Methods of Mathematical Physics

—Lecture 4 Taylor and Laurent series—

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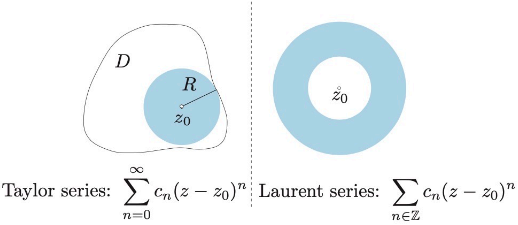
Contents

Introduction

In this lecture we will first learn about the fundamental result which says that a holomorphic function $f(z)$ has a power series expansion around any point in the domain D where it lives. See the picture on the left below.



In the second part of this lecture, we will learn about Laurent series, which are like power series, except that negative integer powers of the terms zz_0 also occur in the expansion. This will be useful to study functions that are holomorphic in annuli (and in particular punctured discs). See the picture on the right above. They are also useful to classify "singularities", and to evaluate some real integrals.

Definition

Just like with real series, given a sequence $(a_n)_{n\in\mathbb{N}}$ of complex numbers, one can form a new sequence $(s_n)_{n\in\mathbb{N}}$ of its partial sums:

$$\begin{aligned} s_1 &:= a_1, \\ s_2 &:= a_1 + a_2, \\ s_3 &:= a_1 + a_2 + a_3, \\ &\vdots \end{aligned}$$

Definition

1. The series $\sum_{n=1}^{\infty} a_n$ converges if $(s_n)_{n\in\mathbb{N}}$ converges, and $\sum_{n=1}^{\infty} a_n := \lim_{n\to\infty} s_n$.
2. The series $\sum_{n=1}^{\infty} a_n$ diverges if $(s_n)_{n\in\mathbb{N}}$ diverges.
3. $\sum_{n=1}^{\infty} a_n$ converges absolutely if the real series $\sum_{n=1}^{\infty} |a_n|$ converges.

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Theorem

A complex sequence converges if and only if the sequences of its real and imaginary parts converge, $\sum_{n=1}^{\infty} a_n$ *converges* $\Leftrightarrow$ *the real series* $\sum_{n=1}^{\infty} \mathrm{Re}(a_n)$ *and* $\sum_{n=1}^{\infty} \mathrm{Im}(a_n)$ *converge.*

Properties of series

Thus the results from real analysis lend themselves for use in testing the convergence of complex series. For example, it is easy to prove the following two facts.

1. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.
2. If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ converges.

Power series and their region of convergence

Definition

Let $(c_n)_{n\in\mathbb{N}}$ be a complex sequence (thought of as a sequence of "coefficients"). An expression of the type

$$\sum_{n=0}^{\infty} c_n z^n$$

is called a power series in the complex variable z.

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is called a power series in the complex variable z.

- All polynomial expressions are power series, with only finitely many nonzero coefficients. Polynomials converge for all $z \in \mathbb{C}$.
- The power series $\sum_{n=0}^{\infty} z^n$ converges whenever $|z| < 1$, but diverges if $|z| \geq 1$.

Power series and their region of convergence

A fundamental question is: For what values of $z \in \mathbb{C}$ does the power series $\sum_{n=0}^{\infty} c_n z^n$ converge? The following result gives the answer to this question.

Theorem

For $\sum_{n=0}^{\infty} c_n z^n$, exactly one of the following hold:

1. *Either it is absolutely convergent for all $z \in \mathbb{C}$.*
2. *Or there is a unique nonnegative real number R such that*
   - *$\sum_{n=0}^{\infty} c_n z^n$ is absolutely convergent for all $z \in \mathbb{C}$ with $|z| < R$, and*
   - *$\sum_{n=0}^{\infty} c_n z^n$ is divergent for all $z \in \mathbb{C}$ with $|z| > R$.*

(The unique $R > 0$ in the above theorem is called the radius of convergence of the power series, and if the power series converges for all $z \in \mathbb{C}$, we say that the power series has an infinite radius of convergence, and write "$R = \infty$".)

Power series and their region of convergence

The calculation of the radius of convergence is facilitated in some cases by the following result.

Theorem

Consider the power series

$$\sum_{n=0}^{\infty} c_n z^n.$$

If $\rho := \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|$ *exists, then*

- *the radius of convergence is* $1/\rho$*, i.e.,* $R = 1/\rho$ *if* $\rho \neq 0$.
- *the radius of convergence is infinite, i.e.,* $R = \infty$ *if* $\rho = 0$.

