Developable surface(可展曲面)

Theorem: For local, developable surface can build an isometric correspondence between plane.

1. cylindrical surface (柱面)

Developable surface(可展曲面)

2. conical surface (锥面) $\boldsymbol{r}(u,v)=\boldsymbol{a}+v\boldsymbol{l}(u)$, where a is a constant vector, $|\boldsymbol{l}(u)|=1, (u,v)\in(a,b)\times(0,\infty)$

Since $|\boldsymbol{l}(u)|=1$, we get $\boldsymbol{l}(u).\boldsymbol{l}'(u)=0$, then

$$\boldsymbol{r}_u=v\boldsymbol{l}'(u), \boldsymbol{r}_v=\boldsymbol{l}(u)$$

$$\Rightarrow E=v^2|\boldsymbol{l}'(u)|^2, F=v\boldsymbol{l}(u).\boldsymbol{l}'(u)=0, G=|\boldsymbol{l}(u)|^2=1$$

So the 1st fundamental form is

$$I=v^2|\boldsymbol{l}'(u)|^2du^2+dv^2$$

Since $|\boldsymbol{l}'(u)|\neq 0$ (otherwise **r**(u,v) is a line).

After the following parameter transformation

$$\bar{u}=\int|\boldsymbol{l}'(u)|du\,,\bar{v}=v$$

We obtain the following 1st fundamental form

$$I=\bar{v}^2d\bar{u}^2+d\bar{v}^2$$

It is the same as the 1st fundamental form of xOy plane

$$\boldsymbol{r}=(\bar{v}\cos(\bar{u})\,,\bar{v}\sin(\bar{u})\,,0)$$

Developable surface(可展曲面)

3. tangent surface (切线曲面) $\boldsymbol{r}(u,v)=\boldsymbol{a}(u)+v\boldsymbol{a}'(u)$, where $(u,v)\in(a,b)\times(0,\infty), \boldsymbol{a}'(u)\times\boldsymbol{a}''(u)\neq 0$

Choose u as the arc length parameter of the directrix
C: $\boldsymbol{a}=\boldsymbol{a}(u)$,
its Frenet frame is $\{\boldsymbol{a};\boldsymbol{T},\boldsymbol{N},\boldsymbol{B}\}$, curvature is k.
Then $\boldsymbol{r}_u=\boldsymbol{a}'(u)+v\boldsymbol{a}''(u)=\boldsymbol{T}(u)+vk\boldsymbol{N}, \boldsymbol{r}_v=\boldsymbol{T}(u)$

