Set Theory

Sets II

Set operations

Content

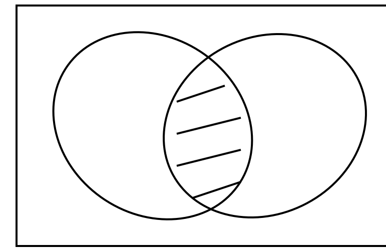
- Set Operations
  - Union (并)
  - Intersection (交)
  - Difference (差)
  - Complement (补)
  - Symmetric Difference (对称差) (Option)

Basic operations on sets

Let A, B be two subsets of a *universal* set U

(depending on the context U could be R, Z, or other sets).

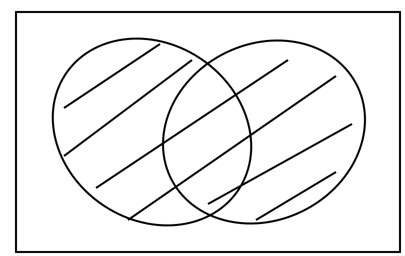
intersection: $A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$



Defintion: Two sets are said to be **disjoint** if their intersection is an empty set.

e.g. Let A be the set of odd numbers, and B be the set of even numbers. Then A and B are disjoint.

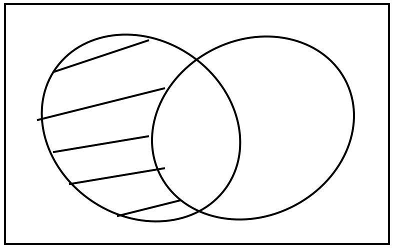
union: $A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$



Fact: $|A \cup B| = |A| + |B| - |A \cap B|$

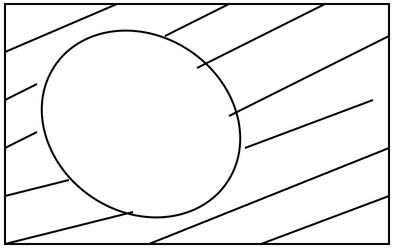
Basic operations on sets

difference: $A - B = \{x \in U \mid x \in A \text{ and } x \notin B\}$



Fact: $|A - B| = |A| - |A \cap B|$

complement: $\overline{A} = A^c = \{x \in U \mid x \notin A\}$



e.g. Let U = Z and A be the set of odd numbers.

Then $\overline{A}$ is the set of even numbers.

Fact: If $A \subseteq B$, then $\overline{B} \subseteq \overline{A}$

Examples

$A = \{1, 3, 6, 8, 10\}$ $B = \{2, 4, 6, 7, 10\}$

$A \cap B = \{6, 10\}$, $A \cup B = \{1, 2, 3, 4, 6, 7, 8, 10\}$ $A-B = \{1, 3, 8\}$

Let $U = \{ x \in Z \mid 1 <= x <= 100\}$.

$A = \{ x \in U \mid x$ is divisible by $3\}$, $B = \{ x \in U \mid x$ is divisible by $5\}$

$A \cap B = \{ x \in U \mid x$ is divisible by $15\}$

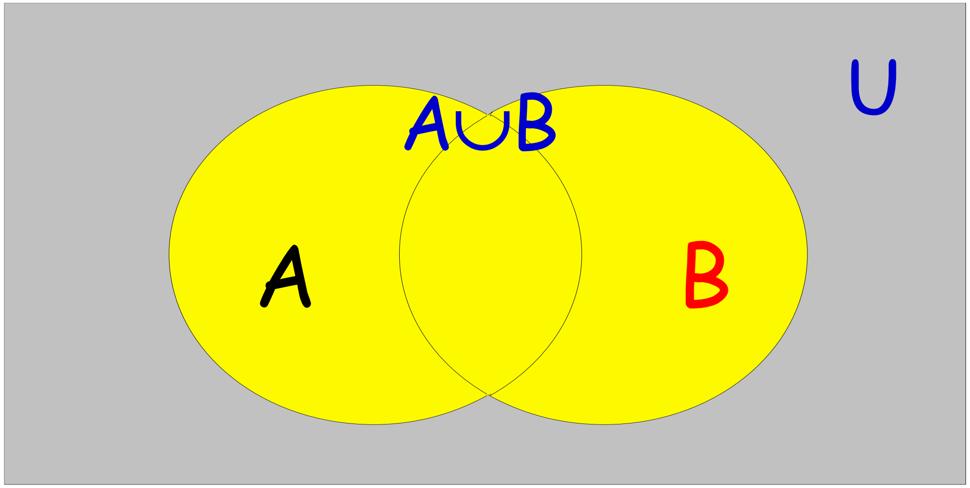
$A \cup B = \{ x \in U \mid x$ is divisible by 3 or is divisible by 5 (or both)$\}$

$A - B = \{ x \in U \mid x$ is divisible by 3 but is not divisible by $5 \}$

Exercise: compute $|A|$, $|B|$, $|A \cap B|$, $|A \cup B|$, $|A - B|$.

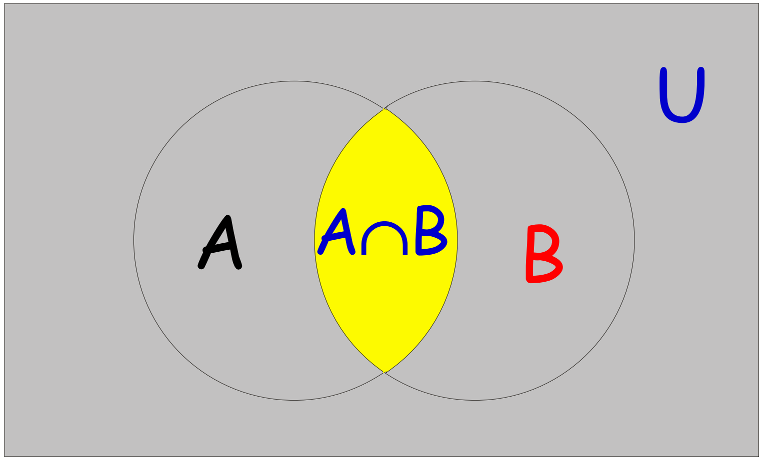
Set operations

- Union: Elements in at least one of the two sets.
  - $A \cup B = \{x \mid x \in A \vee x \in B\}$
  - Example:
    - $A = \{a, b\}$, $B = \{b, c, d\}$
    - $A \cup B = \{a, b, c, d\}$