Power series and their region of convergence

Theorem

Consider the power series $\sum_{n=0}^{\infty} c_n x^n$. *If* $L := \lim_{n\to\infty} \sqrt[n]{|c_n|}$ *exists, then*

- *the radius of convergence is* $1/L$ *if* $L \neq 0$.
- *the radius of convergence is infinite if* $L = 0$.

Power series are holomorphic

We have seen that polynomials are power series with an infinite radius of convergence, that is, they converge in the whole of $\mathbb{C}$. They are of course also holomorphic there. This is not a coincidence. Now we will see, more generally, that a power series

$$f(z) := \sum_{n=0}^{\infty} c_n z^n$$

that converges for $|z| < R$ is actually holomorphic there, and for $|z| < R$, there holds that $f'(z) = \frac{d}{dz}\left(c_0 + c_1 z + c_2 z^2 + \cdots\right) = c_1 + 2c_2 z + 3c_3 z^2 + \cdots = \sum_{n=1}^{\infty} c_n n z^{n-1}$.

Theorem

Let $R > 0$ and $f(z) := \sum_{n=0}^{\infty} c_n z^n$ converge for $|z| < R$. Then $f'(z) = \sum_{n=1}^{\infty} n c_n z^{n-1}$ for $|z| < R$.

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By a repeated application of the previous result, we have the following.

Corollary

Let $R > 0$ and let $f(z) := \sum_{n=0}^{\infty} c_n z^n$ converge for $|z| < R$. Then for $k \geq 1$,

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-k+1) c_n z^{n-k} \text{ for } |z| < R.$$

In particular, for $n \geq 0$, $c_n = \frac{1}{n!} f^{(n)}(0)$.

Power series are holomorphic

There is nothing special about taking power series centered at 0. One can also consider

$$\sum_{n=0}^{\infty} c_n (z - z_0)^n ,$$

where z_0 is a fixed complex number. The following results follow immediately from previous theorems.

Corollary

For $\sum_{n=0}^{\infty} c_n (z - z_0)^n$, exactly one of the following hold:

1. *Either it is absolutely convergent for all $z \in \mathbb{C}$.*
2. *Or there is a unique nonnegative real number R such that*
   - *$\sum_{n=0}^{\infty} c_n (z - z_0)^n$ is absolutely convergent for $|z - z_0| < R$, and*
   - *$\sum_{n=0}^{\infty} c_n (z - z_0)^n$ is divergent for $|z - z_0| > R$.*

Corollary

Let $z_0 \in \mathbb{C}, R > 0$ and $f(z) := \sum_{n=0}^{\infty} c_n (z - z_0)^n$ converge for $|z - z_0| < R$. Then

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)c_n (z - z_0)^{n-k} \quad \text{for } |z - z_0| < R, k \geq 1.$$

In particular, for $n \geq 0$, $c_n = \frac{1}{n!} f^{(n)}(z_0)$.

Taylor series

We have seen in the last section that complex power series

$$\sum_{n=0}^{\infty} c_n (z - z_0)^n$$

are holomorphic in their region of convergence $|z - z_0| < R$, where R is the radius of convergence. In this section, we will show that conversely, if f is holomorphic in the disc $|z - z_0| < R$, then

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \text{ whenever } |z - z_0| < R,$$

where the coefficients can be determined from the f. Thus every holomorphic function f defined in a domain D possesses a power series expansion in a disc around any point $z_0 \in D$.

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Theorem

If f is holomorphic in $D(z_0, R) := \{z \in \mathbb{C} : |z - z_0| < R\}$, then $f(z) = c_0 + c_1 (z - z_0) + c_2 (z - z_0)^2 + c_3 (z - z_0)^3 + \cdots$ for $z \in D(z_0, R)$, where for $n \geq 0$,

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

and C is the circular path with center z_0 and radius r, where $0 < r < R$ traversed in the anticlockwise direction.

Taylor Series

Corollary (Taylor Series)

If

1. *D be a domain,*
2. *$f : D \to \mathbb{C}$ is holomorphic, and*
3. *$z_0 \in D$,*

then

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \dots, \quad |z - z_0| < R,$$

where R is the radius of the largest open disk with center z_0 contained in D. Also,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz,$$

where C is the circular path with center z_0 and radius r, where $0 < r < R$ traversed in the anticlockwise direction.