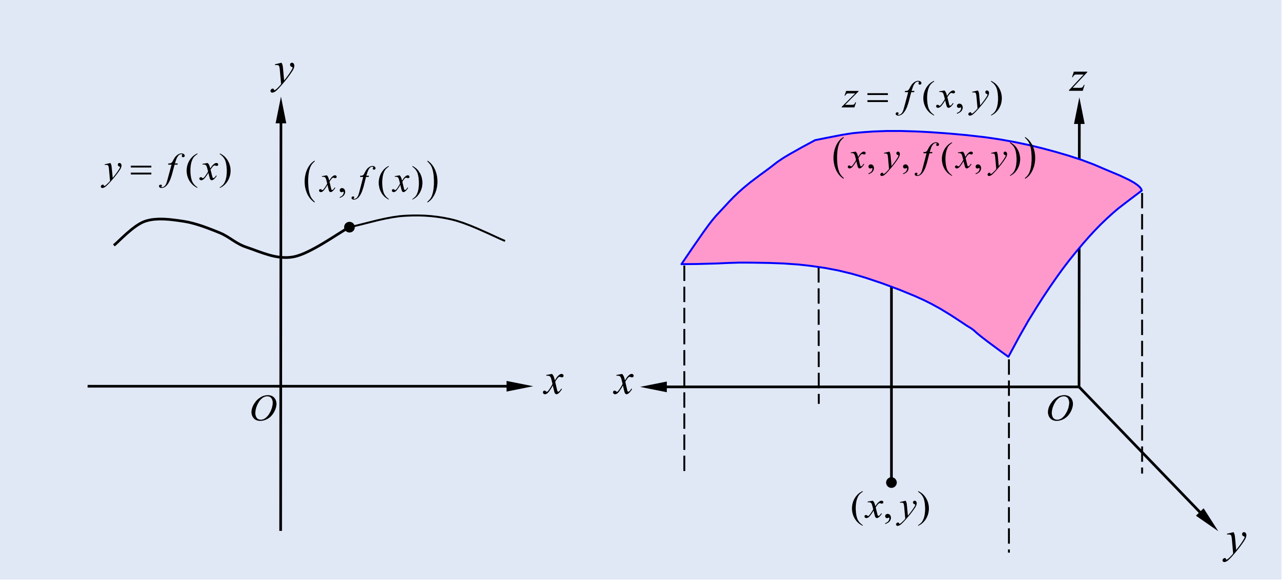
Differential Geometry

Preliminaries

presentation of curves and surfaces

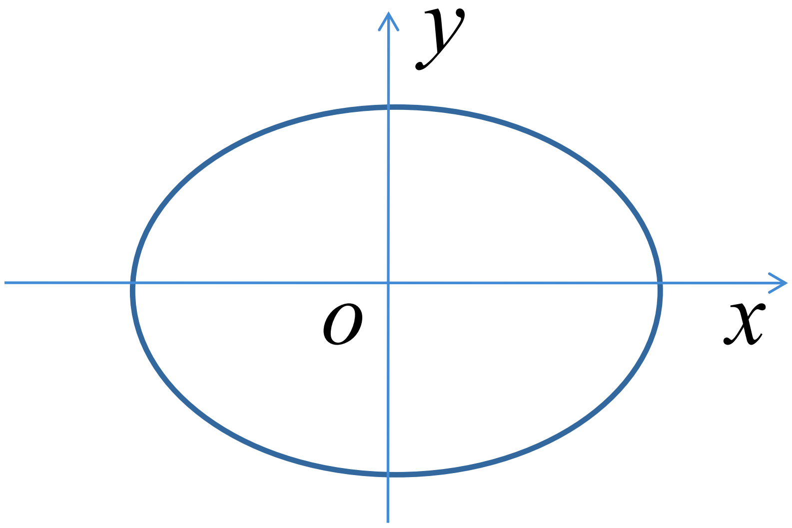
❖ Explicit function



presentation of curves and surfaces

- **Implicit function**

$$f(x,y)=0, f(x,y,z)=0$$



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

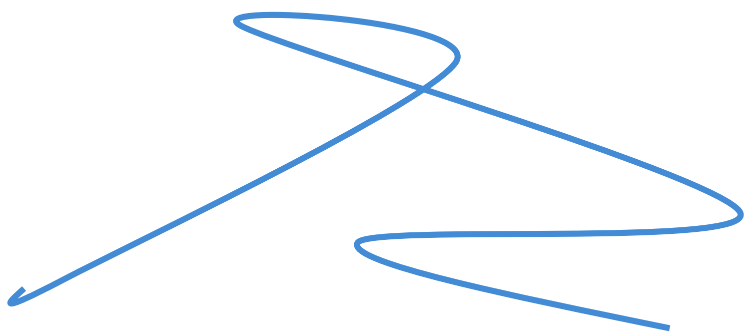
Algebraic geometry： zeros set of a polynomial

presentation of curves and surfaces

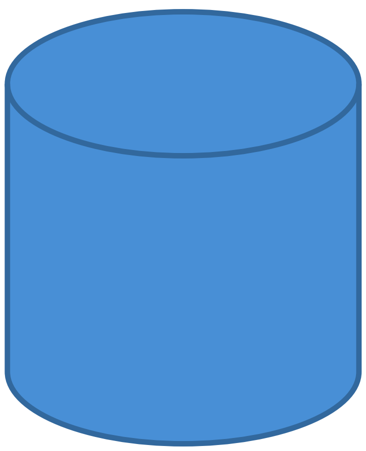
- **Parametric curve & surface（Euler）**

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \qquad \vec{r} = \vec{r}(t) = (x(t), y(t), z(t))$$

