

Let f be analytic on an open set U and let $|f| =$ constant. Show that $f =$ constant.

Polar Form of the Cauchy-Riemann Equations

Theorem

If $f(z) = u + iv$ is an analytic function and $z = re^{i\theta}$, where u, v, r and θ are all real, show that the Cauchy-Riemann equations are as follows:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

Method of Constructing Analytic Functions

Power Series

Definition

A power series is an infinite series of the type

$$\sum_{n=0}^{\infty} a_n z^n \text{ or } \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where variable z and the constants a_0, z_0 are, in general, complex numbers and a_n in independent of z.

Power series

Theorem

The power series $\sum a_n z^n$ either

1. *converges for all values of z;*
2. *converges only for $z = 0$;*
3. *for z in some region in the complex plane.*

Theorem (Abel's Theorem)

If the power series $\sum_{n=0}^{\infty} a_n z^n$ converges for a particular value z_0 of z, then it converges absolutely for all values of z for which $|z| < |z_0|$.

Theorem (Cauchy-Hadamard's Theorem)

For all power series $\sum_{n=0}^{\infty} a_n z^n$, there exists a number $R, 0 \leq R \leq \infty$, called the radius of convergence with the following properties:

1. *The series converges absolutely for all $|z| < R$.*
2. *If $0 \leq \rho < R$, then the series converges uniformly for $|z| \leq \rho$.*
3. *The series diverges if $|z| > R$.*

Theorem

The power series $\sum_{n=0}^{\infty} na_n z^{n-1}$*, obtained by differentiating the power series* $\sum_{n=0}^{\infty} a_n z^n$*, has the same radius of convergence as the original series.*

Definition

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then $f(z)$ is called the sum function of the power series $\sum_{n=0}^{\infty} a_n z^n$.

Theorem

The function $f(z)$ *of the series* $\sum_{n=0}^{\infty} a_n z^n$ *represents an analytic function inside its circle of convergence.*

Exponential Functions

Definition

The exponential function $f(z)$ of a complex variable z is defined as the solution of the differential equation: $f'(z) = f(z)$, with initial value $f(0) = 1$.

Let us solve by setting

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n + \cdots .$$

Then we have

$$f'(z) = a_1 + 2a_2 z + \cdots + n a_n z^{n-1} + \cdots .$$

Hence, if $f'(z) = f(z)$ satisfied, then we must have

$$a_0 = a_1, \quad a_1 = 2a_2, \quad \cdots .$$

In general, $a_{n-1} = n a_n$. Since $f(0) = 1$, we have $a_0 = 1$. If follows from the above relation that

$$a_1 = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = \frac{a_2}{3} = \frac{1}{2 \cdot 3} = \frac{1}{3!}, \quad a_n = \frac{1}{n!}.$$

We denote the solution by $\exp z$ or e^z. Thus we have

$$\exp z = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!}.$$

Addition Theorems for Exponential Functions

Theorem

For all z_1 and z_2,

$$\exp(z_1 + z_2) = \exp z_1 \exp z_2.$$

Proof.

We have

$$\exp z_1 = 1 + \sum_{n=1}^{\infty} \frac{z_1^n}{n!}, \quad \exp z_2 = 1 + \sum_{n=1}^{\infty} \frac{z_2^n}{n!}$$

Since the above two series are absolutely convergent, it follows from Cauchy's Theorem on multiplication of absolutely convergent series that

$$\begin{aligned}
\exp z_1 \cdot \exp z_2 &= \left(1 + \sum_{n=1}^{\infty} \frac{z_1^n}{n!}\right) \cdot \left(1 + \sum_{n=1}^{\infty} \frac{z_2^n}{n!}\right) \\
&= 1 + \frac{z_1 + z_2}{1!} + \frac{z_1^2 + 2z_1 z_2 + z_2^2}{2!} + \cdots \\
&= 1 + \frac{(z_1 + z_2)}{1!} + \frac{(z_1 + z_2)^2}{2!} + \cdots \\
&= \exp(z_1 + z_2).
\end{aligned}$$

Addition Theorems for Exponential Functions

By induction, we have

$$\exp z_1 \cdot \exp z_2 \cdots \exp z_n = \exp\left(z_1 + z_2 + \cdots + z_n\right).$$

In particular, we have

$$\exp z \cdot \exp(-z) = \exp(0) = 1$$

An interesting consequence of this theorem is that $\exp z$ never vanishes. For, if $\exp z_1 = 0$, the identity

$$\exp z_1 \cdot \exp\left(-z_1\right) = 1$$

would yield the conclusion that $\exp\left(-z_1\right)$ is not finite. But this is impossible on account of the fact that $\exp z$ is an analytic function in every bounded domain of the z-plane.

