Nested Quantifiers

Content

- Nested quantifiers (嵌套量词)
- Order of quantifiers
- Translating from nested quantifiers into English
- Translating mathematical statements into statements involving nested quantifiers.
- Translated English sentences into logical expressions.
- Negating nested quantifiers.

Nested quantifiers

- Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.

  Example: “Every real number has an inverse” is

  $$\forall x\ \exists y(x + y = 0)$$

  where the domains of x and y are the real numbers.

- We can also think of nested propositional functions:

  $\forall x\ \exists y(x + y = 0)$ can be viewed as $\forall x\ Q(x)$ where $Q(x)$ is $\exists y\ P(x, y)$
  where $P(x, y)$ is $(x + y = 0)$

Thinking of nested quantification

- Nested loops
  - To see if $\forall x \forall y P(x,y)$ is true, loop through the values of x :
    - At each step, loop through the values for y.
    - If for some pair of x and y, $P(x,y)$ is false, then $\forall x \forall y P(x,y)$ is false and both the outer and inner loop terminate.

    $\forall x\ \forall y\ P(x,y)$ is true if the outer loop ends after stepping through each x.
  - To see if $\forall x\ \exists y P(x,y)$ is true, loop through the values of x:
    - At each step, loop through the values for y.
    - The inner loop ends when a pair x and y is found such that $P(x, y)$ is true.
    - If no y is found such that $P(x, y)$ is true the outer loop terminates as $\forall x\ \exists y P(x,y)$ has been shown to be false.

    $\forall x\ \exists y\ P(x,y)$ is true if the outer loop ends after stepping through each x.
- If the domains of the variables are infinite, then this process can not actually be carried out.

Order of quantifiers

Examples:

1. Let $P(x,y)$ be the statement "$x + y = y + x$." Assume that U is the real numbers. Then $\forall x \, \forall y P(x,y)$ and $\forall y \, \forall x P(x,y)$ have the same truth value.
2. Let $Q(x,y)$ be the statement "$x + y = 0$." Assume that U is the real numbers. Then $\forall x \, \exists y P(x,y)$ is true, but $\exists y \, \forall x P(x,y)$ is false.

Questions on order of quantifiers

Example 1: Let U be the real numbers,
Define $P(x,y) : x \cdot y = 0$
What is the truth value of the following:

1. $\forall x \forall y P(x,y)$
   Answer: False
2. $\forall x \exists y P(x,y)$
   Answer: True
3. $\exists x \forall y\, P(x,y)$
   Answer: True
4. $\exists x\, \exists\, y\, P(x,y)$
   Answer: True

Questions on order of quantifiers

Example 2: Let U be the real numbers,
Define P(x,y) : x / y = 1
What is the truth value of the following:

1. $\forall x \forall y P(x,y)$
   Answer: False
2. $\forall x \exists y P(x,y)$
   Answer: False (x=0)
3. $\exists x \forall y\ P(x,y)$
   Answer: False
4. $\exists x\ \exists\ y\ P(x,y)$
   Answer: True

Quantifications of two variables

Statement	When True?	When False?
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	P(x,y) is true for every pair x,y.	There is a pair x, y for which P(x,y) is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which P(x,y) is true.	There is an x such that P(x,y) is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which P(x,y) is true for every y.	For every x there is a y for which P(x,y) is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which P(x,y) is true.	P(x,y) is false for every pair x,y

Translating nested quantifiers into English

Example 1: Translate the statement

$$\forall x \ (C(x) \lor \exists y \ (C(y) \land F(x, y)))$$

where $C(x)$ is "x has a computer," and $F(x,y)$ is "x and y are friends," and the domain for both x and y consists of all students in your school.

Solution: Every student in your school has a computer or has a friend who has a computer.

Example 2: Translate the statement

$$\exists x \forall y \ \forall z \ ((F(x, y) \land F(x,z) \land (y \neq z)) \rightarrow \neg F(y,z))$$

Solution: There is a student none of whose friends are also friends with each other.

Translating mathematical statements into predicate logic

Example : Translate "The sum of two positive integers is always positive" into a logical expression.

