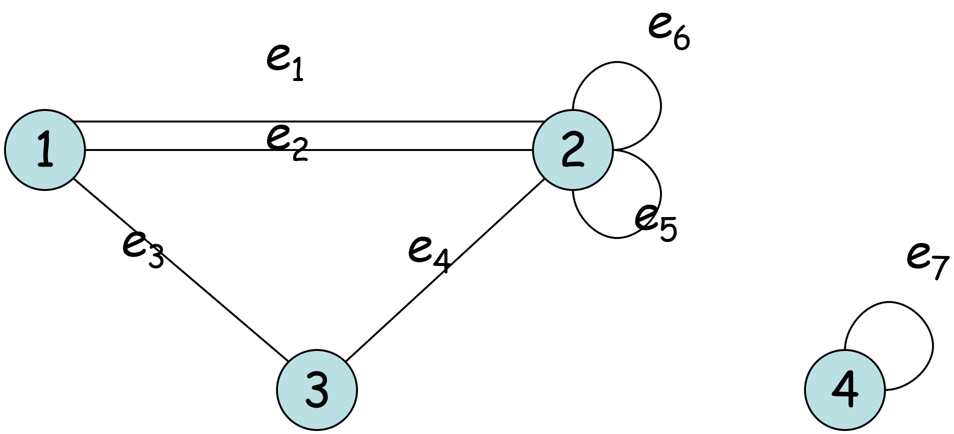
Paths and Connectivity

Content

- Paths
- Connectivity

Paths

A path in a graph is a continuous way of getting from one vertex to another by using a sequence of edges.



EG: could get from 1 to 3 circuitously as follows:

$$1\text{-}e_1\rightarrow 2\text{-}e_1\rightarrow 1\text{-}e_3\rightarrow 3\text{-}e_4\rightarrow 2\text{-}e_6\rightarrow 2\text{-}e_5\rightarrow 2\text{-}e_4\rightarrow 3$$

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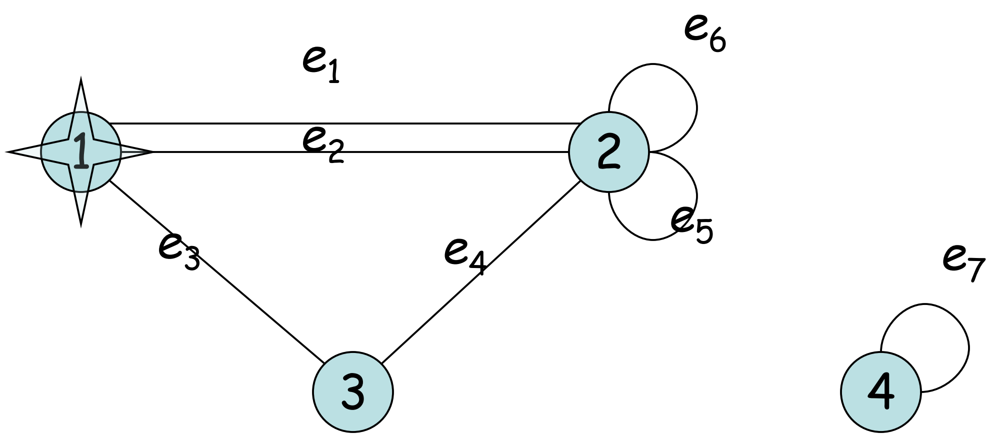

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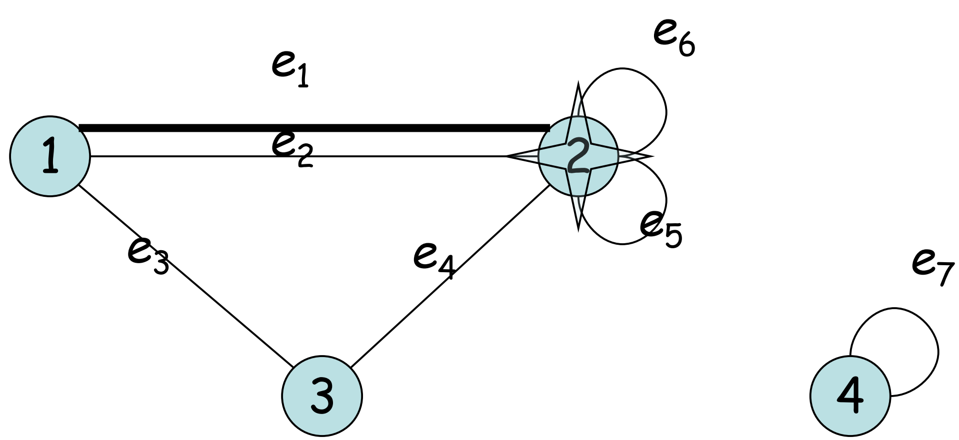

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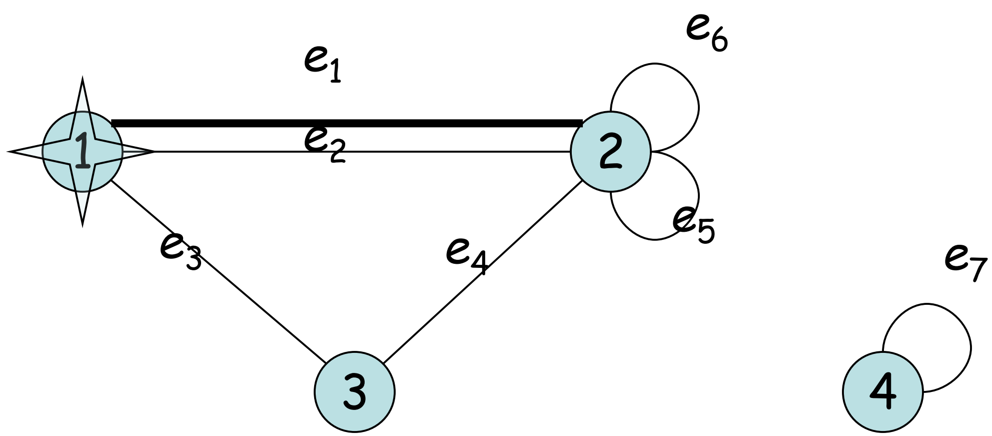