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases} \qquad \vec{r} = \vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$$



Space curve

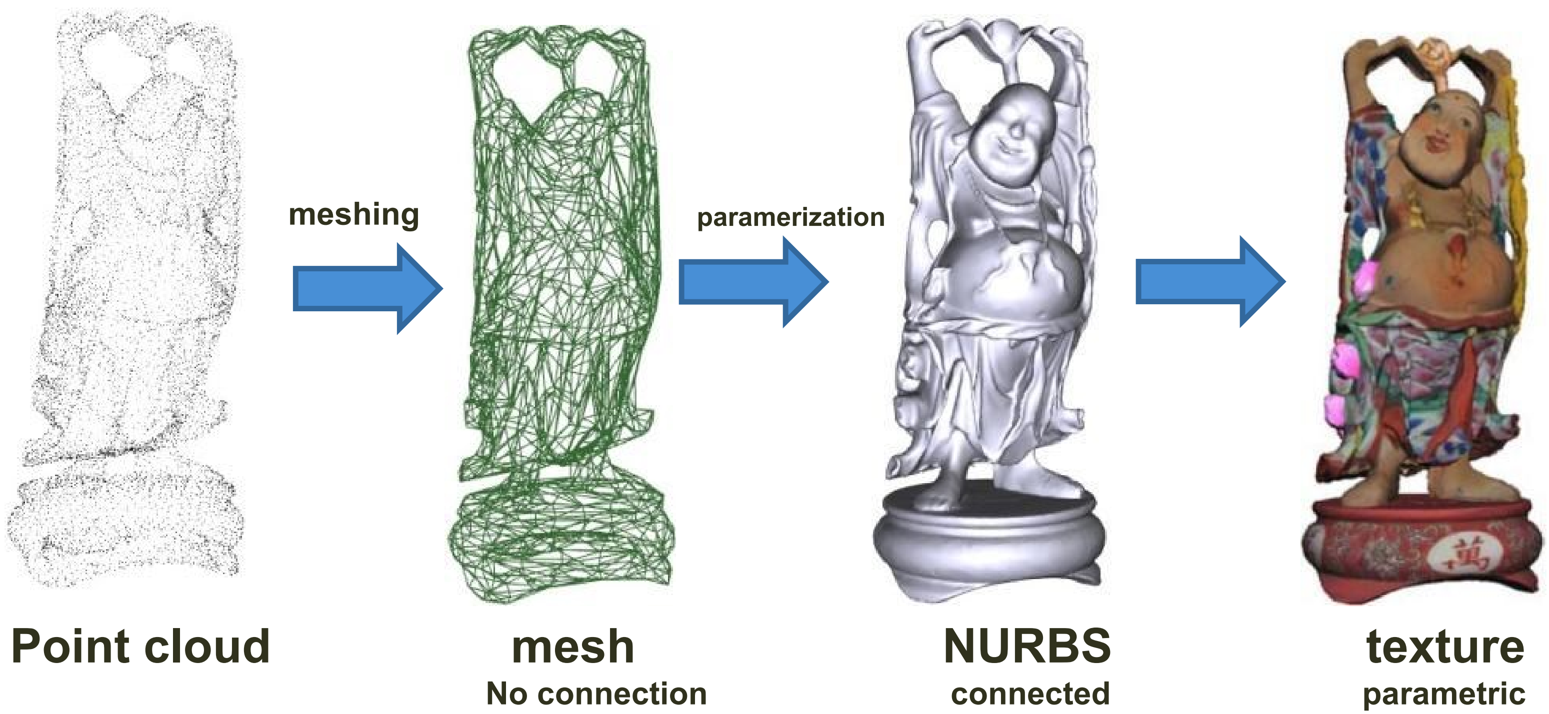


Cylinder

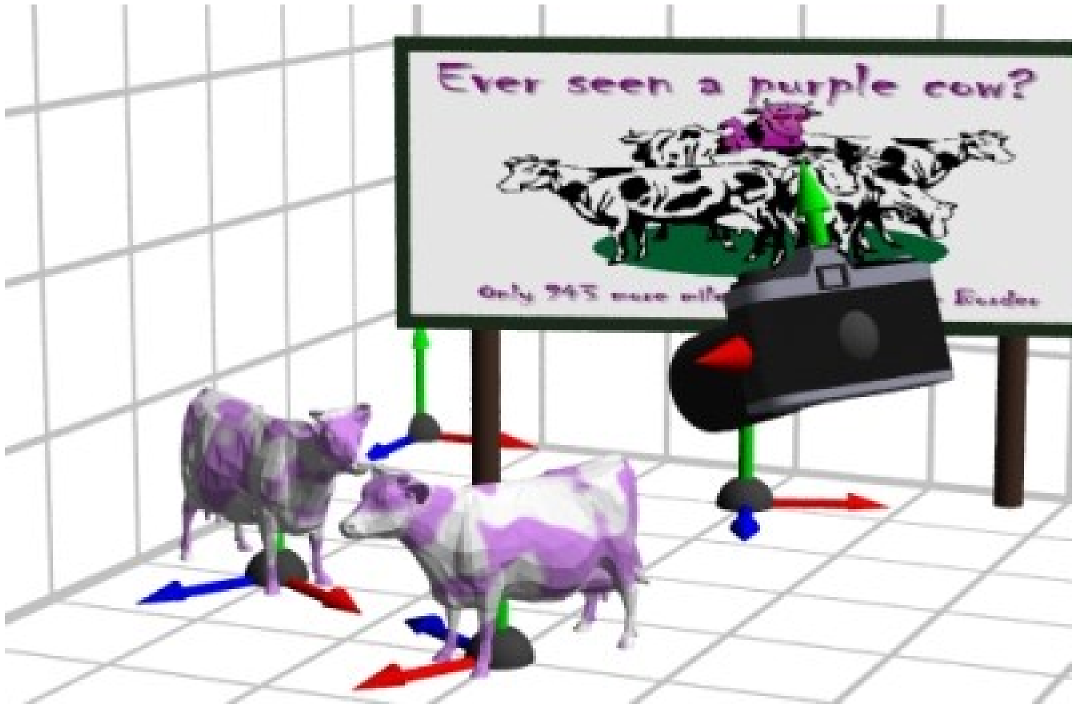
Shape modeling

❖ Surface reconstruction(static)

- From CT or optical images, raw point data, …
- Data repairing, registration, resampling, smoothing



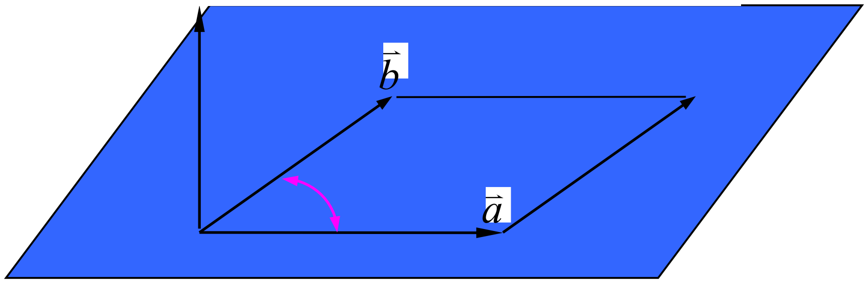
Coordinate system



The frame and coordinate are the bridge to establish the connection between form and number

Vector

- Vector is a directed line segment, it has size and direction, represented as $\overrightarrow{AB}, \mathrm{r}, \vec{r}$
- It is written in Cartesian coordinates as $(x_1, x_2, \ldots, x_k)$