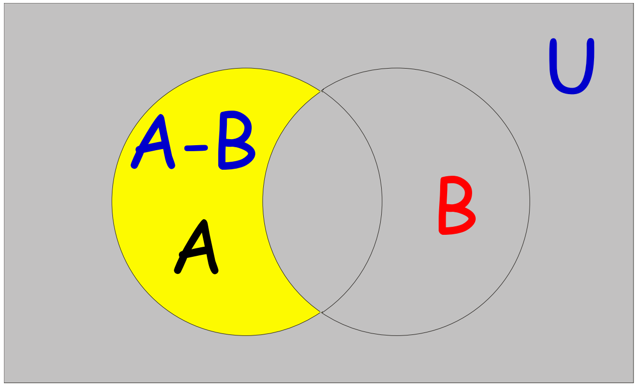
Set operations

- Intersection: Elements in exactly one of the two sets.
  - $A \cap B = \{x \mid x \in A \land x \in B\}$
  - Example:
    - $A = \{a, b\}$, $B = \{b, c, d\}$
    - $A \cap B = \{b\}$



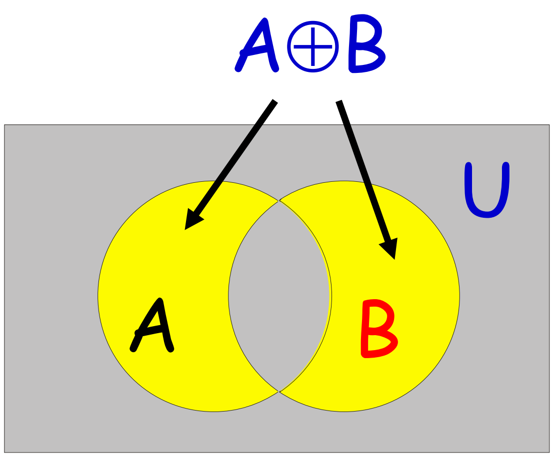
Set operations

- Difference: Elements in first set but not second. Difference is also called the relative complement （相对补） of B in A.
  - $A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap B^c$
  - Example
    - $A = \{a, b\}$, $B = \{b, c, d\}$
    - $A - B = \{a\}$



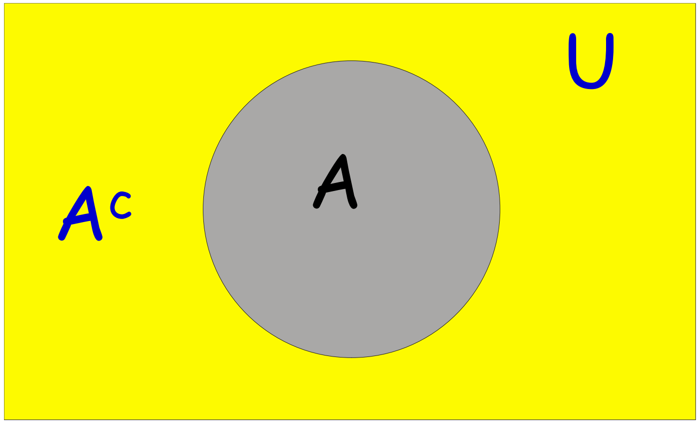
Set operations

- Symmetric Difference: Elements in exactly one of the two sets.
  - $A \oplus B = \{ x \mid x \in A \oplus x \in B \} = (A-B) \cup (B-A) = (A \cup B)-(A \cap B)$
  - Example:
    - $A = \{a, b\}$, $B = \{b, c, d\}$
    - $A \oplus B = \{a,c,d\}$



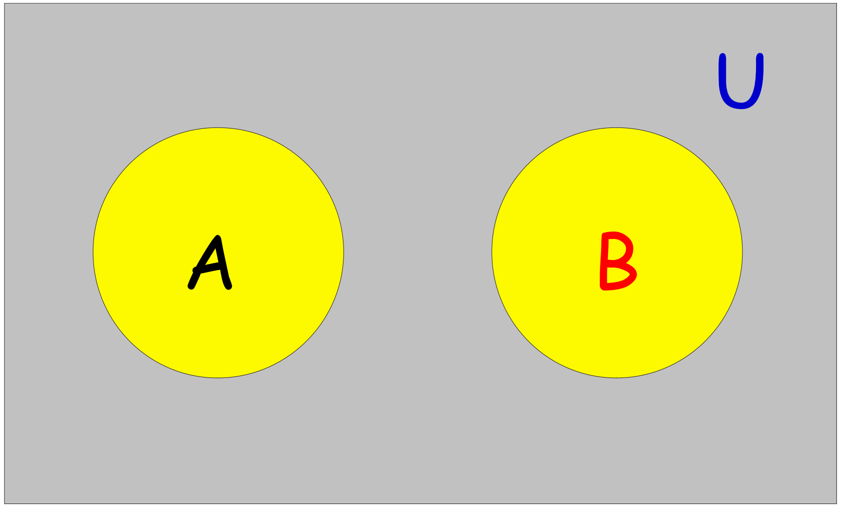
Set operations

- Complement: Elements not in the set (unary operator).
  - $A^c = \{ x \mid x \notin A \}$
  - Example:
    - $U = N$, $A = \{250, 251, 252, \ldots\}$
    - $A^c = \{0, 1, 2, \ldots, 248, 249\}$



Disjoint sets

- Disjoint: If A and B have no common elements, they are said to be disjoint.
  - $A \cap B = \varnothing$



Examples for set operations

- If A={1, 4, 7, 10}, B={1, 2, 3, 4, 5}
- $A \cup B$ =?
- $A \cap B$ =?
- $A - B$ =?
- $B - A$ =?
- $A \oplus B$ =?