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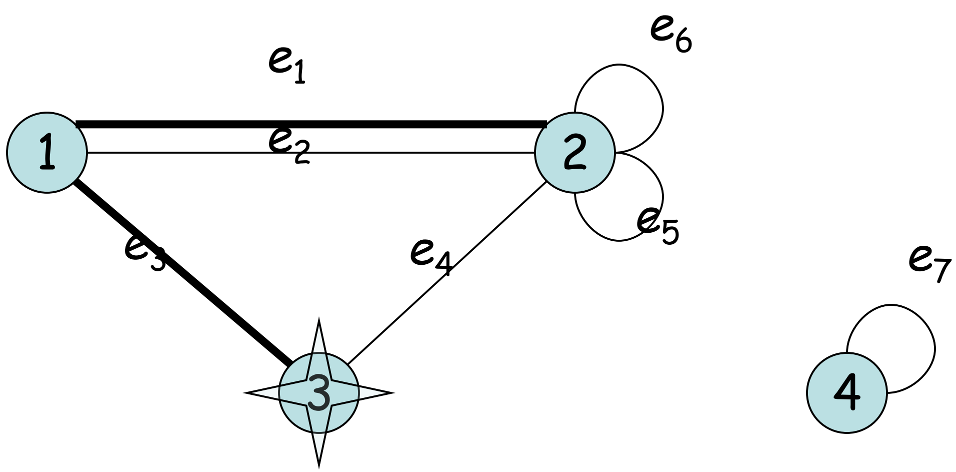

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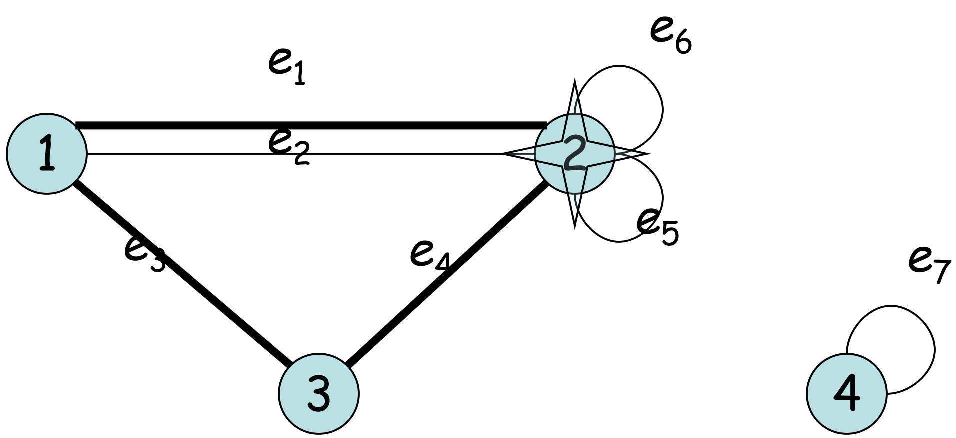

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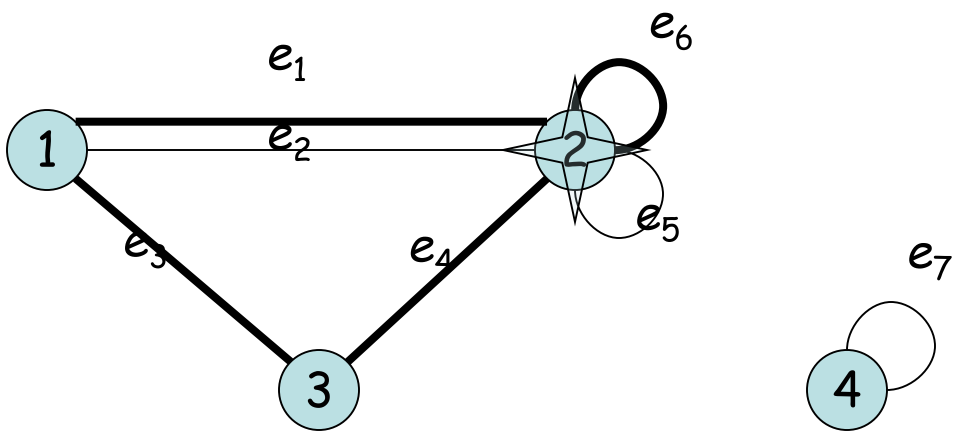

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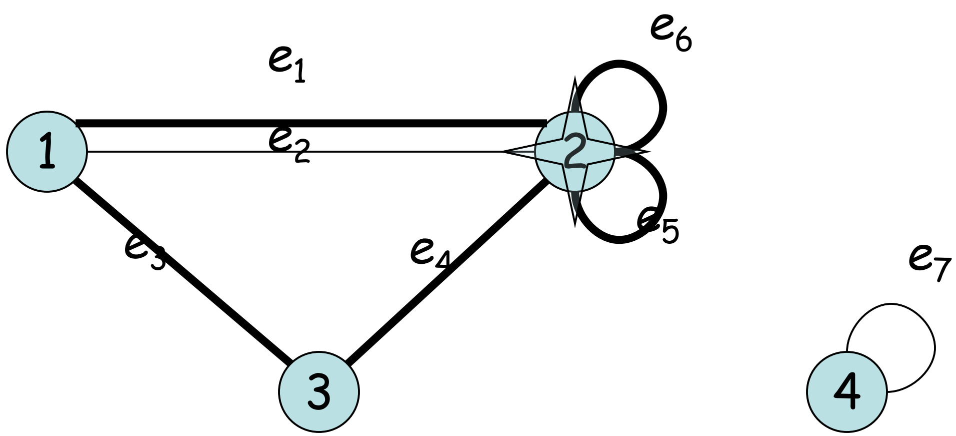

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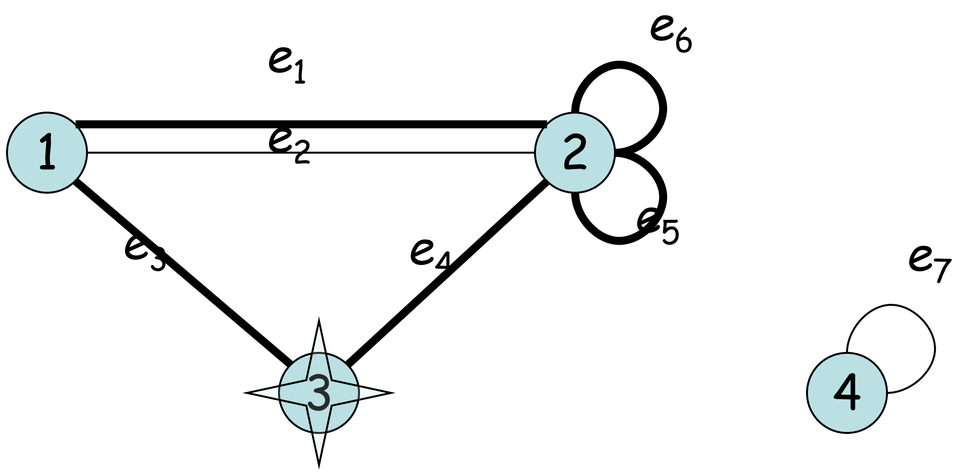

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Paths

Definition: A **path** of length n in an undirected graph is a sequence of n edges $e_1, e_2, \ldots, e_n$ such that each consecutive pair e_i , e_{i+1} share a common vertex. In a simple graph, one may instead define a path of length n as a sequence of n+1 vertices $v_0, v_1, v_2, \ldots, v_n$ such that each consecutive pair v_i , v_{i+1} are adjacent. Paths of length 0 are also allowed according to this definition.