A brief summary of Exponential Functions

1. $\exp 0 = e^0(\cos 0 + i \sin 0) = 1 \cdot (1 + i0) = 1.$
2. For $z_1, z_2 \in \mathbb{C}, \exp(z_1 + z_2) = (\exp z_1)(\exp z_2).$
3. For $z \in \mathbb{C}, \exp z \neq 0,$ and $(\exp z)^{-1} = \exp(-z).$
4. For $z \in \mathbb{C}, \exp(z + 2\pi i) = \exp z.$
5. For $z \in \mathbb{C}, |\exp z| = e^{\mathrm{Re}(z)}.$

Trigonometrical Functions

The functions $\sin z$ and $\cos z$ for complex z are defined as in the case of a real variable by means of the formulae

$$\begin{aligned} \sin z &= \frac{\exp(iz) - \exp(-iz)}{2i}, \\ \cos z &= \frac{\exp(iz) + \exp(-iz)}{2}. \end{aligned} \tag{2}$$

Using the power series for exponential functions on the right-hand side of (2), we can easily see that

$$\begin{aligned} \sin z &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + \frac{(-1)^n \cdot z^{(2n+1)}}{(2n+1)!} + \cdots, \\ \cos z &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots + \frac{(-1)^n \cdot z^{2n}}{(2n)!} + \cdots. \end{aligned} \tag{3}$$

Since each of these power series has an infinite radius of convergence, it follows that $\sin z$ and $\cos z$ are analytic in every bounded domain of the z-plane.

Trigonometrical Functions

The remaining trigonometrical functions are defined, in strict analogy with the case of a real variable, by

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z},$$
$$\sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}.$$

The term-by-term differentiation of the series (3) show that

$$\frac{d}{dz}\sin z = \cos z, \quad \frac{d}{dz}\cos z = -\sin z,$$
$$\sin(-z) = -\sin z, \quad \cos(-z) = \cos z,$$
$$\sin 0 = 0, \quad \cos 0 = 1.$$

These results are in strict parallelism with what we had in the calculus of functions of real variable.

Euler's Equation

From (2), we have

$$\exp(iz) = \cos z + i \sin z,$$

which is known as Euler's equation. Also, we have

$$\begin{aligned} \exp(z) &= \exp(x + iy) \\ &= \exp x \cdot \exp(iy) \\ &= \exp x \cdot (\cos y + i \sin y). \end{aligned} \tag{4}$$

Thus $\exp x$ is the modulus and y the argument of $\exp(x + iy)$.

Addition Theorems for $\sin z$ and $\cos z$

Theorem

For all z_1 and z_2,

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2,$$

and

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2.$$

Proof.

We have

$$\exp\{i(z_1 + z_2)\} = \exp(iz_1)\exp(iz_2). \tag{5}$$

Using Euler's theorem, it follows from (5) that

$$\begin{aligned} \cos(z_1 + z_2) + i\sin(z_1 + z_2) &= (\cos z_1 + i\sin z_1)(\cos z_2 + i\sin z_2) \\ &= (\cos z_1 \cos z_2 - \sin z_1 \sin z_2) + i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2). \end{aligned} \tag{6}$$

Replacing z_1 and z_2 with $-z_1$ and $-z_2$, respectively, it follows from (6) that

$$\begin{aligned} \cos(z_1 + z_2) - i\sin(z_1 + z_2) &= (\cos z_1 - i\sin z_1)(\cos z_2 - i\sin z_2) \\ &= (\cos z_1 \cos z_2 - \sin z_1 \sin z_2) - i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2). \end{aligned} \tag{7}$$

□

Addition Theorems for $\sin z$ and $\cos z$

Adding (6) and (7), we have

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2,$$

and, subtracting (7) from (6), we have

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2.$$

□

By virtue of formulae (5) and (6) and Euler's equation, we obtain the identity

$$\sin^2 z + \cos^2 z = 1.$$

We can also show that all the elementary identities of trigonometry still hold for the trigonometrical functions a complex variable.

