DUT – RU International School
of Information Science & Engineering

Topic # 5

Vibrations and Waves / Optics

Lecture Course (online)
Dalian, 2021

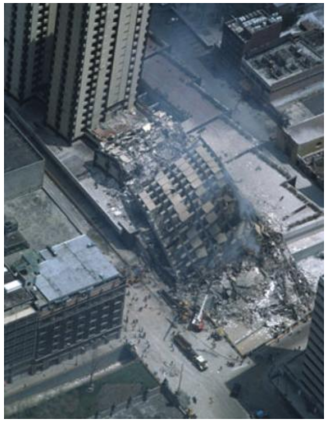
Contents

The Subject of Vibrations and Waves

Our world is filled with **oscillations** in which objects move back and forth repeatedly. Many oscillations are merely amusing or annoying, but many others are dangerous or financially important. The study and control of oscillations are two of the primary goals of both physics and engineering.

Periodic vibrations can cause disturbances that move through a medium in the form of **waves**. Many kinds of waves occur in nature, such as sound waves, water waves, seismic waves, and electromagnetic waves. However, these very different physical phenomena can be described by common terms and concepts.

wings oscillate in the turbulence

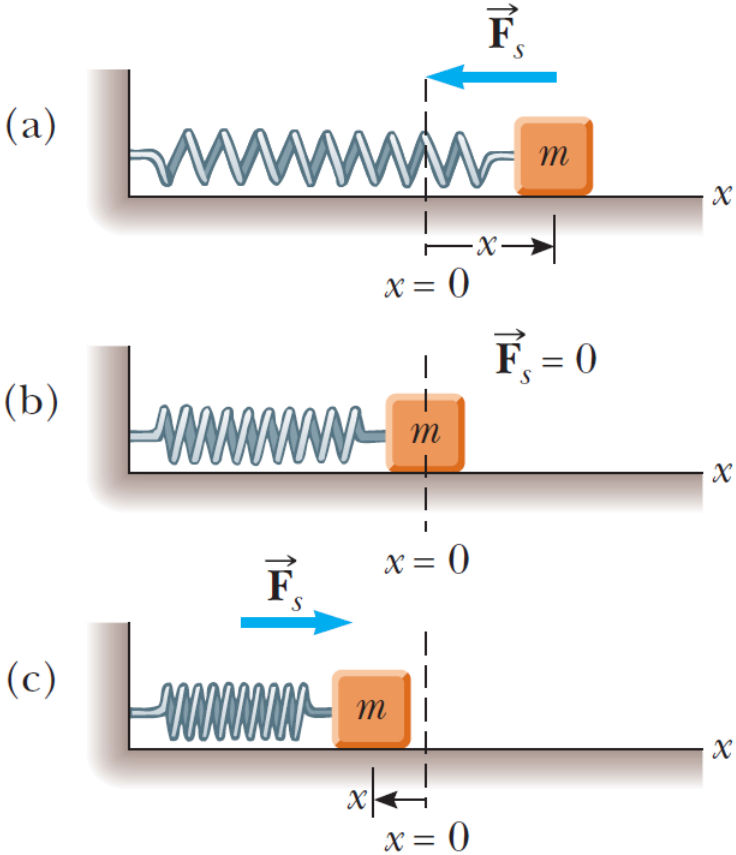


earthquakes

oscillations of power lines due to wind

Simple Harmonic Motion

One of the simplest types of vibrational motion is that of an object attached to a spring. We assume that the object moves on a frictionless horizontal surface.



Hooke's law:
(deduced experimentally)

$$F_s = -kx$$

Note: the force is always directed opposite to the displacement of an object, i.e. it pushes or pulls it towards the equilibrium position.

Projecting the equation of motion onto the *x*-axis:

$$m\ddot{x} = -kx \quad \text{or} \quad \boxed{m\ddot{x} + kx = 0}$$

Solving this ODE results in: EXERCISE

$$\boxed{x(t) = x_m \cos(\omega t + \phi)} \qquad \omega = \sqrt{\frac{k}{m}}$$

Note: it is also possible to use "*sin*" in the solution.

simple harmonic motion (SHM)

Simple Harmonic Motion

$$\boxed{\overbrace{x(t)}^{\text{Displacement at time } t} = \underbrace{x_m}_{\text{Amplitude}} \cos(\overbrace{\omega t + \phi}^{\text{Phase}})}$$

Time

Angular frequency

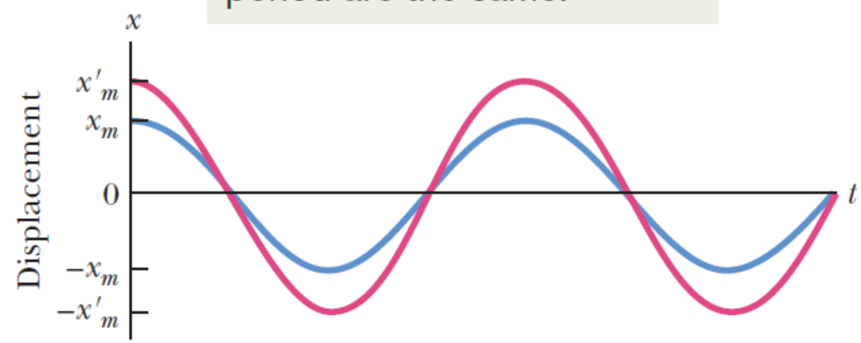
Phase constant or phase angle

$$x_m \cos \omega t = x_m \cos \omega(t + T)$$

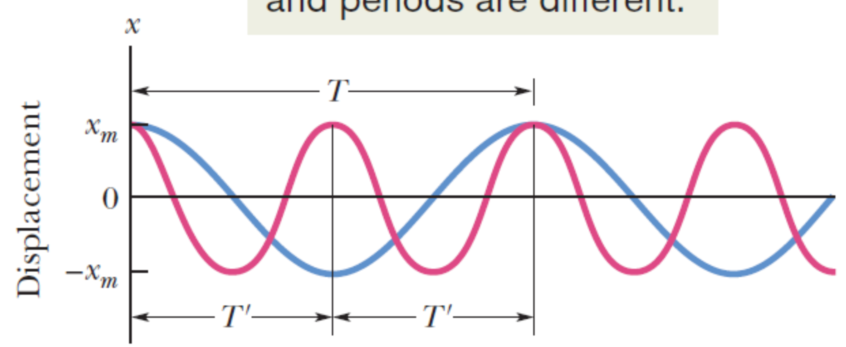
$$\omega(t + T) = \omega t + 2\pi$$

$\boxed{T = \dfrac{2\pi}{\omega}}$ or $\boxed{T = \dfrac{1}{f}}$

The amplitudes are different, but the frequency and period are the same.



