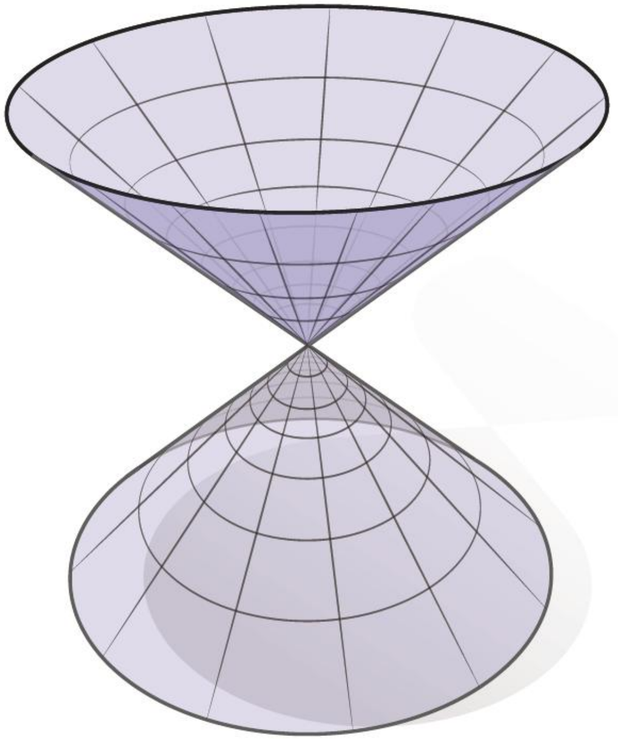
2. Regular Surfaces



Regular Parameterized surface

A continuous map from a domain in R^2 to a domain in E^2

$$r: D \to S = r(D) \subset E^3$$

Image set $S = r(D)$ is called as a ***parameterized surface*** in E^3

Choose a orthogonal frame $\{O; i, j, k\}\ in\ E^3$, we build a Cartesian right-hand coordinate system. Then, parameterized surface S can be represented with parameter (u, v):

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases} \quad (u,v) \in D \subseteq R^2$$

Vector parametric equation is

$$r = r(u,v) = x(u,v)i + y(u,v)j + z(u,v)k$$

Regular Parameterized surface

Assume $S: r = r(u, v)$ is a parameterized surface in E^3.

If the tangent vector of two parameterized curves at point (u_0, v_0) are:

$$r_u(u_0, v_0) = \frac{\partial r}{\partial u}\bigg|_{(u_0,v_0)},$$
$$r_v(u_0, v_0) = \frac{\partial r}{\partial v}\bigg|_{(u_0,v_0)}$$

linearly independent, then

$$r_u \times r_v = \left(\frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)}\right) \neq 0$$

we call (u_0, v_0) is a **regular point** of S.

If all points of S are regular points, then S is a **regular parameterized surface**.

Definition

A subset $M \subset \mathbb{R}^3$ is said to be a regular surface if for any $p \in M$, there is an open neighborhood U of p in M, an open set D in $\mathbb{R}^2$ and a map $\mathbf{X} : D \to M \cap U$ such that the following are true:

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(rs2) $d\mathbf{X}$ *is full rank:* $\mathbf{X}_u = \frac{\partial \mathbf{X}}{\partial u}$ *and* $\mathbf{X}_v = \frac{\partial \mathbf{X}}{\partial v}$ *are linearly independent, for any* $(u, v) \in D$.

