Topic # 2

Mechanics

(part 3: Other Concepts of Mechanics)

Lecture Course (online)
Dalian, 2021

Contents

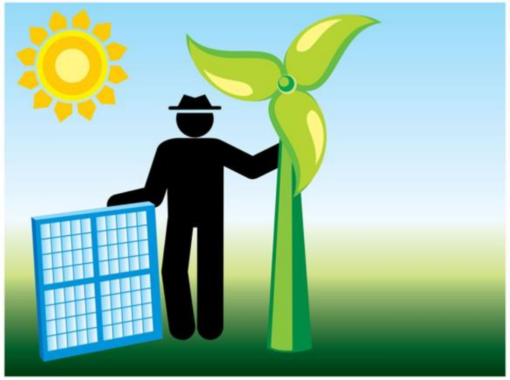
What Is Energy

One of the fundamental goals of physics is to investigate something that everyone talks about: **energy**.

What is energy?

Energy is a **scalar** quantity (i.e., a number) associated with the state (or condition) of one or more objects.

After countless experiments, scientists and engineers realized that if the scheme by which we assign energy numbers is planned carefully, the numbers **can** be used to **predict** the outcomes of experiments and, even more important, to build machines, such as flying machines.



Energy can be transformed from one type to another and transferred from one object to another, but the total amount is always the same (energy is *conserved*).

NO EXCEPTIONS !

Work

Everyday meaning: any activity that requires muscular of mental effort.

- in physics, work has a much more precise definition
- the simplest case: work done by a constant force, the object moves along a straight line

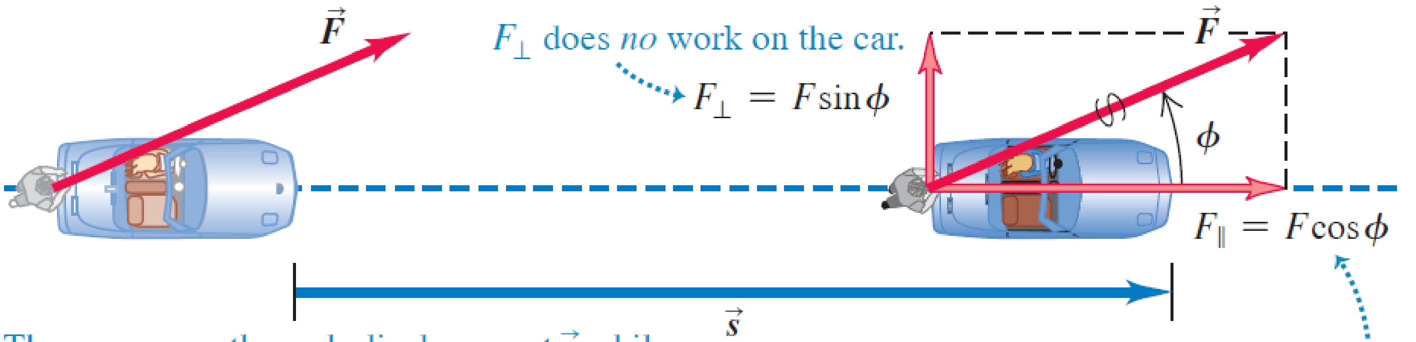
scalar quantity

SI unit : (J)

$1\ \mathrm{J} = 1\ \mathrm{N \cdot m}$

Work done on a particle by **constant force** $\vec{F}$ during **straight-line displacement** $\vec{s}$

$$W = \underbrace{\vec{F} \cdot \vec{s}}_{\text{Scalar product (dot product) of vectors } \vec{F} \text{ and } \vec{s}}$$

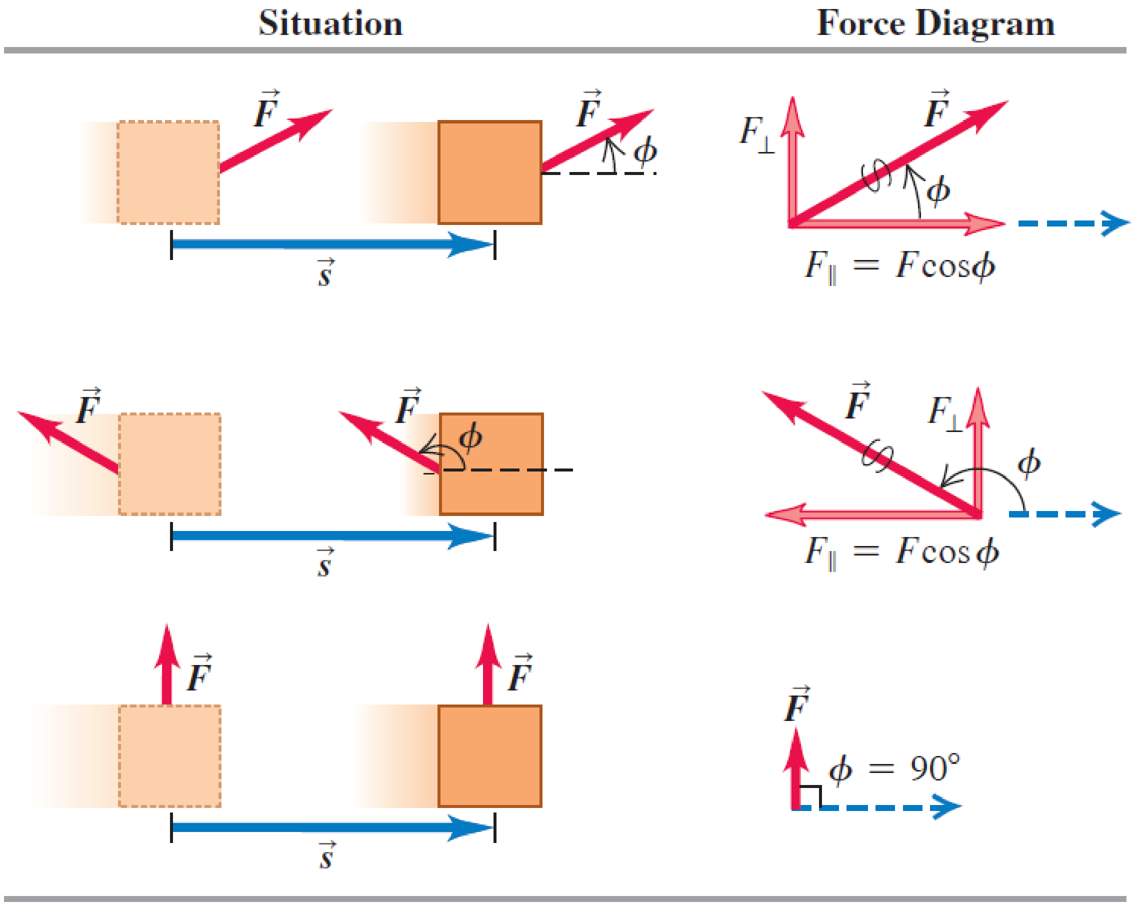


The car moves through displacement $\vec{s}$ while a constant force $\vec{F}$ acts on it at an angle ϕ to the displacement.

