Normal forms

Normal forms

- It will be convenient to use the word **product** in place of **conjunction** ($\wedge$) and **sum** in place of **disjunction** ($\vee$) in the current discussion.
- **Elementary product (基本积):** A product of the variables and their negations in a statement formula is called an **elementary product.**

  such as, $P \wedge \neg Q$

- **Elementary sum (基本和):** A sum of the variables and their negations in a statement formula is called an **elementary sum.**

  such as, $\neg P \vee R \vee \neg Q$

Disjunctive normal form (DNF)

- **Definition: Disjunctive normal form (DNF):** A statement formula which is equivalent to a given formula and which consists of a sum of elementary products is called a **disjunctive normal form (DNF)** of the given formula.

- **Every proposition can be put in an equivalent DNF, but not unique**

Disjunctive normal form (DNF)

Example: Find the Disjunctive Normal Form (DNF) of

$$(p \vee q) \rightarrow \neg r$$

Solution: This proposition is true when r is false or when both p and q are false.

$$(\neg p \wedge \neg q) \vee \neg r$$

Principal disjunctive normal form (PDNF):

- **Minterm (小项):** *A* product (or conjunction) in which each variable or its negation, but not both, occurs only once is called a **minterm.**
- The number of possible minterms for n variables are 2^n. For two variables *p* and *q*, there are $2^2 = 4$ possible minterms : $p \wedge q$, $p \wedge \neg q$, $\neg p \wedge q$ and $\neg p \wedge \neg q$.
- *A* formula consists of a sum of minterms is called a **principal disjunctive normal form (PDNF)**
- **Every proposition (not include contradictions) can be put in an unique equivalent PDNF**

Suppose we have three variables p, q, and r, all minterms are listed as follows

minterms	values	index	notes
$\neg p \wedge \neg q \wedge \neg r$	000	0	m_0
$\neg p \wedge \neg q \wedge r$	001	1	m_1
$\neg p \wedge q \wedge \neg r$	010	2	m_2
$\neg p \wedge q \wedge r$	011	3	m_3
$p \wedge \neg q \wedge \neg r$	100	4	m_4
$p \wedge \neg q \wedge r$	101	5	m_5
$p \wedge q \wedge \neg r$	110	6	m_6
$p \wedge q \wedge r$	111	7	m_7

- **求主析取范式的方法：**
  - 先化成与其等价的析取范式；
  - 若析取范式的基本积中同一命题变元出现多次，则将其化成只出现一次；
  - 去掉析取范式中所有为永假式的基本积，即去掉基本积中含有形如

  $p \wedge \neg p$的子公式的那些基本积；
  - 若析取范式中缺少某一命题变元如p，则可用公式$(p \vee \neg p) \wedge q \Leftrightarrow q$将命题变元P补充进去，并利用分配律展开，然后合并相同的基本积

$A \Leftrightarrow p \wedge q \vee r$

$\Leftrightarrow (p \wedge q) \wedge (r \vee \neg r) \vee (p \vee \neg p) \wedge r$

$\Leftrightarrow (p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge r) \vee (\neg p \wedge r)$

$\Leftrightarrow (p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge r) \wedge (q \vee \neg q) \vee (\neg p \wedge r) \wedge (q \vee \neg q)$

$\Leftrightarrow (p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge r \wedge q) \vee (p \wedge r \wedge \neg q) \vee (\neg p \wedge r \wedge q) \vee (\neg p \wedge r \wedge \neg q)$

$\Leftrightarrow (p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge r \wedge \neg q) \vee (\neg p \wedge r \wedge q) \vee (\neg p \wedge r \wedge \neg q)$