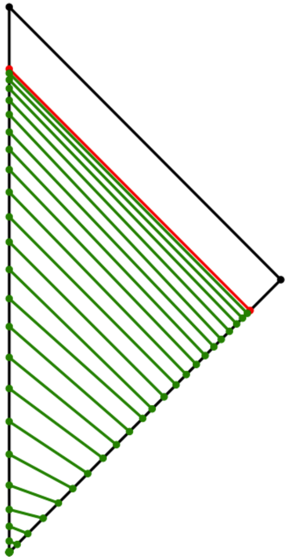
$$\Rightarrow E=1+v^2k^2, F=G=1$$

So the 1st fundamental form is

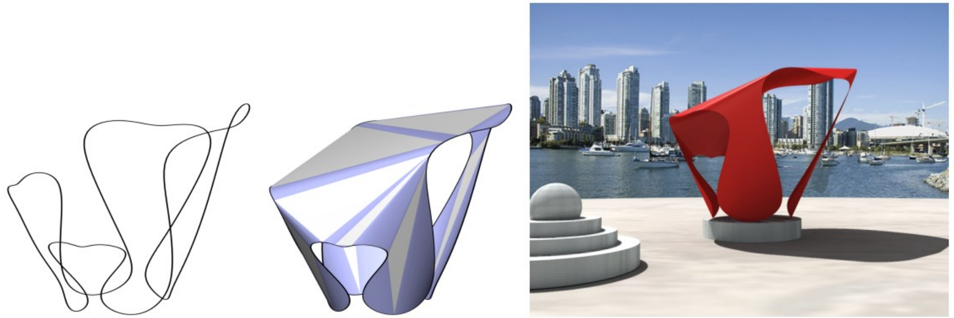
$$I=(1+v^2k^2)du^2+2dudv+dv^2$$

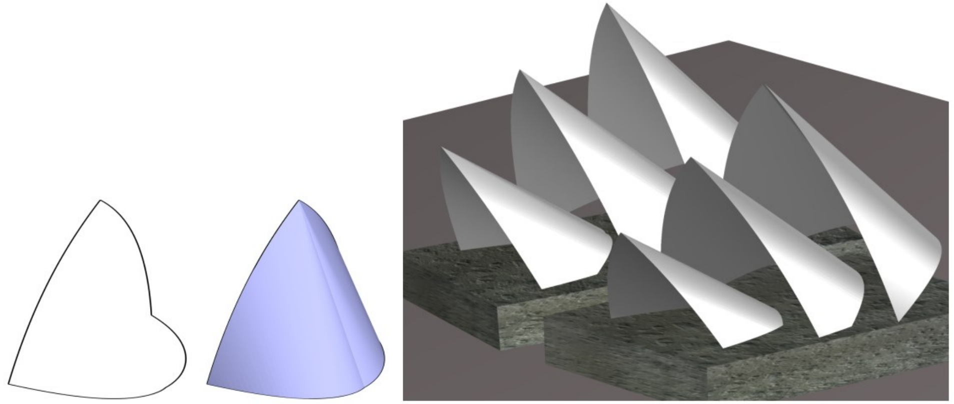
According the fundamental theorem of curve, there is a plane curve C_1: $\boldsymbol{a}_1(u)=(x(u),y(u),0))$, where u is its arc length parameter, $k(u)$ is its curvature, and torsion is zero.
Obviously, tangent surface S_1 of C_1 is part of the plane, and the 1st fundamental form of S_1 and S are the same.

