Methods of Mathematical Physics

—Lecture 3 Complex Integrations—

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Contents

Introduction

In the theory of real variables, the integration is considered from two perspectives: the indefinite integration as an operation inverse to that of differentiation and the definite integration as the limit of a sum. The concept of the indefinite integral as the process of inverse differentiation in case a function of a real variable is extended to a function of a complex variable if the complex function $f(z)$ is analytic. It means that, if $f(z)$ is an analytic function of a complex variable z and

$$\int f(z)dz = F(z),$$

then the differential of $F(z)$ is equal to $f(z)$, i.e., $F'(z) = f(z)$.

However, the concept of the definite integral of a function of a real variable does not extend out, rightly to the domain of complex variables. For example, in the case of real variable, the path of integration of $\int_a^b f(x)dx$ is always along the real axis from $x = a$ to $x = b$. But, in the case of a complex function $f(z)$, the path of the definite integral

$$\int_a^b f(z)dz,$$

may be along any curve joining the points $z = a$ and $z = b$ and so its value depends upon the path (curve) of integration.

Some Definitions

- Let $[a, b]$ be a closed interval where a and b are real numbers. Subdivide the interval $[a, b]$ into n sub-intervals:

  $$[t_0, t_1], \quad [t_1, t_2], \quad [t_2, t_3], \quad \cdots, \quad [t_{n-1}, t_n]$$

  by inserting $n-1$ intermediate points $t_1, t_2, \cdots, t_{n-1}$ satisfying the inequalities:

  $$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b.$$

  Then the set

  $$P = \{t_0, t_1, t_2, \cdots .t_n\}$$

  is called a partition of the interval $[a, b]$ and the greatest of the numbers

  $$t_1 - t_0, t_2 - t_1, \cdots, t_n - t_{n-1}$$

  is called the norm of the partition P, which is denoted by $|P|$.
- Suppose that a point z lies on an arc L is defined by

  $$z = z(t) = x(t) + iy(t),$$

  where t runs through the interval $a \leq t \leq b$ and $x(t), y(t)$ are continuous functions of t. Then the arc L is said to be a continuous arc.

Some Definitions

- Arc L is said to be continuously differentiable or simply differentiable if $z'(t)$ exists and is continuous. If, in addition to the existence of $z'(t)$, we also have $z'(t) \neq 0$, then we say that L is a regular arc (or a smooth arc). Thus a regular arc is characterized by the property that it has, at every point, a tangent whose direction is determined by $\arg z'(t)$. In fact, as t increases from a to b, z continuously traces out the arc L and, at the same time, arg $z'(t)$ varies continuously since $z'(t)$ changes continuously without vanishing.
- An arc L is said to be simple or a Jordan arc if $z(t_1) = z(t_2)$ only when $t_1 = t_2$. If $z(a) = z(b)$, then the arc L is said to be a closed curve. If L is the arc defined by $z = z(t)(a \leq t \leq b)$, then the arc defined by

$$z = z(-t) \quad (-b \leq t \leq -a)$$

is said to be the opposite arc of L and is defined by $-L$.

Complex Integrals

Let L be a Jordan arc defined by

$$z = z(t) = x(t) + iy(t) \quad (a \leq t \leq b)$$

and let $f(z)$ be a function of a complex variable z which has a definite value at each point of a rectifiable arc L. Consider an arbitrary partition

$$P = \{a = t_0, t_1, t_2, \cdots, t_{n-1}, t_n = b\}$$

of $[a, b]$. We divide the arc L into a small arcs by means of the points $z_0, z_1, z_2, \cdots, z_{n-1}, z_n$, which correspond to the values

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$$

of the parameter t, and form the sum

$$\sum = \sum_{k=1}^{n} f(\zeta_k)(z_k - z_{k-1}) = \sum_{k=1}^{n} f(\zeta_k)\,\Delta z_k,$$

where $z_k = z(t_k), \zeta_k = z(\alpha_k)$ and $t_{k-1} \leq \alpha_k \leq t_k$, is a point of L between z_{k-1} and z_k, $\Delta z_k = z_k - z_{k-1}$.