Q: Why does the second definition work for simple graphs?

Paths

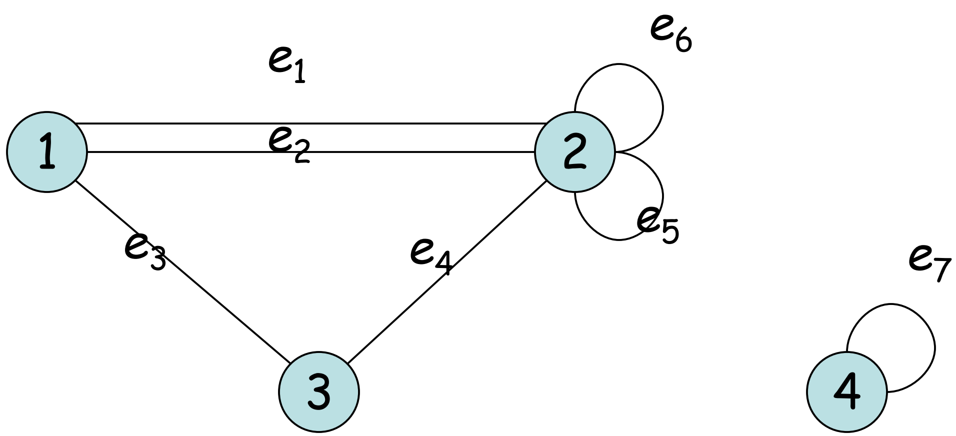
A: For simple graphs, any edge is unique between vertices so listing the vertices gives us the edge-sequence as well.

Definition: A **simple path** contains no duplicate edges (though duplicate vertices are allowed). A **cycle** (or **circuit**) is a path which starts and ends at the same vertex.

Note: Simple paths need not be in simple graphs. E.g., may contain loops.

Paths

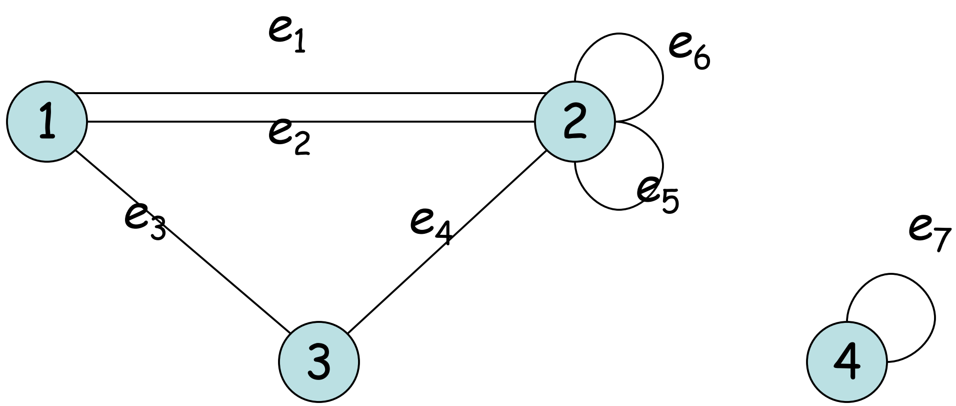
Q: Find a longest possible simple path in the following graph:



Paths

A: The following path from 1 to 2 is a maximal simple path because

- simple: each of its edges appears exactly once
- maximal: because it contains every edge except the unreachable edge e_7



The maximal path: $e_1, e_5, e_6, e_2, e_3, e_4$

Paths in Directed Graphs

One can define paths for directed graphs by insisting that the target of each edge in the path is the source of the next edge:

Definition: A **path** of length n in a directed graph is a sequence of n edges $e_1, e_2, \ldots, e_n$ such that the target of e_i is the source e_{i+1} for each i.

In a digraph, one may instead define a path of length n as a sequence of n+1 vertices $v_0, v_1, v_2, \ldots, v_n$ such that for each consecutive pair v_i , v_{i+1} there is an edge from v_i to v_{i+1} .

Paths in Directed Graphs

Q: Consider digraph adjacency matrix:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

1. Find a path from 1 to 4.
2. Is there a path from 4 to 1?

Paths in Directed Graphs

A:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

1. 1→3 .

Paths in Directed Graphs

A:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

1. 1→3→2 .

Paths in Directed Graphs

A:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

1. $1 \to 3 \to 2 \to 4.$

Paths in Directed Graphs

A:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

1. $1 \rightarrow 3 \rightarrow 2 \rightarrow 4$.

2. There's no path from 4 to 1. From 4 must go to 2, from 2 must stay at 2 or return to 4. In other words 2 and 4 are *disconnected* from 1.

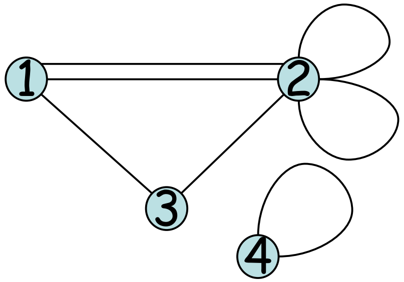
Connectivity

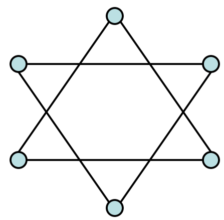
Definition: Let G be a pseudograph. Let u and v be vertices. u and v are **connected** to each other if there is a path in G which starts at u and ends at v. G is said to be **connected** if all vertices are connected to each other.

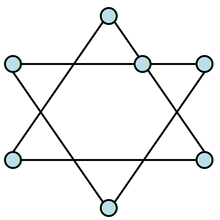
1. Note: Any vertex is automatically connected to itself via the empty path.
2. Note: A suitable definition for directed graphs will follow later.