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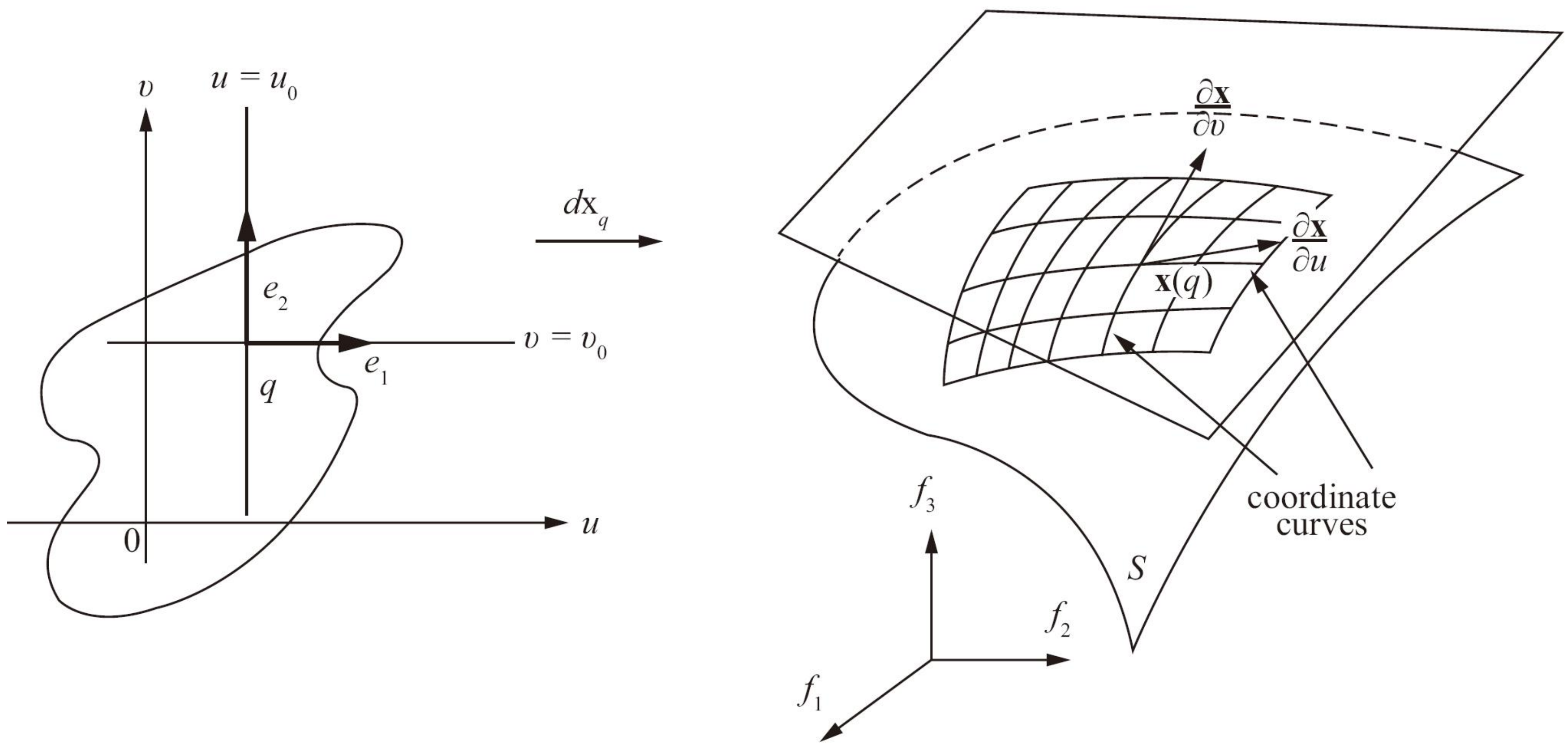
$$d\mathbf{x}_q = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

or

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}, \quad \frac{\partial(y, z)}{\partial(u, v)}, \quad \frac{\partial(x, z)}{\partial(u, v)},$$

be different from zero at q.



v
$u = u_0$
dx_q
e_2
$v = v_0$
e_1
q
0
u
$\frac{\partial \mathbf{x}}{\partial v}$
$\frac{\partial \mathbf{x}}{\partial u}$
$\mathbf{x}(q)$
coordinate
curves
f_3
S
f_2
f_1

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(rs3) $\mathbf{X}$ *is a homeomorphism from* D *onto* $M \cap U$. *(That is:* $\mathbf{X}$ *is bijective,* $\mathbf{X}$ *and* $\mathbf{X}^{-1}$ *are continuous).*