- Vector operations
  - Inner product
    - Symmetry, bilinear, positive definiteness
  - Outer product
    - Antisymmetric, bilinear



Inner product

- **Definition**

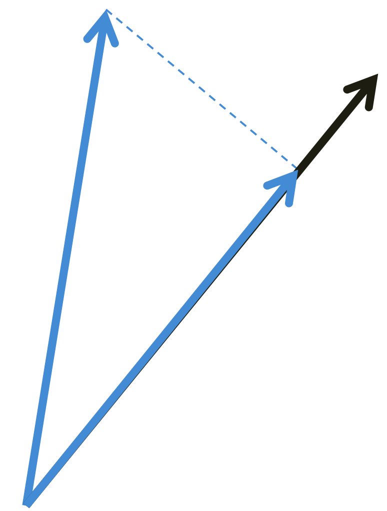
$$\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\angle(\mathbf{a},\mathbf{b})$$

- **Algorithms**

$$\mathbf{c}\cdot(\mathbf{a}+\mathbf{b})=\mathbf{c}\cdot\mathbf{a}+\mathbf{c}\cdot\mathbf{b}$$

$$(\lambda\mathbf{a})\cdot\mathbf{b}=\lambda(\mathbf{a}\cdot\mathbf{b})$$

$$\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$$



- **Properties**

$$|\mathbf{a}|^2=\mathbf{a}.\mathbf{a}\geq 0$$

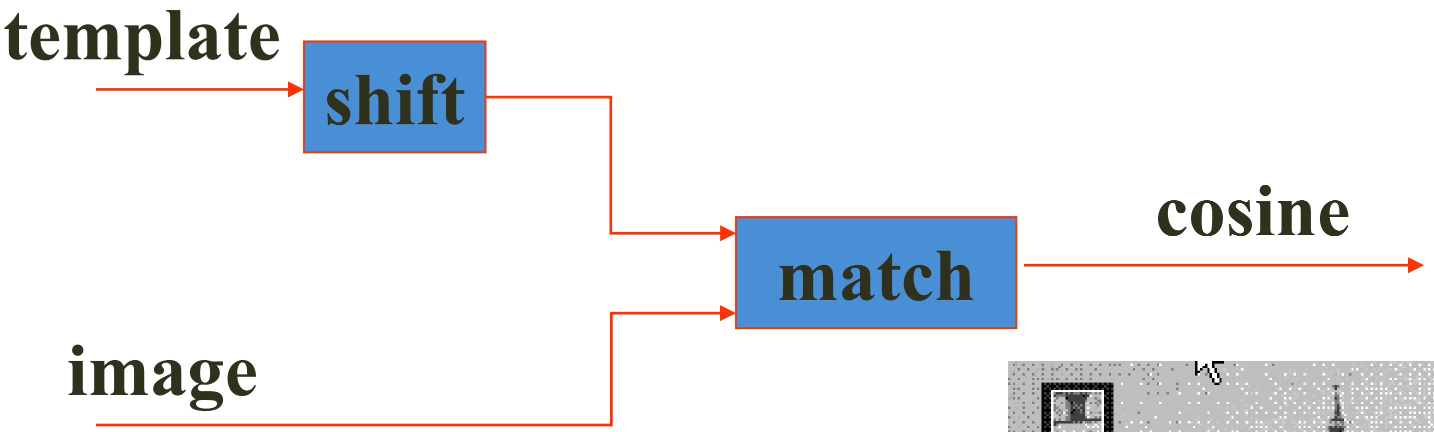
$$\mathbf{a}\perp\mathbf{b}\Leftrightarrow\mathbf{a}.\mathbf{b}=0$$