Connectivity

Q: Which of the following graphs are connected?

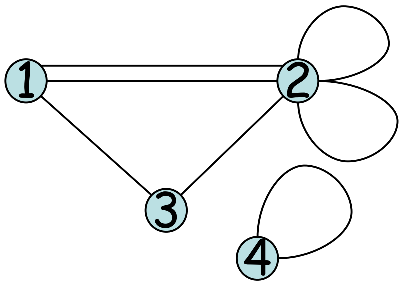


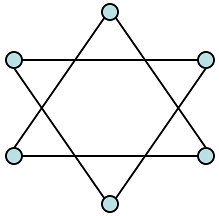


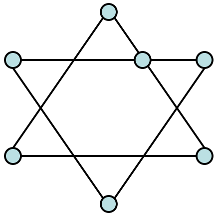


Connectivity

A: First and second are disconnected. Last is connected.







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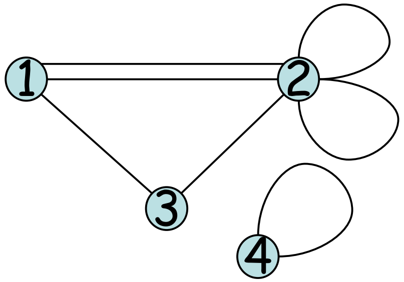

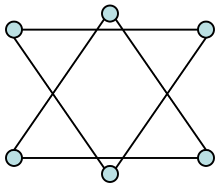

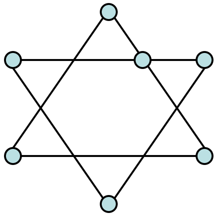

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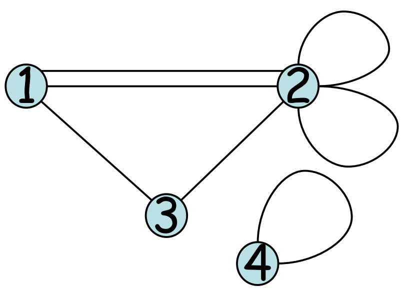

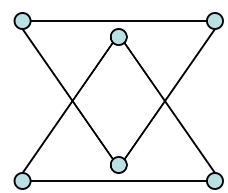

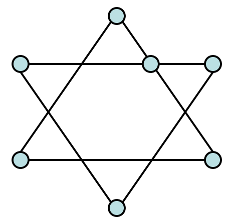

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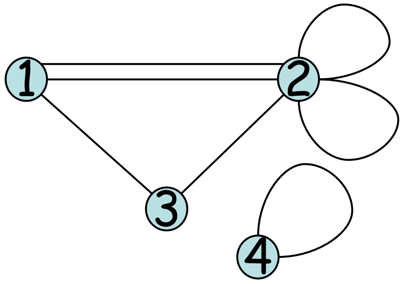
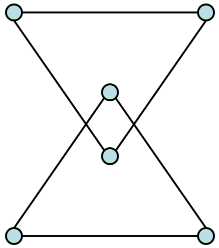
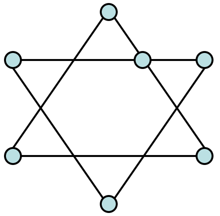
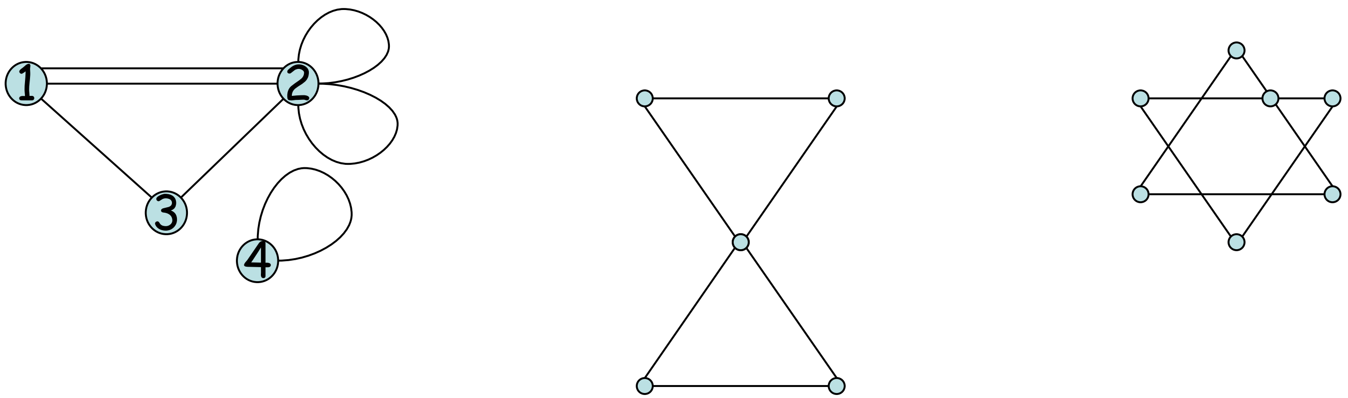






Connectivity

A: First is disconnected. Second and Last are connected.



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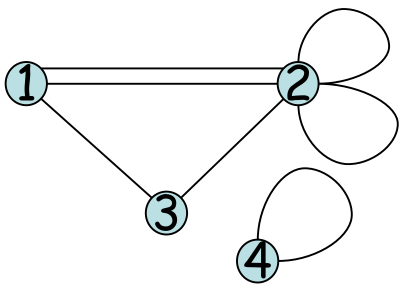
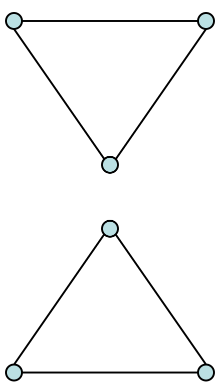
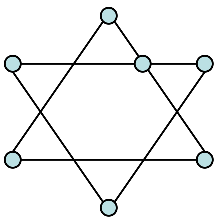






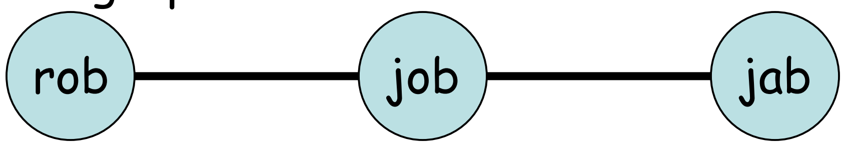
English Connectivity Puzzle

Can define a puzzling graph *G* as follows:

V = {3-letter English words}

E : two words are connected if can get one word from the other by changing a single letter.