Solution:

1. Rewrite the statement to make the implied quantifiers and domains explicit:

   "For every two integers, if these integers are both positive, then the sum of these integers is positive."

2. Introduce the variables x and y, and specify the domain, to obtain:

   "For all positive integers x and y, x + y is positive."

3. The result is:

   $$\forall x \, \forall y \, ((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$$

   where the domain of both variables consists of all integers

Translating English into logical expressions example

Example: Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."

Solution:

1. Let $P(w,f)$ be "w has taken f " and $Q(f,a)$ be "f is a flight on a ."
2. The domain of w is all women, the domain of f is all flights, and the domain of a is all airlines.
3. Then the statement can be expressed as:

$$\exists w \, \forall a \, \exists f \; (P(w,f) \wedge Q(f,a))$$

Questions on translation from English

Choose the obvious predicates and express in predicate logic.

Example 1: "Brothers are siblings."

Solution: $\forall x \, \forall y \, (B(x,y) \rightarrow S(x,y))$

Example 2: "Siblinghood is symmetric."

Solution: $\forall x \, \forall y \, (S(x,y) \rightarrow S(y,x))$

Example 3: "Everybody loves somebody."

Solution: $\forall x \, \exists y \, L(x,y)$

Example 4: "There is someone who is loved by everyone."

Solution: $\exists y \, \forall x \, L(x,y)$

Example 5: "There is someone who loves someone."

Solution: $\exists x \, \exists y \, L(x,y)$

Example 6: "Everyone loves himself"

Solution: $\forall x \, L(x,x)$

Negating nested quantifiers

Example 1: Recall the logical expression developed three slides back:

$$\exists w \ \forall a \ \exists f \ \ (P(w,f) \land Q(f,a))$$

Part 1: Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."

Solution: $\neg\exists w \ \forall a \ \exists f \ \ (P(w,f) \land Q(f,a))$

Part 2: Now use De Morgan's Laws to move the negation as far inwards as possible.

Solution:

1. $\neg\exists w \ \forall a \ \exists f \ \ (P(w,f) \land Q(f,a))$
2. $\forall w \ \neg \ \forall a \ \exists f \ \ (P(w,f) \land Q(f,a))$ by De Morgan's for $\exists$
3. $\forall w \ \exists \ a \ \neg \ \exists f \ \ (P(w,f) \land Q(f,a))$ by De Morgan's for $\forall$
4. $\forall w \ \exists \ a \ \forall f \ \neg \ (P(w,f) \land Q(f,a))$ by De Morgan's for $\exists$
5. $\forall w \ \exists \ a \ \forall f \ (\neg \ P(w,f) \lor \neg \ Q(f,a))$ by De Morgan's for $\land$.

Part 3: Can you translate the result back into English?

Solution:

"For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline"

Some questions about quantifiers

- Can you switch the order of quantifiers?
  - Is this a valid equivalence?

    Solution: Yes! The left and the right side will always have the same truth value. The order in which x and y are picked does not matter.

$$\forall x \forall y P(x,y) \equiv \forall y \forall x P(x,y)$$

  - Is this a valid equivalence?

    Solution: No! The left and the right side may have different truth values for some propositional functions for P. Try "x + y = 0" for P(x,y) with U being the integers. The order in which the values of x and y are picked does matter.

$$\forall x \exists y P(x,y) \equiv \exists y \forall x P(x,y)$$

Some questions about quantifiers

- Can you distribute quantifiers over logical connectives?
  - Is this a valid equivalence?

    Solution: Yes! The left and the right side will always have the same truth value no matter what propositional functions are denoted by P(x) and Q(x).

$$\forall x(P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x)$$

  - Is this a valid equivalence?