Only $F_\parallel$ does work on the car:
$W = F_\parallel s = (F \cos\phi)s = Fs \cos\phi$

Work

In physics, the work can be positive, negative, or zero (depending on φ).



work is **positive**

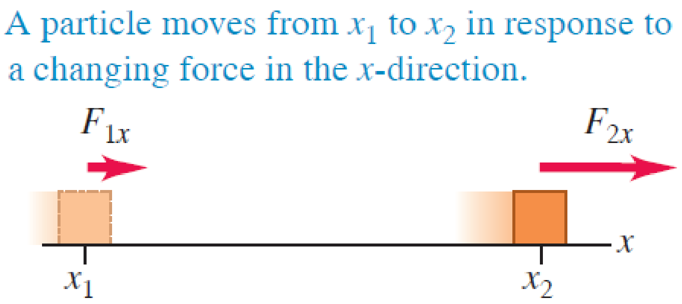
work is **negative**

work is **zero**

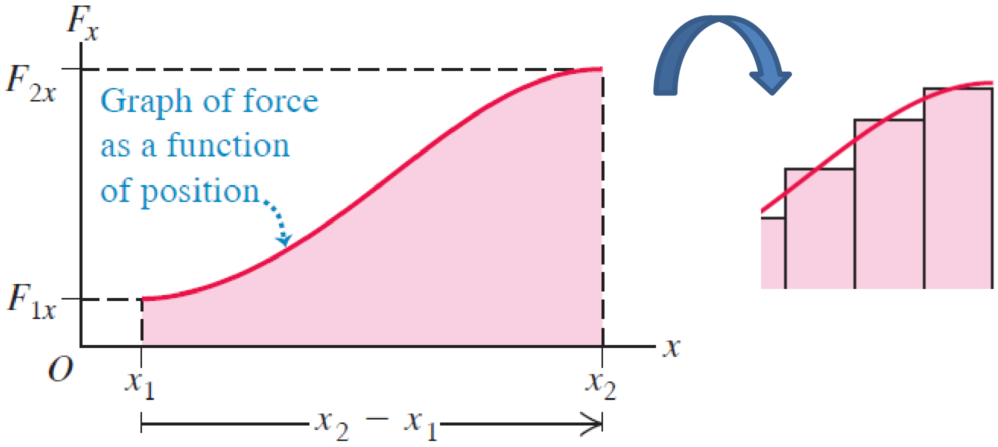
Work Done by a Varying Force

Let us consider a more general case when the force is not constant during the motion.

⇨ the motion still takes place along a straight line



A particle moves from x_1 to x_2 in response to a changing force in the x-direction.



Graph of force as a function of position.

Work done on a particle by a **varying *x*-component of force** F_x during **straight-line displacement** along x-axis

Upper limit = final position

$$W = \int_{x_1}^{x_2} F_x \, dx$$

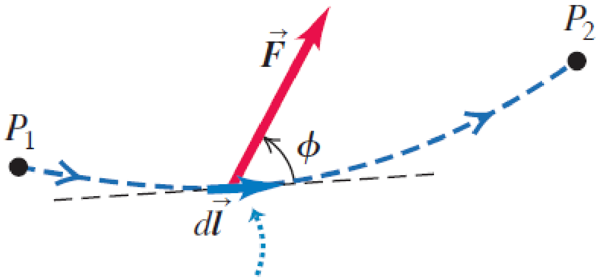
Integral of x-component of force

Lower limit = initial position

Note: the work is equal to the area under the *F*-*x* curve between the initial and final positions.

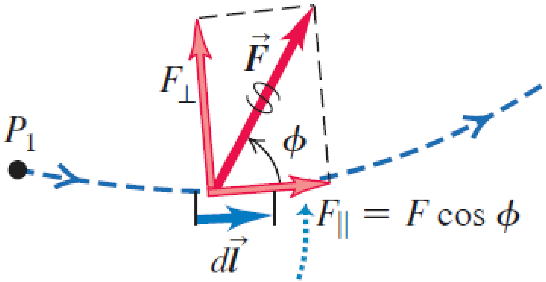
Work Done by a Varying Force

Let us generalize our definition of work further to include a force that varies not only in magnitude but also in direction, and a displacement lies along a curved path.



During an infinitesimal displacement $d\vec{l}$, the force $\vec{F}$ does work dW on the particle:

$$dW = \vec{F} \cdot d\vec{l} = F\cos\phi\, dl = F_{\|}\, dl$$



Only the component of $\vec{F}$ parallel to the displacement, $F_{\|} = F\cos\phi$, contributes to the work done by $\vec{F}$.

Upper limit = final position

Scalar product (dot product) of $\vec{F}$ and displacement $d\vec{l}$

Work done on a particle by a **varying force $\vec{F}$** along a **curved path**

$$W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{l} = \int_{P_1}^{P_2} F\cos\phi\, dl = \int_{P_1}^{P_2} F_{\|}\, dl$$

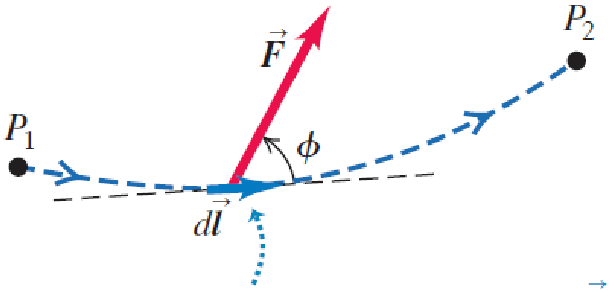
Lower limit = initial position

Angle between $\vec{F}$ and $d\vec{l}$

Component of $\vec{F}$ parallel to $d\vec{l}$

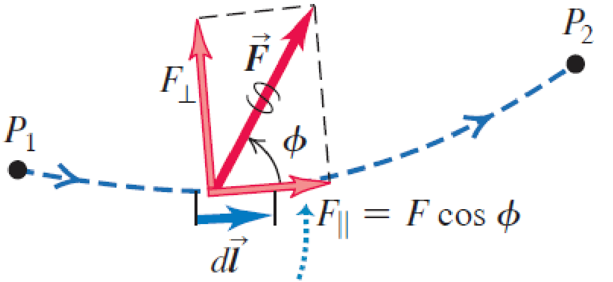
Work Done by a Varying Force

Let us generalize our definition of work further to include a force that varies not only in magnitude but also in direction, and a displacement lies along a curved path.



During an infinitesimal displacement $d\vec{l}$, the force $\vec{F}$ does work dW on the particle:

$$dW = \vec{F} \cdot d\vec{l} = F \cos\phi \, dl = F_{\|} \, dl$$



Only the component of $\vec{F}$ parallel to the displacement, $F_{\|} = F \cos \phi$, contributes to the work done by $\vec{F}$.

In Cartesian coordinates:

$$dW = \vec{F} \cdot d\vec{r} = F_x \, dx + F_y \, dy + F_z \, dz$$

$$W = \int_{r_i}^{r_f} dW = \int_{x_i}^{x_f} F_x \, dx + \int_{y_i}^{y_f} F_y \, dy + \int_{z_i}^{z_f} F_z \, dz$$

Power

The definition of work makes no reference to the passage of time. But often we need to know how quickly work is done. This can be described in terms of **power**.