Complex Integrals

If this $\mathrm{sum} \sum$ tends to a unique limit I as $n \to \infty$ and the norm of P, i.e., $|P|$ tends to zero, then we say that $f(z)$ is integrable from a to b along the arc L and we write

$$I = \int_L f(x)dt.$$

We also call $\int_L f(z)dz$ the complex line integral or, simply, the line integral of $f(z)$ along the arc L or the definite integral of $f(z)$ from a to b along L. The sense of direction of integration is from a to b, since the points $x(t) + iy(t)$, for increasing values of t, are oriented in the very sense on the arc L. In fact, the value of t depends not only on the end points of arc L, but also on the actual form. Thus we have

$$\int_L f(z)dz = \lim_{|P|\to 0, n\to\infty} \sum_{k=1}^{n} f(\zeta_k)\left(z_k - z_{k-1}\right).$$

Note that an integral of this type exists under pretty general conditions. However, we may do without the assumption that $x'(t)$ and $y'(t)$ exist at each point of L. In fact, the continuity of $f(z)$ on L is a sufficient condition.

Complex Integrals

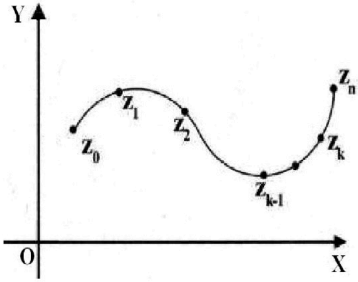

Figure: 1. The complex line integral of $f(z)$ along the arc L.

Evaluation of Integrals by the Direct Definition

Example

1. $\int_L dz$;
2. $\int_L |dz|$;
3. $\int_L z dz$, where L is any rectifiable[a] joining the points $z = \alpha$ and $z = \beta$.

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[a] The length of the polygonal arc, obtained by joining successively z_0 and z_1, z_1 and $z_2, \cdots, z_{n-1}$ and z_n, by straight line segments is given by

$$\Sigma = \sum_{k=1}^{n} L_k = \sum_{k=1}^{n} |z_k - z_{k-1}|$$

where $L_k = \text{Arc}\, z_{k-1} z_k (k = 1, 2, \cdots, n)$ and $z_l = z(t_l)\, (l = 0, 1, \cdots, n)$. If this sum $\sum$ tends to a unique limit l, say, as $n \to \infty$ and the norm of the partition P tends to zero, then we say that the arc L defined by $z = x(t) + iy(t)$ $(a \leq t \leq b)$ is rectifiable and its length is l. Rectifiable Jordan arcs with continuously turning tangents are called regular arcs.

Properties of Complex Integrals

Some elementary properties of complex integrals are as follows:

1. $\int_L [f(z) + g(z)]dz = \int_L f(z)dz + \int_L g(z)dz$.
2. $\int_L f(z)dz = -\int_{-L} f(z)dz$, where by $-L$ we mean the curve L traversed in the opposite direction.
3. $\int_{L_1+L_2} f(z)dz = \int_{L_1} f(z)dz + \int_{L_2} f(z)dz$, where the terminal point of L_1 coincides with the initial point of L_2.
4. $\int_L cf(z)dz = c\int_L f(z)dz$, where c is any complex constant.
5. We have
$$\int_L [c_1 f_1(z) + c_2 f_2(z) + \cdots + c_n f_n(z)]\, dz$$
$$= c_1 \int_L f_1(z)dz + c_2 \int_L f_2(z)dz + \cdots + c_n \int_L f_n(z)dz.$$
6. $\left|\int_L f(z)dz\right| \leq \int_L |f(z)||dz|$.