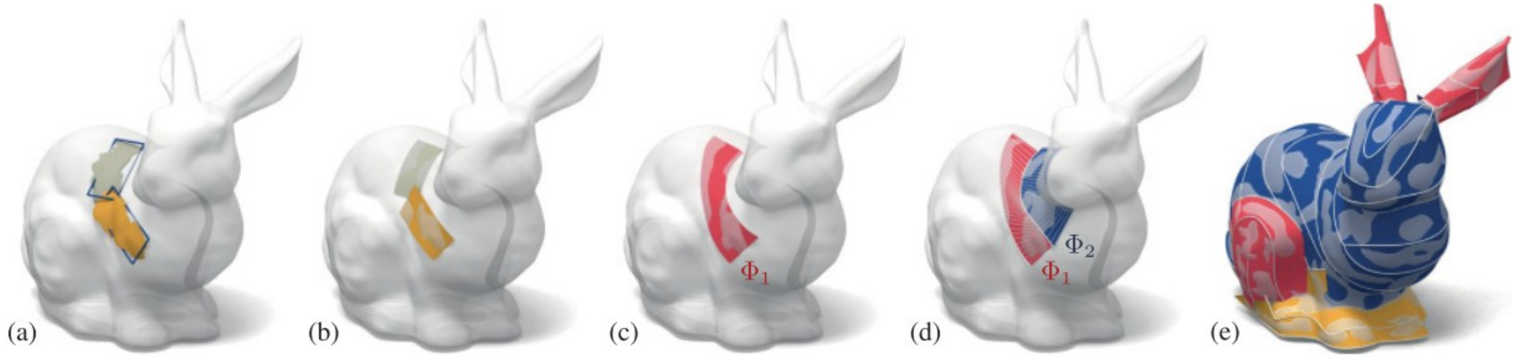
Applications of developable surface



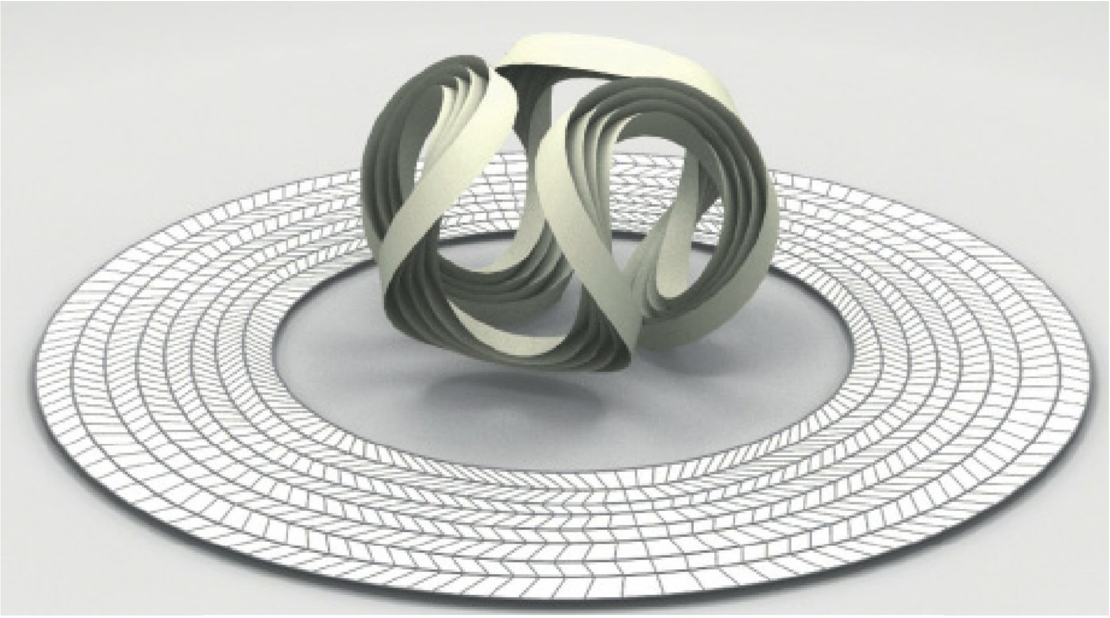
Applications of developable surface

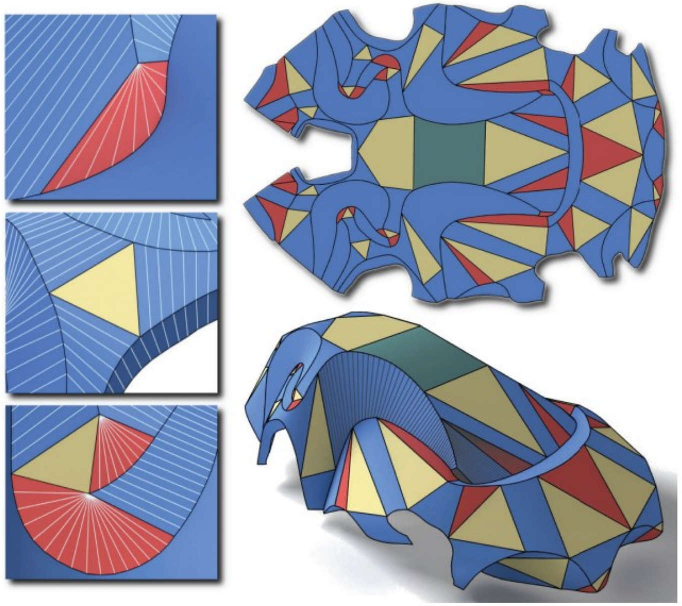






Applications of developable surface





Envelope surface(包络面)

Assume $\{S_\alpha\}$ is family of regular parametric surfaces which are dependent on $\alpha \in (a,b)$. If a regular surface S satisfies

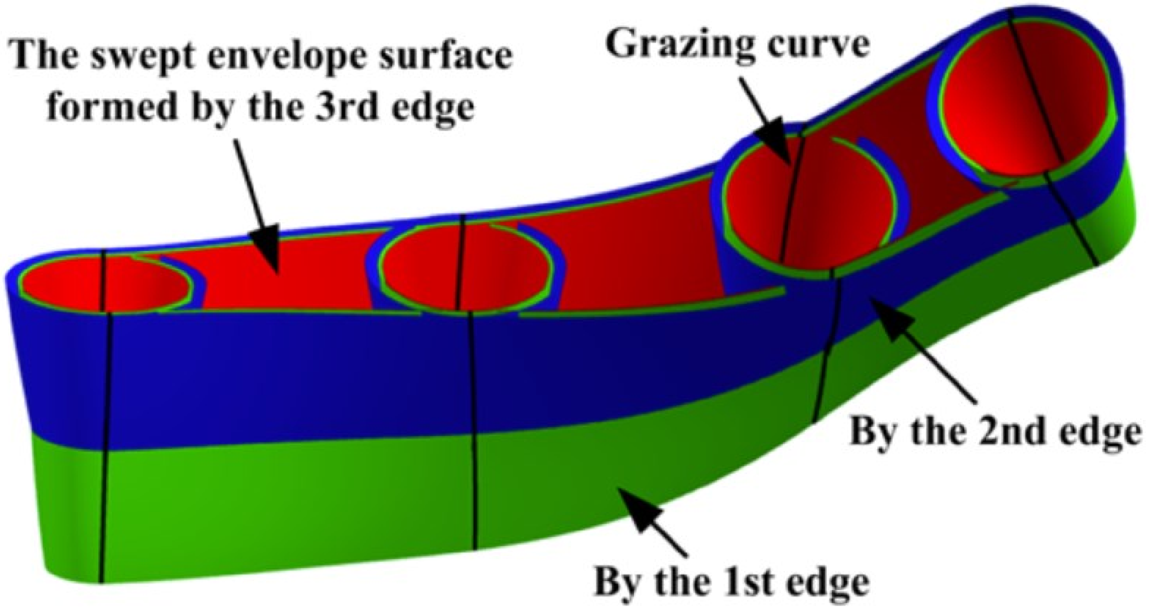
1. $\forall p \in S$, there is a unique $\alpha \in (a,b)$, so that $p \in S_\alpha$ and $S \; and \; S_\alpha$ have the same tangent plane at point p;
2. $\forall \alpha \in (a,b), \exists\, p \in S \cap S_\alpha$, so that S and S_α have the same tangent plane at point p;