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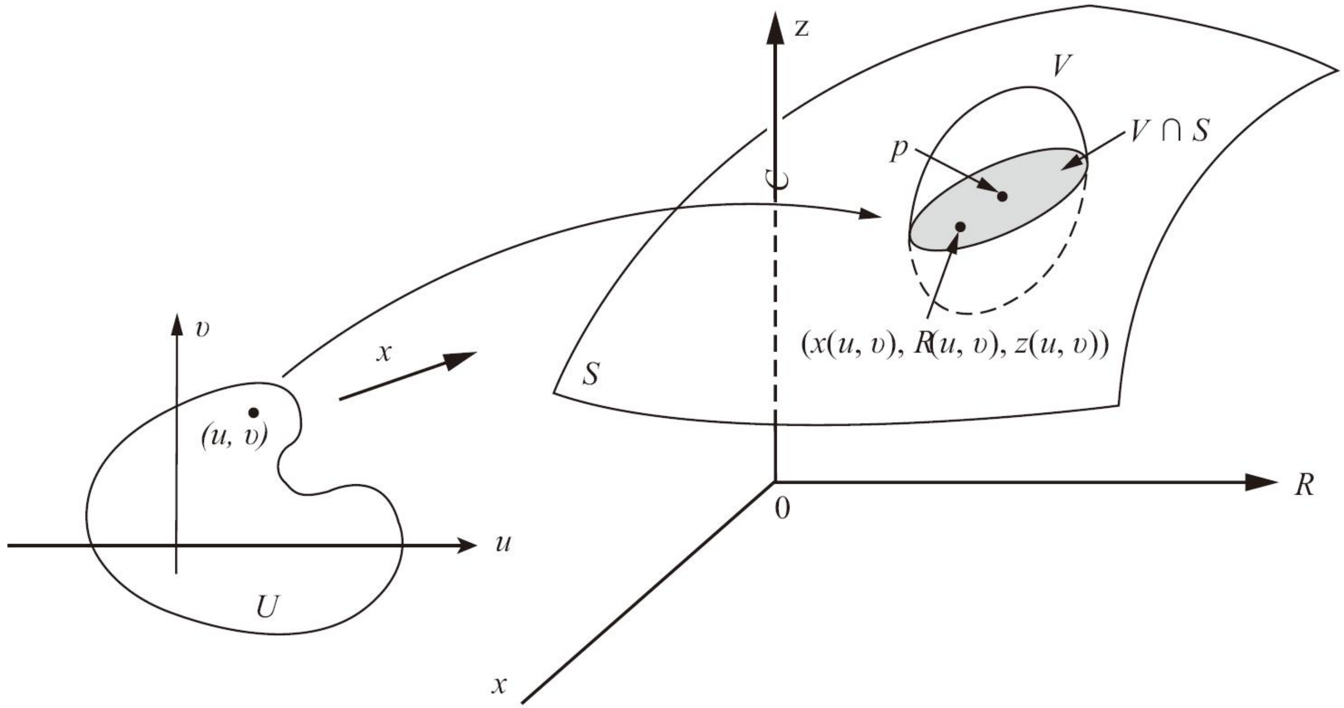
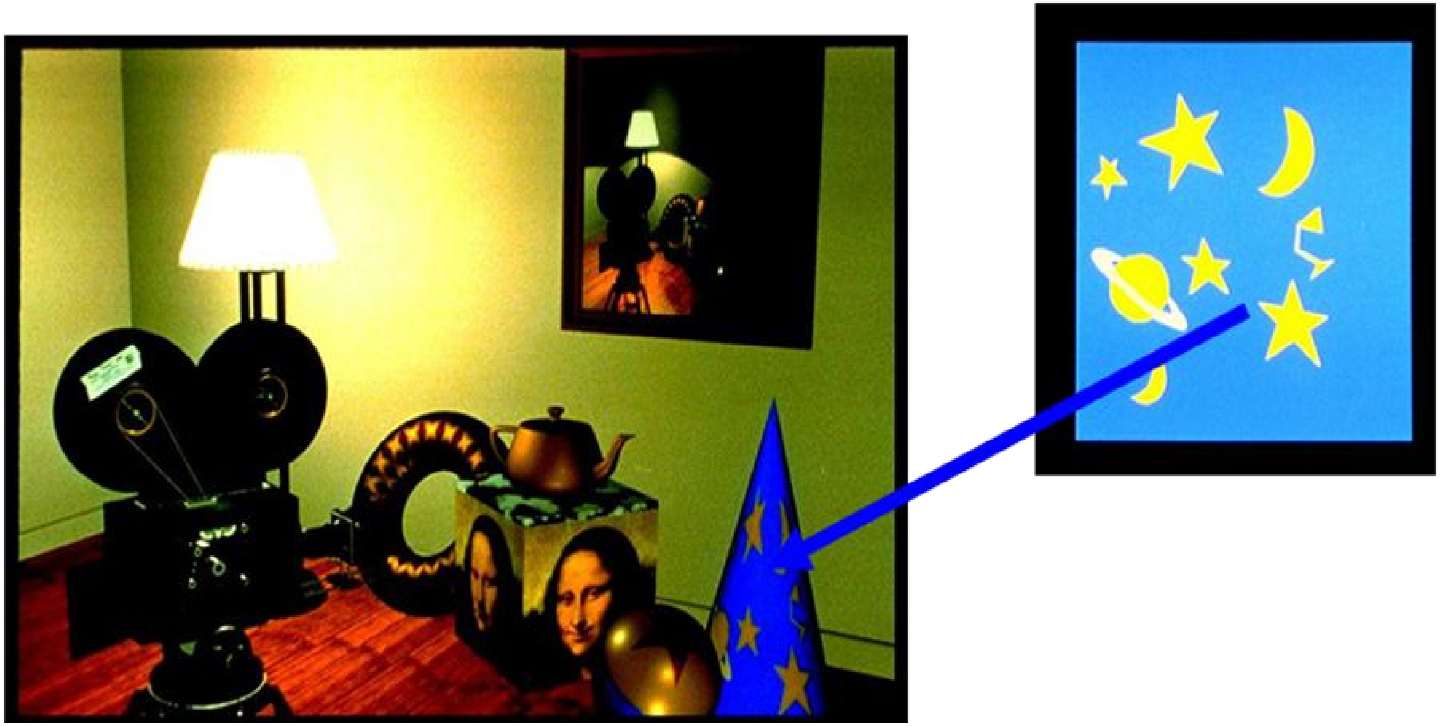
So a regular surface is a set M in $\mathbb{R}^3$ which can be covered by a family of coordinate charts.