$\Leftrightarrow m_7 \vee m_6 \vee m_5 \vee m_3 \vee m_1$

$\Leftrightarrow \sum(1,3,5,6,7)$

DNF vs Truth table

- The truth table for $p \wedge q \vee r$
- The DNR of $p \wedge q \vee r$ is just

$$m_1 \vee m_3 \vee m_5 \vee m_6 \vee m_7$$

p,q,r	**minterms**	**$p \wedge q \vee r$**	
0,0,0	$\neg p \wedge \neg q \wedge \neg r$	0	m_0
0,0,1	$\neg p \wedge \neg q \wedge r$	1	m_1
0,1,0	$\neg p \wedge q \wedge \neg r$	0	m_2
0,1,1	$\neg p \wedge q \wedge r$	1	m_3
1,0,0	$p \wedge \neg q \wedge \neg r$	0	m_4
1,0,1	$p \wedge \neg q \wedge r$	1	m_5
1,1,0	$p \wedge q \wedge \neg r$	1	m_6
1,1,1	$p \wedge q \wedge r$	1	m_7

DNF vs Truth table

Example: Show that every compound proposition can be put in disjunctive normal form.

Solution:

- Construct the truth table for the proposition.
- Then an equivalent proposition is the disjunction with n disjuncts (where n is the number of rows for which the formula evaluates to T).
- Each disjunct has m conjuncts where m is the number of distinct propositional variables.
- Each conjunct includes the positive form of the propositional variable if the variable is assigned T in that row and the negated form if the variable is assigned F in that row.
- This proposition is in disjunctive normal from.

Conjunctive normal form (CNF)

- **Definition: Conjunctive normal form (CNF):** A statement formula which is equivalent to a given formula and which consists of a product of elementary sums is called a **conjunctive normal form (CNF)** of the given formula.

$$(p \vee \neg r \vee p) \wedge (\neg q \vee \neg r \vee p)$$

- **Every proposition can be put in an equivalent CNF, but not unique**

Conjunctive normal form

- A compound proposition is in Conjunctive Normal Form (CNF) if it is a conjunction of disjunctions.

$$(p \vee \neg r \vee p) \wedge (\neg q \vee \neg r \vee p)$$

- Conjunctive Normal Form (CNF) can be obtained by eliminating implications, moving negation inwards and using the distributive and associative laws.

$$\neg(p \rightarrow q) \vee (r \rightarrow p)$$

- Important in resolution theorem proving used in artificial Intelligence (AI).
- A compound proposition can be put in conjunctive normal form through repeated application of the logical equivalences covered earlier.

Conjunctive normal form

Example: Put the following into CNF:

$$\neg(p \to q) \vee (r \to p)$$

Solution:

- Eliminate implication signs:

$$\neg(\neg p \vee q) \vee (\neg r \vee p)$$

- Move negation inwards; eliminate double negation:

$$(p \wedge \neg q) \vee (\neg r \vee p)$$

- Convert to CNF using associative/distributive laws

$$(p \vee \neg r \vee p) \wedge (\neg q \vee \neg r \vee p)$$

Principal conjunctive normal form (PCNF):

- **Maxterm (大项):** A sum (or disjunction) in which each variable or its negation, but not both, occurs only once is called a **maxterm.**
- The number of possible maxterms for n variables are 2^n. For two variables *p* and *q*, there are $2^2 = 4$ possible maxterms : $p \vee q$, $p \vee \neg q$, $\neg p \vee q$ and $\neg p \vee \neg q$.
- A formula consists of a product of maxterms is called a **principal conjunctive normal form (PCNF)**

- **Every proposition (not include tautologies) can be put in an unique equivalent PCNF**

Suppose we have three variables p, q, and r, all maxterms are listed as follows

maxterms	values	index	notes
$p \vee q \vee r$	000	0	M_0
$p \vee q \vee \neg r$	001	1	M_1
$p \vee \neg q \vee r$	010	2	M_2
$p \vee \neg q \vee \neg r$	011	3	M_3
$\neg p \vee q \vee r$	100	4	M_4
$\neg p \vee q \vee \neg r$	101	5	M_5
$\neg p \vee \neg q \vee r$	110	6	M_6
$\neg p \vee \neg q \vee \neg r$	111	7	M_7

$$A \Leftrightarrow p \wedge q \vee r$$

$$\Leftrightarrow (p \vee r) \wedge (q \vee r)$$

$$\Leftrightarrow ((p \vee r) \vee (q \wedge \neg q)) \wedge ((q \vee r) \vee (p \wedge \neg p))$$