- power is the time rate at which work is done

scalar quantity

SI unit : (W)

$1\ \mathrm{W} = 1\ \mathrm{J/s}$

Average power during time interval Δt

$$P_{\mathrm{av}} = \frac{\Delta W}{\Delta t}$$

Work done during time interval

Duration of time interval

scalar quantity

- the rate at which work is done might not be constant

Instantaneous power

$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt}$$

Time rate of doing work

Average power over infinitesimally short time interval

EXERCISE

Instantaneous power for a force doing work on a particle

$$P = \vec{F} \cdot \vec{v}$$

Force that acts on particle

Velocity of particle

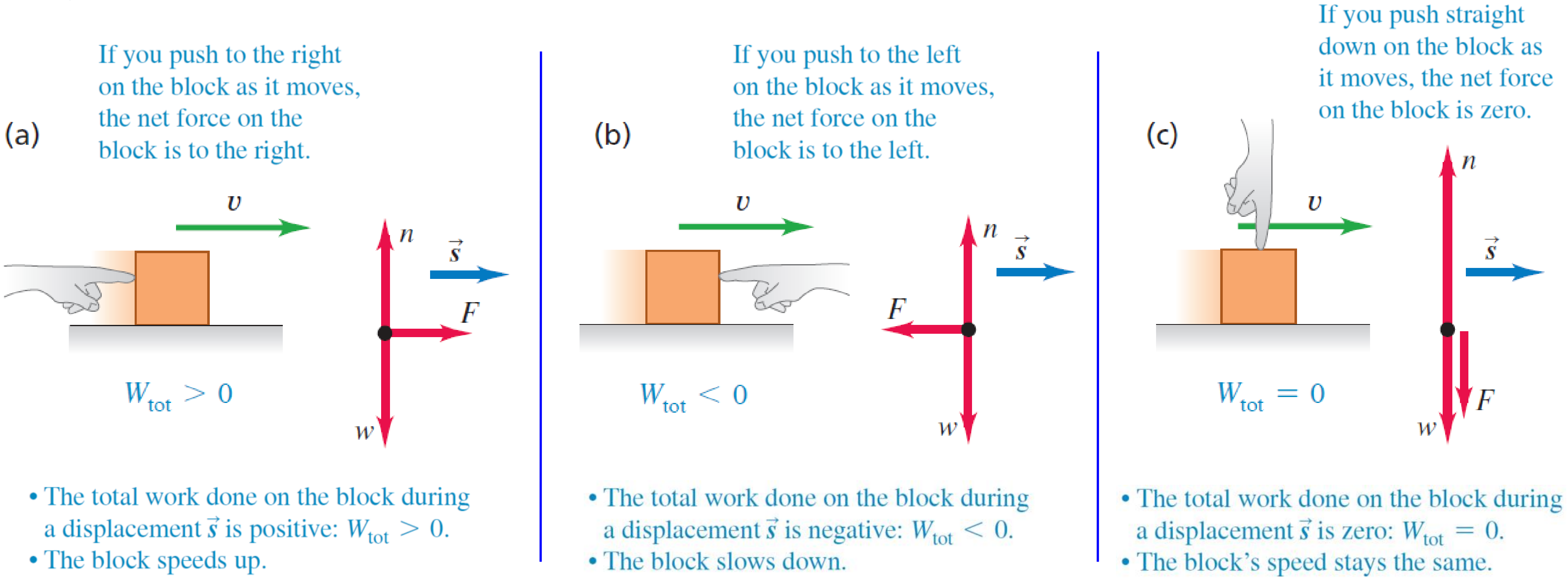

LOW POWER


HIGH POWER

Kinetic Energy and Work

The total work done on an object is related not only to the object's displacement, but also to the changes in its speed.

consider a block sliding to the right on a frictionless surface



If you push to the right on the block as it moves, the net force on the block is to the right.

- The total work done on the block during a displacement $\vec{s}$ is positive: $W_{tot} > 0$.
- The block speeds up.

If you push to the left on the block as it moves, the net force on the block is to the left.

- The total work done on the block during a displacement $\vec{s}$ is negative: $W_{tot} < 0$.
- The block slows down.

If you push straight down on the block as it moves, the net force on the block is zero.

- The total work done on the block during a displacement $\vec{s}$ is zero: $W_{tot} = 0$.
- The block's speed stays the same.

Note: when a particle undergoes a displacement, it speeds up if $W_{tot} > 0$, slows down if $W_{tot} < 0$, and maintains the same speed if $W_{tot} = 0$.

Kinetic Energy and Work

EXERCISE

$$F = ma_x = m\frac{v_2^2 - v_1^2}{2s} \Rightarrow$$

Kinetic energy of a particle

$$K = \tfrac{1}{2}mv^2$$

Mass of particle

Speed of particle

Work–energy theorem: Work done by the net force on a particle equals the change in the particle's kinetic energy.

Total work done on particle = work done by net force

$$W_{\text{tot}} = K_2 - K_1 = \Delta K$$

Change in kinetic energy

Final kinetic energy

Initial kinetic energy

Note: kinetic energy can never be negative, and it is zero only when the particle is at rest.

Particular case: $W_{\text{tot}} = K - 0 = K$

$$\boxed{1\ \text{J} = 1\ \text{N}\cdot\text{m} = 1\ \text{kg}\cdot\text{m}^2/\text{s}^2}$$

- ➡ the kinetic energy of a particle is equal to the total work that was done to accelerate it from rest to its present speed
- ➡ the kinetic energy of a particle is equal to the total work that particle can do in the process of being brought to rested

Kinetic Energy and Work

Work – Kinetic Energy Theorem with a Variable Force

EXERCISE

$$W = \int_{x_i}^{x_f} F(x)\, dx = \int_{x_i}^{x_f} ma\, dx$$

$$ma\, dx = m\frac{dv}{dt}\, dx \qquad \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}\, v$$

$$W = \int_{v_i}^{v_f} mv\, dv = m\int_{v_i}^{v_f} v\, dv = \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2$$

$$\boxed{W = K_f - K_i = \Delta K}$$

work-kinetic energy theorem

Note: this result can be easily generalized for any arbitrary 3-D case.

Kinetic Energy and Work

QUIZ Check your understanding:

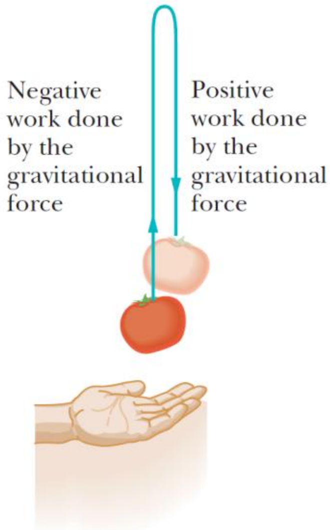
An electron moves in a straight line toward the east with a constant speed of $8*10^7$m/s. It has electric, magnetic, and gravitational forces acting on it. During a 1 m displacement, the total work done on the electron is (i) positive; (ii) negative; (iii) zero; (iv) not enough information is given.