Hyperbolic Functions $\sinh z$ and $\cosh z$

The hyperbolic functions of a complex variable are defined in the same way as for real variables. The fundamental formulae are

$$\sinh z = \frac{1}{2}\left(e^z - e^{-z}\right), \tag{8}$$

$$\cosh z = \frac{1}{2}\left(e^z + e^{-z}\right) \tag{9}$$

Using power series for e^x and e^{-x} on the R.H.S. of (8) and (9), we can easily see that

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!},$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}.$$

Evidently, $\sinh z$ and $\cosh z$ are regular in every bounded domain of the z plane. Relation between Hyperbolic and Trigonometric Functions It can be easily verified that

$$\sin iz = i \sinh z, \quad \cos iz = \cosh z,$$

$$\sinh iz = i \sin z, \quad \cosh iz = \cos z.$$

The logarithm of a complex variable w denoted by $\log w$ is defined as the solution of the equation

$$\exp z = w, \tag{10}$$

where

$$z = \log w. \tag{11}$$

Observe that the number 0 has no logarithm since $\exp z$ is never zero. If $z = x + iy$, then, for $w \neq 0$, (10) can be written

$$\exp(x + iy) = w$$

or

$$\exp x \cdot \exp(iy) = w.$$

Since x and y are real, we have

$$\exp x = |w| \tag{12}$$

and

$$\exp iy = \frac{w}{|w|}. \tag{13}$$

Evidently, the equation (12) has a unique solution $x = \log |w|$, the real logarithm of the positive number $|w|$ and the R.H.S. of (13) is a complex number of the magnitude 1.

It follows that it has one and only one solution, say y_0, such that $0 \leq y_0 < 2\pi$. Now, we have

$$\begin{aligned}\exp(iy) &= \cos y + i \sin y \\ &= \cos(2n\pi + y) + i \sin(2n\pi + y)\end{aligned}$$

where $n = 0, 1, 2, \cdots$. Thus every non-zero complex number has infinitely many logarithms which differ from one another by an integral multiple of 2π.

The imaginary part of $\log w$ is called the argument of w. We denote it by $\arg w$ and it is geometrically interpreted as the angle between the positive real axis and the semi-line from 0 through the point w. This implies that $\arg w$ has infinitely many values which differ by an integral multiple of 2π. A value of $\arg w$ satisfying the inequality $0 \leq \arg w < 2\pi$ is called its principal value. Thus we have

$$\log w = \log |w| + i \arg w.$$

Let $|w| = r$ and $\arg w = \theta$. Then we have

$$\log w = \log r + i\theta, \quad r > 0. \tag{14}$$

If θ_0 denotes the principle value of θ, we may write

$$\log w = \log r + i\left(\theta_0 + 2n\pi\right),$$

where $n = 0, \pm 1, \pm 2, \cdots$. We notice from (10) and (11) that

$$\log(\exp z) = z. \tag{15}$$

When emphasis is sought to be placed on a many-valued character of the logarithm of any complex number w, it is denoted by Log w rather than by log w. The symbol $\log w$ is then reserved for the value of a logarithm corresponding to the principal value of w, namely, $\log |w| + i\theta_0$, where θ_0 is the principal to the principal value of $\arg w$. Thus we have

$$\mathrm{Log}\, w = \log w \pm 2n\pi i$$

for all $n = 0, 1, 2, \cdots$.

Each Log w, obtained by taking a special value of n, is called a branch of the logarithm. The most important branch is, of course, the branch corresponding to $n = 0$, which is identical to $\log w$, the principal value of the logarithm of w.