Then S is the **envelope surface** of $\{S_\alpha\}$.

Envelope surface of $\{S_\alpha\} = \{\mathrm{A}(\alpha)x + \mathrm{B}(\alpha)y + \mathrm{C}(\alpha)z + D(\alpha) =$

$$\begin{cases} F(x,y,z,\alpha) = 0, \\ F_\alpha(x,y,z,\alpha) = 0 \end{cases}$$

Envelope Surface Modeling and Tool Path Optimization for Five-Axis Flank Milling Considering Cutter Runout



Analysis of improved positioning in five-axis ruled surface milling using envelope surface

Johanna Senatore, Frédéric Monies, Jean-Max Redonnet, Walter Rubio*

Homework

1. Prove surface $\boldsymbol{r}(u,v)=\left(u^2+\frac{1}{3}v, 2u^3+uv, u^4+\frac{2}{3}u^2v\right)$ is a developable surface.
2. Prove surface

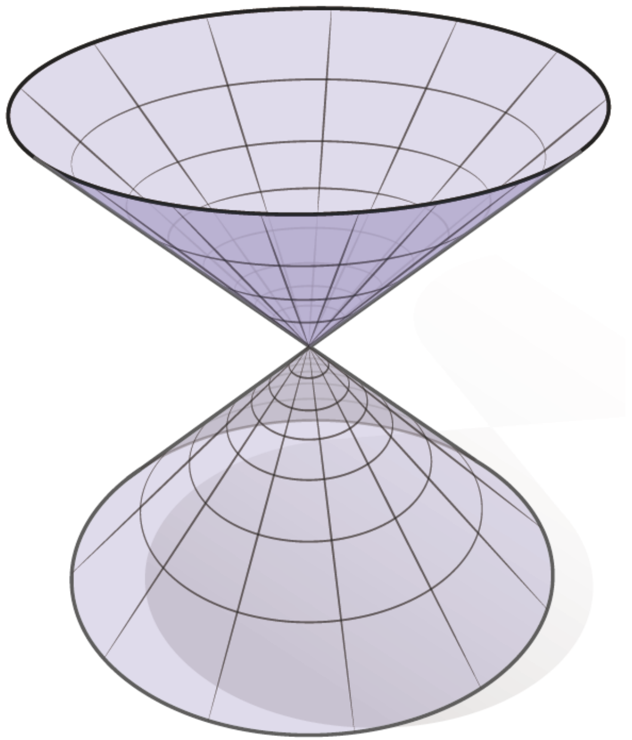
$$\boldsymbol{r}(u,v)=(ucos(v), usin(v), av+b)\ (a, b\ are\ constants)$$

is not a developable surface.

3. Calculate the envelope surface of the family of planes

$\{xcos(\alpha)+ysin(\alpha\ )-zsin(\alpha)=2b\}, b\ is\ a\ constant.$

Orientation



Definition

Let M be a regular surface in $\mathbb{R}^3$. M is said to be orientable if there is a unit vector field $\mathbf{N}$ on M such that

(i) $\mathbf{N}$ *is smooth;*

Definition

Let M be a regular surface in $\mathbb{R}^3$. M is said to be orientable if there is a unit vector field $\mathbf{N}$ on M such that

(i) $\mathbf{N}$ *is smooth;*

(ii) $\mathbf{N}$ *has unit length;*

Definition

Let M be a regular surface in $\mathbb{R}^3$. M is said to be orientable if there is a unit vector field $\mathbf{N}$ on M such that

(i) $\mathbf{N}$ *is smooth;*

(ii) $\mathbf{N}$ *has unit length;*

(iii) $\mathbf{N}$ *is orthogonal to $T_p(M)$ at all point.*

Orientation of regular surfaces

Definition

Let M be a regular surface in $\mathbb{R}^3$. M is said to be orientable if there is a unit vector field $\mathbf{N}$ on M such that

(i) $\mathbf{N}$ *is smooth;*

(ii) $\mathbf{N}$ *has unit length;*

(iii) $\mathbf{N}$ *is orthogonal to $T_p(M)$ at all point.*

If such $\mathbf{N}$ exists, then it is called an orientation of M.