The amplitudes are the same, but the frequencies and periods are different.



f – frequency

SI unit: (Hz) = (s^{-1})

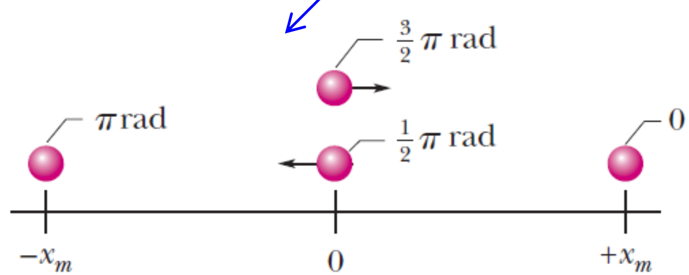
Simple Harmonic Motion

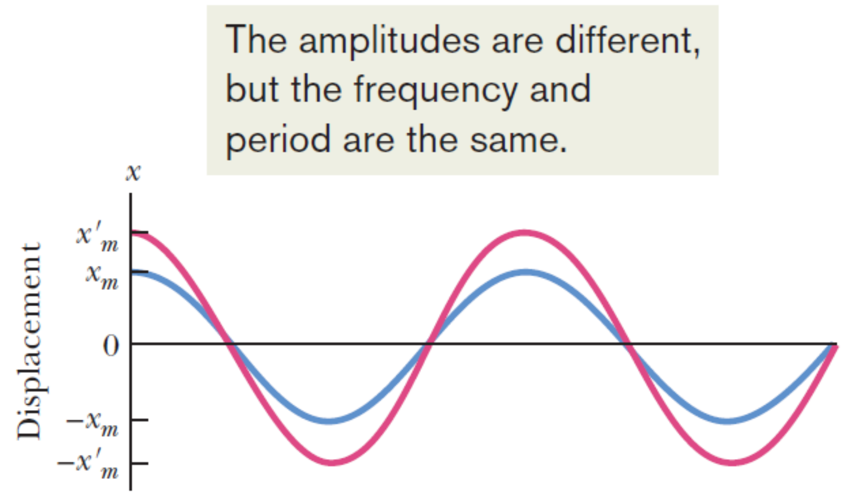
$$\overbrace{x(t)}^{\text{Displacement at time } t} = \underbrace{x_m}_{\text{Amplitude}} \cos(\overbrace{\omega t + \phi}^{\text{Phase}})$$

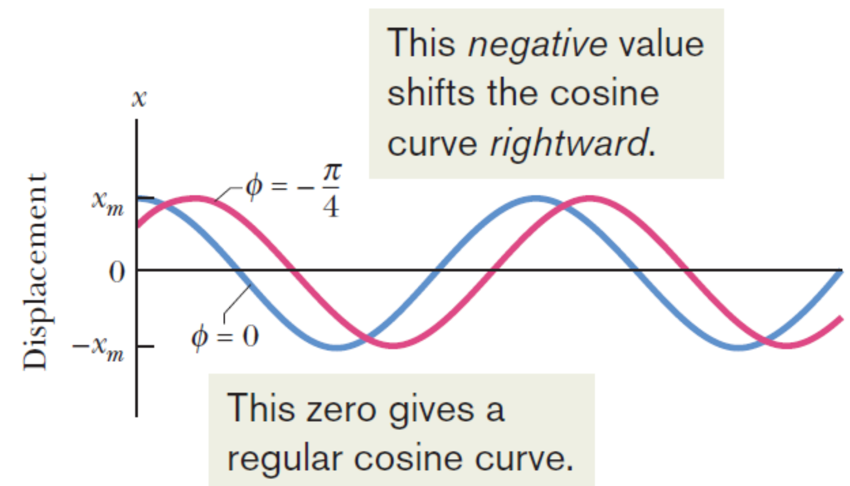
Angular frequency (ω)

Time (t)

Phase constant or phase angle (ϕ)







f – frequency

$\boxed{\text{SI unit: (Hz) = (s}^{-1}\text{)}}$

Velocity and Acceleration of SHM

Let us derive the expressions for both **velocity** and **acceleration** (based on their general definitions) of an object during the SHM.

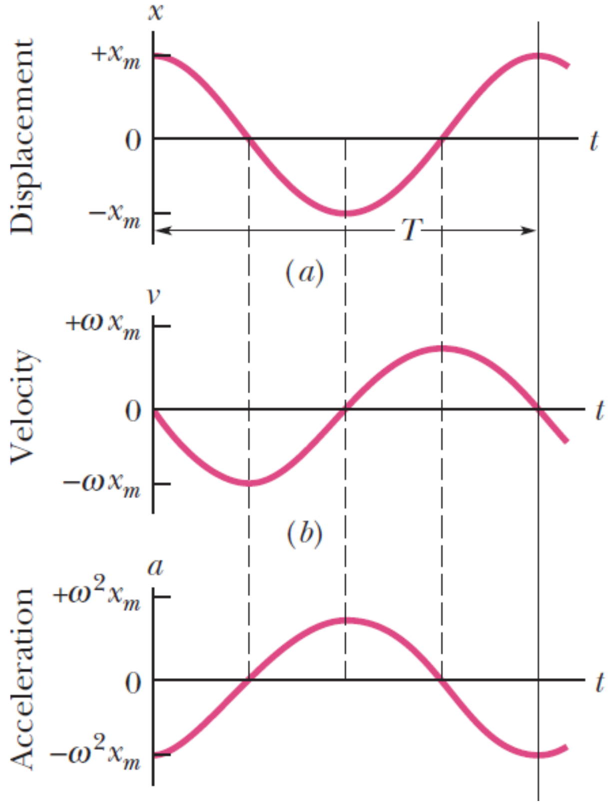
$$v(t) = \frac{dx(t)}{dt} \qquad a(t) = \frac{dv(t)}{dt}$$

$$\boxed{x(t) = x_m \cos(\omega t + \phi)}$$

$$\boxed{v(t) = -\omega x_m \sin(\omega t + \phi)}$$

$$\boxed{a(t) = -\omega^2 x_m \cos(\omega t + \phi)}$$

$$\Rightarrow \quad a(t) = -\omega^2 x(t)$$



Extreme values here mean ...

zero values here and ...

extreme values here.