One small subgraph of *G* is:



Q: Is "fun" connected to "car" ?

A: Yes: fun→fan→far→car

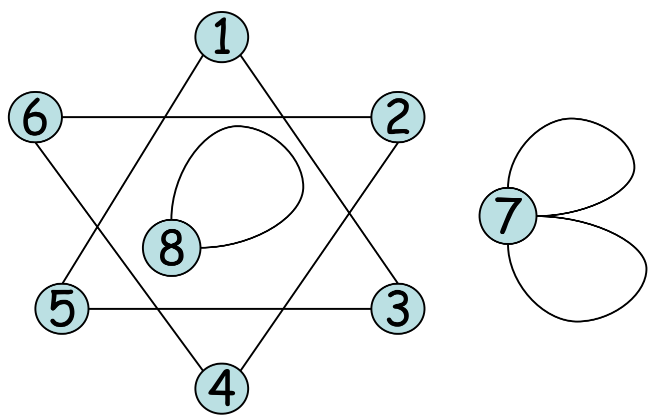
Or:

fun→fin→bin→ban→bar→car

Connected Components

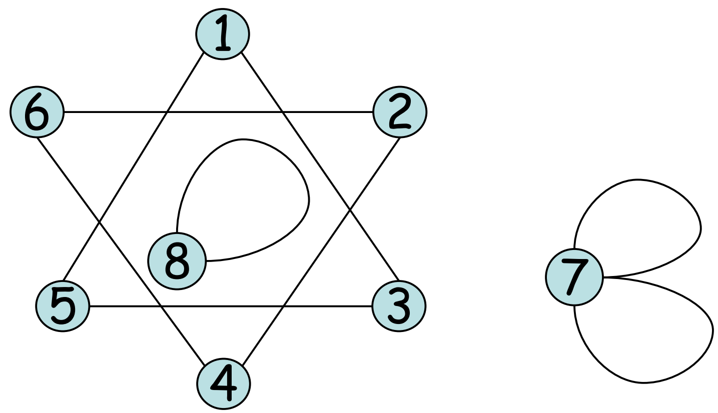
Definition: *A* **connected component** (or just **component**) in a graph *G* is a set of vertices such that all vertices in the set are connected to each other and every possible connected vertex is included.

Q: What are the connected components of the following graph?



Connected Components

A: The components are {1,3,5},{2,4,6},{7} and {8} as one can see visually by pulling components apart:



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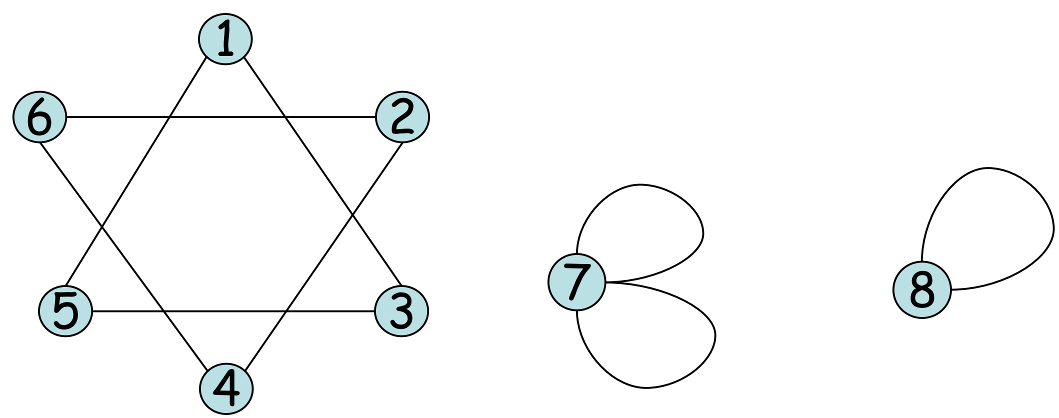

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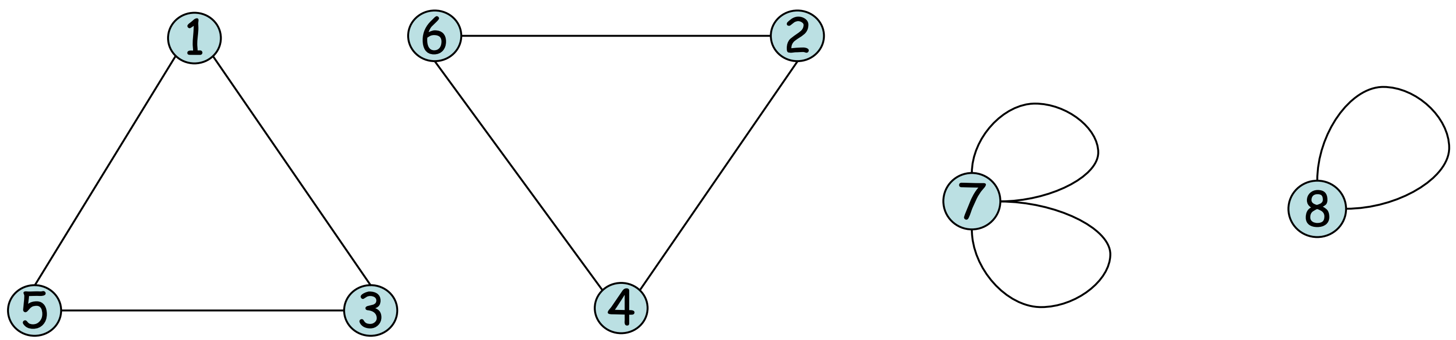
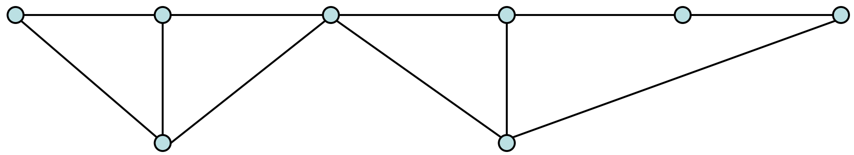


N-Connectivity

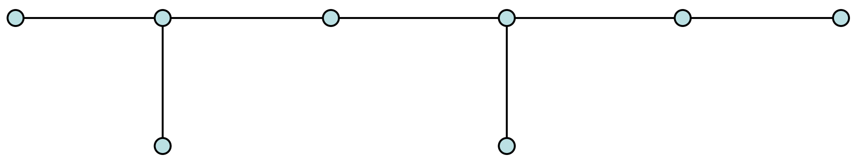
Not all connected graphs are created equal!