$$\Leftrightarrow (p \vee r \vee q) \wedge (p \vee r \vee \neg q) \wedge (q \vee r \vee p)$$

$$\wedge (q \vee r \vee \neg p)$$

$$\Leftrightarrow (p \vee r \vee q) \wedge (p \vee r \vee \neg q) \wedge (q \vee r \vee \neg p)$$

$$\Leftrightarrow M_0 \wedge M_2 \wedge M_4$$

$$\Leftrightarrow \prod(0,2,4)$$

DNR vs Truth table

- The truth table for $p \wedge q \vee r$
- The CNR of $p \wedge q \vee r$ is just

$$M_0 \wedge M_2 \wedge M_4$$

p,q,r	**maxterms**	**$p \wedge q \vee r$**	
0,0,0	$p \vee q \vee r$	0	M_0
0,0,1	$p \vee q \vee \neg r$	1	M_1
0,1,0	$p \vee \neg q \vee r$	0	M_2
0,1,1	$p \vee \neg q \vee \neg r$	1	M_3
1,0,0	$\neg p \vee q \vee r$	0	M_4
1,0,1	$\neg p \vee q \vee \neg r$	1	M_5
1,1,0	$\neg p \vee \neg q \vee r$	1	M_6
1,1,1	$\neg p \vee \neg q \vee \neg r$	1	M_7

Minterm vs Maxterm

- The relations between m_i and M_i are

$$M_i \Leftrightarrow \neg m_i \qquad m_i \Leftrightarrow \neg M_i$$

p,q,r	maxterms	$p \wedge q \vee r$		p,q,r	minterms	$p \wedge q \vee r$	
0,0,0	$p \vee q \vee r$	0	M_0	0,0,0	$\neg p \wedge \neg q \wedge \neg r$	0	m_0
0,0,1	$p \vee q \vee \neg r$	1	M_1	0,0,1	$\neg p \wedge \neg q \wedge r$	1	m_1
0,1,0	$p \vee \neg q \vee r$	0	M_2	0,1,0	$\neg p \wedge q \wedge \neg r$	0	m_2
0,1,1	$p \vee \neg q \vee \neg r$	1	M_3	0,1,1	$\neg p \wedge q \wedge r$	1	m_3
1,0,0	$\neg p \vee q \vee r$	0	M_4	1,0,0	$p \wedge \neg q \wedge \neg r$	0	m_4
1,0,1	$\neg p \vee q \vee \neg r$	1	M_5	1,0,1	$p \wedge \neg q \wedge r$	1	m_5
1,1,0	$\neg p \vee \neg q \vee r$	1	M_6	1,1,0	$p \wedge q \wedge \neg r$	1	m_6
1,1,1	$\neg p \vee \neg q \vee \neg r$	1	M_7	1,1,1	$p \wedge q \wedge r$	1	m_7

Translate CNF to DNF

Let CNF of A be

$$(P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee \neg R)$$

Find the DNF of A.

Solution:

The CNF of A is $M_1 \wedge M_3$, so the DNF can be written as

$$\sum(0,2,4,5,6,7)$$

And thus we have

$$(\neg P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge \neg R)$$
$$\vee (P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R)$$

主析取范式和主合取范式

- 一个命题公式是永真式，它的命题变元的所有极小项均出现在其主析取范式中，不存在与其等价的主合取范式；
- 一个命题公式是永假式，它的命题变元的所有极大项均出现在其主合取范式中，不存在与其等价的主析取范式；
- 一个命题公式是可满足的，它既有与其等价的主析取范式，也有与其等价的主合取范式.