Note: If you ever see **such a relationship** in an oscillating situation (such as with, say, the current in an electrical circuit, or the rise and fall of water in a tidal bay), you can immediately say that the motion is **SHM** and immediately identify the angular frequency ω of the motion.

Simple Harmonic Motion

QUIZ Check your understanding:

A particle undergoing simple harmonic oscillation of period T is at $-x_m$ at time $t = 0$. Is it at $-x_m$, at $+x_m$, at 0, between $-x_m$ and 0, or between 0 and $+x_m$ when (a) $t = 2.00T$, (b) $t = 3.50T$, and (c) $t = 5.25T$?

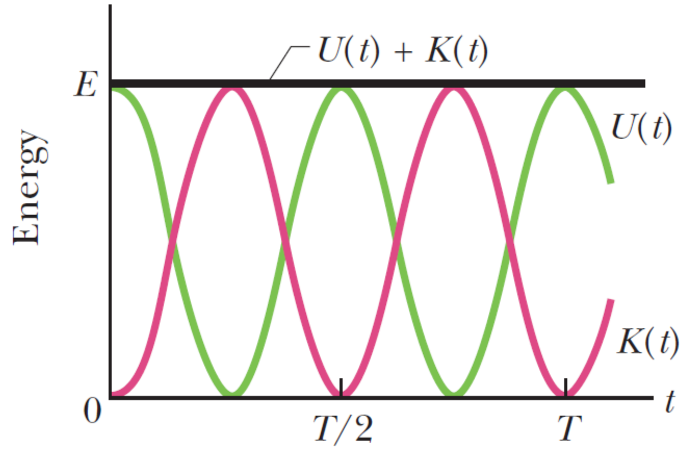
Which of the following relationships between a particle's acceleration a and its position x indicates simple harmonic oscillation: (a) $a = 3x^2$, (b) $a = 5x$, (c) $a = -4x$, (d) $a = -2/x$? For the SHM, what is the angular frequency (assume the unit of rad/s)?

Energy in Simple Harmonic Motion

If **no friction** is present, the energy transfers back and forth between the kinetic energy and potential energy, while the sum of the two – the mechanical energy *E* of the oscillator – remains constant.

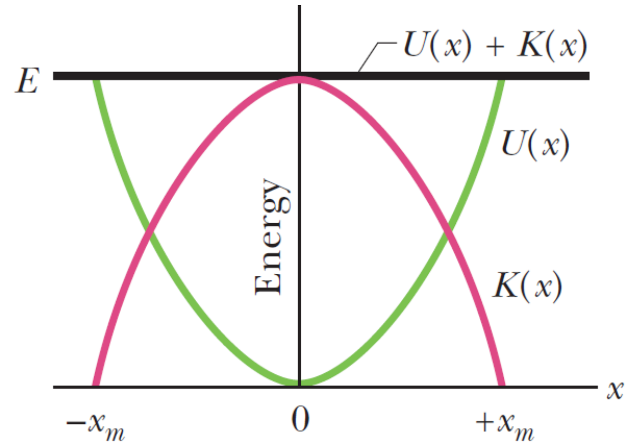
$$\boxed{U(t) = \tfrac{1}{2}kx^2 = \tfrac{1}{2}kx_m^2\cos^2(\omega t + \phi)}$$

$$\boxed{K(t) = \tfrac{1}{2}mv^2 = \tfrac{1}{2}kx_m^2\sin^2(\omega t + \phi)}$$



$$\begin{aligned} E &= U + K \\ &= \tfrac{1}{2}kx_m^2\cos^2(\omega t + \phi) + \tfrac{1}{2}kx_m^2\sin^2(\omega t + \phi) \\ &= \tfrac{1}{2}kx_m^2\left[\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)\right] \end{aligned}$$

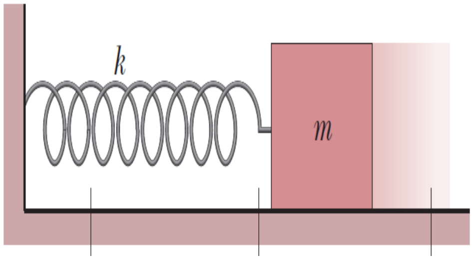
$$\Rightarrow \quad \boxed{E = U + K = \tfrac{1}{2}kx_m^2}$$



Simple Harmonic Motion Systems

Block-spring system

$$\ddot{x} + \omega^2 x = 0, \qquad \omega = \sqrt{\frac{k}{m}}, \qquad T = 2\pi\sqrt{\frac{m}{k}}$$

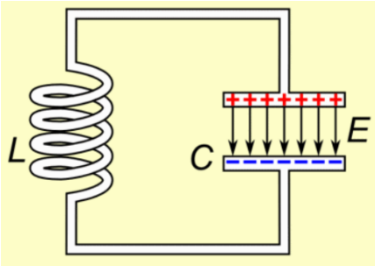


Mathematical pendulum

$$\ddot{\theta} + \omega^2 \theta = 0, \qquad \omega = \sqrt{\frac{g}{L}}, \qquad T = 2\pi\sqrt{\frac{L}{g}}$$

***LC* circuit**

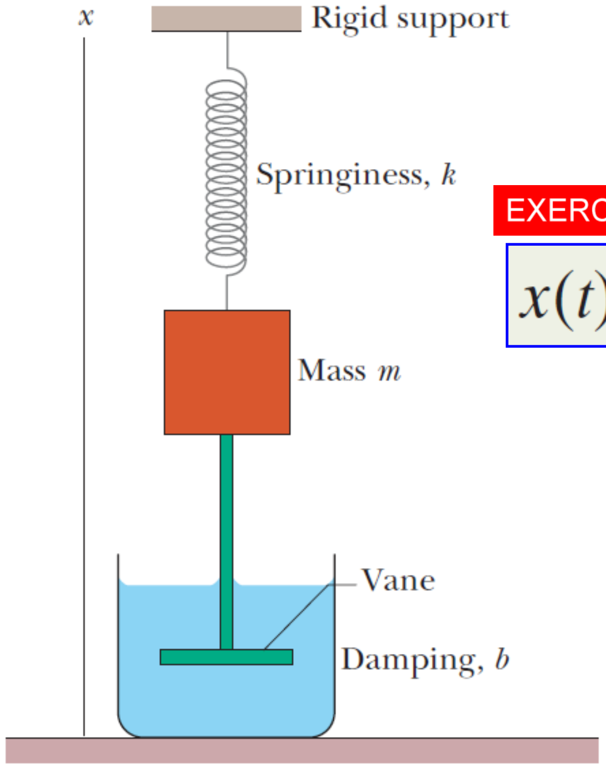
$$\ddot{q} + \omega^2 q = 0, \qquad \omega = \frac{1}{\sqrt{LC}}, \qquad T = 2\pi\sqrt{LC}$$



$$\boxed{\ddot{\heartsuit} + \omega^2 \heartsuit = 0 \quad \Rightarrow \quad \heartsuit(t) = \heartsuit_m \cos(\omega t + \phi)}$$

Damped Simple Harmonic Motion

The mechanical energy E in a real oscillating system decreases during the oscillations because external forces, such as drag forces or friction forces, inhibit the oscillations and **transfer** mechanical energy to **thermal** energy. The real oscillator and its motion are then said to be **damped**.