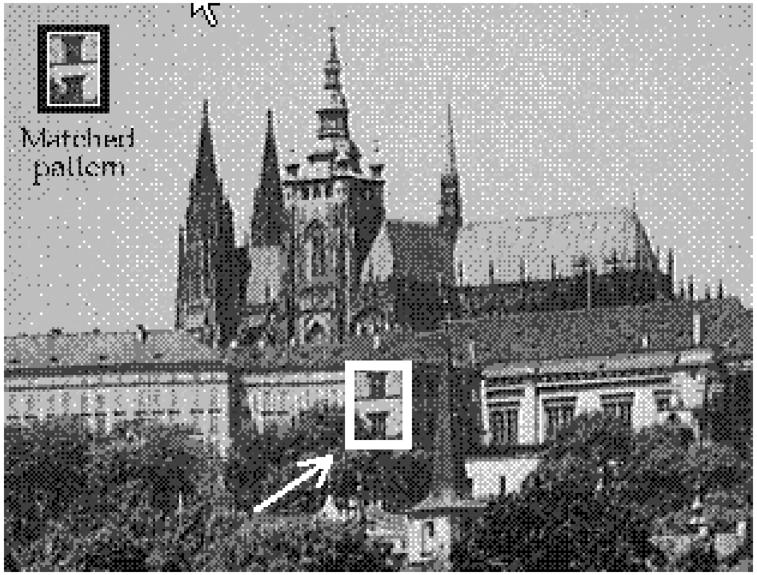
Template matching

$$\cos\alpha = \frac{x \bullet y}{\|x\| \cdot \|y\|}$$

- **bounded** $-1 \leq \cos\alpha \leq 1$
- **linear dependent** $\cos\alpha = \pm 1$ **equivalent**
- **orthogonal** $\cos\alpha = 0$ **unequal**

Template Matching





Outer product

- Definition

  $\mathbf{a} \times \mathbf{b}$ is a vector which perpendicular to both $\mathbf{a}$ and $\mathbf{b}$

  It forms a right-hand system with $\mathbf{a}$ and $\mathbf{b}$

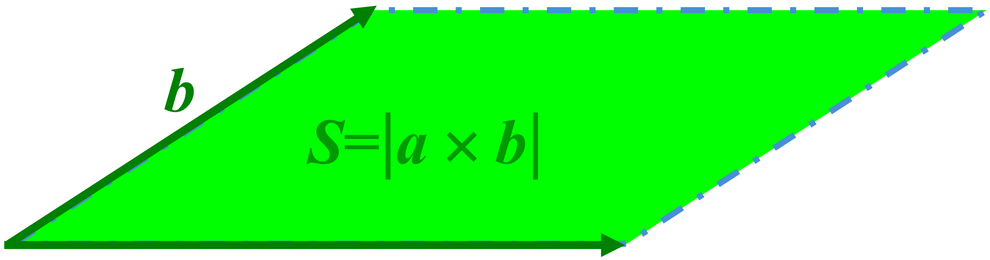
$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| sin\angle(\mathbf{a}, \mathbf{b})$$

- Algorithms

$$\mathbf{c} \times (\mathbf{a} + \mathbf{b}) = \mathbf{c} \times \mathbf{a} + \mathbf{c} \times \mathbf{b}$$

$$(\lambda \mathbf{a}) \times \mathbf{b} = \lambda (\mathbf{a} \times \mathbf{b})$$