Integrations along Regular Arcs

Theorem

Let $f(z)$ be continuous on the regular arc L which is defined by

$$z = z(t) = x(t) + iy(t) \quad (z \leq t \leq b).$$

Then $f(z)$ is integrable along L and

$$\int_L f(z)dz = \int_a^b F(t)\{\dot{x}(t) + i\dot{y}(t)\}dt,$$

where $F(t)$ denotes the value of $f(z)$ at the point $z = x(t) + iy(t)$ of L corresponding to the parameter value t.

Proof.

Consider the sum

$$\sum = \sum_{k=1}^{n} f(\zeta_k)(z_k - z_{k-1})$$

where ζ_k is a point of L between z_{k-1} and z_k and $\zeta_k = z(\tau_k), t_{k-1} \leq \tau_k \leq t_k$. Write

$$F(t) = \phi(t) + i\psi(t),$$

Integrations along Regular Arcs

where $\phi(t)$ and $\psi(t)$ are real-valued functions of parameter t. Then we have

$$\begin{aligned}\sum =&\sum_{k=1}^{n}\phi(\tau_k)(x_k-x_{k-1})+i\sum_{k=1}^{n}\psi(\tau_k)(x_k-x_{k-1})+i\sum_{k=1}^{n}\phi(\tau_k)(y_k-y_{k-1})-\sum_{k=1}^{n}\psi(\tau_k)(y_k-y_{k-1})\\ =&\Sigma_1+i\Sigma_2+i\Sigma_3-\Sigma_4,\ \text{say.}\end{aligned}$$

By the mean value theorem of differential calculus, the first sum is

$$\Sigma_1=\sum_{k=1}^{n}\phi(\tau_k)\dot{x}(\tau_k')(t_k-t_{k-1}),$$

where $t_{k-1}\le\tau_k'\le t_k$. Let $\Sigma_1'=\sum_{k=1}^{n}\phi(t_k)\dot{x}(t_k)(t_k-t_{k-1})$, by making the norm of P, i.e., $|P|$ sufficiently small, we show that

$$|\Sigma_1-\Sigma_1'|<M(\epsilon)\epsilon.$$

Now, by the hypothesis, $\phi(t)$ and $\dot{x}(t)$ are continuous and, since every continuous function is bounded, there exists a positive number M such that the inequalities:

$$|\phi(t)|\le M,\quad |\dot{x}(t)|\le M$$

hold for $a\le t\le b$.

Integrations along Regular Arcs

Again, since a continuous function is necessarily uniformly continuous, we can preassign an arbitrary positive number ϵ and then we can choose a positive number $\delta = \delta(\epsilon)$ such that

$$\left|\phi(t) - \phi\left(t'\right)\right| < \epsilon, \quad \left|\dot{x}(t) - \dot{x}\left(t'\right)\right| < \epsilon,$$

where $|t - t'| < \delta$. Hence, if $|P| < \delta$, then we have

$$\begin{aligned}&|\ \phi\left(\tau_k\right)\dot{x}\left(\tau_k'\right) - \phi\left(t_k\right)\dot{x}\left(t_k\right) = \left|\phi\left(\tau_k\right)\left\{\dot{x}\left(\tau_k'\right) - \dot{x}\left(t_k\right)\right\} + \dot{x}\left(t_k\right)\left\{\phi\left(\tau_k\right) - \phi\left(t_k\right)\right\}\right| \\ &\leq \left|\phi\left(\tau_k\right)\right|\left|\dot{x}\left(\tau_k'\right) - \dot{x}\left(t_k\right)\right| + \left|\dot{x}\left(t_k\right)\right|\left|\phi\left(\tau_k\right) - \phi\left(t_k\right)\right| < 2M\epsilon\end{aligned}$$

and so it follows that $\left|\Sigma_1 - \Sigma_1'\right| < 2M\epsilon(b-a)$.
By the definition of the integral of functions of a real variable, Σ_1' tends to the limit