Example for set operations

- If $A=\{1, 4, 7, 10\}$, $B=\{1, 2, 3, 4, 5\}$
- $A \cup B = \{1, 2, 3, 4, 5, 7, 10\}$
- $A \cap B = \{1, 4\}$
- $A - B = \{7, 10\}$
- $B - A = \{2, 3, 5\}$
- $A \oplus B = (A \cup B) - (A \cap B) = \{2, 3, 5, 7, 10\}$

Properties of set operations (1)

- Theorem: Let U be a universal set, and A, B and C subsets of U. The following properties hold:
- a) Associativity（结合律）:

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

- b) Commutativity（交换律）:

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

Properties of set operations (2)

- c) Distributive laws（分配律）：

  $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$

  $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$

- d) Identity laws（恒等律）：

  $A\cap U=A$ $\qquad$ $A\cup\varnothing = A$

- e) Complement laws （补集律）：

  $A\cup A^c = U$ $\qquad$ $A\cap A^c = \varnothing$

Properties of set operations (3)

- f) Idempotent laws（幂等律）：

  $A \cup A = A \qquad A \cap A = A$

- g) Bound laws （绑定律）：

  $A \cup U = U \qquad A \cap \varnothing = \varnothing$

- h) Absorption laws （吸收率）：

  $A \cup (A \cap B) = A \qquad A \cap (A \cup B) = A$

Properties of set operations (4)

- i) Double complementation /Involution law（退化律）:

  $(A^c)^c = A$

- j) 0/1 laws:

  $\varnothing^c = U \qquad U^c = \varnothing$

- k) De Morgan's laws:

  $(A \cup B)^c = A^c \cap B^c$

  $(A \cap B)^c = A^c \cup B^c$

  $A - (B \cup C) = (A-B) \cap (A-C)$

  $A - (B \cap C) = (A-B) \cup (A-C)$

Proof for set properties

- In fact, the logical identities create the set identities by applying the definitions of the various set operations.

- For example: $(A \cup B) \cup C = A \cup (B \cup C)$
- Proof :
  - $(A \cup B) \cup C = \{x \mid x \in A \cup B \lor x \in C\}$ (by def.)
  - $= \{x \mid (x \in A \lor x \in B) \lor x \in C\}$ (by def.)
  - $= \{x \mid x \in A \lor (x \in B \lor x \in C)\}$ (logic law)
  - $= \{x \mid x \in A \lor (x \in B \cup C)\}$ (by def.)
  - $= A \cup (B \cup C)$ (by def.)

- Other identities are derived similarly.

Proof for set properties

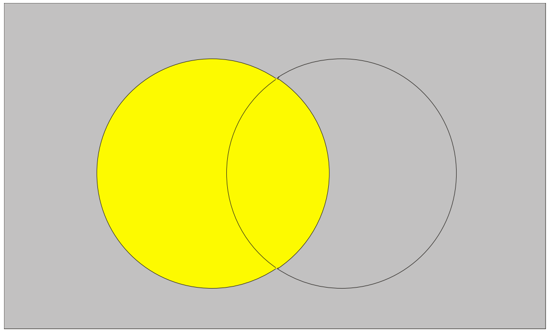
- It's often simpler to understand an identity by drawing a Venn Diagram.
- For example:
- DeMorgan's first law

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

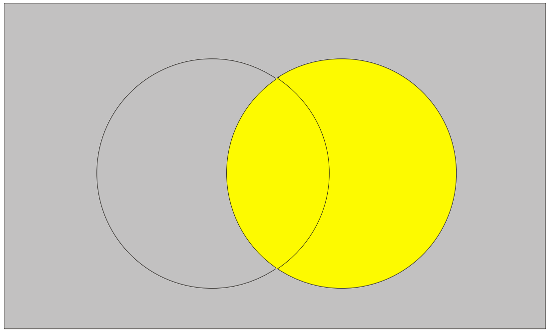
can be visualized as follows:

Visual De Morgan

A:

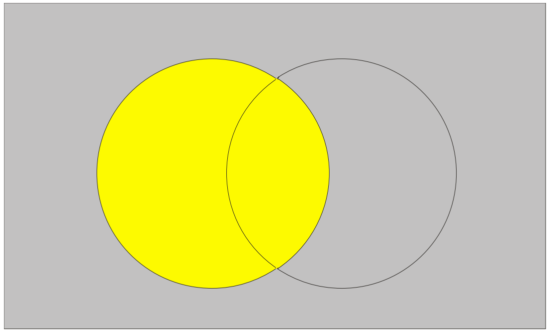


B:



Visual DeMorgan

A:



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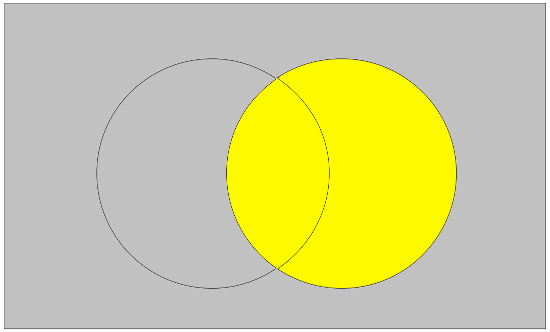
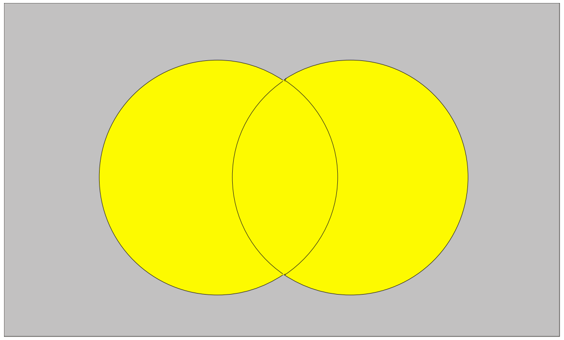
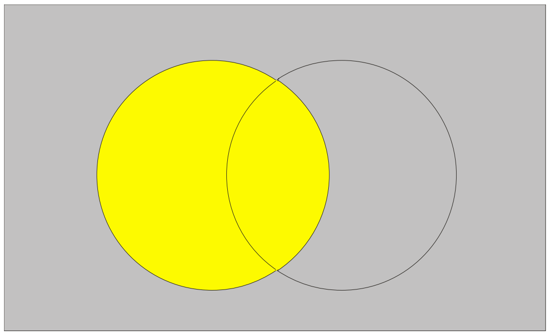


$A \cup B$:

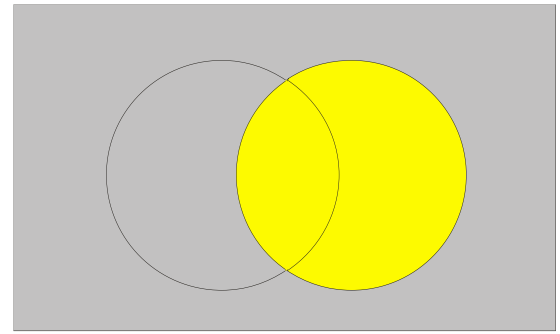


Visual De Morgan

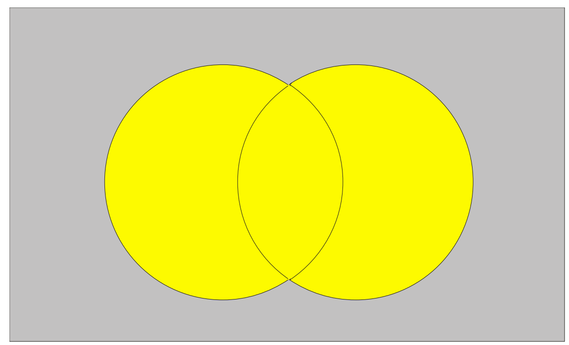
A:



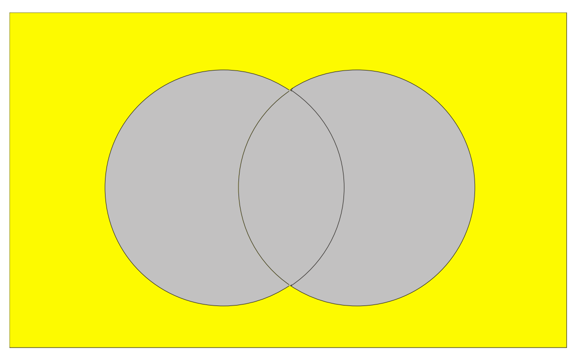
B:



$A \cup B$:

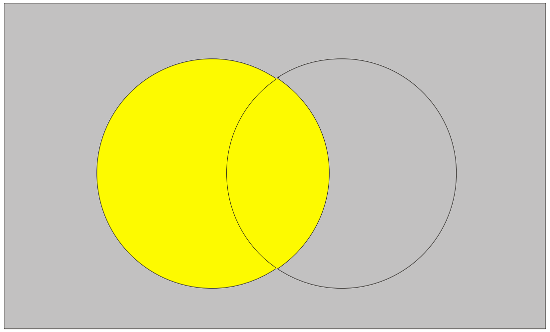


$\overline{A \cup B}$:

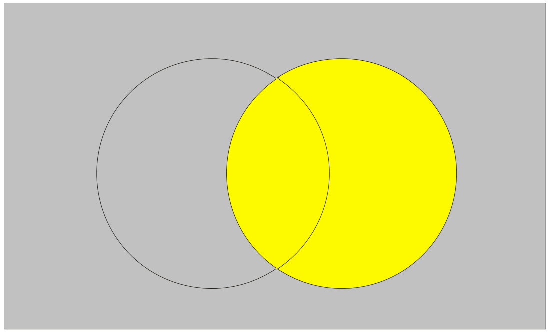


Visual De Morgan

A:

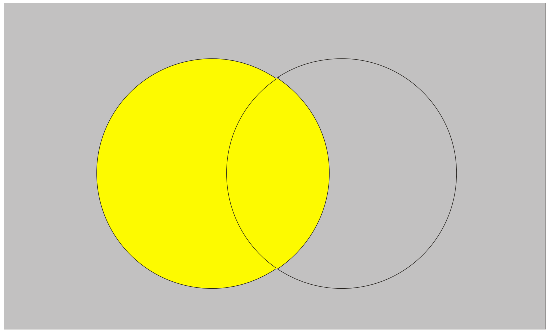


B:

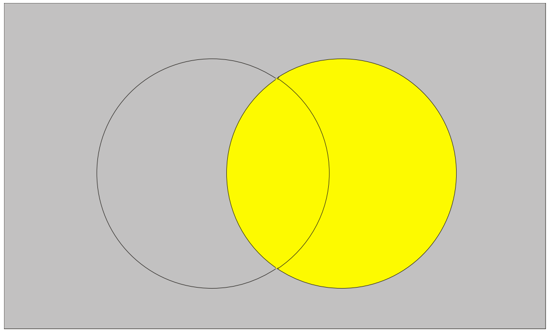


Visual De Morgan

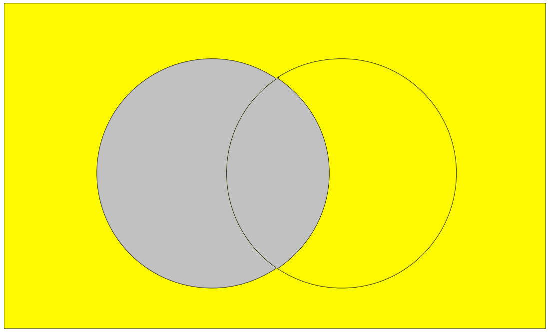
A:



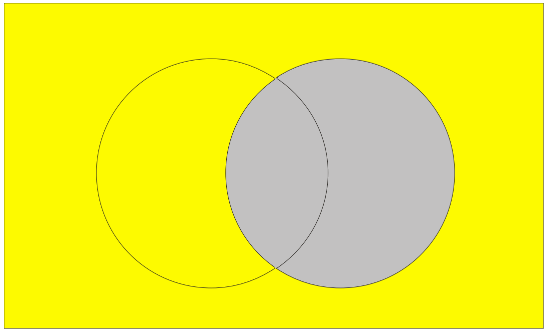
B:



$\overline{A}$:

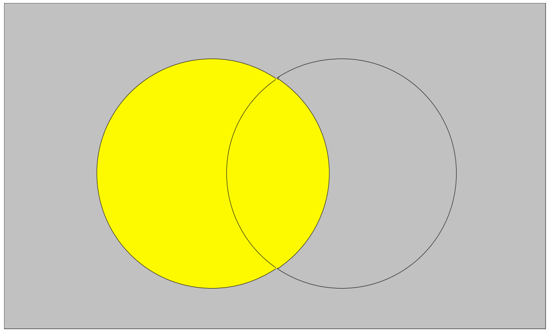


$\overline{B}$:

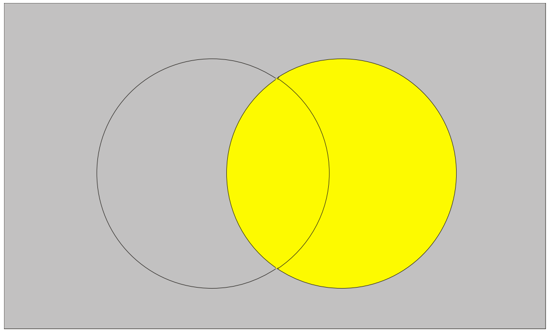


Visual De Morgan

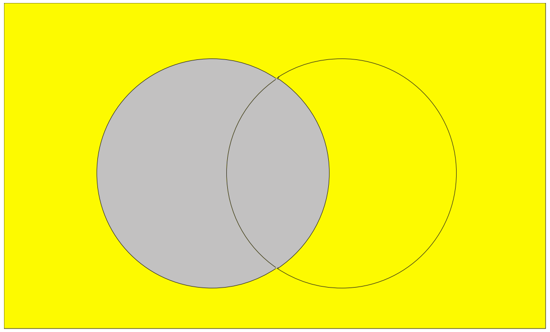
A:



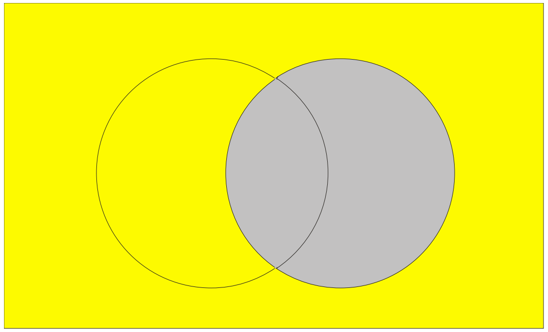
B:



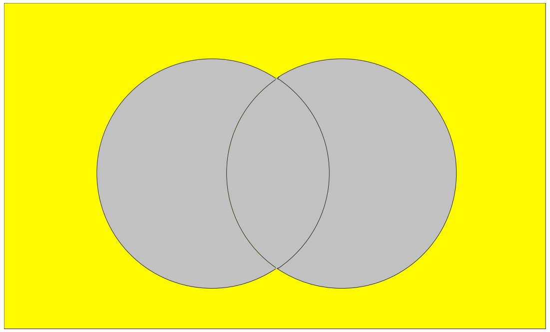
$\overline{A}$:



$\overline{B}$:

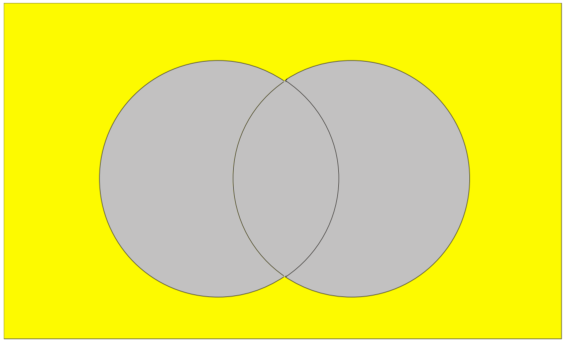


$\overline{A} \cap \overline{B}$:

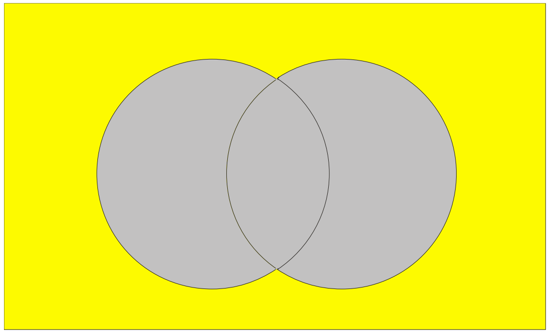


Visual De Morgan

$\overline{A \cup B} =$



$\overline{A} \cap \overline{B} =$



Set identities

Some basic properties of sets, which are true for all sets.

$$A \cap B \subseteq A$$

$$A \subseteq A \cup B$$

if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

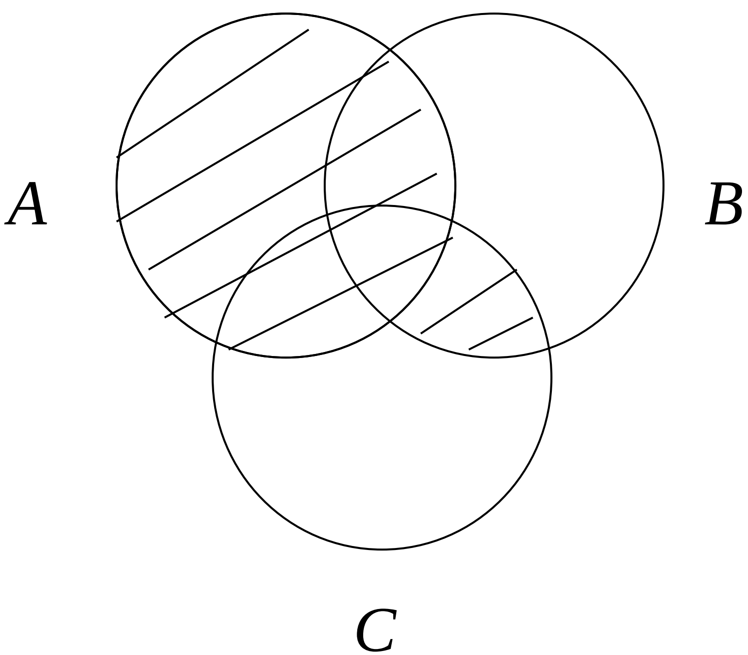
$$A \cap \overline{A} = \emptyset$$

$$\overline{\overline{A}} = A$$

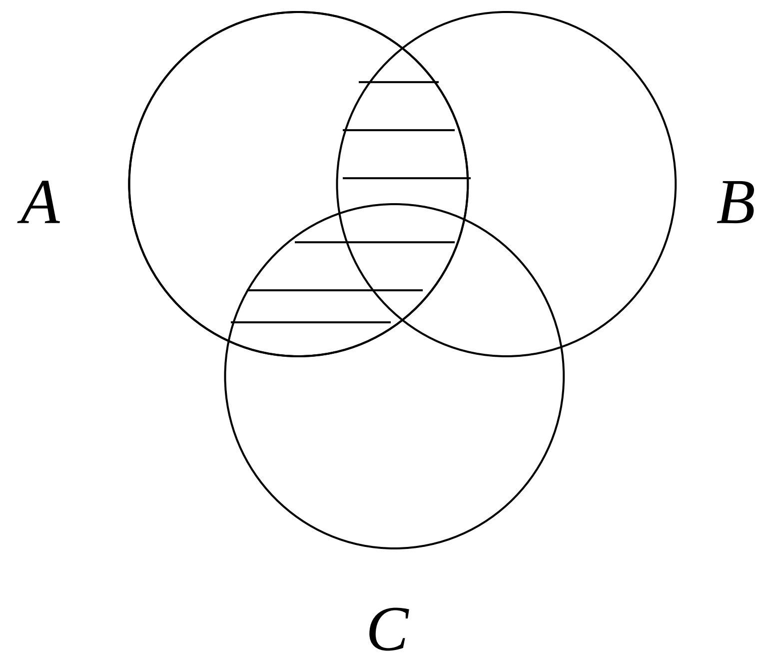
$$A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$$