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

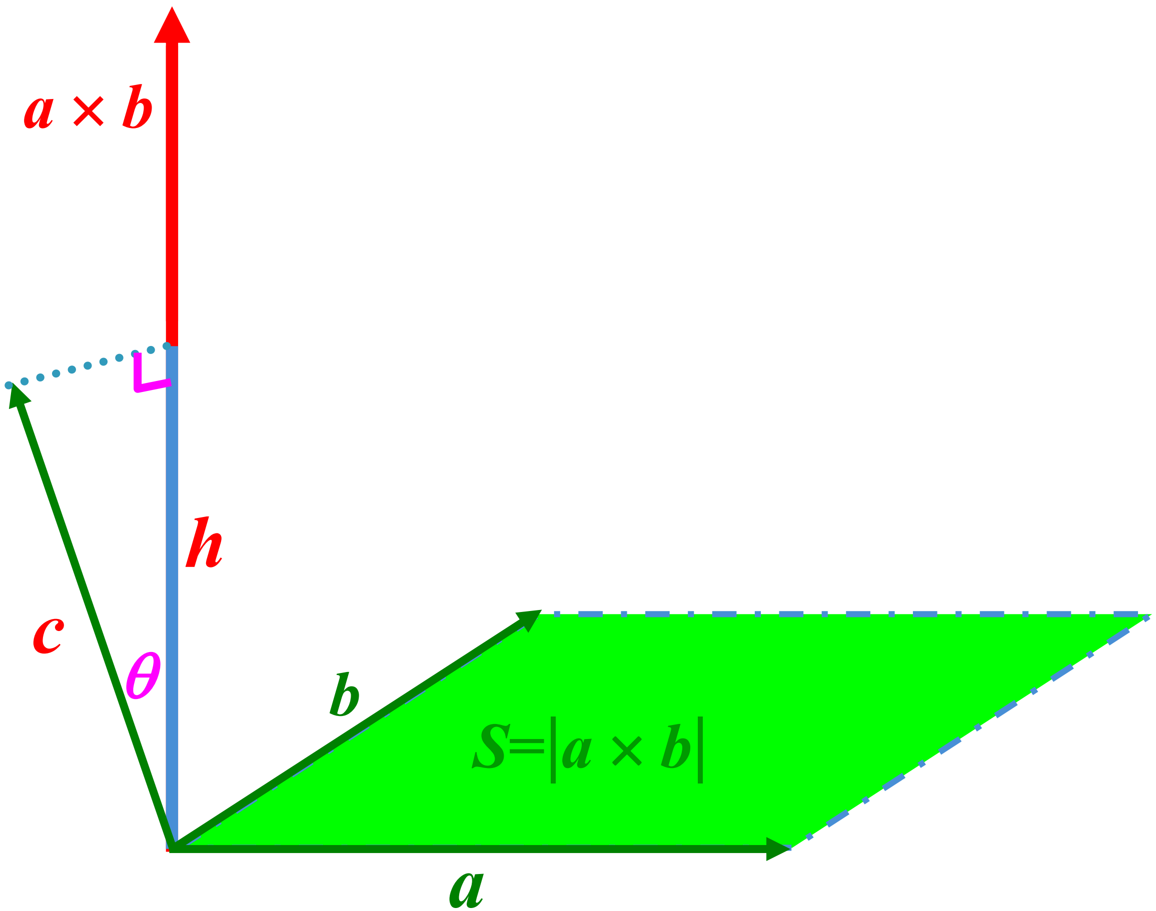


- Properties

$$\mathbf{a} \parallel \mathbf{b} \Leftrightarrow \mathbf{a} \times \mathbf{b} = \mathbf{0}$$

Mixed products of vectors

$$|[abc]| = |a \times b \cdot c| = |a \times b| \cdot |\Pr j_{a \times b} c| = S\,h = V$$

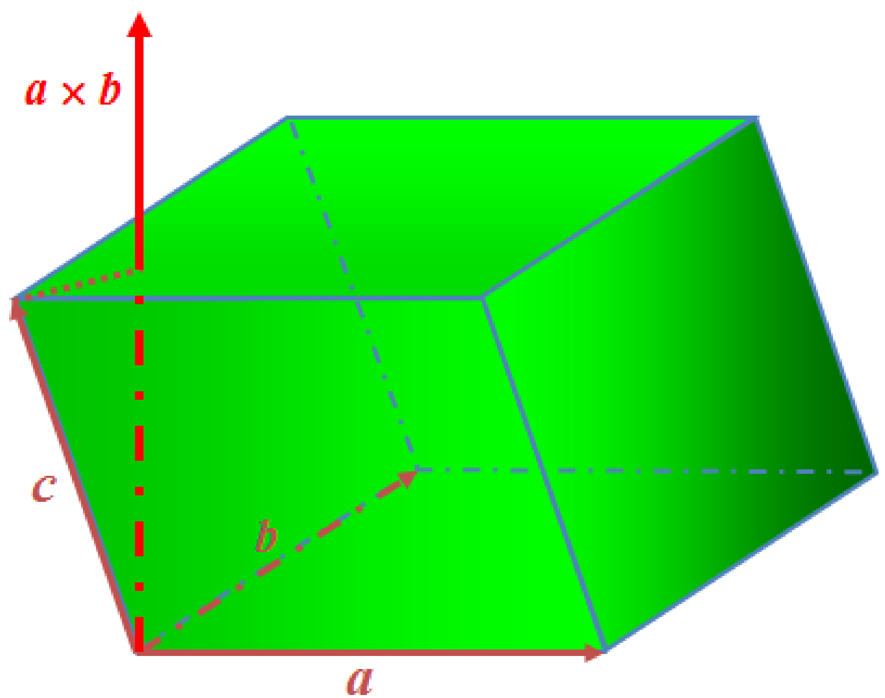


Mixed products of vectors

❖ **Geometry meanning**

$$|[abc]| = |a \times b \cdot c| = |a \times b| \cdot |\mathrm{Prj}_{a \times b} c| = S\,h = V$$