    Solution: No! The left and the right side may have different truth values. Pick "x is a fish" for P(x) and "x has scales" for Q(x) with the domain of discourse being all animals. Then the left side is false, because there are some fish that do not have scales. But the right side is true since not all animals are fish.

$$\forall x(P(x) \to Q(x)) \equiv \forall x P(x) \to \forall x Q(x)$$

量词演算规则

- 命题公式推广
  - 谓词公式的等价、蕴涵是在谓词公式进行赋值、转化为命题之后才讨论，所以可以把命题演算中的等价式、蕴涵式推广到谓词演算中
  - 命题演算中的永真式的变元用谓词演算中的公式代替，所得的谓词公式仍然有效

$$\forall x(A(x) \rightarrow B(x)) \Leftrightarrow \forall x(\neg A(x) \vee B(x))$$

$$\exists xH(x,y) \wedge \neg \exists xH(x,y) \Leftrightarrow F$$

量词演算规则

- 量词的否定

  $\neg\forall x A(x) \Leftrightarrow \exists x \neg A(x)$, $\neg\exists x A(x) \Leftrightarrow \forall x \neg A(x)$,

  注：出现在量词前的否定运算符，是否定被量化了的整个命题。量词外边的否定可以深入到辖域内，辖域内的否定也可以移到辖域外，但要注意量词的变化

Example：

P(x): x is a boy. The domain is all students in the class. Then

$\neg\forall x P(x)$?

$\exists x \neg P(x)$?

$\neg\exists x P(x)$?

$\forall x \neg P(x)$?

量词演算规则

- 量词辖域的收缩与扩张

$\forall x(A(x) \vee B) \Leftrightarrow \forall xA(x) \vee B$, $\quad \forall x(A(x) \wedge B) \Leftrightarrow \forall xA(x) \wedge B$

$\forall x(A(x) \rightarrow B) \Leftrightarrow \exists xA(x) \rightarrow B$, $\quad \forall x(B \rightarrow A(x)) \Leftrightarrow B \rightarrow \forall xA(x)$

$\exists x(A(x) \vee B) \Leftrightarrow \exists xA(x) \vee B$, $\quad \exists x(A(x) \wedge B) \Leftrightarrow \exists xA(x) \wedge B$

$\exists x(A(x) \rightarrow B) \Leftrightarrow \forall xA(x) \rightarrow B$, $\quad \exists x(B \rightarrow A(x)) \Leftrightarrow B \rightarrow \exists xA(x)$

$$A(x) \rightarrow B \Leftrightarrow \neg A(x) \vee B$$

量词演算规则

- 量词对∧、∨的分配律

  $\forall x(A(x) \wedge B(x)) \Leftrightarrow \forall xA(x) \wedge \forall xB(x)$,

  $\exists x(A(x) \vee B(x)) \Leftrightarrow \exists xA(x) \vee \exists xB(x)$

  $\forall xA(x) \vee \forall xB(x) \Rightarrow \forall x(A(x) \vee B(x))$,

  $\exists x(A(x) \wedge B(x)) \Rightarrow \exists xA(x) \wedge \exists xB(x)$

  (⇒永真蕴含式)

逻辑等价式	两个命题公式等价，当且仅当在同一赋值下的真值均相同，它们只是形式不同，内涵完全相同。
永真蕴含式	若A为真，则B必为真，就有A⇒B。
逻辑等价与永真蕴涵的关系	A⇔B，当且仅当A⇒B且B⇒A。

量词演算规则

- 多个量词的使用
  - 各量词按照从左到右的顺序读出，不得随意颠倒顺序，但是当出现的量词相同时，量词的顺序可以颠倒

Example:

$A(x,y)$: x read y. The domains of x and y are all human and books, respectively

$\forall x \forall y A(x,y)$

$\forall y \forall x A(x,y)$

$\exists x \exists y A(x,y)$

$\exists y \exists x A(x,y)$

$\forall y \exists x A(x,y)$: T

$\exists x \forall y A(x,y)$: F

$\exists x \forall y A(x,y) \Rightarrow \forall y \exists x A(x,y)$

Example

- Let $Q(x, y)$ denote "$x + y = 0$." What are the truth values of the quantifications $\exists y \forall x Q(x, y)$ and $\forall x \exists y Q(x, y)$, where the domain for all variables consists of all real numbers?