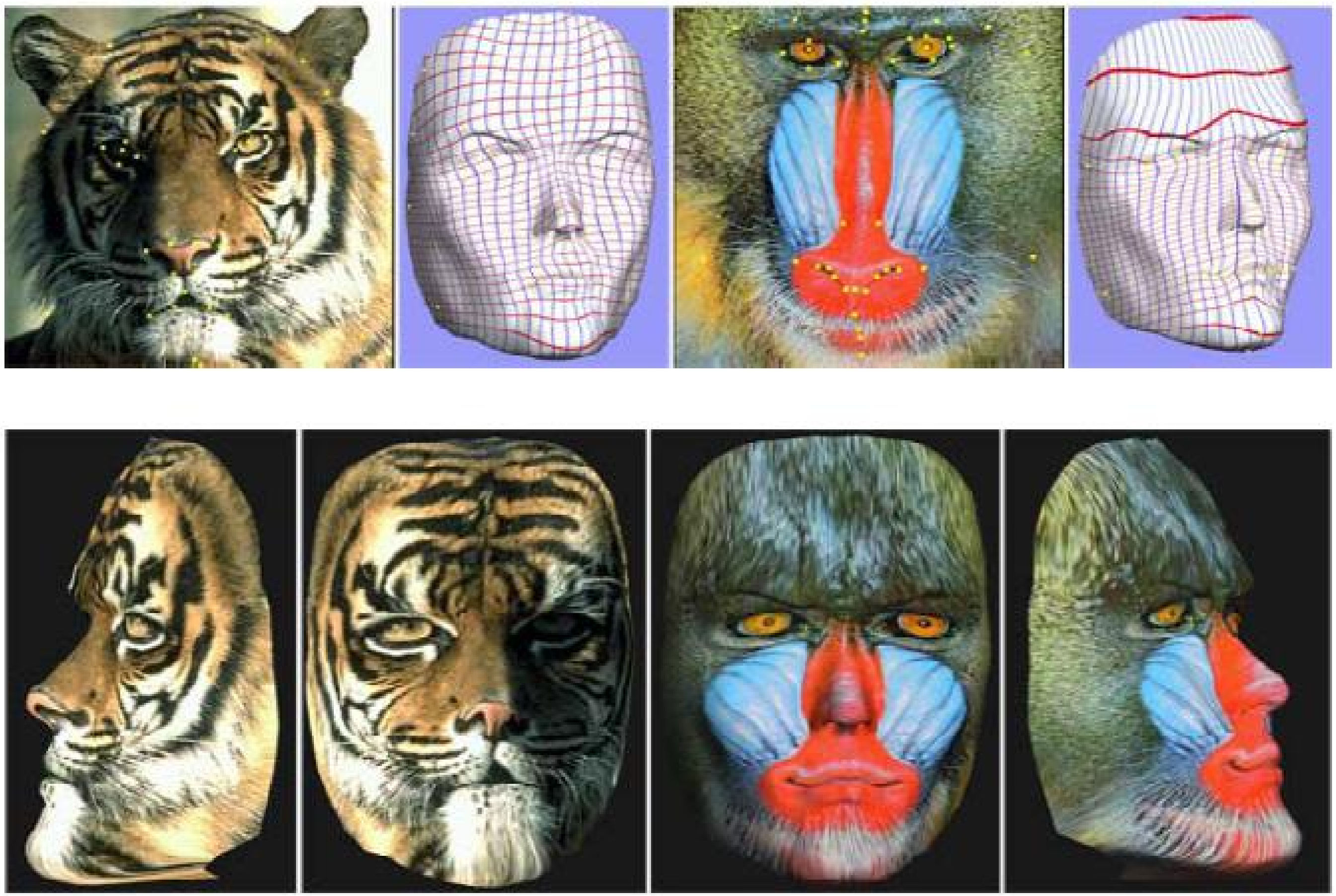
Intuitively, parametric surface S is generated by embedding a plane region D into E^3 which is deformed with several continuous operations such as scaling and distortion, etc.