Q: Rate following graphs in terms of their design value for computer networks:

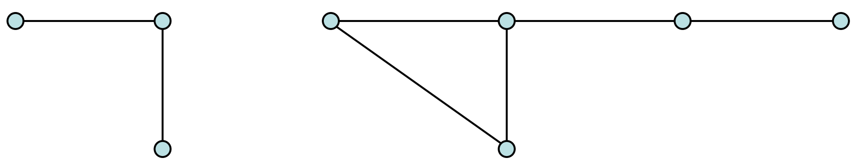
1)



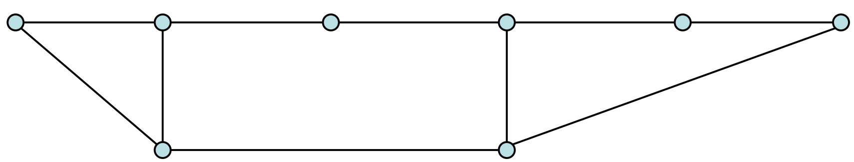
2)



3)



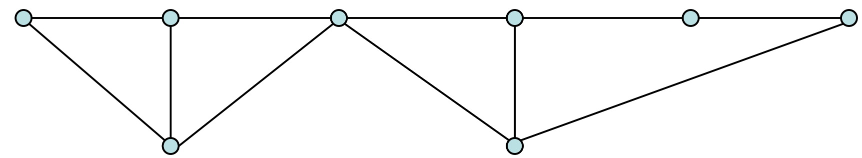
4)



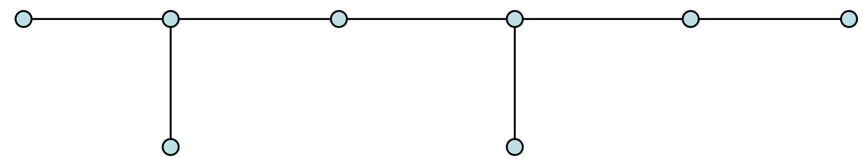
N-Connectivity

A: Want all computers to be connected, even if 1 computer goes down:

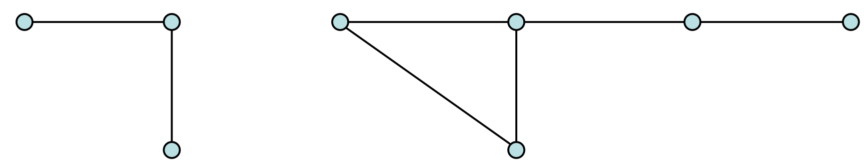
1) 2nd best. However, there's
a weak link— "cut vertex"



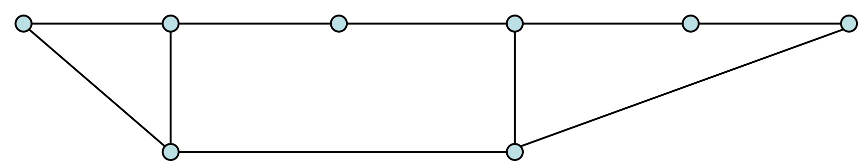
2) 3rd best. Connected
but any computer can disconnect



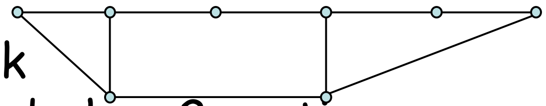
3) Worst!
Already disconnected



4) Best! Network dies
only with 2 bad computers



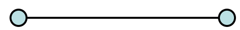
N-Connectivity

The network  is best because it can only become disconnected when 2 vertices are removed. In other words, it is 2-connected. Formally:

Definition: A connected simple graph with 3 or more vertices is **2-connected** if it remains connected when any vertex is removed. When the graph is not 2-connected, we call the disconnecting vertex a **cut vertex**.

Q: Why the condition on the number of vertices?

N-Connectivity

A: To avoid  being 2-connected.

There is also a notion of *N*-Connectivity where we require at least *N* vertices to be removed to disconnect the graph.

Connectivity in Directed Graphs

In directed graphs may be able to find a path from *a* to *b* but not from *b* to *a*. However, Connectivity was a symmetric concept for undirected graphs. So how to define directed Connectivity is non-obvious:

1. Should we ignore directions?
2. Should we insist that can get from *a* to *b* in actual digraph?
3. Should we insist that can get from *a* to *b* and that can get from *b* to *a*?

Connectivity in Directed Graphs

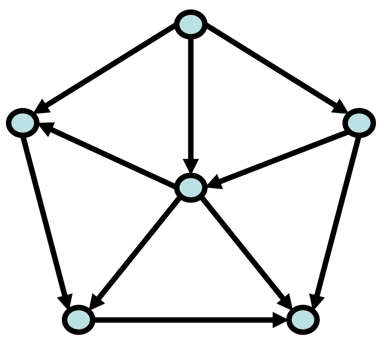
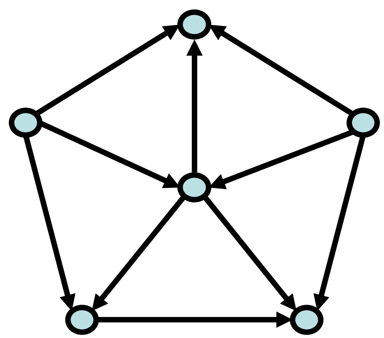
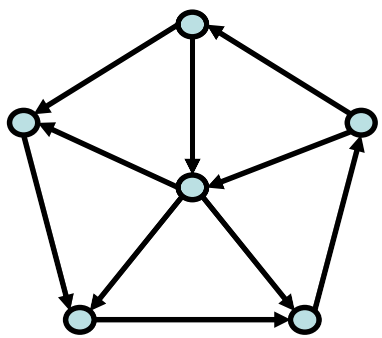
Resolution: Don't bother choosing which definition is better. Just define to separate concepts:

1. **Weakly connected**: can get from a to b in underlying undirected graph
2. **Semi-connected** (my terminology): can get from a to b OR from b to a in digraph
3. **Strongly connected**: can get from a to b AND from b to a in the digraph

Definition: A graph is **strongly** (resp. **semi**, resp. **weakly**) connected if every pair of vertices is connected in the same sense.