QUIZ Check your understanding:

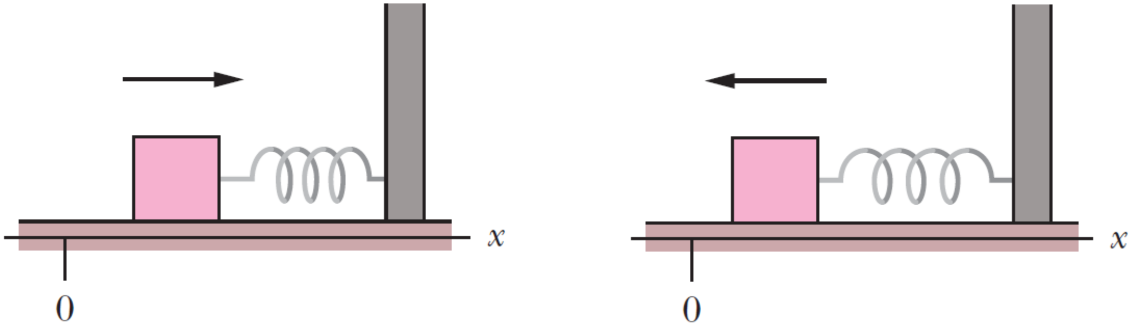
Rank the following objects in order of their kinetic energy, from least to greatest.
(i) A 2.0 kg object moving at 5.0 m/s; (ii) a 1.0 kg object that initially was at rest and then had 30 J of work done on it; (iii) a 1.0 kg object that initially was moving at 4.0 m/s and then had 20 J of work done on it; (iv) a 2.0 kg object that initially was moving at 10 m/s and then did 80 J of work on another object.

Potential Energy

There is a useful alternative way to think about work and kinetic energy. This new approach uses the idea of **potential energy**, which is associated with the **position** of a system rather than with its motion.



two simplest cases: **gravitational** potential energy and **elastic** potential energy



Example: the kinetic energy is transferred by the gravitational force to the gravitational potential energy of the tomato-Earth system and vice versa, or to the elastic potential energy of the spring in the block-spring system.

for an arbitrary **conservative force** the following relation-ship is valid

$$\boxed{\Delta U = -W}$$

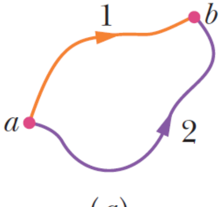
Note: the minus sign is essential!

Conservative Forces

An essential feature of conservative forces is that their work is always **reversible** (anything that we deposit in the energy “bank” can later be withdrawn without loss).

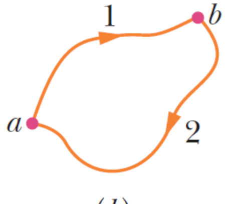
$$\boxed{\oint \vec{F}\cdot d\vec{r} = 0} \quad \Leftrightarrow \quad \vec{F} \quad \text{is conservative}$$

the net work done by a **conservative** force on a particle moving around any **closed** path is **zero**



(a)

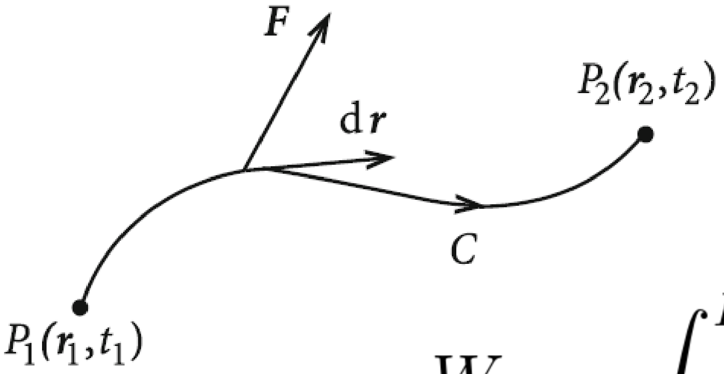
The force is conservative. Any choice of path between the points gives the same amount of work.



(b)

And a round trip gives a total work of zero.

$$W_{ab,1} = W_{ab,2}$$



$$W_{12} = \int_{P_1}^{P_2} \vec{F}\cdot d\vec{r}$$

curvilinear (line) integral

Conservative Forces

EXERCISE

Task #1: There are two stationary fields of force: (a) ***F*** = ay***i*** ; (b) ***F*** = ax***i*** + *by**j***. Are these fields of force conservative or not?

Solution:

Let us find work done by the force of each field over the path from a certain point (x_1,y_1) to a certain point (x_2,y_2)!

(a)

$$dW = \vec{F} \cdot d\vec{r} = aydx \quad \rightarrow \quad W = a \int_{x_1}^{x_2} ydx$$

path dependent → non-conservative

(b)

$$dW = \vec{F} \cdot d\vec{r} = axdx + bydy$$

$$W = a \int_{x_1}^{x_2} xdx + b \int_{y_1}^{y_2} ydy$$

path independent → conservative

Gravitational Potential Energy

Let us derive a general relation between a conservative force and the associated potential energy.

$$W = \int_{x_i}^{x_f} F(x)\, dx \quad \Rightarrow \quad \Delta U = -\int_{x_i}^{x_f} F(x)\, dx$$

Note: the force is assumed to be conservative !

1. Gravitational Potential Energy

EXERCISE

$$\Delta U = -\int_{y_i}^{y_f} (-mg)\, dy = mg \int_{y_i}^{y_f} dy = mg\Big[\, y \,\Big]_{y_i}^{y_f},$$

$$\Delta U = mg(y_f - y_i) = mg\, \Delta y$$

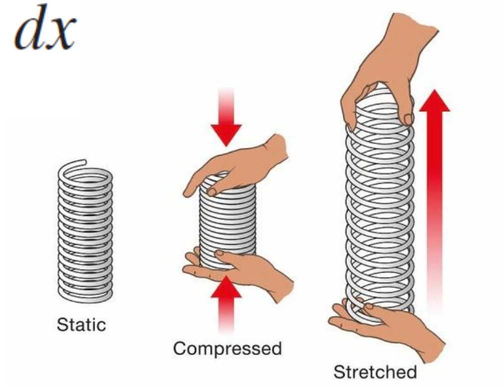
$$U - U_i = mg(y - y_i) \qquad \boxed{U(y) = mgy \qquad \text{(gravitational potential energy)}}$$

Elastic Potential Energy

Let us derive a general relation between a conservative force and the associated potential energy.

$$W = \int_{x_i}^{x_f} F(x)\,dx \quad \Rightarrow \quad \Delta U = -\int_{x_i}^{x_f} F(x)\,dx$$

Note: the force is assumed to be conservative !



2. Elastic Potential Energy

EXERCISE

$$\Delta U = -\int_{x_i}^{x_f} (-kx)\,dx = k\int_{x_i}^{x_f} x\,dx = \tfrac{1}{2}k\Big[x^2\Big]_{x_i}^{x_f},$$

$$\Delta U = \tfrac{1}{2}kx_f^2 - \tfrac{1}{2}kx_i^2$$

$$\boxed{U(x) = \tfrac{1}{2}kx^2 \qquad \text{(elastic potential energy)}}$$

Conservation of Mechanical Energy

The **mechanical energy** of a system is the sum of its potential energy *U* and the kinetic energy *K* of the objects within it.

