Propositional Equivalences

Logical equivalence

- Tautologies (重言式，永真式), Contradictions (矛盾式，永假式), and Contingencies (可满足式).
- Logical equivalence (逻辑等价)
  - Important logical equivalences
  - Showing logical equivalence
- Normal forms (范式)
  - Disjunctive normal form (析取范式)
  - Conjunctive normal form (合取范式)

Tautologies, Contradictions, and Contingencies

- A tautology is a (compound) proposition which is always true.
  - Example: $p \vee \neg p$
- A contradiction is a (compound) proposition which is always false. (negation of a tautology)
  - Example: $p \wedge \neg p$
- A contingency is a (compound) proposition which is neither a tautology nor a contradiction, such as p

P	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

In general it is "difficult" to tell whether a statement is a contradiction.

It is one of the most important problems in CS – the satisfiability problem.

Logically equivalent

- Two compound propositions p and q are logically equivalent if $p \boxed{\leftrightarrow} q$ is a tautology.
- We write this as $p \Leftrightarrow q$ or as $p \equiv q$ where p and q are compound propositions.
- Two compound propositions p and q are equivalent if and only if the columns in a truth table giving their truth values agree.
- This truth table show $\neg p \vee q$ is equivalent to $p \rightarrow q$.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Logical equivalence

$$p \to q \equiv ?$$

If you see a question in the above form, there are usually 3 ways to deal with it.

(1) Truth table

(2) Use logical rules

(3) Intuition

De Morgan's Laws

Augustus De Morgan, 1806-1871

Logical equivalence: Two statements have the same truth table

Statement: Tom is in the football team and the basketball team.

Negation: Tom is not in the football team or not in the basketball team.

De Morgan's Law

$$\boxed{\neg(p \wedge q) \equiv \neg p \vee \neg q}$$

Why the negation of the above statement is not the following
"Tom is not in the football team <u>and</u> not in the basketball team"?

The definition of the negation is that exactly one of P or ¬P is true, but it could be the case that both the above statement and the original statement are false (e.g. Tom is in the football team but not in the basketball team).

De Morgan's Laws

Logical equivalence: Two statements have the same truth table

Statement: The number 783477841 is divisible by 7 or 11.

Negation: The number 783477841 is not divisible by 7 and not divisible by 11.

De Morgan's Law

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Again, the negation of the above statement is not
"The number 783477841 is not divisible by 7 or not divisible by 11".

In either case, we"flip" the inside operator from OR to AND or from AND to OR.

De Morgan's Laws

De Morgan's Law

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

p	q	¬p	¬q	p∧q	¬(p∧q)	¬p∨¬q
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

De Morgan's Law

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

p	q	¬p	¬q	(p∨q)	¬(p∨q)	¬p∧¬q
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Key logical equivalences

- Identity Laws (恒等律):

$$p \wedge T \equiv p \qquad p \vee F \equiv p$$

- Domination Laws (支配律):

$$p \vee T \equiv T \qquad p \wedge F \equiv F$$

- Idempotent laws (幂等律):

$$p \vee p \equiv p \qquad p \wedge p \equiv p$$

- Double Negation Law (双非律):

$$\neg(\neg p) \equiv p$$

- Negation Laws (否定律):

$$p \vee \neg p \equiv T \qquad p \wedge \neg p \equiv F$$

Key logical equivalences

- Commutative Laws (交换律):

$$p \vee q \equiv q \vee p \qquad p \wedge q \equiv q \wedge p$$

- Associative Laws (结合律):

$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$
$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$

- Distributive Laws (分配律):

$$(p \vee (q \wedge r) \equiv (p \vee q)) \wedge (p \vee r)$$
$$(p \wedge (q \vee r)) \equiv (p \wedge q) \vee (p \wedge r)$$

- Absorption Laws (吸收率):

$$p \vee (p \wedge q) \equiv p \quad p \wedge (p \vee q) \equiv p$$

Homework->proof these using the aforementioned Key logical equivalences!

More logical equivalences

TABLE 7 Logical Equivalences Involving Conditional Statements.

$$\boxed{p \to q \equiv \neg p \vee q}$$

$$p \to q \equiv \neg q \to \neg p$$

$$p \vee q \equiv \neg p \to q$$

$$p \wedge q \equiv \neg(p \to \neg q)$$

$$\neg(p \to q) \equiv p \wedge \neg q$$

$$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$$

$$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$$

$$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$$

$$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$$\boxed{p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)}$$

$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$

$$\boxed{p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)}$$

$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

Constructing new logical equivalences

- We can show that two expressions are logically equivalent by developing a series of logically equivalent statements.
- To prove that $A \equiv B$ we produce a series of equivalences beginning with A and ending with B.

$$\begin{aligned} A &\equiv A_1 \\ &\vdots \\ A_n &\equiv B \end{aligned}$$

- Keep in mind that whenever a proposition (represented by a propositional variable) occurs in the equivalences listed earlier, it may be replaced by an arbitrarily complex compound proposition.

Equivalence proofs

Example: Show that $\neg(p \vee (\neg p \wedge q))$

is logically equivalent to $\neg p \wedge \neg q$

Solution:

$$
\begin{array}{rcll}
\neg(p \vee (\neg p \wedge q)) & \equiv & \neg p \wedge \neg(\neg p \wedge q) & \text{by the second De Morgan law} \\
& \equiv & \neg p \wedge [\neg(\neg p) \vee \neg q] & \text{by the first De Morgan law} \\
& \equiv & \neg p \wedge (p \vee \neg q) & \text{by the double negation law} \\
& \equiv & (\neg p \wedge p) \vee (\neg p \wedge \neg q) & \text{by the second distributive law} \\
& \equiv & F \vee (\neg p \wedge \neg q) & \text{because } \neg p \wedge p \equiv F \\
& \equiv & (\neg p \wedge \neg q) \vee F & \text{by the commutative law} \\
& & & \text{for disjunction} \\
& \equiv & (\neg p \wedge \neg q) & \text{by the identity law for } \mathbf{F}
\end{array}
$$

Equivalence proofs

Example: Show that $$(p \wedge q) \rightarrow (p \vee q)$$ is a tautology.

Solution:

$$\begin{array}{lcll}
(p \wedge q) \rightarrow (p \vee q) & \equiv & \neg(p \wedge q) \vee (p \vee q) & \text{by truth table for } \rightarrow \\
& \equiv & (\neg p \vee \neg q) \vee (p \vee q) & \text{by the first De Morgan law} \\
& \equiv & (\neg p \vee p) \vee (\neg p \vee \neg q) & \text{by associative and} \\
& & & \text{commutative laws} \\
& & & \text{laws for disjunction} \\
& \equiv & T \vee T & \text{by truth tables} \\
& \equiv & T & \text{by the domination law}
\end{array}$$

Simplifying statement

We can use logical rules to simplify a logical formula.

$$\neg(\neg p \land q) \land (p \lor q)$$

$$\equiv (\neg\neg p \lor \neg q) \land (p \lor q)$$ De Morgan

$$\equiv (p \lor \neg q) \land (p \lor q)$$

$$\equiv p \lor (\neg q \land q)$$ Distributive law

$$\equiv p \lor \text{False}$$

$$\equiv p$$

The De Morgan's Law allows us to always "**move the NOT inside**".

Checkpoint

Key points to know.

1. Write a logical formula from a truth table.
2. Check logical equivalence of two logical formulas.
3. De Morgan's rule and other logical rules.
4. Use simple logical rules to simplify a logical formula.