Set identities

Distributive Law:
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad (1)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad (2)$$



(1)

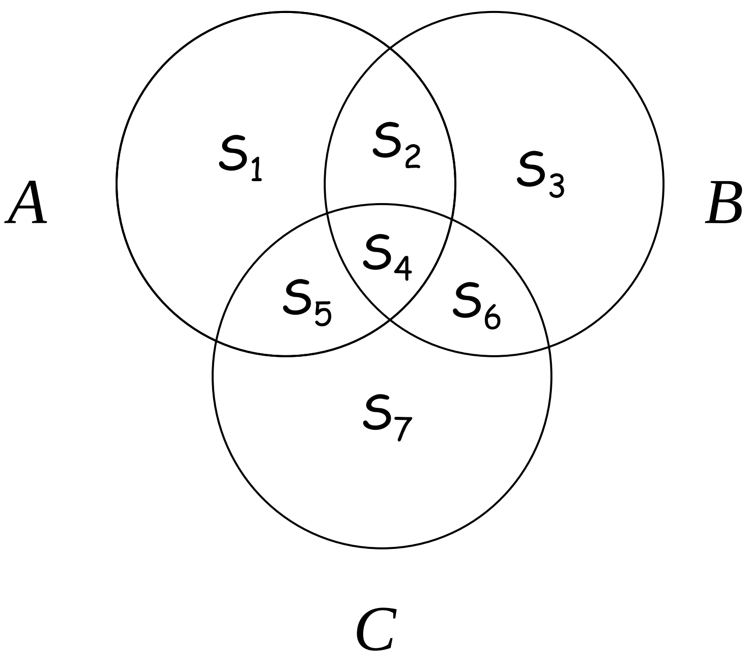


(2)

Set identities

Distributive Law: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

We can also verify this law more carefully



L.H.S

$A = S_1 \cup S_2 \cup S_4 \cup S_5$

$B \cap C = S_4 \cup S_6$

$A \cup (B \cap C) = S_1 \cup S_2 \cup S_4 \cup S_5 \cup S_6$

R.H.S.

$(A \cup B) = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6$

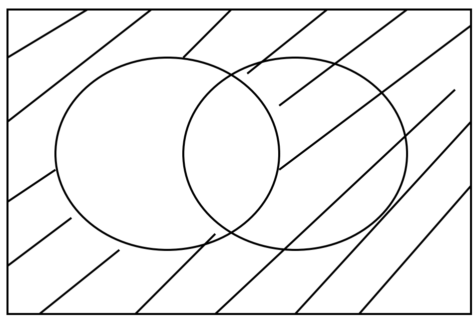
$(A \cup C) = S_1 \cup S_2 \cup S_4 \cup S_5 \cup S_6 \cup S_7$

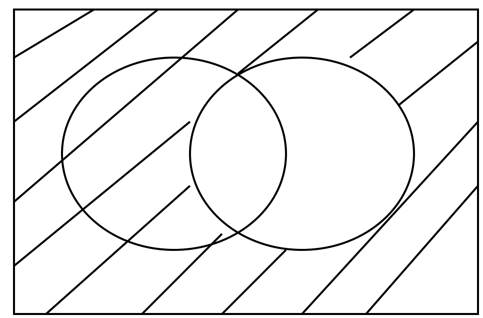
$(A \cup B) \cap (A \cup C) = S_1 \cup S_2 \cup S_4 \cup S_5 \cup S_6$

Set identities

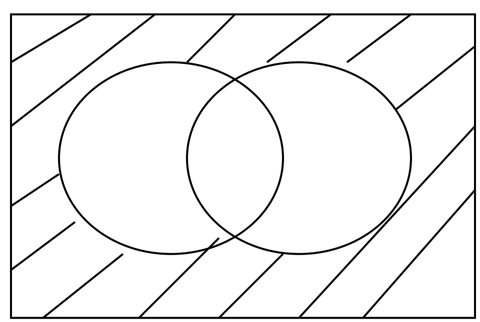
De Morgan's Law: $\overline{A \cup B} = \overline{A} \cap \overline{B}$

$\overline{A}$





$\overline{B}$

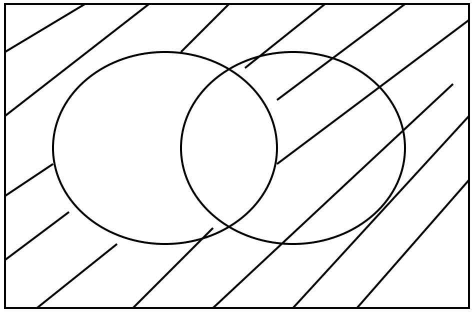


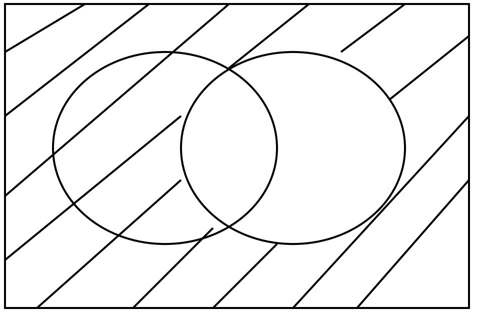
$\overline{A \cup B} = \overline{A} \cap \overline{B}$

Set identities

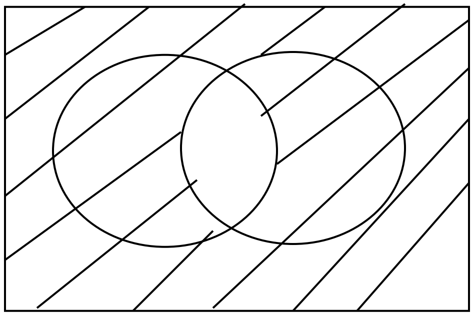
De Morgan's Law: $$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