$$E_{\text{mec}} = K + U$$

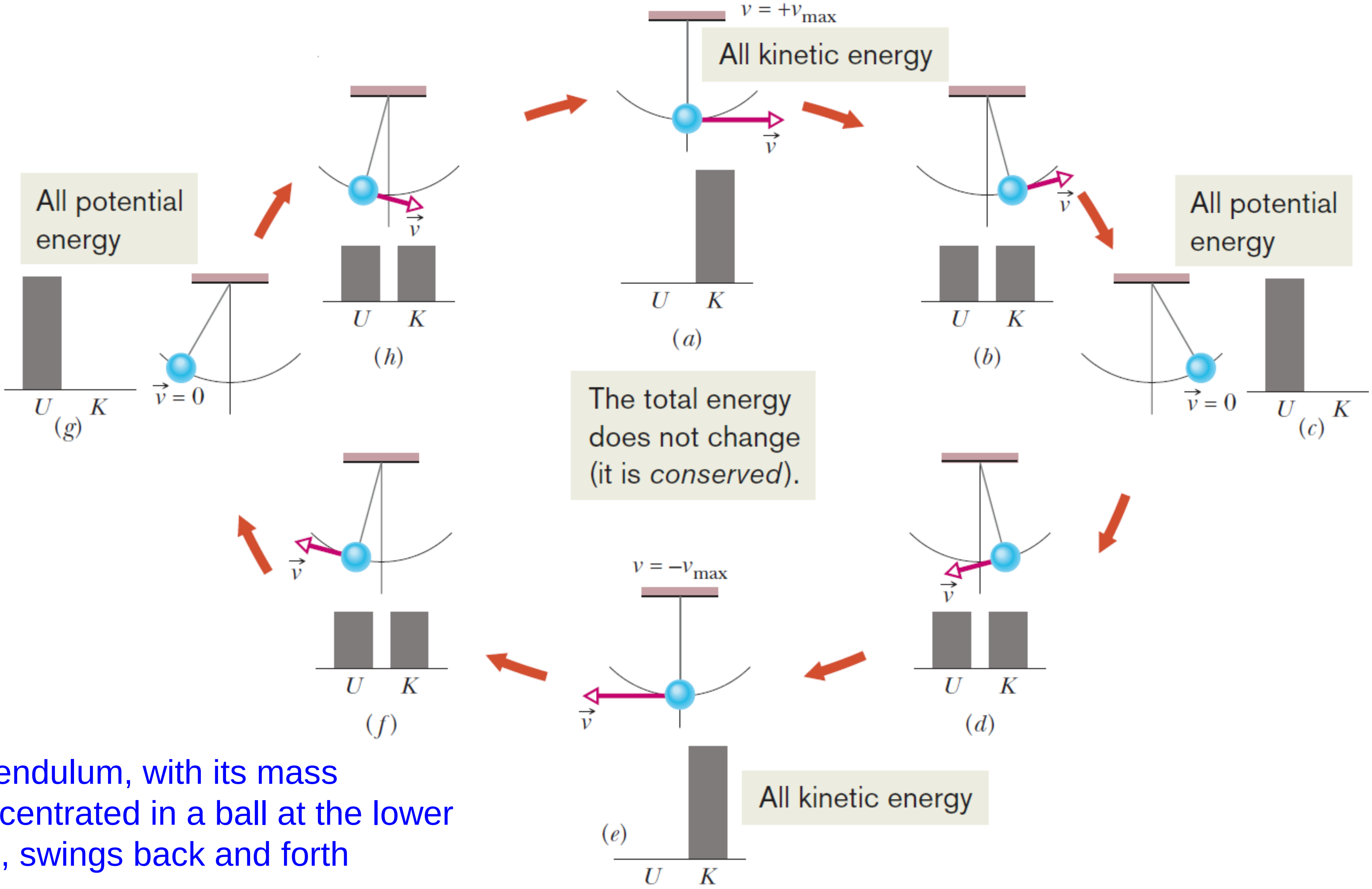
In an isolated system where only conservative forces cause energy changes, the kinetic energy and potential energy can change, but their sum, the mechanical energy E_{mec} of the system, cannot change.

$$\Delta K = W \qquad \Delta U = -W \qquad \Rightarrow \qquad \begin{gathered} \Delta K = -\Delta U \\ K_2 - K_1 = -(U_2 - U_1) \end{gathered}$$

$$K_2 + U_2 = K_1 + U_1 \qquad \text{(conservation of mechanical energy)}$$

Note: when the mechanical energy is conserved, we can relate the sum of kinetic and potential energy at one instant to that at another instant *without considering the intermediate motion* and *without finding the work done by the forces involved*.

Conservation of Mechanical Energy



All kinetic energy

All potential energy

All potential energy

The total energy does not change (it is *conserved*).

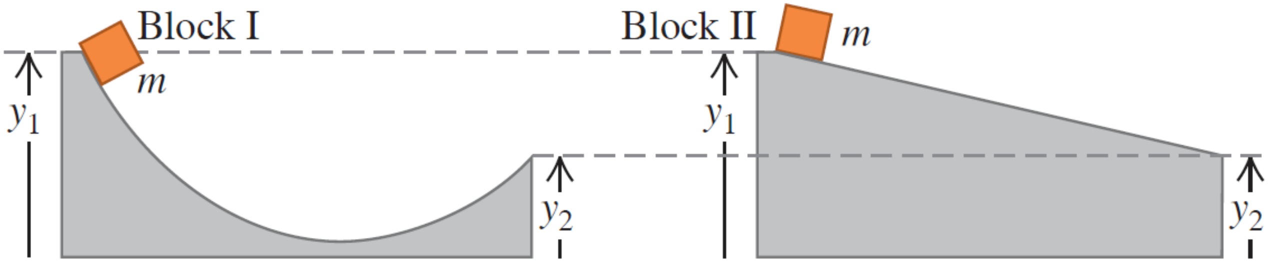
All kinetic energy

A pendulum, with its mass concentrated in a ball at the lower end, swings back and forth

Conservation of Mechanical Energy

QUIZ Check your understanding:

The figure shows two frictionless ramps. The heights y_1 and y_2 are the same for both ramps. If a block of mass m is released from rest at the left-hand end of each ramp, which block arrives at the right-hand end with the greater speed?
(i) Block I; (ii) block II; (iii) the speed is the same for both blocks.