Basic facts

- If $\mathbf{N}$ is an orientation, then $-\mathbf{N}$ is also an orientation. There are exactly two orientations on an orientable surface.

- If $\mathbf{N}$ is an orientation, then $-\mathbf{N}$ is also an orientation. There are exactly two orientations on an orientable surface.
- $\mathbf{N}$ is smooth means that if $\mathbf{N} = (N_1, N_2, N_3)$ then each N_i is a smooth function.

Basic facts

- If $\mathbf{N}$ is an orientation, then $-\mathbf{N}$ is also an orientation. There are exactly two orientations on an orientable surface.
- $\mathbf{N}$ is smooth means that if $\mathbf{N} = (N_1, N_2, N_3)$ then each N_i is a smooth function.
- $\mathbf{N}$ is continuous and satisfies (ii), (iii) above that $\mathbf{N}$ is smooth.

We have the following intrinsic characterization of orientable surface.

Proposition

M is orientable if and only if there exist coordinate charts covering M so that the change of coordinate matrices have positive determinant.

Example

Let prove cylindrical surface is orientable.

Proof:

The regular parametric function of cylindrical surface is

$$r(u,v) = (\mathrm{acos}(u)\,, \mathrm{asin}(u)\,, bv),$$

where a>0 and b are constant.

If let $-\pi < u < \pi, -\infty < v < \infty$, then this coordinate chart covers most of the surface except the following line

$$x = -a, y = 0, z = bv$$

If let 0< $u < 2\pi, -\infty < v < \infty$, then this coordinate chart covers most of the surface except the following line

$$x = a, y = 0, z = bv$$

These two coordinate charts cover the whole surface.

Example

If the parameter of the first one is denoted as (u, v), and the second one is denoted as $(\tilde{u}, \tilde{v})$, then they have the following transformation

$$\tilde{u} = \begin{cases} u, & 0 < u < \pi \\ u + 2\pi, & -\pi < u < 0 \end{cases}, \qquad v = \tilde{v}$$

The determinant of Jacobi matrix of the parameter transformation is

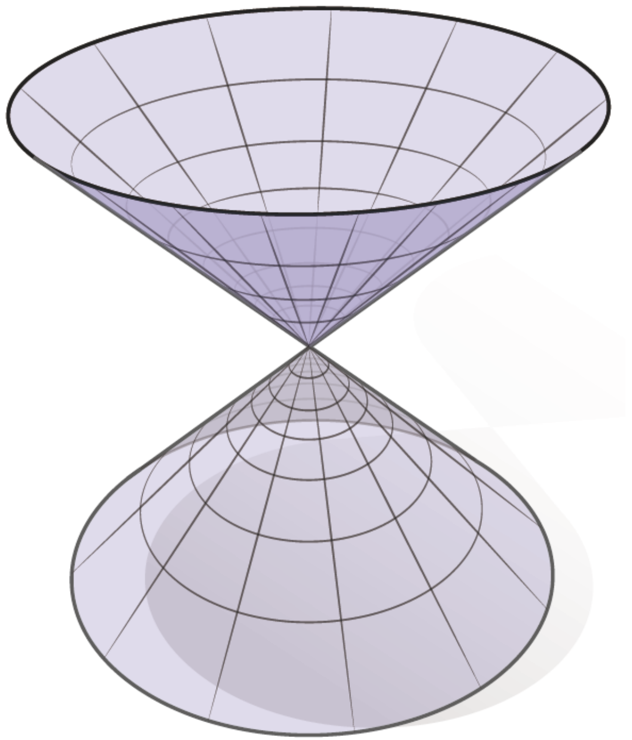
$$\frac{\partial(\tilde{u}, \tilde{v})}{\partial(u, v)} = 1 > 0$$

Therefore, cylindrical surface is orientable.