$$\int_a^b \phi(t)\dot{x}(t)dt = \lim_{n\to\infty, |P|\to 0} \Sigma_1'.$$

The remaining $\Sigma's$ tend to corresponding limits in the same manner. Then Σ tends to the limit

$$\int_a^b \{\phi(t)\dot{x}(t) - \psi(t)\dot{y}(t)\}dt + i\int_a^b \{\psi(t)\dot{x}(t) + \phi(t)\dot{y}(t)\}dt = \int_a^b F(t)\{\dot{x}(t) + i\dot{y}(t)\}dt.$$

Examples

Example

1. Evaluate $\int_C \frac{dz}{z}$, where C is the circle with center at the origin and radius r.
2. Evaluate $\int_C \frac{dz}{z-\alpha}$, where C represents a circle $|z-\alpha| = r$.
3. Evaluate $\int_C f(z)dz$, if $f(z) \equiv 1$, and C is any smooth curve.

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Solutions:

1. $$\int_C \frac{dz}{z} = \int_0^{2\pi} \frac{1}{re^{it}} rie^{it} dt = i \int_0^{2\pi} dt = 2\pi i.$$

2. $$\int_C \frac{dz}{z-\alpha} = \int_0^{2\pi} \frac{1}{re^{it}} rie^{it} dt = i \int_0^{2\pi} dt = 2\pi i.$$

3. $$\int_C f(z)dz = \int_a^b \dot{z}(t)dt = z(b) - z(a).$$

An important integral

Theorem

Let C be a circular path with center z_0 and radius $r > 0$ traversed in the anticlockwise direction. Then

$$\int_C (z - z_0)^n \, dz = \begin{cases} 2\pi i & \text{if } n = -1, \\ 0 & \text{if } n \neq -1. \end{cases}$$

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Proof.

We have $C(t) = z_0 + r\exp(it), t \in [0, 2\pi]$, and so $C'(t) = ir\exp(it), t \in [0, 2\pi]$.
1 : When $n = -1$, we have

$$\int_C (z - z_0)^n \, dz = \int_C (z - z_0)^{-1} \, dz = \int_0^{2\pi} \frac{1}{r\exp(it)} \cdot ir\exp(it)dt = \int_0^{2\pi} idt = 2\pi i.$$

2 : When $n \neq -1$, we have

$$\begin{aligned} \int_C (z - z_0)^n \, dz &= \int_0^{2\pi} r^n \exp(nit) \cdot ir\exp(it)dt = \int_0^{2\pi} ir^{n+1} \exp(i(n+1)t)dt \\ &= -r^{n+1} \int_0^{2\pi} \sin((n+1)t)dt + ir^{n+1} \int_0^{2\pi} \cos((n+1)t)dt = 0 + 0 = 0. \end{aligned}$$

$\square$

Integrations along Regular Arcs

Theorem

Let C be the given curve. Then

$$\int_{-C} f(z)dz = -\int_C f(z)dz.$$

Proof.

We have

$$\int_{-C} f(z)dz = -\int_a^b f(z(b+a-t))\dot{z}(b+a-t)dt$$

Now expanding the integral into real and imaginary parts and applying the change of variable theorem to each real integral, we obtain

$$\int_{-C} f(z)dz = \int_b^a f(z(t))\dot{z}(t)dt = -\int_C f(z)dz.$$

□

Complex Integrals as Sum of Two Real Line Integrals

Example

1. Prove that the value of the integral of $\frac{1}{z}$ along a semi-circular arc $|z| = a$ from $-a$ to $+a$ is $-\pi i$ or πi if the arc lies above or below the real axis.
2. Find the value of the integral $\int_0^{1+i} (x - y + ix^2)\, dz$
   - along the straight line from $z = 0$ as $z = 1 + i$; $\quad (\frac{-1+i}{3})$
   - along the real axis from $z = 0$ to $z = 1$ and then along a line parallel to the imaginary axis from $z = 1$ to $z = 1 + i$. $\quad (\frac{-1}{2} + \frac{5i}{6})$
3. Evaluate the integral $\int_0^{1+i} z^2 dz$.
4. Evaluate the integral $\int_{-2+i}^{5+3i} z^3 dz$.