Branches of Log w

As we have discussed above, Log w is an infinitely many-valued function, but it can be easily decomposed into "branches" all of which are single valued. The only thing that we have to do is to restrict the value of θ in an interval of length 2π.

By imposing limitations on r and θ so that $r > 0$ and $\theta_0 < \theta < \theta_0 + 2\pi$, the function $\log w$ defined by (14) can be made single-valued and continuous, where θ_0 is any fixed angle in radians. Then we may write

$$\log w = \log r + i\theta, \tag{16}$$

where

$$r > 0, \quad \theta_0 < \theta < \theta_0 + 2\pi.$$

Note that, for each fixed θ_0, the function defined by (16) is a branch of the multi-valued function Log w.

Addition Theorem for $\log w$

Theorem

If w_1 and w_2 are two complex numbers, then

$$\log(w_1 w_2) = \log w_1 + \log w_2,$$
$$\arg(w_1 w_2) = \arg w_1 + \arg w_2.$$

Proof.

Suppose that $\log w_1 = z_1$ and $\log w_2 = z_2$. Then, by the definition, we have

$$\exp z_1 = w_1, \quad \exp z_2 = w_2.$$

Hence, by the addition theorem for exponential functions and (15), we have

$$\log(w_1 w_2) = \log(\exp z_1 \exp z_2) = \log[\exp(z_1 + z_2)] = z_1 + z_2 = \log w_1 + \log w_2.$$

Thus we have

$$\log(w_1 w_2) = \log w_1 + \log w_2. \tag{17}$$

The second result, i.e.,

$$\arg(w_1 w_2) = \arg w_1 + \arg w_2 \tag{18}$$

follows in the usual sense. □

Analyticity of $\log w$

If we make a cut along the negative half of the real axis from $-\infty$ to 0 and stipulate that the variable does not cross it, then, if D is any bounded domain in this cut plane, so that no point of the cut belongs to D, $\log w$ is one-valued and continuous in D. Let w and w' be any two distinct points in a domain D and, if $z = \log w$ and $z' = \log w'$, then $z' \to z$ as $w' \to w$ along any path in D. Hence, as $w' \to w$

$$\begin{aligned}
\frac{\log w - \log w'}{w - w'} &= \frac{z - z'}{\exp(z) - \exp(z')} \\
&= \frac{z - z'}{(z - z') + \frac{1}{2}(z^2 - z'^2) + \frac{1}{3!}(z^3 - z'^3) + \cdots} \\
&= \frac{1}{1 + \frac{1}{2!}(z + z') + \frac{1}{3!}(z^2 + zz' + z'^2) + \cdots} \\
&= \frac{1}{1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots}
\end{aligned}$$

as $z' \to z$. Then we have

$$\lim_{w \to w'} \frac{z - z'}{w - w'} = \frac{1}{\exp(z)} = \frac{1}{w}$$

whence we conclude that $\ln w$ is an analytic function and regular in D with $\frac{1}{w}$ as its derivative.

Power Series for $\log(1+z)$

Since $\log(1+z)$ is an analytic function, regular in the z-plane and cut along the real axis from $-\infty$ to -1 with the derivative $\frac{1}{1+z}$. It can be easily seen that, in the domain $|z|<1$, the function $\frac{1}{1+z}$ can be expanded as a convergent power series given by

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots$$

Evidently, the sum of this power series is the derivative of the function

$$f(z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots + (-1)^n \frac{z^{n+1}}{n+1} + \cdots \tag{19}$$

for all those values of z for which the series $\sum_{n=0}^{\infty}(-1)^n \frac{z^{n+1}}{n+1}$ converges. We find that the radius of convergence of the power series (19) defining $f(z)$ is unity. Thus $F(z) = f(z) - \log(1+z)$ is an analytic function and regular for $|z|<1$ with differential coefficient equal to zero. Therefore, for $|z|<1$, $F(z)$ is a constant. Let us put $z=0$. Then we find that $F(0)=0$. Hence $\log(1+z)$ can be represented by a power series given by

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots + (-1)^n \frac{z^{n+1}}{n+1} + \cdots$$

which converges for $|z|<1$.