Let us assume the liquid exerts a damping force that is proportional to the **velocity** of the vane and block

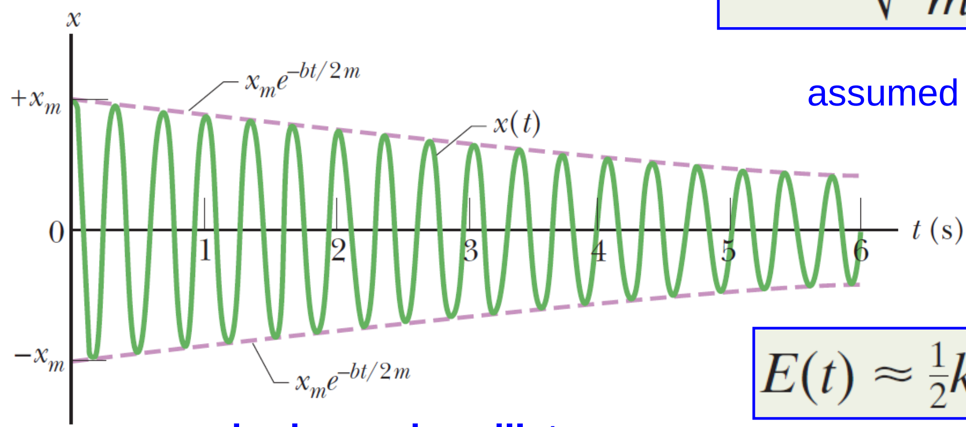
$$F_d = -bv \quad \Rightarrow \quad m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

EXERCISE

$$\boxed{x(t) = x_m e^{-bt/2m}\cos(\omega' t + \phi)}$$

$$\boxed{\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}}$$

assumed to be positive

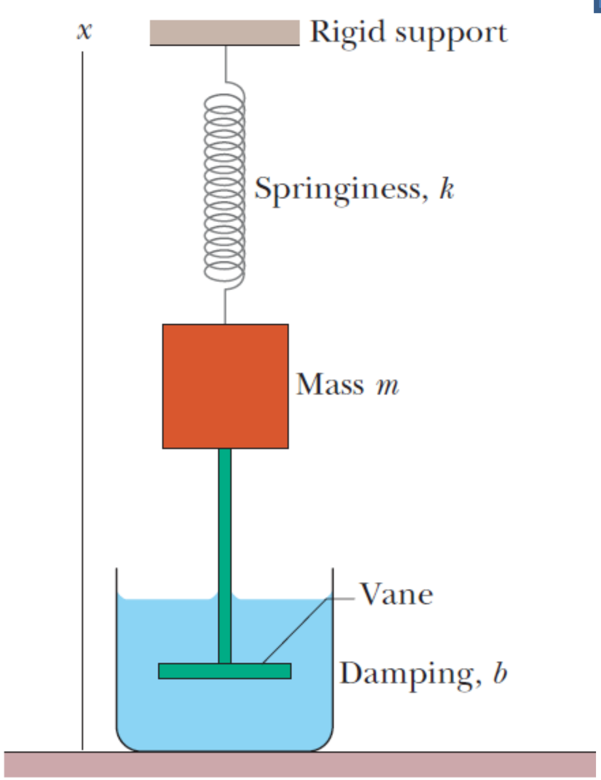


underdamped oscillator

$$\boxed{E(t) \approx \tfrac{1}{2}kx_m^2 e^{-bt/m}}$$

Damped Simple Harmonic Motion

The mechanical energy E in a real oscillating system decreases during the oscillations because external forces, such as drag forces or friction forces, inhibit the oscillations and **transfer** mechanical energy to **thermal** energy. The real oscillator and its motion are then said to be **damped**.



Let us assume the liquid exerts a damping force that is proportional to the **velocity** of the vane and block

$$F_d = -bv \quad \Rightarrow \quad m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

$$\boxed{\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}}$$

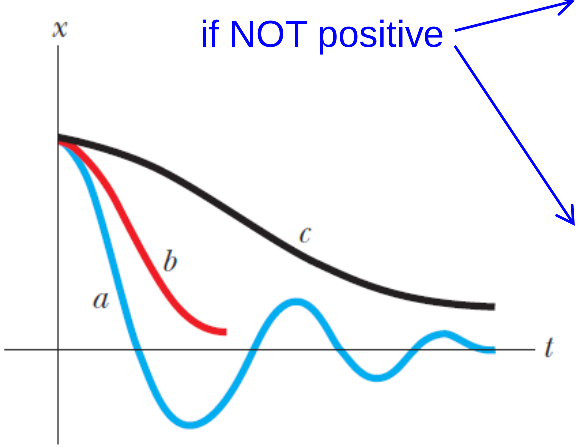
if NOT positive

$$\boxed{\frac{k}{m} < \frac{b^2}{4m^2} \quad \text{or} \quad \frac{b^2}{4km} > 1}$$

overdamped oscillator

$$\boxed{\frac{k}{m} = \frac{b^2}{4m^2} \quad \text{or} \quad \frac{b^2}{4km} = 1}$$

critically damped oscillator



(a) underdamped oscillator, (b) critically damped oscillator, and (c) overdamped oscillator

Damped Simple Harmonic Motion

QUIZ Check your understanding:

Here are three sets of values for the spring constant, damping constant, and mass for the damped oscillator. Rank the sets according to the time required for the mechanical energy to decrease to one-fourth of its initial value, greatest first.