Proposition

PROPOSITION 2. *If a regular surface is given by* $S = \{(x, y, z) \in R^3;\ f(x, y, z) = a\}$, *where* $f\colon U \subset R^3 \to R$ *is differentiable and* a *is a regular value of* f, *then* S *is orientable.*

The second fundamental form

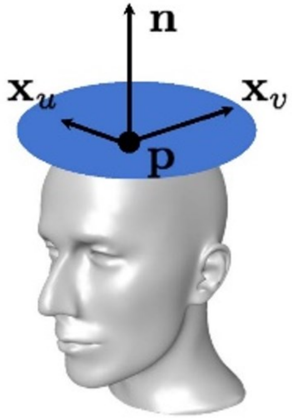


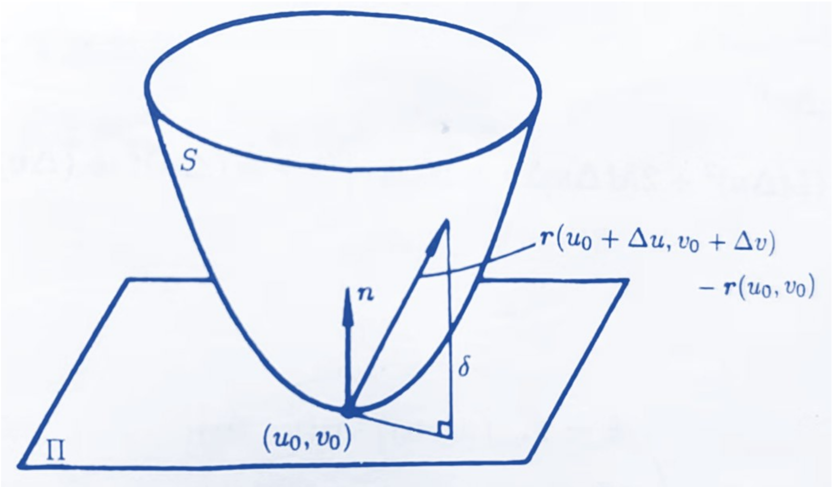
The second fundamental form of surface

Assume $S\colon \boldsymbol{r}=\boldsymbol{r}(u,v)$ is a regular parametric surface.
The normal of tangent plane Π at any point (u_0, v_0) is represented as

$$\boldsymbol{n}=\frac{\boldsymbol{r}_u\times\boldsymbol{r}_v}{|\boldsymbol{r}_u\times\boldsymbol{r}_v|}\bigg|_{(u_0,v_0)}$$

Intuitively, the bending condition of S at point (u_0, v_0) can be measured with the directed distance δ from the neighboring points of (u_0, v_0) to Π.





The second fundamental form of surface

Assume the neighboring point of (u_0, v_0) is denoted as $(u_0 + \Delta u, v_0 + \Delta v)$, then its directed distance from tangent plane Π is

$$\delta(\Delta u, \Delta v) = \left(\boldsymbol{r}(u_0 + \Delta u, v_0 + \Delta v) - \boldsymbol{r}(u_0, v_0)\right).\boldsymbol{n}$$

According to Taylor expansion, we have

$$\begin{aligned} &\boldsymbol{r}(u_0 + \Delta u, v_0 + \Delta v) - \boldsymbol{r}(u_0, v_0) \\ &= (\boldsymbol{r}_u \Delta u + \boldsymbol{r}_v \Delta v) + \frac{1}{2}(\boldsymbol{r}_{uu}(\Delta u)^2 + 2\boldsymbol{r}_{uv}\Delta u \Delta v + \boldsymbol{r}_{vv}(\Delta v)^2) \\ &+ \boldsymbol{o}\left((\Delta u)^2 + (\Delta v)^2\right) \end{aligned}$$

where $\boldsymbol{r}_u, \boldsymbol{r}_v, \boldsymbol{r}_{uu}, \boldsymbol{r}_{uv}, \boldsymbol{r}_{vv}$ are evaluated at (u_0, v_0), and

$$\lim_{(\Delta u)^2 + (\Delta v)^2 \to 0} \frac{\boldsymbol{o}\left((\Delta u)^2 + (\Delta v)^2\right)}{(\Delta u)^2 + (\Delta v)^2} = 0$$

The second fundamental form of surface

$$\boldsymbol{r}(u_0+\Delta u, v_0+\Delta v)-\boldsymbol{r}(u_0,v_0)$$
$$=(\boldsymbol{r}_u\Delta u+\boldsymbol{r}_v\Delta v)+\frac{1}{2}(\boldsymbol{r}_{uu}(\Delta u)^2+2\boldsymbol{r}_{uv}\Delta u\Delta v+\boldsymbol{r}_{vv}(\Delta v)^2)$$
$$+\boldsymbol{o}\big((\Delta u)^2+(\Delta v)^2\big)$$