The Functions a^z and z^a

We define the principal value of the function a^z as the number uniquely determined by the equation

$$a^z = e^{z \log a}$$

where $\log a$ is the principal value of Log a, and we permit both a and z to be complex. Taking the other values of Log a in place of $\log a$, we can obtain other values of a^z, which may be called its subsidiary values. Of course, all these are contained in the expression

$$\exp\{z(2\pi i + \log a)\}.$$

Suppose that a is real or complex. Then we defined z^a by the equation

$$z^a = \exp(a \log z),$$

where $\log z$ is the principal value of Log z. Also, we observe that, for any values of α and β, real and complex,

$$\begin{aligned} z^\alpha z^\beta &= \exp(\alpha \log z) \exp(\beta \log z) \\ &= \exp\{(\alpha + \beta) \log z\}. \end{aligned}$$

Therefore, we have

$$z^\alpha z^\beta = z^{\alpha+\beta}.$$

The Functions a^z and z^a

Since the value of $\log z$ changes suddenly by $2\pi i$ as a crosses the negative half of the real axis and so we have

$$\log(x+iy) - \log(x-iy) \to 2\pi i$$

as $y \to 0$ and so

$$\frac{(x+iy)^a}{(x-iy)^a} \to e^{2a\pi i}$$

as $y \to 0$. The above limit shows that z^a is also discontinuous for real negative z except for integral values of a because in such cases $e^{2a\pi i} = 1$.

Evidently, z^a is one-valued and continuous in every bounded domain D of the z-plane and cut along the real negative axis from $-\infty$ to the origin. We can calculate the derivative of z^a for a value of z in such a domain D. Thus we have

$$\begin{aligned}\frac{d}{dz}z^a = \frac{d}{dz}\exp(a\log z) = \exp(a\log z)\frac{a}{z} = a\frac{\exp(a\log z)}{\exp(\log z)} \\ = a\exp[(a-1)\log z] \\ = az^{a-1}.\end{aligned}$$

Remark: In general, we have

$$z_1^a z_2^a \neq (z_1 z_2)^a.$$

Inverse Trigonometrical Functions

We define the inverse function $z = \cos^{-1} w$ of the cosine function $\cos z$ by the equation

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = w. \tag{20}$$

Solving (20), we have

$$e^{2iz} - 2we^{iz} + 1 = 0.$$

This is a quadratic equation in e^{iz} with roots

$$e^{iz} = \frac{2w \pm \sqrt{4w^2 - 4}}{2} = w \pm \sqrt{w^2 - 1}.$$

Hence $z = -i \ln \left(w \pm \sqrt{w^2 - 1} \right)$. Since

$$\frac{1}{w + \sqrt{w^2 - 1}} = \frac{w - \sqrt{w^2 - 1}}{w^2 - (w^2 - 1)} = w - \sqrt{w^2 - 1},$$

the numbers $w + \sqrt{w^2 - 1}$ and $w - \sqrt{w^2 - 1}$ are reciprocal.

Inverse Trigonometrical Functions

Therefore, we have

$$\ln\left(w-\sqrt{w^2-1}\right)=-\ln\left(w+\sqrt{w^2-1}\right).$$

Hence we can write

$$\cos^{-1} w = z = \pm i \ln\left(w+\sqrt{w^2-1}\right).$$

Since Logarithm is a many-valued function, it follows that $\cos^{-1} w$ becomes single-valued and analytic since it is a composite of analytic function. Now, we define the inverse function $\sin^{-1} w$ by

$$\sin^{-1} w = \frac{\pi}{2} - \cos^{-1} w.$$

It may be proved easily that

$$\tan^{-1} w = \frac{i}{2}\log\left(\frac{1-iw}{1+iw}\right),$$

i.e.,

$$\arctan w = \frac{i}{2}\log\left(\frac{i-w}{i+w}\right).$$

We also define the inverse function $\cot^{-1} w$ by

$$\cot^{-1} w = \frac{\pi}{2} - \tan^{-1} w,$$

i.e.,

$$\mathrm{arccot}\, w = \frac{\pi}{2} - \frac{i}{2}\log\left(\frac{i-w}{i+w}\right).$$