***a*, *b*, *c* are coplane** $\Leftrightarrow$ $[abc] = 0$



Frame

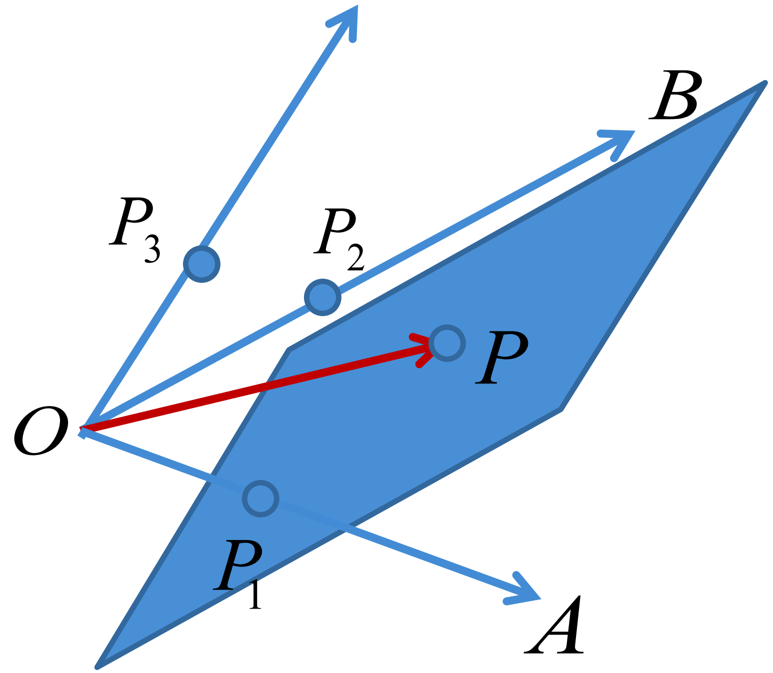
- **Affine frame** $\{O; \overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}\}$

- **For any point** $P \in E^3$

$$\overrightarrow{OP} = \overrightarrow{OP_1} + \overrightarrow{OP_2} + \overrightarrow{OP_3}$$

$$= x\overrightarrow{OA} + y\overrightarrow{OB} + z\overrightarrow{OC}$$

(x, y, z) is called the coordinate of point P



ectangular Cartesian coordinate system

- Let $\{O, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ be a frame in E^3, i, j, and k are unit vectors and they are perpendicular to each other, this frame is called as Right hand unit orthogonal frame, Orthogonal Frame for short.

- The coordinate system given by an orthogonal frame is called a **rectangular Cartesian coordinate system**

Rectangular Cartesian coordinate system

- **Point** $P = O + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, the coordinate is $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$

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- Let $\mathbf{a} = (x_1, y_1, z_1), \mathbf{b} = (x_2, y_2, z_2)$
- Inner product $\mathbf{a} \cdot \mathbf{b} = x_1x_2 + y_1y_2 + z_1z_2$
- Outer product $\mathbf{a} \times \mathbf{b} = \left(\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix}, \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \right)$

Rectangular Cartesian coordinate system

- **Distance**

$$|AB| = \sqrt{\overrightarrow{AB} \cdot \overrightarrow{AB}} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Rectangular Cartesian coordinate system

- **Distance**

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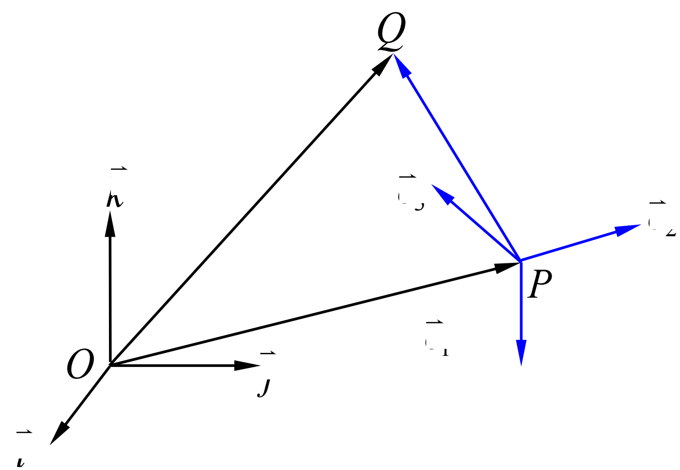
- 3D Euclidean space is denoted as R^3, vector (x, y, z) has length $\sqrt{x^2 + y^2 + z^2}$

Coordinate transformation

- Choose a Orthogonal frame. $\{O; \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, and any other orthogonal frame $\{P; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is uniquely determined represented as

$$\begin{cases} \overrightarrow{Op} = a_1 \boldsymbol{i} + a_2 \boldsymbol{j} + a_3 \boldsymbol{k}, \\ \boldsymbol{e}_1 = a_{11} \boldsymbol{i} + a_{12} \boldsymbol{j} + a_{13} \boldsymbol{k}, \\ \boldsymbol{e}_2 = a_{21} \boldsymbol{i} + a_{22} \boldsymbol{j} + a_{23} \boldsymbol{k}, \\ \boldsymbol{e}_3 = a_{31} \boldsymbol{i} + a_{32} \boldsymbol{j} + a_{33} \boldsymbol{k}. \end{cases}$$



$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \sum_{k=1}^{3} a_{ik} a_{jk} = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Coordinate transformation

❖ **Transformation matrix**

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

❖ **Orthogonal、|A|=1、rotation transformation**

Coordinate transformation

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- **Orthogonal、|A|=1、rotation transformation**
- **For a point** $q(x, y, z)$ **in** $\{O; \mathbf{i}, \mathbf{j}, \mathbf{k}\}$**, its coordinate in** $\{P; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ **can be represented as (**

$$\begin{cases} x = a_1 + a_{11}\tilde{x} + a_{21}\tilde{y} + a_{31}\tilde{z}, \\ y = a_2 + a_{12}\tilde{x} + a_{22}\tilde{y} + a_{32}\tilde{z}, \\ z = a_3 + a_{13}\tilde{x} + a_{23}\tilde{y} + a_{33}\tilde{z}. \end{cases}$$