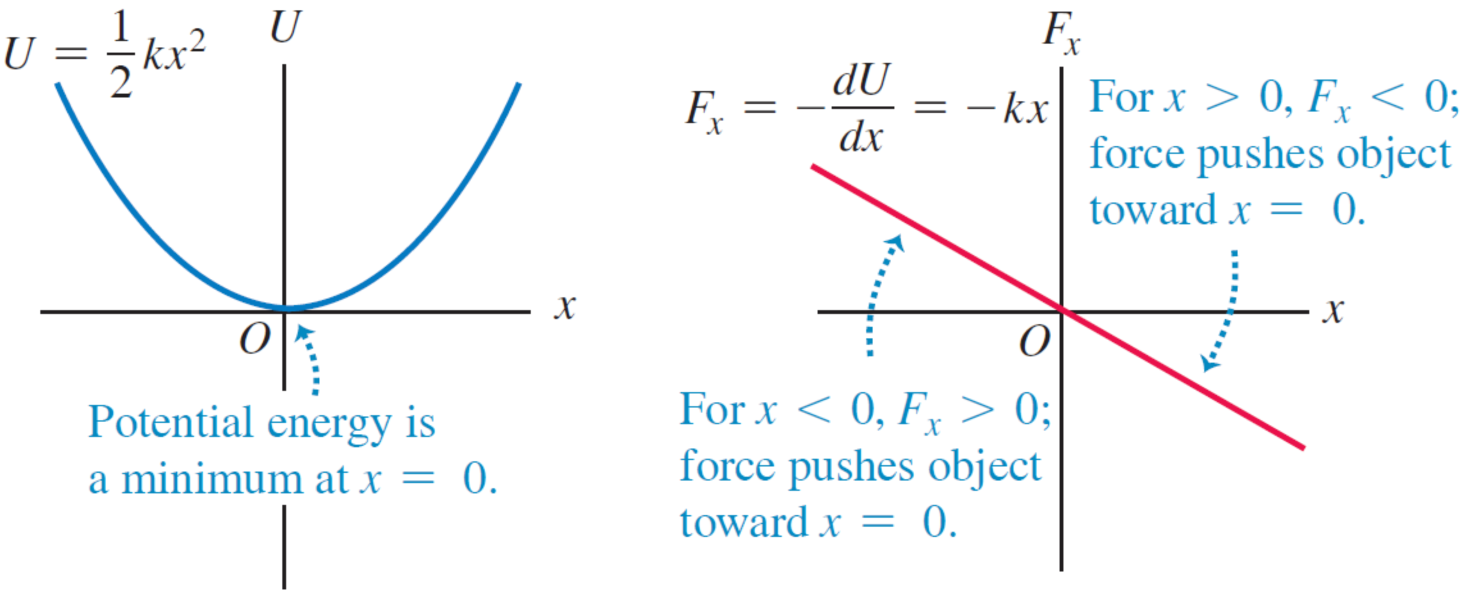


Force and Potential Energy

In studying physics sometimes there are situations when one has to find the force from the known potential energy as a function of position.

let us consider the 1-D case:

$$\Delta U(x) = -W = -F(x)\,\Delta x \quad \Rightarrow \quad \boxed{F(x) = -\frac{dU(x)}{dx}}$$


(a) Elastic potential energy and force as functions of x

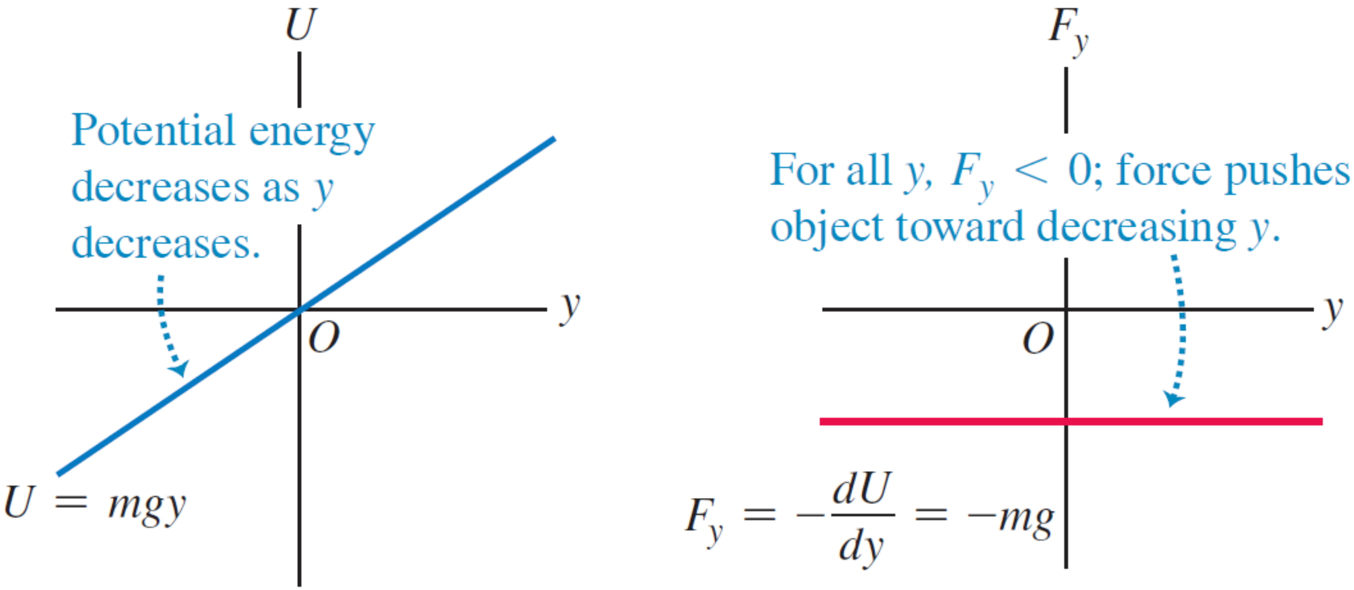
Force and Potential Energy

In studying physics sometimes there are situations when one has to find the force from the known potential energy as a function of position.

➡ let us consider the 1-D case:

$$\Delta U(x) = -W = -F(x)\,\Delta x \quad \Rightarrow \quad \boxed{F(x) = -\frac{dU(x)}{dx}}$$

(b) Gravitational potential energy and force as functions of *y*

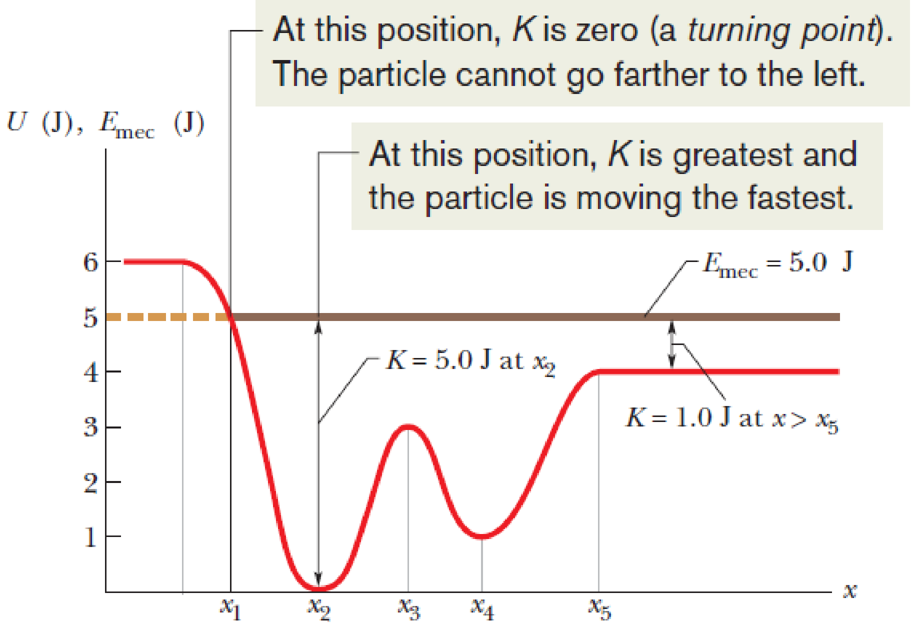


Reading a Potential Energy Curve

In the absence of non-conservative forces, the mechanical energy of a system has the constant value given by

$$U(x) + K(x) = E_{\text{mec}} \Rightarrow K(x) = E_{\text{mec}} - U(x)$$

Since K can never be negative, the particle moves only in the regions with $E_{\text{mec}} > U$.


At this position, K is zero (a *turning point*). The particle cannot go farther to the left.

At this position, K is greatest and the particle is moving the fastest.

Points, which satisfy the condition

$$E_{\text{mec}} = U$$

are called **turning** points (they correspond to the case $K = 0$ – hence, the particle stops and begins to move in the opposite direction).