So $\delta(\Delta u,\Delta v)$

$$=\frac{1}{2}(L(\Delta u)^2+2M\Delta u\Delta v+N(\Delta v)^2)+\boldsymbol{o}\big((\Delta u)^2+(\Delta v)^2\big)$$

where

$$L=\boldsymbol{r}_{uu}(u_0,v_0).\boldsymbol{n}(u_0,v_0)$$
$$M=\boldsymbol{r}_{uv}(u_0,v_0).\boldsymbol{n}(u_0,v_0)$$
$$N=\boldsymbol{r}_{vv}(u_0,v_0).\boldsymbol{n}(u_0,v_0)$$

Since $\boldsymbol{r}_u.\boldsymbol{n}=\boldsymbol{r}_v.\boldsymbol{n}=0$, so

$$\boldsymbol{r}_{uu}.\boldsymbol{n}+\boldsymbol{r}_u.\boldsymbol{n}_u=0$$
$$\boldsymbol{r}_{uv}.\boldsymbol{n}+\boldsymbol{r}_u.\boldsymbol{n}_v=0$$
$$\boldsymbol{r}_{vu}.\boldsymbol{n}+\boldsymbol{r}_v.\boldsymbol{n}_u=0$$
$$\boldsymbol{r}_{vv}.\boldsymbol{n}+\boldsymbol{r}_v.\boldsymbol{n}_v=0$$

The second fundamental form of surface

So L, M, N can also be represented as

$$L = -\boldsymbol{r}_u . \boldsymbol{n}_u$$
$$M = -\boldsymbol{r}_u . \boldsymbol{n}_v = -\boldsymbol{r}_v . \boldsymbol{n}_u$$
$$N = -\boldsymbol{r}_v . \boldsymbol{n}_v$$

The main part of the directed distance $\delta(\Delta u, \Delta v)$ is a quadric differential form, that is

$$\frac{1}{2}(L(\Delta u)^2 + 2M\Delta u\Delta v + N(\Delta v)^2)$$

$$\boldsymbol{r}_{uu} . \boldsymbol{n} + \boldsymbol{r}_u . \boldsymbol{n}_u = 0$$
$$\boldsymbol{r}_{uv} . \boldsymbol{n} + \boldsymbol{r}_u . \boldsymbol{n}_v = 0$$
$$\boldsymbol{r}_{vu} . \boldsymbol{n} + \boldsymbol{r}_v . \boldsymbol{n}_u = 0$$
$$\boldsymbol{r}_{vv} . \boldsymbol{n} + \boldsymbol{r}_v . \boldsymbol{n}_v = 0$$

Then we consider a quadric differential form

$$\begin{aligned}
\Pi &= d^2\boldsymbol{r} . \boldsymbol{n} \\
&= d(d\boldsymbol{r}) . \boldsymbol{n} \\
&= d(\boldsymbol{r}_u du + \boldsymbol{r}_v dv) . \boldsymbol{n} \\
&= (\boldsymbol{r}_{uu}(du)^2 + 2\boldsymbol{r}_{uv} du dv + \boldsymbol{r}_{vv}(dv)^2) . \boldsymbol{n} \\
&= -d\boldsymbol{r} . d\boldsymbol{n} \\
&= L(du)^2 + 2M du dv + N(dv)^2
\end{aligned}$$

The second fundamental form of surface

We call

$$\Pi = L(du)^2 + 2Mdudv + N(dv)^2$$

as the **second fundamental form** (第二基本形式) of surface S, L, M, N are coefficients of the second fundamental form.

Similar to the first fundamental form I, Π is independent of the selection of allowable parameter transformation that maintains orientation (保持定向的参数变换).