The Absolute Value of Complex Integrals

Theorem

Let $f(z)$ be continuous on a contour L of length l and let $|f(z)| \leq M$ for every point z on L. Then we have

$$\left|\int_L f(z)dz\right| \leq Ml.$$

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$$\left| \int_L f(z) dz \right| \leq Ml.$$

Proof.

Without loss of generality, we may assume that L is a regular arc. Now, we have

$$\Sigma = \sum_{k=1}^{n} f(z_k)\left(z_k - z_{k-1}\right).$$

Since the modulus of the sum is less than or equal to the sum of the moduli, we have

$$|\Sigma| = \left| \sum_{k=1}^{n} f(z_k)\left(z_k - z_{k-1}\right) \right| \leq \sum_{k=1}^{n} \left| f(z_k)\left(z_k - z_{k-1}\right) \right| = \sum_{k=1}^{n} |f(z_k)| \left| \left(z_k - z_{k-1}\right) \right| \leq M \sum_{k=1}^{n} \left| \left(z_k - z_{k-1}\right) \right|.$$

Therefore, we have

$$\lim_{n\to\infty} \left| \sum_{k=1}^{n} f(z_k)\left(z_k - z_{k-1}\right) \right| \leq M \lim_{n\to\infty} \sum_{k=1}^{n} \left| \left(z_k - z_{k-1}\right) \right| \Rightarrow \left| \int_L f(z) dz \right| \leq M \int_L |dz| = Ml.$$

Line Integrals as Functions of Arcs

Observe that a line integral $\int_L f(z)dz$ over an arc L can be put in the form, i.e.,

$$\int_L (u+iv)(dx+idy), \quad \text{or} \int_L pdx+qdy.$$

General line integrals of the form $\int_L pdx+qdy$ are often studied as functions (or functionals) of the arc L under the assumption that p, q are defined and continuous in a domain D such that L is free to vary in D. An important class of integrals is characterized by the property that the integral over an arc depends only on its end points. This means that, if the two arcs L_1 and L_2 have the same initial point and the same end point, then we have

$$\int_{L_1} pdx+qdy = \int_{L_2} pdx+qdy.$$

Notice that an integral depends only on the end points is equivalent to saying that the integral over any closed curve is zero. Indeed, if L is a closed curve, then L and $-L$ have the same end points and, if the integral depends only on the end points, then we obtain

$$\int_L = \int_{-L} = -\int_L$$

and, consequently, $\int_L = 0$. Conversely, if L_1 and L_2 have the same end points, then $L_1 - L_2$ is a closed curve and, if the integral over any closed curve vanishes, then we see that

$$\int_{L_1} = \int_{L_2}.$$

Line Integrals as Functions of Arcs

The following theorem gives a necessary and sufficient condition under which a line integral depends only on the end points.

Theorem

The line integral $\int_L pdx + qdy$, defined in a domain D, depends only on the end points of L if and only if there exists a function $U(x, y)$ in D with the partial derivatives $\frac{\partial U}{\partial x} = p$ and $\frac{\partial U}{\partial y} = q$.

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Proof.

Sufficiency: For, if the condition is fulfilled and a, b are the end points of L, then we can write, with the usual notations,

$$\begin{aligned}\int_L pdx + qdy &= \int_L \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy \\ &= \int_a^b \left(\frac{\partial U}{\partial x} x'(t) + \frac{\partial U}{\partial y} y'(t) \right) dt \\ &= \int_a^b \left(\frac{d}{dt} U(x(t), y(t)) \right) dt \\ &= U(x(b), y(b)) - U(x(a), y(a))\end{aligned}$$

and the value of the difference depends only on the end points.