Set 1	$2k_0$	b_0	m_0
Set 2	k_0	$6b_0$	$4m_0$
Set 3	$3k_0$	$3b_0$	m_0

Driven Oscillations and Resonance

A person sitting in a swing without anyone pushing it is an example of free oscillation. However, if someone pushes the swing periodically, the swing has **forced**, or **driven**, oscillations.

it is of great interest to consider the particular case when the **external** driving force is also periodic (**harmonic**)

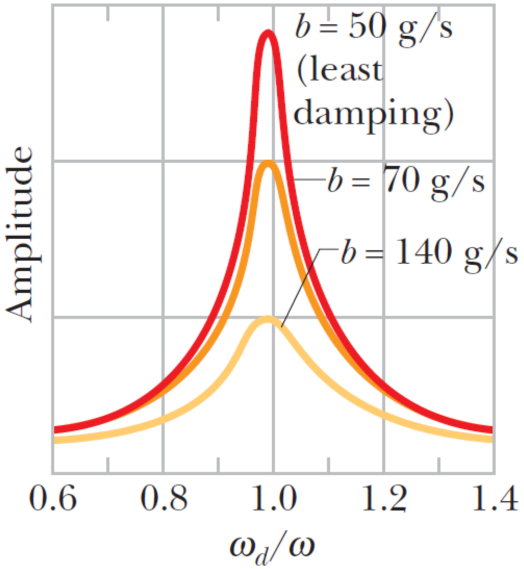
EXERCISE

$$\boxed{m\ddot{x} + b\dot{x} + kx = f_d \sin \omega_d t}$$

$$\boxed{x(t) = x_m e^{-\frac{bt}{2m}} \cos\left(\omega' t + \phi\right) + \tilde{x}}$$

particular solution of non-homogeneous ODE

$$\boxed{\tilde{x} = A_d \sin\left(\omega_d t + \delta\right)}$$



$$A_d = \frac{f_d/m}{\sqrt{(\omega^2 - \omega_d^2)^2 + \left(\frac{b}{m}\right)^2 \omega_d^2}}$$

$$\boxed{\omega_d \approx \omega}$$

resonance condition

$$\tan \delta = \frac{b}{m}\left(\frac{\omega_d}{\omega_d^2 - \omega^2}\right)$$

Note: at resonance the displacement amplitude of the oscillations is greatest.

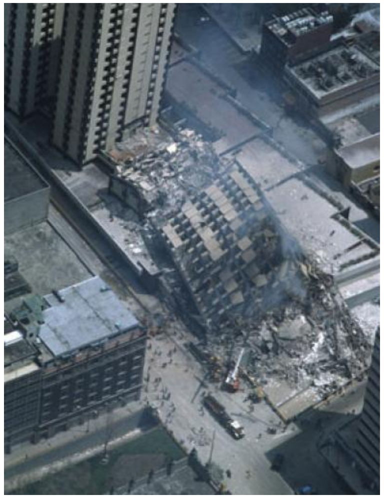
Driven Oscillations and Resonance

A person sitting in a swing without anyone pushing it is an example of free oscillation. However, if someone pushes the swing periodically, the swing has **forced**, or **driven**, oscillations.

→ it is of great interest to consider the particular case when the **external** driving force is also periodic (**harmonic**)

$$m\ddot{x} + b\dot{x} + kx = f_d \sin \omega_d t$$

$$x(t) = x_m e^{-\frac{bt}{2m}} \cos\left(\omega' t + \phi\right) + \tilde{x}$$



All **mechanical structures** have one or more **natural** angular frequencies, and if a structure is subjected to a strong external driving force that **matches** one of these angular frequencies, the **resulting oscillations** of the structure **may rupture** it.

→ for example, aircraft designers must make sure that none of the natural angular frequencies at which a wing can oscillate matches the angular frequency of the engines in flight. A wing that flaps violently at certain engine speeds would obviously be dangerous.

Waves

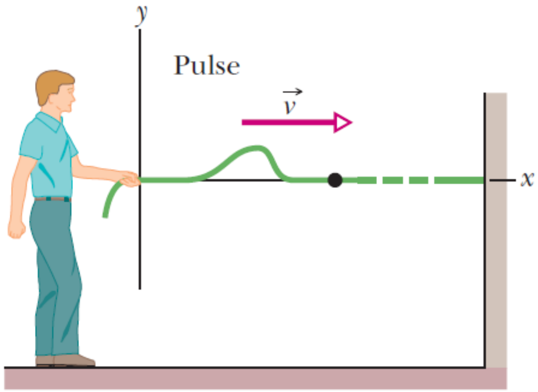
The world is full of **waves**: sound waves, waves on a string, seismic waves, and electromagnetic waves, such as visible light, radio waves, television signals, and x-rays. All of these waves have as their source a vibrating object, so we can apply the concepts of simple harmonic motion in describing them.

Types of waves

- **mechanical waves**: all these waves have two central features: (i) they are governed by Newton's laws, and (ii) they can **exist only within a material medium**, such as water, air, and rock (examples: water waves, sound waves, and seismic waves).
- **electromagnetic waves**: these waves require **no material medium** to exist. Light waves from stars, for example, travel through the vacuum of space to reach us. All electromagnetic waves travel through a vacuum at the same speed 299 792 458 m/s. (examples: water waves, sound waves, and seismic waves).
- **matter waves**: these waves are associated with electrons, protons, and other fundamental particles, and even atoms and molecules. Because we commonly think of these particles as constituting matter, such waves are called matter waves.

Transverse and Longitudinal Waves

One of the simplest ways to demonstrate wave motion is to flip one end of a long rope that is under tension and has its opposite end fixed.

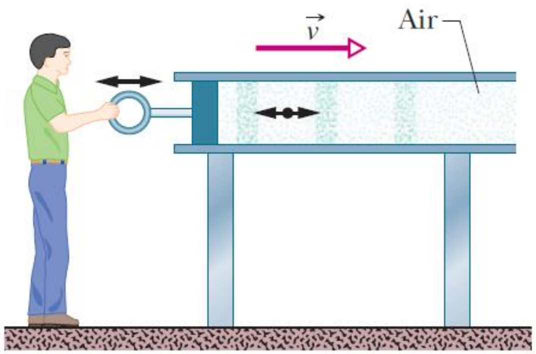


The bump (called a **pulse**) travels to the right with a definite speed. A disturbance of this type is called a **traveling wave**.

Each segment of the rope that is disturbed moves in a direction **perpendicular** to the wave motion.

> A traveling wave in which the particles of the disturbed medium move in a direction **perpendicular** to the wave velocity is called a **transverse wave**.

Let us consider how sound waves can be produced. If you suddenly move the piston rightward and then leftward, you can send a pulse of sound along the pipe.