Assume surface S has an allowable parameter transformation

$$u = u(\tilde{u}, \tilde{v}), v = v(\tilde{u}, \tilde{v})$$

and

$$\frac{\partial(u,v)}{\partial(\tilde{u},\tilde{v})} > 0$$

The second fundamental form of surface

So

$$\boldsymbol{r}_{\tilde{u}} = \boldsymbol{r}_u \frac{\partial u}{\partial \tilde{u}} + \boldsymbol{r}_v \frac{\partial v}{\partial \tilde{u}},$$
$$\boldsymbol{r}_{\tilde{v}} = \boldsymbol{r}_u \frac{\partial u}{\partial \tilde{v}} + \boldsymbol{r}_v \frac{\partial v}{\partial \tilde{v}}$$

Then

$$\boldsymbol{r}_{\tilde{u}} \times \boldsymbol{r}_{\tilde{v}} = \frac{\partial(u, v)}{\partial(\tilde{u}, \tilde{v})} \boldsymbol{r}_u \times \boldsymbol{r}_v,$$
$$\tilde{\boldsymbol{n}} = \boldsymbol{n}$$

and

$$\boldsymbol{n}_{\tilde{u}} = \boldsymbol{n}_u \frac{\partial u}{\partial \tilde{u}} + \boldsymbol{n}_v \frac{\partial v}{\partial \tilde{u}},$$
$$\boldsymbol{n}_{\tilde{v}} = \boldsymbol{n}_u \frac{\partial u}{\partial \tilde{v}} + \boldsymbol{n}_v \frac{\partial v}{\partial \tilde{v}}$$

The second fundamental form of surface

That is

$$\begin{pmatrix} \boldsymbol{r}_{\tilde{u}} \\ \boldsymbol{r}_{\tilde{v}} \end{pmatrix} = J.\begin{pmatrix} \boldsymbol{r}_{u} \\ \boldsymbol{r}_{v} \end{pmatrix}$$

and

$$\begin{pmatrix} \boldsymbol{n}_{\tilde{u}} \\ \boldsymbol{n}_{\tilde{v}} \end{pmatrix} = J.\begin{pmatrix} \boldsymbol{n}_{u} \\ \boldsymbol{n}_{v} \end{pmatrix}$$

where

$$\boldsymbol{J} = \begin{pmatrix} \dfrac{\partial u}{\partial \tilde{u}} & \dfrac{\partial u}{\partial \tilde{v}} \\ \dfrac{\partial v}{\partial \tilde{u}} & \dfrac{\partial v}{\partial \tilde{v}} \end{pmatrix}$$

Since

$$\begin{pmatrix} \tilde{L} & \widetilde{M} \\ \widetilde{M} & \widetilde{N} \end{pmatrix} = -\begin{pmatrix} \boldsymbol{r}_{\tilde{u}} \\ \boldsymbol{r}_{\tilde{v}} \end{pmatrix}.\begin{pmatrix} \boldsymbol{n}_{\tilde{u}} & \boldsymbol{n}_{\tilde{v}} \end{pmatrix}$$

The second fundamental form of surface

We get

$$\begin{pmatrix} \tilde{L} & \widetilde{M} \\ \widetilde{M} & \widetilde{N} \end{pmatrix} = -J.\begin{pmatrix} \boldsymbol{r}_u \\ \boldsymbol{r}_v \end{pmatrix}.(\boldsymbol{n}_u, \boldsymbol{n}_v).J^T$$

$$= J\begin{pmatrix} L & M \\ M & N \end{pmatrix} J^T$$

Then

$$\Pi = L(du)^2 + 2Mdudv + N(dv)^2$$

$$= (du, dv)\begin{pmatrix} L & M \\ M & N \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix}$$

Sine $(du, dv) = (d\tilde{u}, d\tilde{v})J$, so

$$\Pi = L(du)^2 + 2Mdudv + N(dv)^2$$

$$= (d\tilde{u}, d\tilde{v})J\begin{pmatrix} L & M \\ M & N \end{pmatrix} J^T \begin{pmatrix} d\tilde{u} \\ d\tilde{v} \end{pmatrix}$$

$$= (d\tilde{u}, d\tilde{v})\begin{pmatrix} \tilde{L} & \widetilde{M} \\ \widetilde{M} & \widetilde{N} \end{pmatrix}\begin{pmatrix} d\tilde{u} \\ d\tilde{v} \end{pmatrix}$$

$$= L(d\tilde{u})^2 + 2Md\tilde{u}d\tilde{v} + N(d\tilde{v})^2$$

$$= \widetilde{\Pi}$$

Direct geometry meaning of Π

$$\delta(\Delta u, \Delta v)$$
$$= \frac{1}{2}\left(L(\Delta u)^2 + 2M\Delta u\Delta v + N(\Delta v)^2\right) + \boldsymbol{o}\left((\Delta u)^2 + (\Delta v)^2\right)$$

when

$$\sqrt{(\Delta \mathrm{u})^2 + (\Delta v)^2} \to 0$$
$$\Longrightarrow \Pi \approx 2\delta(\mathrm{d}u, \mathrm{d}v)$$