Line Integrals as Functions of Arcs

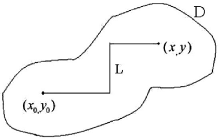
Necessity: we choose a fixed point $(x_0, y_0) \in D$, join it to (x, y) by a polygon L, contained in D, whose sides are parallel to the coordinate axes (see Fig. 2). Now, we define a function U by

$$U(x,y) = \int_L p dx + q dy.$$

By the hypothesis, the integral depends only on the end points and so it is well defined. Further, if we choose the last segment of L horizontal, we can keep y constant and let x vary without changing the other segments. Choosing x as a parameter on the last segment, we obtain

$$U(x,y) = \int^x p(x,y) dx + \text{ constant}, \tag{1}$$

the lower limit of the integral being irrelevant. From (1). it follows at once that $\frac{\partial U}{\partial x} = p$. In the same way, by choosing the last segment vertical, we can show that $\frac{\partial U}{\partial y} = q$. $\square$



Line Integrals as Functions of Arcs

It is customary to write $dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy$, and an expression $pdx + qdy$ which can be written in this form is an exact differential. Using this terminology, the above theorem can be stated as:

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Now, we determine the conditions under which

$$f(z)dz = f(z)dx + if(z)dy$$

is an exact differential.

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By the definition of an exact differential, there must exist a function $F(z)$ in D with the partial derivatives

$$\frac{\partial F(z)}{\partial x} = f(z), \quad \frac{\partial F(z)}{\partial y} = if(z).$$

It follows that

$$\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y},$$

which is a Cauchy-Riemann equation.

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Also, $f(z)$ is, by the assumption, continuous (otherwise, $\int_L f(z)dz$ would not be defined). Hence $F(z)$ is analytic with the derivative $f(z)$.

Line Integrals as Functions of Arcs

From the above discussion, we conclude:

Theorem

The integral $\int_L f(z)dz$, with continuous f, depends only on the end points of L if and only if f is the derivative of an analytic function in D.

Green's Theorem for Two Dimensions

Theorem (Green's Theorem)

Let $P(x, y)$ and $Q(x, y)$ be single-valued and continuous functions of x and y, possessing continuous partial derivatives with respect to both the variables x and y in a simply connected region containing a closed contour C. Then the double integral taken over the simply connected region D enclosed by C, i.e.,

$$\iint_D \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) dxdy,$$

equals the curvilinear integrals taken round C, i.e.,

$$\int_C (Pdx + Qdy).$$

Elementary Form of Cauchy's Theorem

Theorem (Cauchy's Theorem)

Let $f(z)$ be a regular function and let $f'(z)$ be continuous at each point within and on a closed contour C. Then

$$\int_C f(z)dz = 0.$$

Elementary Form of Cauchy's Theorem

Theorem (Cauchy's Theorem)

Let $f(z)$ be a regular function and let $f'(z)$ be continuous at each point within and on a closed contour C. Then

$$\int_C f(z)dz = 0.$$

Proof.

Let D denote the closed domain consisting of all points within and on C. Assume that $f(z) = u(x, y) + iv(x, y)$. Then we see that

$$\int_C f(z)dz = \int_C (udx - vdy) + i \int_C (vdx + udy). \tag{2}$$

By the Cauchy-Riemann equations, we have $f'(z) = u_x + iv_x = v_y - iu_y$. Since $f'(z)$ is continuous in D, the four partial derivatives u_x, u_y, v_x and v_y exist are all continuous in D. Thus the conditions of Green's theorem are satisfied. Hence, from (2) and the Cauchy-Riemann equations, it follows that

$$\begin{aligned}\int_C f(z)dz &= -\iint_D \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dxdy + i \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy \\ &= \iint_D \left(-\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\right) dxdy + i \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dxdy\end{aligned}$$

Index of Closed Curves with Respect to a Point

It is well known that, if $\gamma(t) = a + e^{2\pi int}$, then

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} = n.$$

Let γ be a closed rectifiable curve and $\{\gamma\}$ be its graph. Then we have the following result.