Every proposition (not include tautologies) can be put in an unique equivalent PCNF

Every proposition (not include contradictions) can be put in an unique equivalent PDNF

Find the normal forms of $(p\rightarrow\neg q)\rightarrow\neg r$

Solution:

$(p\rightarrow\neg q)\rightarrow\neg r$

$\Leftrightarrow \neg(\neg p \vee \neg q)\vee\neg r$

$\Leftrightarrow(p\wedge q)\vee\neg r$

$\Leftrightarrow(p \vee\neg r) \wedge(q \vee\neg r)$

$\Leftrightarrow(p\vee\neg r\vee(q\wedge\neg q))\wedge(q \vee\neg r \vee(p\wedge\neg p))$

$\Leftrightarrow(p\vee q\vee\neg r)\wedge(p\vee\neg q \vee\neg r))\wedge(p\vee q\vee\neg r)\wedge(\neg p\vee q\vee\neg r)$

$\Leftrightarrow\prod(1,3,5)$

/*其中∏表示求合取*/

$\Leftrightarrow\sum(0,2,4,6,7)$

/*即该公式是可满足的，应存在与其等价的主析取范式*/

Find the normal forms of $(p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge \neg s)$

Solution:

$(p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge \neg s)$

$\Leftrightarrow (p \wedge \neg q \wedge r) \wedge (s \vee \neg s) \vee (\neg p \wedge q \wedge \neg s) \wedge (r \vee \neg r)$

$\Leftrightarrow (p \wedge \neg q \wedge r \wedge s) \vee (p \wedge \neg q \wedge r \wedge \neg s) \vee (\neg p \wedge q \wedge r \wedge \neg s) \vee (\neg p \wedge q \wedge \neg r \wedge \neg s)$

$\Leftrightarrow \sum(11,10,6,4)$

/*这里∑代表析取*/

$\Leftrightarrow \prod(0,1,2,3,5,7,8,9,12,13,15,15)$

If one kind of normal form have been found, the other kind of normal form can be directly founded, unless the tautologies or contradictions!!

Example

Put $(\neg p \wedge q) \leftrightarrow (p \rightarrow q)$ into CNF

$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$

Solution:

$(\neg p \wedge q) \leftrightarrow (p \rightarrow q)$

$\Leftrightarrow ((\neg p \wedge q) \wedge (p \rightarrow q)) \vee (\neg(\neg p \wedge q) \wedge \neg(p \rightarrow q))$

/*消$\leftrightarrow$*/

$\Leftrightarrow ((\neg p \wedge q) \wedge (\neg p \vee q)) \vee ((p \vee \neg q) \wedge (p \wedge \neg q))$

/*消$\rightarrow$ 并且否定深入到单个变元前*/

$\Leftrightarrow (\neg p \wedge q) \vee (p \wedge \neg q)$ /*析取范式*/

$\Leftrightarrow ((\neg p \wedge q) \vee p) \wedge ((\neg p \wedge q) \vee \neg q)$

$\Leftrightarrow (p \vee q) \wedge (\neg p \vee \neg q)$

/*使用或对与的分配律及补余律，现在是合取范式的形式*/

Example

Put $\neg(p \vee q) \leftrightarrow (p \wedge q)$ into DNF

$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$

Solution:

$$
\begin{aligned}
& \neg(p \vee q) \leftrightarrow (p \wedge q) \\
\Leftrightarrow\ & \neg(p \vee q) \wedge (p \wedge q) \vee \neg(\neg(p \vee q)) \wedge \neg(p \wedge q) \\
\Leftrightarrow\ & (\neg p \wedge \neg q \wedge p \wedge q) \vee ((p \vee q) \wedge (\neg p \vee \neg q)) \\
\Leftrightarrow\ & F \vee (p \vee q) \wedge (\neg p \vee \neg q) \\
\Leftrightarrow\ & (p \vee q) \wedge (\neg p \vee \neg q) \\
\Leftrightarrow\ & ((p \vee q) \wedge \neg p) \vee ((p \vee q) \wedge \neg q) \\
\Leftrightarrow\ & p \wedge \neg p \vee \neg p \wedge q \vee p \wedge \neg q \vee q \wedge \neg q \\
\Leftrightarrow\ & F \vee \neg p \wedge q \vee p \wedge \neg q \vee F \\
\Leftrightarrow\ & (\neg p \wedge q) \vee (p \wedge \neg q)
\end{aligned}
$$

The End