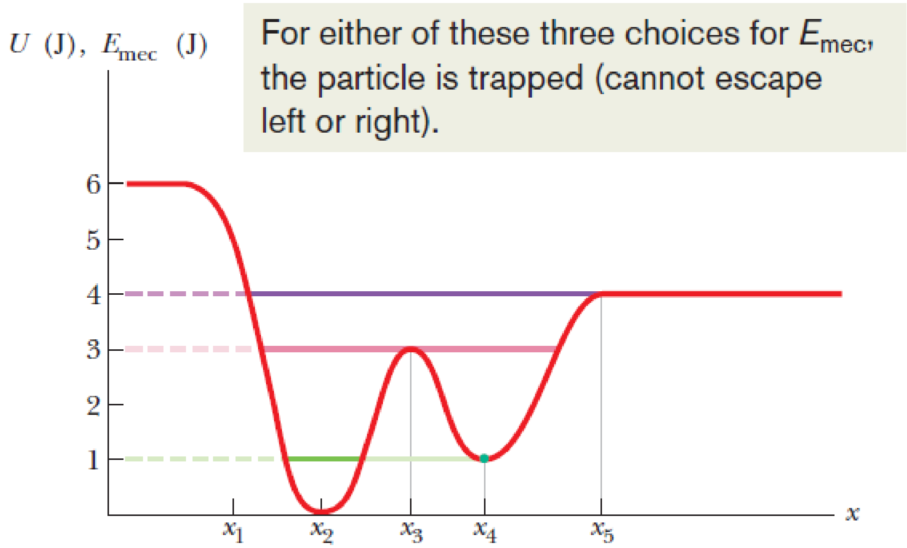
Infinite motion

Reading a Potential Energy Curve

In the absence of a non-conservative force, the mechanical energy of a system has the constant value given by

$$U(x) + K(x) = E_{\mathrm{mec}} \quad \Rightarrow \quad K(x) = E_{\mathrm{mec}} - U(x)$$

Since *K* can never be negative, the particle moves only in the regions with $E_{\mathrm{mec}} > U$.



For either of these three choices for E_{mec}, the particle is trapped (cannot escape left or right).

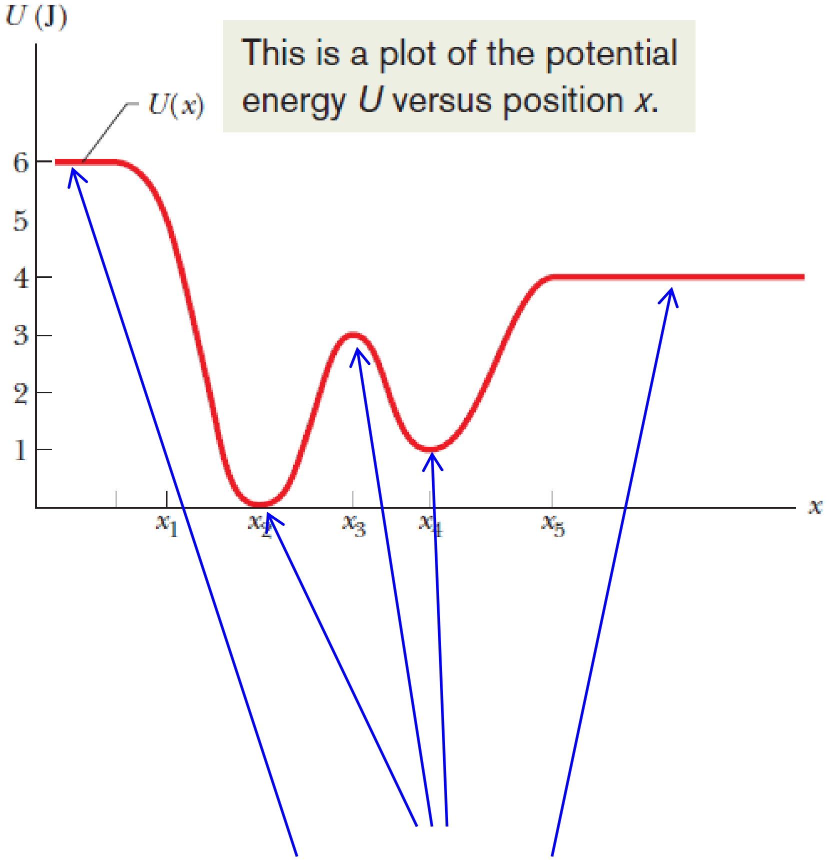
Points, which satisfy the condition

$$E_{\mathrm{mec}} = U$$

are called **turning** points (they correspond to the case $K = 0$ – hence, the particle stops and begins to move in the opposite direction).

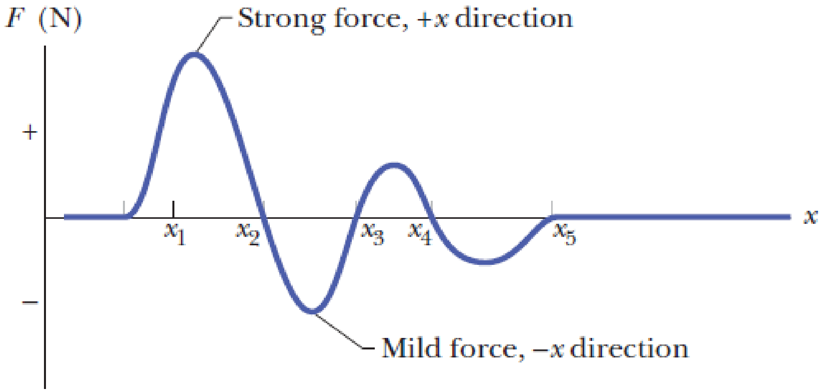
Finite motion

Reading a Potential Energy Curve



This is a plot of the potential energy *U* versus position *x*.

Force is equal to the negative of the slope of the *U*(*x*) plot.



equilibrium points

Types of equilibrium:
- → stable
- → unstable
- → neutral

Force and Potential Energy in 3-D

Generalization on the 3-D case:

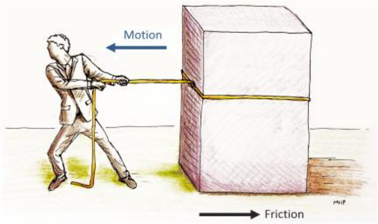
$$dU = -\vec{F} \cdot d\vec{r} \qquad \vec{F} = -\left(\frac{\partial U}{\partial x}\vec{i} + \frac{\partial U}{\partial y}\vec{j} + \frac{\partial U}{\partial z}\vec{k} \right)$$

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

del (nabla) operator

$$\boxed{\vec{F} = -\nabla U}$$

Not all forces are conservative. A force that is not conservative is called **nonconservative**.



the work done by a nonconservative force can not be represented by a potential-energy function

Note: some nonconservative forces cause mechanical energy to be lost (friction, fluid resistance), whereas some others may also increase it (the forces unleashed by chemical reactions via the explosions)

Force and Potential Energy in 3-D

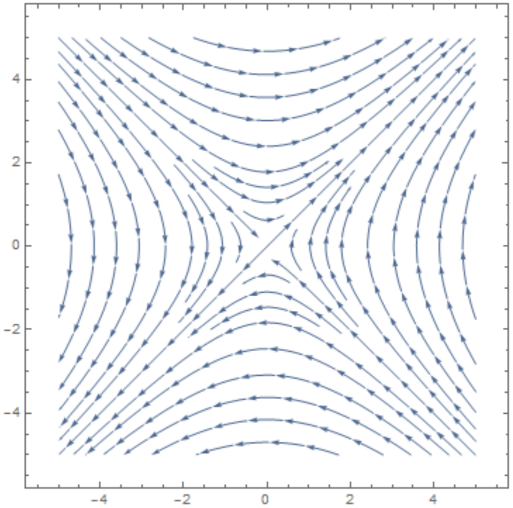
EXERCISE

Task #2: Derive the force from the given potential energy

$$U(x,y) = -\alpha xy$$

Solution:

$$\vec{F} = -\left(\frac{\partial U}{\partial x}\vec{i} + \frac{\partial U}{\partial y}\vec{j}\right) = \alpha(y\vec{i} + x\vec{j})$$



QUIZ

Check your understanding:

A particle moving along the *x*-axis is acted on by a conservative force F_x *. At a certain point, the force is zero. (a) Which of the following* statements about the value of the potential-energy function *U*(*x*) at that point is correct? (i) *U = 0;* (ii) *U > 0;* (iii) *U < 0;* (iv) not enough information is given to decide.