Theorem

If $\gamma : [0,1] \to \mathbb{C}$ is a closed rectifiable curve, n is an integer and $a \notin \{\gamma\}$, then

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} = n.$$

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It is well known that, if $\gamma(t) = a + e^{2\pi int}$, then

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Proof.

We prove this result by assuming that γ is smooth. Define

$$g(t) = \int_0^t \frac{\gamma'(s)}{\gamma(s)-a} ds.$$

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continue to prove.

By the definition of g, we have

$$g(0) = 0, \quad g(1) = \int_\gamma \frac{dz}{z-a}.$$

Also, we have $g'(t) = \frac{\gamma'(t)}{\gamma(t)-a} \quad (0 \le t \le 1)$.
But this gives

$$\frac{d}{dt} e^{-g}(\gamma - a) = e^{-g}\gamma' - g' e^{-g}(\gamma - a) = e^{-g}\left[\gamma' - \gamma'(\gamma - a)^{-1}(\gamma - a)\right] = 0$$

and so $e^{-g}(\gamma - a)$ is the constant function $e^{-g(0)}(\gamma(0) - a) = \gamma(0) - a = e^{-g(1)}(\gamma(1) - a)$.
Since $\gamma(0) = \gamma(1)$, it follows that $e^{-g(1)} = 1$ or $g(1) = 2\pi$ in for some integer n. Thus we have

$$\int_\gamma \frac{dz}{z-a} = 2\pi i n.$$

Therefore, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} = n.$$

Index of Closed Curves with Respect to a Point

Now, we introduce the concept of the index of a closed curve with respect to a point.
Let γ be a closed rectifiable curve and a be a point not on the graph of γ. Then the index or winding number $n(\gamma; a)$ of γ with respect to a point a is defined by the integral

$$n(\gamma; a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - a}$$

Geometrically speaking, the winding number counts the number of rounds of a path around the point. It may easily be observed that the winding number is a positive integer with a positively oriented contour and a negative integer for contours with negative orientation. If the point a is enclosed by a simple closed contour, then $n(\gamma; a) = 1$ and, if a is outside of the path, then $n(\gamma; a) = 0$.
As a direct consequence of the definition, we have the following assertions: If $\{\gamma\} = \{\gamma_1\} \cup \{\gamma_2\}$ is the sum of paths of two curves γ_1 and γ_2 in a complex plane, then, for all $a \notin \{\gamma\}$,

1. $n(\gamma; a) = n(\gamma_1; a) + n(\gamma_2; a)$
2. $n(-\gamma; a) = -n(\gamma; a)$.

Recall that, if $\gamma : [0, 1] \to C$ is a curve, then $-\gamma$ is the curve defined by

$$(-\gamma)(t) = \gamma(1 - t).$$

If γ_1 and γ_2 are curves defined on $[0, 1]$ with $\gamma_1(1) = \gamma_2(0)$, then $\gamma_1 + \gamma_2$ is the curve defined by

$$(\gamma_1 + \gamma_2)(t) = \begin{cases} \gamma(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \gamma(1 - 2t), & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

General Form of Cauchy's Theorem

Theorem (Cauchy-Goursat's Theorem)

If a function $f(z)$ is analytic and single-valued inside and on a simple closed contour C, then

$$\int_C f(z)dz = 0.$$

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Theorem (The Extension of Cauchy's Theorem to Multiply Connected Regions)

Let $f(z)$ be analytic in the multiply connected region D bounded by the closed contour C and the two interior contours C_1, C_2, then

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz,$$

where C, C_1 and C_2 are all the three traversed in the positive sense (i.e., anticlockwise direction.)