Orthogonal transformation

- The relation between the transformed mapping point and the original point under rigid body motion

$$(\tilde{x}, \tilde{y}, \tilde{z}) = a + (x, y, z) \cdot A,$$

$$\begin{cases} \tilde{x} = a_1 + a_{11}x + a_{21}y + a_{31}z, \\ \tilde{y} = a_2 + a_{12}x + a_{22}y + a_{32}z, \\ \tilde{z} = a_3 + a_{13}x + a_{23}y + a_{33}z. \end{cases}$$

Rigid body motion

- Theorem A rigid body motion in E^3 transforms a orthogonal frame into another orthogonal frame; for any two orthogonal frames in E^3, there must be a rigid body motion in E^3 which transforms one frame into the other one:

$$\sigma : E^3 \to E^3$$

- A transformation in E^3 which transforms between itself and keeps the distance between any two points is called as isometric transformation

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Continuous (preserves neighbourhoods)

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One to one, invertible

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Collineation (lines remain lines)
- Perspective

Non-linear (lines become curves)
- Twists, bends, warps, morphs, ...

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Symmetries-x

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Symmetries - x=y

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotations

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Symmetries-y

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Symmetries - x=-y

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scaling

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

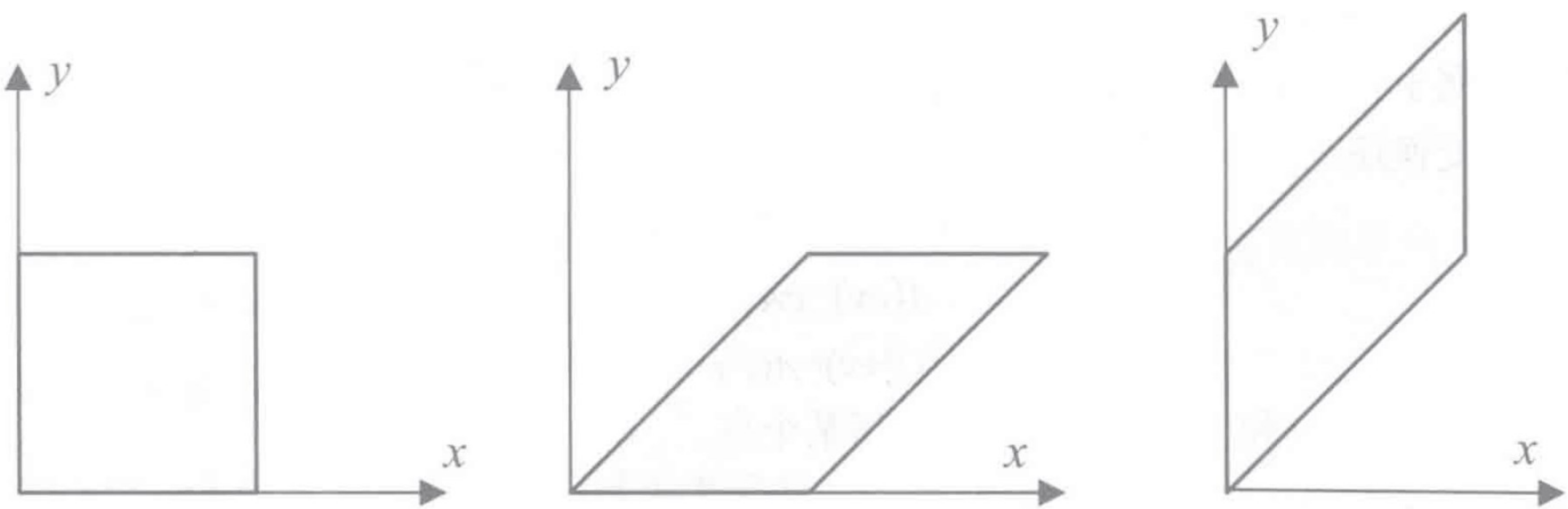
Symmetries-origin

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Shear - x

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Shear - y



（a）原图　（b）沿 x 方向的错切　（c）沿 y 方向的错切

Affine frame

- A point O and three non-coplanar vectors e_1, e_2, e_3 form a frame $\{O; e_1, e_2, e_3\}$. They do not need to be orthogonal.
- Any point P can be represented as

$$P = O + xe_1 + ye_2 + ze_3$$

- How to calculate the vector length?

For instance, the length of a vector $\overrightarrow{OP}$ can be calculated as $\sqrt{\overrightarrow{OP}.\overrightarrow{OP}}$, that is

$$\sqrt{\overrightarrow{OP}.\overrightarrow{OP}} = (\mathrm{x}, \mathrm{y}, \mathrm{z})\begin{pmatrix} e_1e_1 & e_1e_2 & e_1e_3 \\ e_1e_2 & e_2e_2 & e_2e_3 \\ e_1e_3 & e_2e_3 & e_3e_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Affine frame

We call $g_{ij} = e_i . e_j, 1 \le i, j \le 3$

as metric coefficient of affine frame $\{O; e_1, e_2, e_3\}$, matrix

$$(g_{ij}) = \begin{pmatrix} e_1e_1 & e_1e_2 & e_1e_3 \\ e_1e_2 & e_2e_2 & e_2e_3 \\ e_1e_3 & e_2e_3 & e_3e_3 \end{pmatrix}$$

is the metric matrix. It is very important in differential geometry.