Connectivity in Directed Graphs

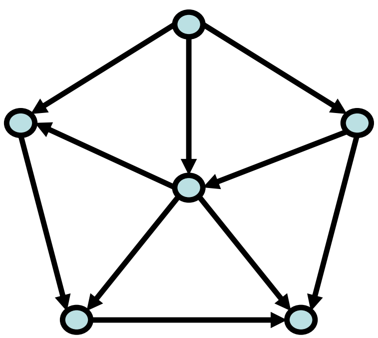
Q: Classify the connectivity of each graph.

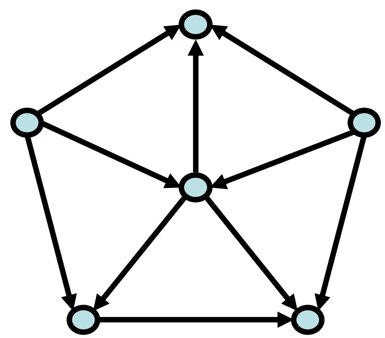
Connectivity in Directed Graphs

A:

semi



weak



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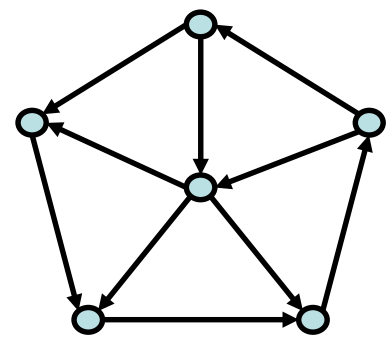


Euler Characteristic

The formula is proved by showing that the quantity (chi) $\chi = r - |E| + |V|$ must equal 2 for planar graphs. χ is called the ***Euler characteristic***. The idea is that any connected planar graph can be built up from a vertex through a sequence of vertex and edge additions. For example, build 3-cube as follows:

Euler Characteristic

Thus to prove that χ is always 2 for planar graphs, one calculate χ for the trivial vertex graph:

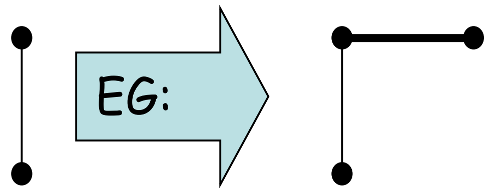
$$\chi = 1\text{-}0\text{+}1 = 2$$

and then checks that each possible move does not change χ .

Euler Characteristic

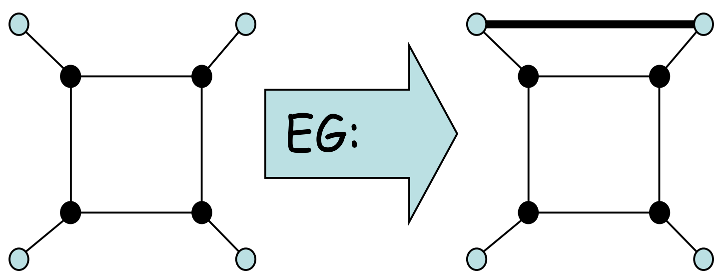
Check that moves don't change χ :

1) Adding a degree 1 vertex:



r is unchanged. $|E|$ increases by 1. $|V|$ increases by 1. χ += (0-1+1)

2) Adding an edge between pre-existing vertices:



r increases by 1. $|E|$ increases by 1. $|V|$ unchanged. χ += (1-1+0)

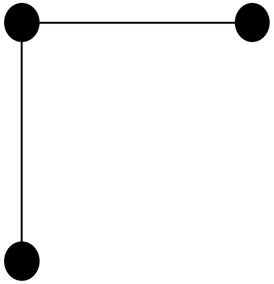
Animated Invariance of Euler Characteristic

$\lvert V \rvert$	$\lvert E \rvert$	r	$\chi =$ $r - \lvert E \rvert + \lvert V \rvert$
1	0	1	2

Animated Invariance of Euler Characteristic

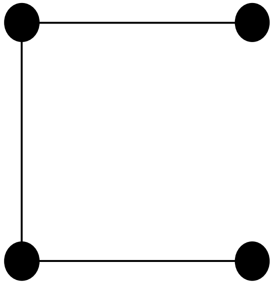
$\|V\|$	$\|E\|$	r	$\chi =$ $r - \|E\| + \|V\|$
2	1	1	2

Animated Invariance of Euler Characteristic



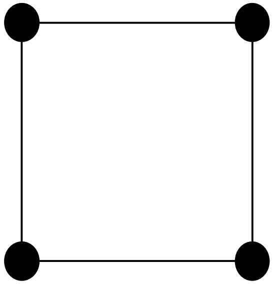
$\lvert V \rvert$	$\lvert E \rvert$	r	$\chi = r - \lvert E \rvert + \lvert V \rvert$
3	2	1	2

Animated Invariance of Euler Characteristic



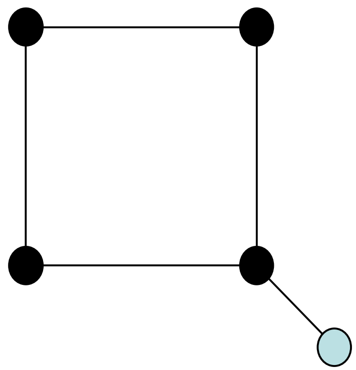
$\|V\|$	$\|E\|$	r	$\chi =$ $r - \|E\| + \|V\|$
4	3	1	2

Animated Invariance of Euler Characteristic



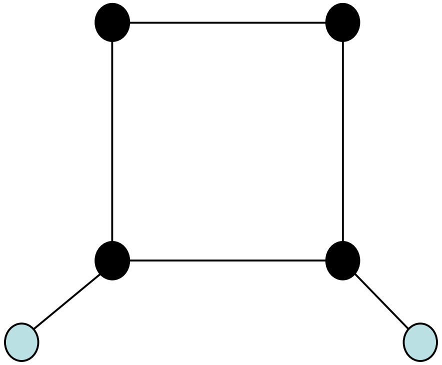
$\|V\|$	$\|E\|$	r	$\chi =$ $r - \|E\| + \|V\|$
4	4	2	2