$\overline{A}$





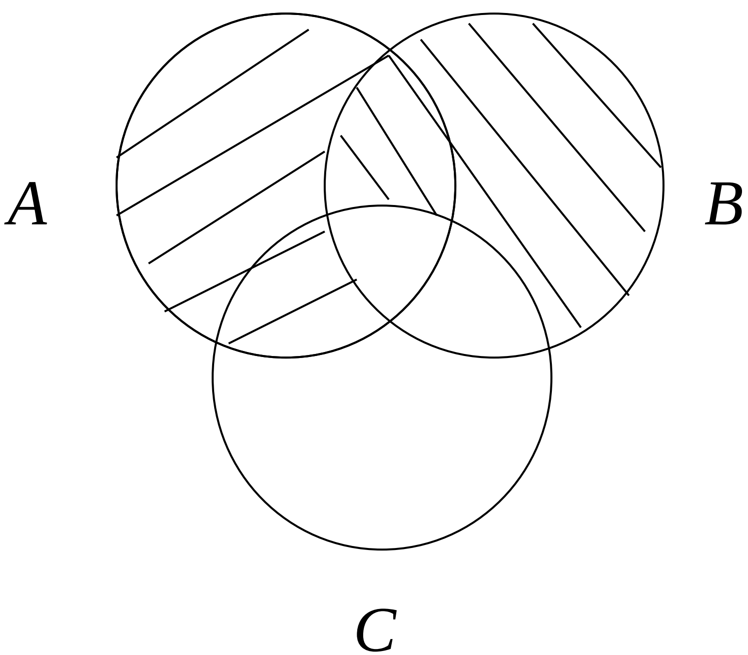
$\overline{B}$



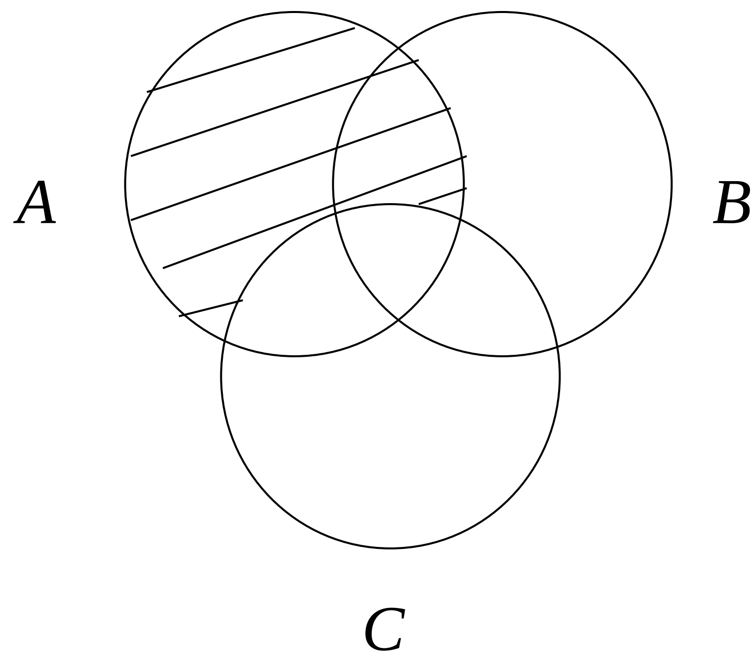
$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Disproof

$$(A - B) \cup (B - C) = A - C?$$



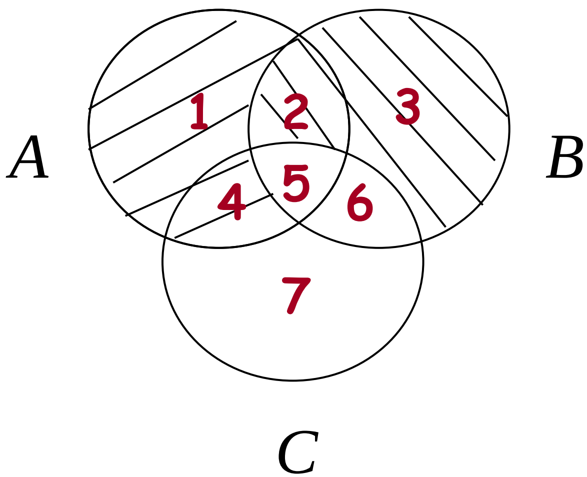
L.H.S

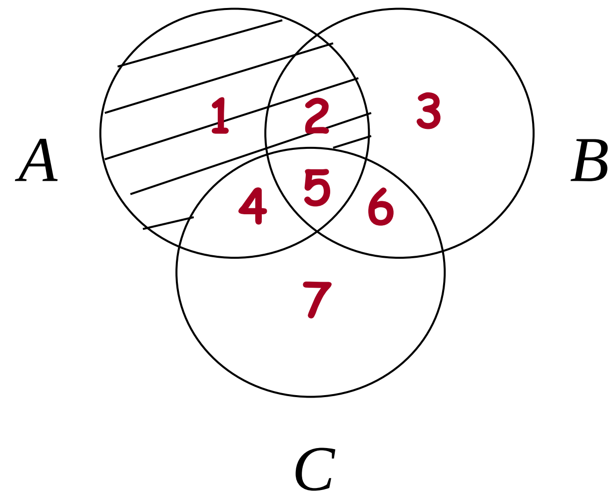


R.H.S

Disproof

$$(A - B) \cup (B - C) = A - C?$$





We can easily construct a **counterexample** to the equality, by putting a number in each region in the figure.

Let $A = \{1,2,4,5\}$, $B = \{2,3,5,6\}$, $C = \{4,5,6,7\}$.

Then we see that L.H.S = $\{1,2,3,4\}$ and R.H.S = $\{1,2\}$.

Algebraic proof

Sometimes when we know some rules, we can use them to prove new rules without drawing figures.

e.g. we can prove $\overline{(\overline{A} \cap \overline{B})} = A \cup B$ without drawing figures.

$$\overline{(\overline{A} \cap \overline{B})} = \overline{\overline{A}} \cup \overline{\overline{B}}$$

by using DeMorgan's rule on $\overline{A}$ and $\overline{B}$

$$= A \cup B$$

Algebraic proof

$$\overline{((A \cup C) \cap (B \cup C))} = (\overline{A} \cup \overline{B}) \cup \overline{C}?$$

$$\overline{((A \cup C) \cap (B \cup C))}$$

$$= \overline{(A \cup C)} \cup \overline{(B \cup C)}$$ by DeMorgan's law on A U C and B U C

$$= (\overline{A} \cap \overline{C}) \cup \overline{(B \cup C)}$$ by DeMorgan's law on the first half

$$= (\overline{A} \cap \overline{C}) \cup (\overline{B} \cap \overline{C})$$ by DeMorgan's law on the second half

$$= (\overline{A} \cup \overline{B}) \cap \overline{C}$$ by distributive law

$$\neq (\overline{A} \cup \overline{B}) \cup \overline{C}$$

Exercises

$$A - (A \cap B) = A - B?$$

$$(A \cup B) - C = (A - C) \cup (B - C)?$$

$$\overline{(A \cup B \cup C)} = \overline{A} \cap \overline{B} \cap \overline{C}?$$