(b) Which of the following statements about the value of the derivative of *U(x) at that point is* correct? (i) *dU/dx = 0;* (ii) *dU/dx > 0;* (iii) *dU/dx < 0;* (iv) not enough information is given to decide.

Conservation of Total Energy

The **total energy** of a system can change only by amounts of energy that are transferred to or from the system.

$$W = \Delta E = \Delta E_{\text{mec}} + \Delta E_{\text{th}} + \Delta E_{\text{int}}$$

work done by **external** force

change in mechanical energy

change in thermal energy

change in any other type of internal energy

Note: this law of conservation of energy is NOT something we have derived from basic physical principles. Rather, it is a law based on countless experiments.

$$\boxed{\frac{dE}{dt} = \vec{F}_{\text{ex}} \cdot \vec{v}}$$

The **total energy** of an **isolated** system can not change.

$$\Delta E_{\text{mec}} + \Delta E_{\text{th}} + \Delta E_{\text{int}} = 0 \qquad \text{(isolated system)}$$

Conservation of Total Energy

QUIZ

Check your understanding:

In a hydroelectric generating station, falling water is used to drive turbines ("water wheels"), which in turn run electric generators. Compared to the amount of gravita-tional potential energy released by the falling water, how much electrical energy is produced? (i) The same; (ii) more; (iii) less.

Systems with Varying Mass

There exist cases when the mass of a body varies in the process of motion due to the continuous separation or addition of matter.

⇨ how should one change the fundamental equation of motion in such cases?

EXERCISE

net external force

$$m d\vec{v} = \vec{F} dt + dm \cdot \vec{u} \quad \Rightarrow \quad \boxed{m\frac{d\vec{v}}{dt} = \vec{F} + \underbrace{\frac{dm}{dt}\vec{u}}_{\vec{R}}}$$

separation (addition) velocity relative to the "main" body

Meshchersky equation
(fundamental equation of dynamics of a mass point with variable mass)

reactive force

$\vec{R}$

Systems with Varying Mass

$$\boxed{m\frac{d\vec{v}}{dt} = \vec{F} + \frac{dm}{dt}\vec{u}} \qquad \vec{R} = \frac{dm}{dt}\vec{u}$$

Meshchersky equation

Special cases:

(i) when ***u*** = 0 (separation or addition of mass with zero velocity relative to a body)

$$\vec{R} = 0 \quad \Rightarrow \quad m(t)\frac{d\vec{v}}{dt} = \vec{F}$$

(ii) when ***u*** = -***v*** (the added mass is stationary in the considered reference frame)

$$m\frac{d\vec{v}}{dt} + \frac{dm}{dt}\vec{v} = \vec{F} \quad \Rightarrow \quad \frac{d}{dt}(m\vec{v}) = \vec{F}$$

Systems with Varying Mass

EXERCISE

Task #3: A rocket stays at rest in the air at a fixed height by sending the vertical jet of gas having velocity ***u***. Find (a) how much time elapses before the rocket starts falling if the initial fuel mass constitutes *η*-part of its hollow mass; (b) what is the mass *μ*(*t*) which should be ejected every second in order to stay at this fixed height if the initial total mass (rocket + fuel) is m_0.

Solution:

(a) $$\frac{d\vec{v}}{dt} = 0$$

$$m\vec{g} + \left(\frac{dm}{dt}\right)\vec{u} = 0$$

$$\frac{dm}{m} = -\left(\frac{g}{u}\right)dt$$

$$\ln\frac{m}{m_0} = -\left(\frac{g}{u}\right)t$$

$$\eta = \frac{m_0 - m}{m}$$

$$t = \left(\frac{u}{g}\right)\ln\frac{m_0}{m} = \left(\frac{u}{g}\right)\ln(1+\eta)$$

Systems with Varying Mass

EXERCISE

Task #3: A rocket stays at rest in the air at a fixed height by sending the vertical jet of gas having velocity ***u***. Find (a) how much time elapses before the rocket starts falling if the initial fuel mass constitutes η-part of its hollow mass; (b) what is the mass $\mu(t)$ which should be ejected every second in order to stay at this fixed height if the initial total mass (rocket + fuel) is m_0.

Solution:

(b)

$$\mu = -\frac{dm}{dt} = \left(\frac{g}{u}\right) m$$

$$m = m_0 e^{-gt/u}$$

$$\mu = \left(\frac{g}{u}\right) m_0 e^{-gt/u}$$

this law is valid for a time span obtained in (a)

Systems with Varying Mass

EXERCISE

Task #4: A flatcar starts rolling at the moment *t* = 0 due to the permanent force ***F***. Ignoring the friction in axes, find the time dependence of its velocity if (a) it is loaded with sand which pours out through a hole in the bottom at the constant rate of *μ* kg/s, and the initial mass of the flatcar with sand at *t* =0 is equal to m_0; (b) the sand is loaded on it from a stationary hopper at the permanent rate *μ* kg/s, starting from the moment *t* = 0, when it had the mass m_0.

Solution:

(a)

$$(m_0 - \mu t)\frac{d\vec{v}}{dt} = \vec{F}$$

$$d\vec{v} = \frac{\vec{F}dt}{(m_0 - \mu t)}$$

➡

$$\vec{v} = \frac{\vec{F}}{\mu}\ln\frac{m_0}{(m_0 - \mu t)}$$

(b)

$$\vec{R} = \mu(-\vec{v})$$

$$d(m\vec{v}) = \vec{F}dt$$

$$m\vec{v} = \vec{F}t$$

$$m = m_0 + \mu t$$

$$\vec{v} = \frac{\vec{F}t}{m_0 + \mu t}$$

Note: both expressions are valid only during the load or unload process.

Systems with Varying Mass

QUIZ

Check your understanding:

(a) If a rocket in gravity-free outer space has the same thrust at all times, is its acceleration constant, increasing, or decreasing? (b) If the rocket has the same acceleration at all times, is the thrust constant, increasing, or decreasing?

Conclusions

- energy is a **scalar** quantity associated with the state (or condition) of one or more objects
- in mechanics we introduce the **kinetic** and **potential** energies as the quantities associated with the motion and configuration of the system, respectively
- **work-energy theorem** establishes useful relation between the work done by the force and the changes occurring in kinetic and potential energies. Based on this, we also derived energy conservation law for the closed isolated systems
- potential energy can be introduced for the **conservative** forces only
- reading potential energy **curves** is important in order to understand the allowed regions of motion as well as some other useful information (type of motion, turning points, equilibrium points, etc.)
- in cases when the mass of a body varies in the process of motion, its dynamics is described by means of **Meshchersky equation**