- **Solution**:
  - The quantification $\exists y \forall x Q(x, y)$ denotes the proposition "There is a real number y such that for every real number x, $Q(x, y)$."
  - No matter what value of y is chosen, there is only one value of x for which $x + y = 0$. Because there is no real number y such that $x + y = 0$ for all real numbers x, the statement $\exists y \forall x Q(x, y)$ is false.
  - The quantification $\forall x \exists y Q(x, y)$ denotes the proposition "For every real number x there is a real number y such that $Q(x, y)$."
  - Given a real number x, there is a real number y such that $x + y = 0$; namely, $y = -x$. Hence, the statement $\forall x \exists y Q(x, y)$ is true.

重要等价式和永真蕴含式

E_{31} $(\exists x)(A(x) \vee B(x)) \Leftrightarrow (\exists x)A(x) \vee (\exists x)B(x)$

E_{32} $(\forall x)(A(x) \wedge B(x)) \Leftrightarrow (\forall x)A(x) \wedge (\forall x)B(x)$

E_{33} $\neg(\exists x)A(x) \Leftrightarrow (\forall x)\neg A(x)$

E_{34} $\neg(\forall x)A(x) \Leftrightarrow (\exists x)\neg A(x)$

E_{35} $(\forall x)A(x) \vee P \Leftrightarrow (\forall x)(A(x) \vee P)$

E_{36} $(\forall x)A(x) \wedge P \Leftrightarrow (\forall x)(A(x) \wedge P)$

重要等价式和永真蕴含式

E_{37} $(\exists x)A(x) \vee P \Leftrightarrow (\exists x)(A(x) \vee P)$

E_{38} $(\exists x)A(x) \wedge P \Leftrightarrow (\exists x)(A(x) \wedge P)$

E_{39} $(\forall x)A(x) \to B \Leftrightarrow (\exists x)(A(x) \to B)$

E_{40} $(\exists x)A(x) \to B \Leftrightarrow (\forall x)(A(x) \to B)$

E_{41} $A \to (\forall x)B(x) \Leftrightarrow (\forall x)(A \to B(x))$

E_{42} $A \to (\exists x)B(x) \Leftrightarrow (\exists x)(A \to B(x))$

E_{43} $(\exists x)(A(x) \to B(x)) \Leftrightarrow (\forall x)A(x) \to (\exists x)B(x)$

重要等价式和永真蕴含式

$$I_{17} \quad (\forall x)A(x) \vee (\forall x)B(x) \Rightarrow (\forall x)(A(x) \vee B(x))$$

$$I_{18} \quad (\exists x)(A(x) \wedge B(x)) \Rightarrow (\exists x)A(x) \wedge (\exists x)B(x)$$

$$I_{19} \quad (\exists x)A(x) \rightarrow (\forall x)B(x) \Rightarrow (\forall x)(A(x) \rightarrow B(x))$$

$$I_{20} \quad (\forall x)(A(x) \rightarrow B(x)) \Rightarrow (\forall x)A(x) \rightarrow (\forall x)B(x)$$

量词交换式

$B_1 \quad (\forall x)(\forall y)P(x,y) \Leftrightarrow (\forall y)(\forall x)P(x,y)$

$B_2 \quad (\forall x)(\forall y)P(x,y) \Rightarrow (\exists y)(\forall x)P(x,y)$

$B_3 \quad (\forall y)(\forall x)P(x,y) \Rightarrow (\exists x)(\forall y)P(x,y)$

$B_4 \quad (\exists y)(\forall x)P(x,y) \Rightarrow (\forall x)(\exists y)P(x,y)$