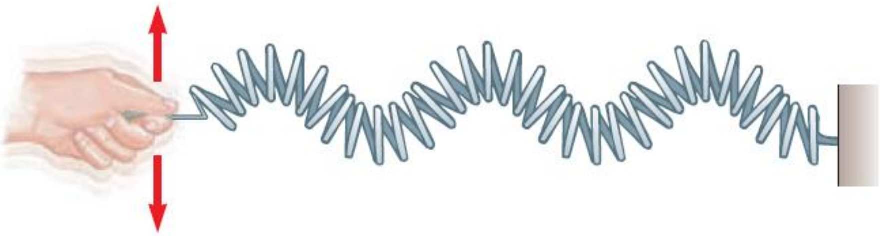


The motion of the elements of air is **parallel** to the direction of the wave's travel.

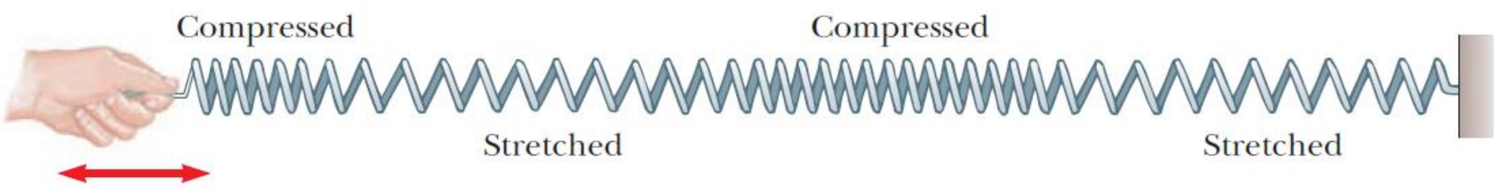
> A traveling wave in which the particles of the disturbed medium undergo displacements **parallel** to the direction of wave motion are called **longitudinal waves**.

Transverse and Longitudinal Waves

Both types of waves can be produced by means of a **spring**.



(a) Transverse wave

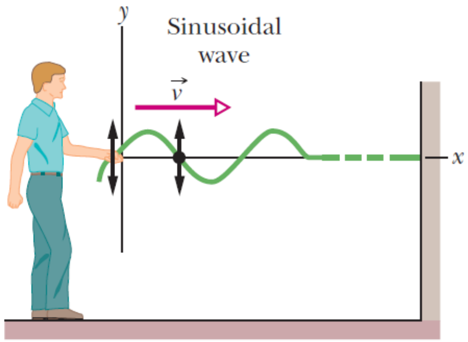


(b) Longitudinal wave

Note: Waves **need not** be purely transverse or purely longitudinal. For example, ocean waves exhibit a **superposition** of **both** types. When an ocean wave encounters a cork, the cork executes a circular motion, going up and down while going forward and back.

Sinusoidal Wave

To completely describe a wave on a string (and the motion of any element along its length), we need a function that gives the shape of the wave.



we need a relation in the form:

$$y = h(x, t)$$

In case of a **sinusoidal** wave:

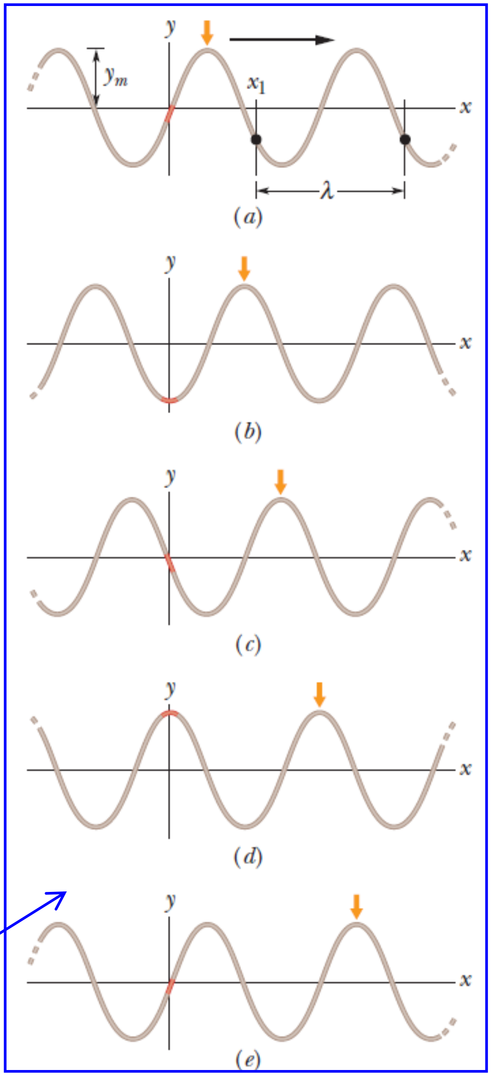
$$\boxed{y(x, t) = y_m \sin(kx - \omega t)}$$

(transverse displacement of any string element)

Note: because this equation is written in terms of position *x*, it can be used to find the displacements of all the elements of the string as a function of time. Thus, it can tell us the shape of the wave at any given time.

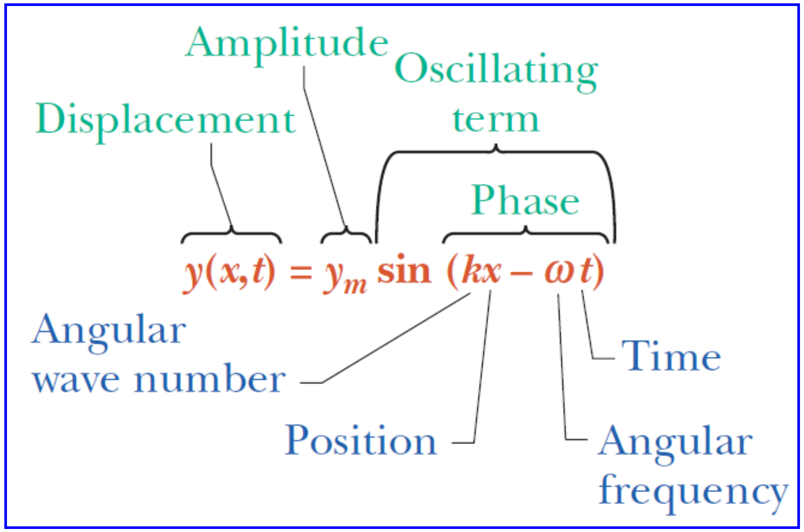
Note: as the wave sweeps through succeeding elements of the string, the elements oscillate parallel to the *y*-axis.