Animated Invariance of Euler Characteristic



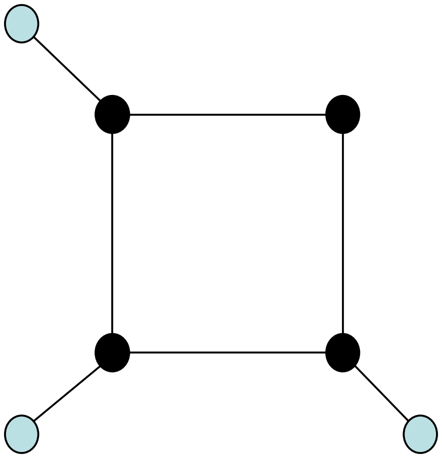
$\|V\|$	$\|E\|$	r	$\chi = r - \|E\| + \|V\|$
5	5	2	2

Animated Invariance of Euler Characteristic



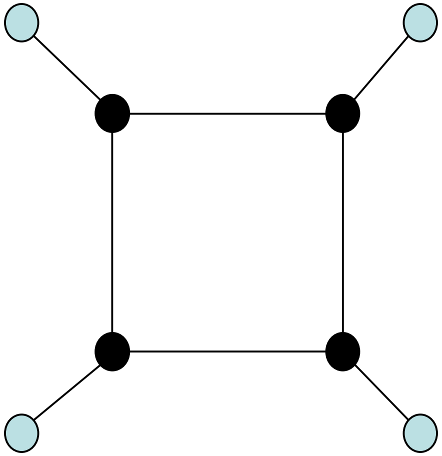
$\lvert V \rvert$	$\lvert E \rvert$	r	$\chi = r - \lvert E \rvert + \lvert V \rvert$
6	6	2	2

Animated Invariance of Euler Characteristic



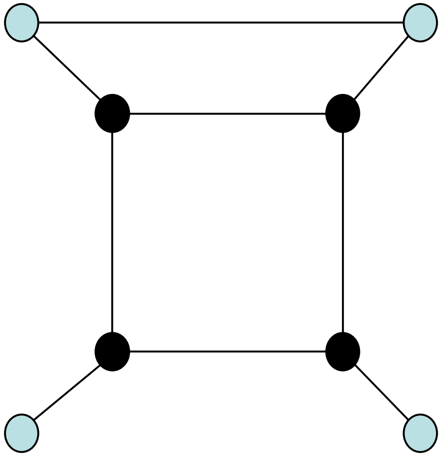
$\|V\|$	$\|E\|$	r	$\chi = r - \|E\| + \|V\|$
7	7	2	2

Animated Invariance of Euler Characteristic



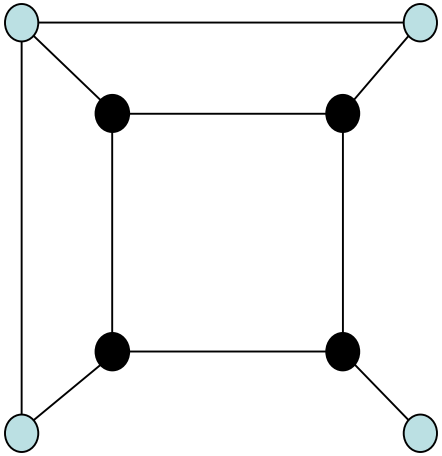
$\|V\|$	$\|E\|$	r	$\chi = r - \|E\| + \|V\|$
8	8	2	2

Animated Invariance of Euler Characteristic



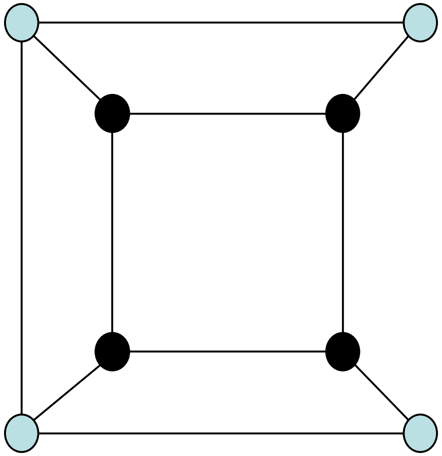
$\lvert V \rvert$	$\lvert E \rvert$	r	$\chi =$ $r - \lvert E \rvert + \lvert V \rvert$
8	9	3	2

Animated Invariance of Euler Characteristic



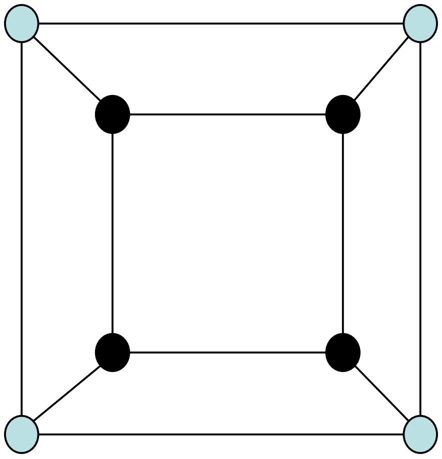
$\lvert V \rvert$	$\lvert E \rvert$	r	$\chi = r - \lvert E \rvert + \lvert V \rvert$
8	10	4	2

Animated Invariance of Euler Characteristic



$\|V\|$	$\|E\|$	r	$\chi = r - \|E\| + \|V\|$
8	11	5	2

Animated Invariance of Euler Characteristic



$\lvert V \rvert$	$\lvert E \rvert$	r	$\chi = r - \lvert E \rvert + \lvert V \rvert$
8	12	6	2

Face-Edge Handshaking

For **all** graphs handshaking theorem relates degrees of *vertices* to number of *edges*.

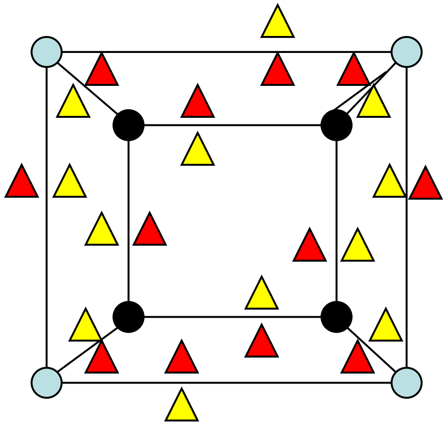
For **planar** graphs, can relate *regions* to *edges* in similar fashion:

EG: There are two ways to count the number of edges in 3-cube:

1) Count directly: 12

2) Count no. of edges around each region; divide by 2:

(4+4+4+4+4+4)/2 = 12 (2 *triangles per edge*)



Face-Edge Handshaking

Definition: The ***degree*** *of a* ***region*** *F* is the number of edges at its boundary, and is denoted by deg(*F*).

Theorem: Let *G* be a planar graph with region set *R*. Then:

$$|E| = \frac{1}{2}\sum_{F\in R}\deg(F)$$

The End