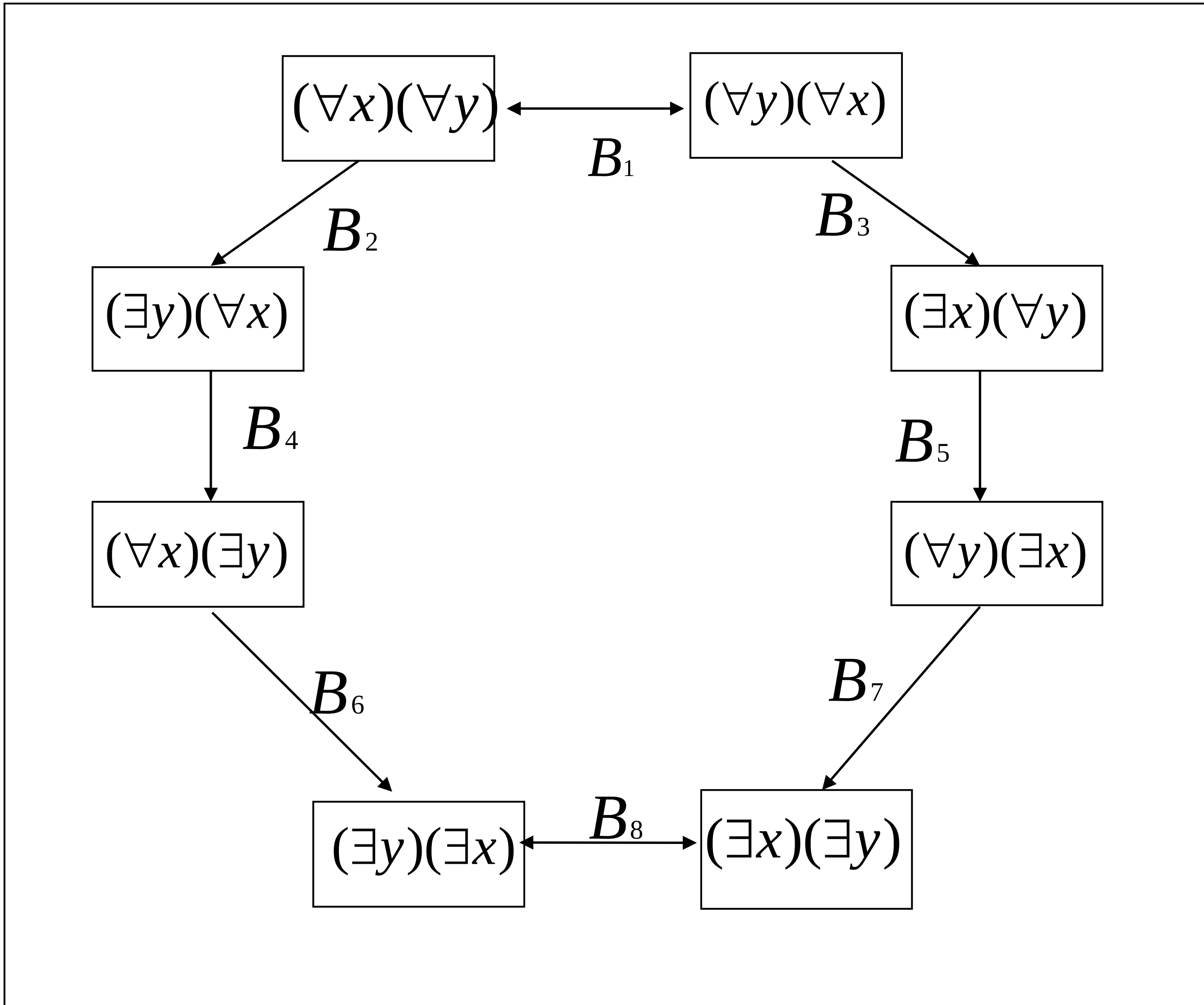
$B_5 \quad (\exists x)(\forall y)P(x,y) \Rightarrow (\forall y)(\exists x)P(x,y)$

$B_6 \quad (\forall x)(\exists y)P(x,y) \Rightarrow (\exists y)(\exists x)P(x,y)$

$B_7 \quad (\forall y)(\exists x)P(x,y) \Rightarrow (\exists x)(\exists y)P(x,y)$

$B_8 \quad (\exists x)(\exists y)P(x,y) \Leftrightarrow (\exists y)(\exists x)P(x,y)$

记忆规律


$(\forall x)(\forall y)$
$(\forall y)(\forall x)$
B_1
B_2
B_3
$(\exists y)(\forall x)$
$(\exists x)(\forall y)$
B_4
B_5
$(\forall x)(\exists y)$
$(\forall y)(\exists x)$
B_6
B_7
B_8
$(\exists y)(\exists x)$
$(\exists x)(\exists y)$

The End