Affine frame forms a 12 dimensional vector space $E^3 \times GL(3)$

Homework

- Assume line L goes through point $P(x_0, y_0, z_0)$, its direction is $v(v_1, v_2, v_3)$ (unit vector), a point $X(x,y,z)$ outside the line rotates an angle θ around L in the right hand direction, try to calculate the coordinate of X after rotating.
- 提示：以向量v为标架的一个向量，再通过点P，X构造另外两个向量，以P为原点，构造新的正交标架；计算X点在新的标架下的坐标，做旋转变换，再计算在原坐标系下的坐标。

Vector-valuedfunction

Vector-valued function

- Vector-valued function **r** is a map from $\mathscr{D}$ to R^3

$$\mathbf{r} : \mathscr{D} \to \mathrm{R}^3 : p \mapsto$$

- Vector-valued function which is defined in [a, b] can be represented as

$$\mathbf{r}(t) = (x(t), y(t), z(t)), \qquad a \le t \le b$$

- $\mathbf{r}(t)$ is continuous and continuous differential mean $x(t)$, $y(t)$, and $z(t)$ are continuous and continuous differential regarding to t.

…entiation and integration of vector-valued functions

❖ **Derivative求导**

$$\left.\frac{d\mathbf{r}}{dt}\right|_{t=t_0} = \left(x'(t_0), y'(t_0), z'(t_0)\right) = \left(\frac{dx(t_0)}{dt}, \frac{dy(t_0)}{dt}, \frac{dz(t_0)}{dt}\right)$$

❖ **Integration积分**

$$\int_a^b \mathbf{r}(t)dt = \lim_{\lambda\to 0}\sum_{i=1}^{n}\mathbf{r}(t_i')\Delta t_i = \left(\int_a^b x(t)dt, \int_a^b y(t)dt, \int_a^b z(t)dt\right)$$

The differentiability and integrability of a vector-valued function come down to the differentiability and integrability of its component functions.

Leibniz rule

Theorem 1: assume $a(t), b(t), and\ c(t)\ are\ differentiable$ Vector functions, then the derivatives of their inner product, outer product, and mixed product are

① $\left(a(t).b(t)\right)' = a'(t).b(t) + a(t).b'(t)$

② $\left(a(t) \times b(t)\right)' = a'(t) \times b(t) + a(t) \times b'(t)$

③ $\left(a(t), b(t), c(t)\right)' = \left(a'(t), b(t), c(t)\right) + \left(a(t), b'(t), c(t)\right) + \left(a(t), b(t), c'(t)\right)$

Theorem 2: assume $a(t)$ is a continuous differentiable vector function that is nonzero everywhere, then

① The length of vector function $a(t)$ is a constant if and only if $a'(t).a(t) = 0$

① The direction of vector function $a(t)$ does not change if and only if $a'(t) \times a(t) = 0$

① If vector function $a(t)$ is perpendicular to a vector, then

$$\left(a(t), a'(t), a''(t)\right) \equiv 0$$

Conversely, if the above equations are true, and $a'(t) \times a(t) \neq 0$ everywhere, then vector function $a(t)$ must be perpendicular to a certain vector

$(1)\ |\boldsymbol{a}(\boldsymbol{t})|^2 = \boldsymbol{a}(\boldsymbol{t}).\boldsymbol{a}(\boldsymbol{t}) = \boldsymbol{c}$

$(2)\ \boldsymbol{a}(\boldsymbol{t}) = f(t).\boldsymbol{b}$

$\Rightarrow\ \boldsymbol{a}'(\boldsymbol{t}) = f'(t).\boldsymbol{b}$

$(3)\ \boldsymbol{a}(\boldsymbol{t}).\boldsymbol{b} = \boldsymbol{0}$

$\Rightarrow\ \boldsymbol{a}'(\boldsymbol{t}).\boldsymbol{b} = \boldsymbol{0}$

$\Rightarrow\ \boldsymbol{a}''(\boldsymbol{t}).\boldsymbol{b} = \boldsymbol{0}$

$\Rightarrow\ \boldsymbol{a}(\boldsymbol{t}), \boldsymbol{a}'(\boldsymbol{t}), \boldsymbol{a}''(\boldsymbol{t})\, \boldsymbol{coplanar}$

$$(1)\ |\boldsymbol{a}(t)|^2 = \boldsymbol{a}(t).\,\boldsymbol{a}(t) = \boldsymbol{c}$$

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$(2)\ \boldsymbol{a}(\boldsymbol{t}) = f(t).\boldsymbol{b}$

➡ $\boldsymbol{a}'(\boldsymbol{t}) = f'(t).\boldsymbol{b}$

$(3)\ \boldsymbol{a}(\boldsymbol{t}).\boldsymbol{b} = \boldsymbol{0}$

➡ $\boldsymbol{a}'(\boldsymbol{t}).\boldsymbol{b} = \boldsymbol{0}$

➡ $\boldsymbol{a}''(\boldsymbol{t}).\boldsymbol{b} = \boldsymbol{0}$

➡ $\boldsymbol{a}(\boldsymbol{t}), \boldsymbol{a}'(\boldsymbol{t}), \boldsymbol{a}''(\boldsymbol{t})\, \boldsymbol{coplanar}$

Homework

Assume vector function r(t) has any derivative of order. Let $r^{(k)}(t)$ denote the k-order derivative, and assume

$$r^{(k)}(t) \times r^{(k+1)}(t) = 0$$

everywhere. Try to calculate the necessary and sufficient conditions of

$$\left(r^{(k)}(t), r^{(k+1)}(t), r^{(k+2)}(t)\right) \equiv 0$$

Thank You !