Using properties of set operations

- How can we prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$?
- Method I:
- $x \in A \cup (B \cap C)$
- $\Leftrightarrow x \in A \lor x \in (B \cap C)$
- $\Leftrightarrow x \in A \lor (x \in B \land x \in C)$
- $\Leftrightarrow (x \in A \lor x \in B) \land (x \in A \lor x \in C)$
- (distributive law for logical expressions)
- $\Leftrightarrow x \in (A \cup B) \land x \in (A \cup C)$
- $(A \cup B) \cap (A \cup C)$

Using properties of set operations

- Method II: Membership table
  - 1 means "x is an element of this set"
  - 0 means "x is not an element of this set"

A B C	B∩C	A∪(B∩C)	A∪B	A∪C	(A∪B) ∩(A∪C)
0 0 0	0	0	0	0	0
0 0 1	0	0	0	1	0
0 1 0	0	0	1	0	0
0 1 1	1	1	1	1	1
1 0 0	0	1	1	1	1
1 0 1	0	1	1	1	1
1 1 0	0	1	1	1	1
1 1 1	1	1	1	1	1

$$S_1 \quad A \cap B \subseteq A$$

$$S_2 \quad A \cap B \subseteq B$$

$$S_3 \quad A \subseteq A \cup B$$

$$S_4 \quad B \subseteq A \cup B$$

$$S_5 \quad A - B \subseteq A$$

$$S_6 \quad A \oplus B \subseteq A \cup B$$

$$\left.\begin{array}{ll} S_7 & A \cup B = B \cup A \\ S_8 & A \cap B = B \cap A \\ S_9 & A \oplus B = B \oplus A \end{array}\right\} \text{交换律}$$

$$\left.\begin{array}{ll} S_{10} & A \cup (B \cup C) = (A \cup B) \cup C \\ S_{11} & (A \cap B) \cap C = A \cap (B \cap C) \\ S_{12} & (A \oplus B) \oplus C = A \oplus (B \oplus C) \end{array}\right\} \text{结合律}$$

$$\left.\begin{array}{ll} S_{13} & A\cap(B\cup C)=(A\cap B)\cup(A\cap C) \\ S_{14} & A\cup(B\cap C)=(A\cup B)\cap(A\cup C) \end{array}\right\}\text{分配律}$$

$$S_{15}\quad \overline{\overline{A}}=A \quad \text{双重否定律}$$

$$\left.\begin{array}{ll} S_{16} & \overline{(A\cap B)}=\overline{A}\cup\overline{B} \\ S_{17} & \overline{(A\cup B)}=\overline{A}\cap\overline{B} \end{array}\right\}\text{德·摩根律}$$

$$\left.\begin{array}{ll} S_{18} & A\cap A=A \\ S_{19} & A\cup A=A \end{array}\right\}\text{等幂律}$$

$$\left.\begin{array}{ll} S_{20} & A\cap\overline{A}=\varnothing \\ S_{21} & A\cup\overline{A}=E \end{array}\right\}\text{补余律}$$

$$\left.\begin{array}{ll} S_{22} & A\cap E=A \\ S_{23} & A\cup\varnothing=A \\ S_{24} & A-\varnothing=A \\ S_{25} & A\oplus\varnothing=A \end{array}\right\}\text{同一律}$$

$$\left.\begin{array}{ll} S_{26} & A \cap \varnothing = \varnothing \\ S_{27} & A \cup E = E \end{array}\right\} \text{零律}$$

$$\left.\begin{array}{ll} S_{28} & A \cup (A \cap B) = A \\ S_{29} & A \cap (A \cup B) = A \end{array}\right\} \text{吸收律}$$

$$S_{30} \quad \overline{\varnothing} = E$$

$$S_{31} \quad \overline{E} = \varnothing$$

$$S_{32} \quad A \oplus A = \varnothing$$

$$S_{33} \quad A \cap (B - A) = \varnothing$$

$$S_{34} \quad A \cup (B - A) = A \cup B$$

$$S_{35} \quad A - (B \cup C) = (A - B) \cap (A - C)$$

S_{36} $A-(B\cap C)=(A-B)\cup(A-C)$

S_{37} $A-B=A\cap\overline{B}$

S_{38} $A\oplus B=(A\cap\overline{B})\cup(\overline{A}\cap B)$

S_{39} $(A\cup B\neq\varnothing)\Rightarrow(A\neq\varnothing)\vee(B\neq\varnothing)$

S_{40} $(A\cap B\neq\varnothing)\Rightarrow(A\neq\varnothing)\wedge(B\neq\varnothing)$

Partitions of sets

Two sets are disjoint if their intersection is empty.

A collection of nonempty sets $\{A_1, A_2, \ldots, A_n\}$ is a partition of a set A if and only if

$$A = A_1 \cup A_2 \cup \cdots \cup A_n$$

$A_1, A_2, \ldots, A_n$ are mutually disjoint (or pairwise disjoint).

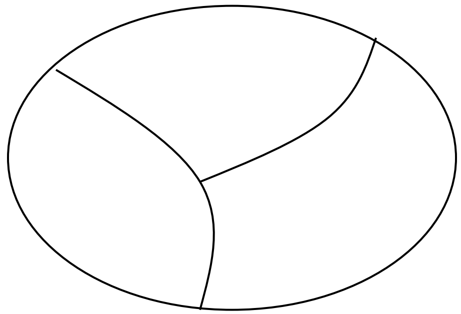
e.g. Let A be the set of integers.

$A_1 = \{x \in A \mid x = 3k+1 \text{ for some integer } k\}$

$A_2 = \{x \in A \mid x = 3k+2 \text{ for some integer } k\}$

$A_3 = \{x \in A \mid x = 3k \text{ for some integer } k\}$

Then $\{A_1, A_2, A_3\}$ is a partition of A



Partitions of sets

e.g. Let A be the set of integers divisible by 6.

A_1 be the set of integers divisible by 2.

A_2 be the set of integers divisible by 3.

Then $\{A_1, A_2\}$ is not a partition of A, because A_1 and A_2 are not disjoint,

and also $A \subset A_1 \cup A_2$ (so both conditions are not satisfied).

e.g. Let A be the set of integers.

Let A_1 be the set of negative integers.

Let A_2 be the set of positive integers.

Then $\{A_1, A_2\}$ is not a partition of A, because $A \neq A_1 \cup A_2$

as 0 is contained in A but not contained in $A_1 \cup A_2$

The End