"snapshots" of a string wave travelling in the positive direction of an *x*-axis



Amplitude, Phase and Wavelength

Let us consider the quantities which define the transversal displacement of the elements of the string caused by the propagation of the sinusoidal wave.



amplitude of the wave is the magnitude of the maximum displacement of the elements from their equilibrium positions as the wave passes through them (always positive).

phase of the wave is the argument of the sine function. As the wave sweeps through a string element at a particular position *x*, the phase changes linearly with time, so that it undergoes harmonic oscillations.

wavelength (λ) of the wave is the distance (parallel to the direction of the wave's travel) between repetitions of the wave shape.

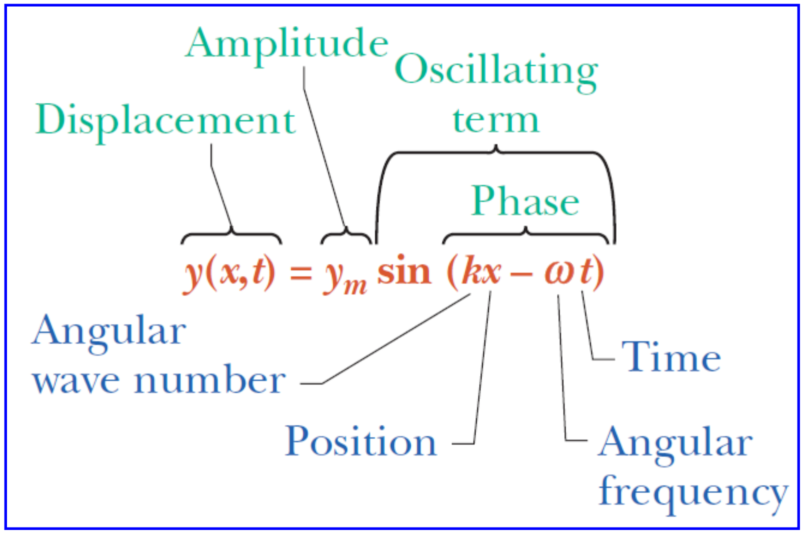
$$y_m \sin kx_1 = y_m \sin k(x_1 + \lambda) = y_m \sin(kx_1 + k\lambda) \rightarrow \qquad k\lambda = 2\pi$$

angular wave number: $\boxed{k = \dfrac{2\pi}{\lambda}}$

SI units: (rad / m) = (m^{-1})

Period, Angular Frequency and Frequency

Let us consider the quantities which define the transversal displacement of the elements of the string caused by the propagation of the sinusoidal wave.



period of oscillation of a wave is the time any string element takes to move through one full oscillation.

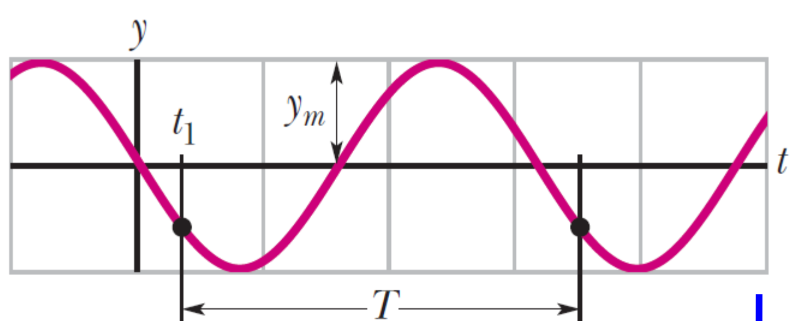
$$(x = 0)$$

$$y(0,t) = y_m \sin(-\omega t) = -y_m \sin \omega t$$

$$-y_m \sin \omega t_1 = -y_m \sin \omega(t_1 + T) =$$
$$= -y_m \sin(\omega t_1 + \omega T) \rightarrow \omega T = 2\pi$$

angular frequency: $\boxed{\omega = \frac{2\pi}{T}}$

frequency: $\boxed{f = \frac{1}{T} = \frac{\omega}{2\pi}}$

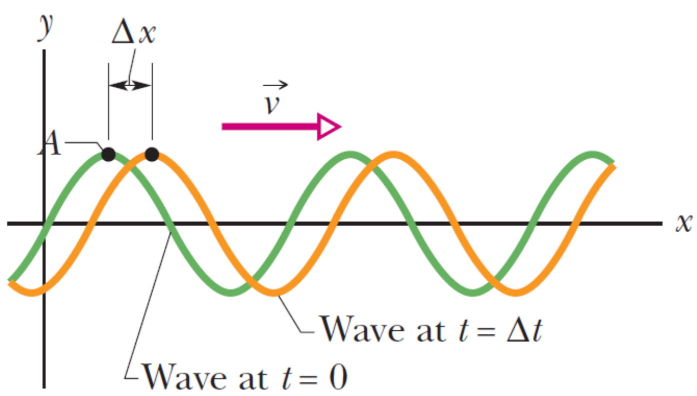


Note: this is a graph, not a snapshot !

(as it was for SHM, this is a number of oscillations per unit time)

The Speed of a Travelling Wave

Consider the sinusoidal wave is traveling in the positive direction of *x*. The entire wave pattern moving a distance Δ*x* in that direction during the interval Δ*t*. The ratio Δ*x*/Δ*t* (or, in the differential limit, d*x*/d*t*) is the **wave speed** *v*. How can we find its value?



as the wave moves, each point of the moving wave form, such as point *A* on a peak, retains its displacement *y* (points on the string do not retain their displacement, but points on the wave form do).

$$kx - \omega t = \text{a constant}$$

$$k\frac{dx}{dt} - \omega = 0$$

$$\frac{dx}{dt} = v = \frac{\omega}{k}$$

$$\boxed{v = \frac{\omega}{k} = \frac{\lambda}{T} = \lambda f}$$

wave speed

Note: the wave travelling in the negative direction can be described by:

$$y(x,t) = y_m \sin(kx + \omega t)$$

$$\frac{dx}{dt} = -\frac{\omega}{k}$$

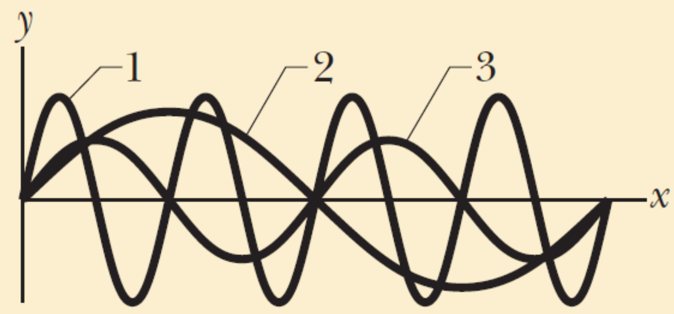
$$\boxed{y(x,t) = h(kx \pm \omega t)}$$

generalization for arbitrary shape

Sinusoidal Wave

QUIZ Check your understanding:

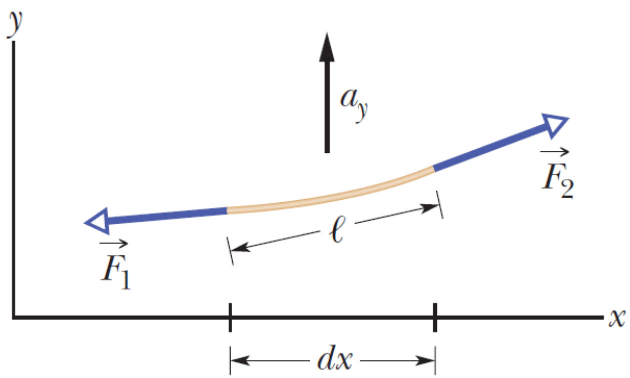
The figure is a composite of three snapshots, each of a wave traveling along a particular string. The phases for the waves are given by (a) $2x - 4t$, (b) $4x - 8t$, and (c) $8x - 16t$. Which phase corresponds to which wave in the figure?

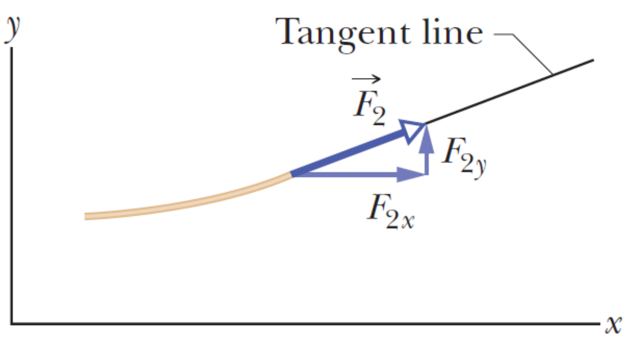


Here are the equations of three waves:
(1) $y(x, t) = 2\sin(4x - 2t)$, (2) $y(x, t) = \sin(3x - 4t)$, (3) $y(x, t) = 2\sin(3x - 3t)$.
Rank the waves according to their (a) wave speed and (b) maximum speed perpendicular to the wave's direction of travel (the transverse speed), greatest first.

The Wave Equation

As a wave passes through any element on a stretched string, the element moves perpendicularly to the wave's direction of travel (we are dealing with a transverse wave). By applying Newton's second law to the element's motion, we can derive a general differential equation, called the **wave equation**, that governs the travel of waves of any type.





$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}}$$

This is the general differential equation that governs the travel of waves of all types.

Note: we switched to the notation of partial derivatives because on the left we differentiate only with respect to x and on the right we differentiate only with respect to t.

The Principle of Superposition of Waves

It often happens that two or more waves pass simultaneously through the same region. Suppose that two waves travel simultaneously along the same stretched string.

$y_1(x, t)$
$y_2(x, t)$
displacements that the string would experience if each wave traveled alone

During the overlap:

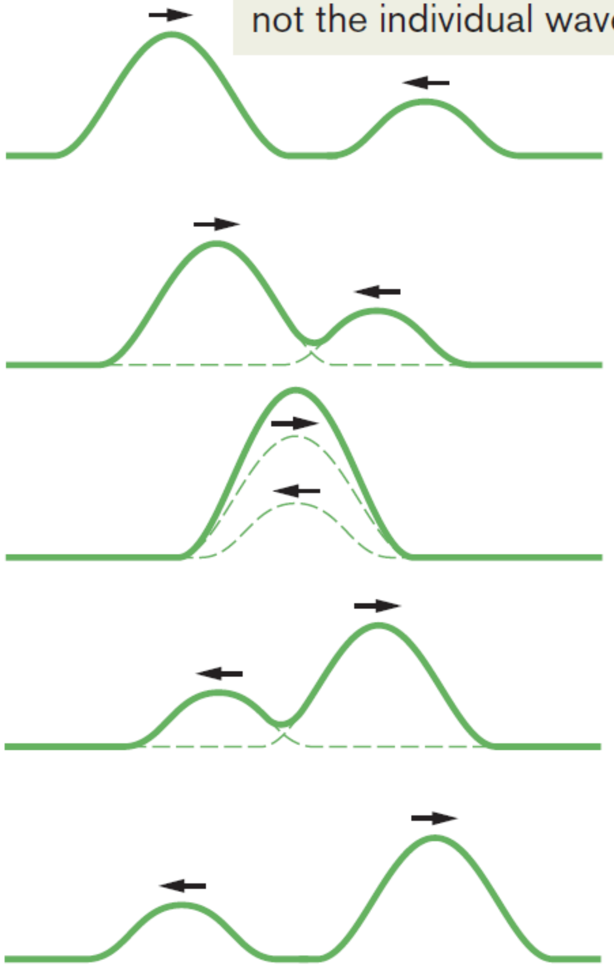
$$\boxed{y'(x,t) = y_1(x,t) + y_2(x,t)}$$

Overlapping waves algebraically add to produce a **resultant wave** (or **net wave**).

principle of superposition

Overlapping waves do not in any way alter the travel of each other.

When two waves overlap, we see the resultant wave, not the individual waves.



Conclusions

- **periodic** motion is a very common type of motion in nature. The systems which undergo such types of motion are called **oscillators**
- **simple harmonic motion** occurs when the net force on an object along the direction of motion is **proportional** to the object's **displacement** and in the **opposite** direction
- the time required for one complete vibration is called the **period** of the motion. The reciprocal of the period is the **frequency** of the motion, which is the number of oscillations per second
- **friction** and **external forces** lead to the so-called damped or damped-driven motion of a system which substantially differs from the motion of a free harmonic oscillator (resulting in damping and resonance effects)
- **wave** is a motion of some disturbance in space. According to the direction of the displacements of the considered objects (quantities) we distinguish between **transverse** and **longitudinal** waves
- waves obey the **superposition principle** which states that if two or more traveling waves are moving through a medium, the resultant wave is found by adding the individual waves together point by point