(u, v) can be seen as the coordinate of the points in surface S, which is called as Grain coordinates (纹理坐标)

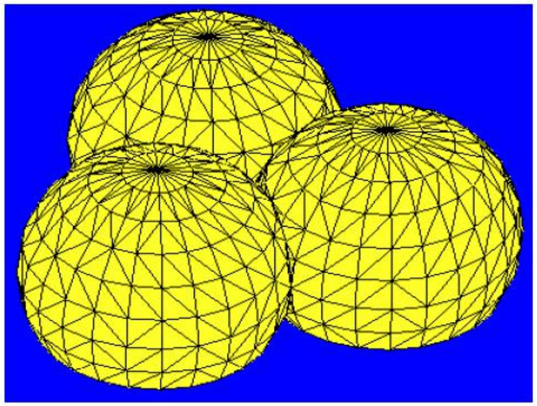
Texture mapping example

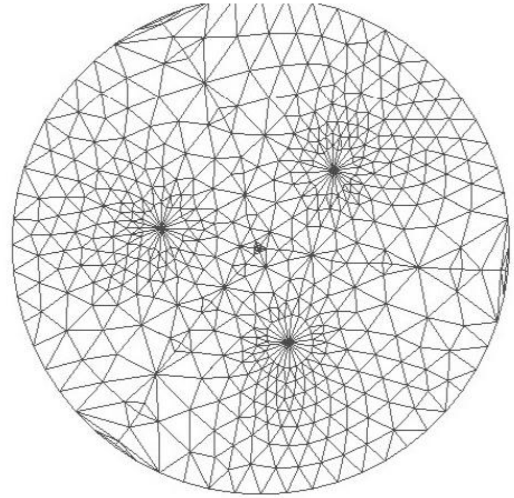


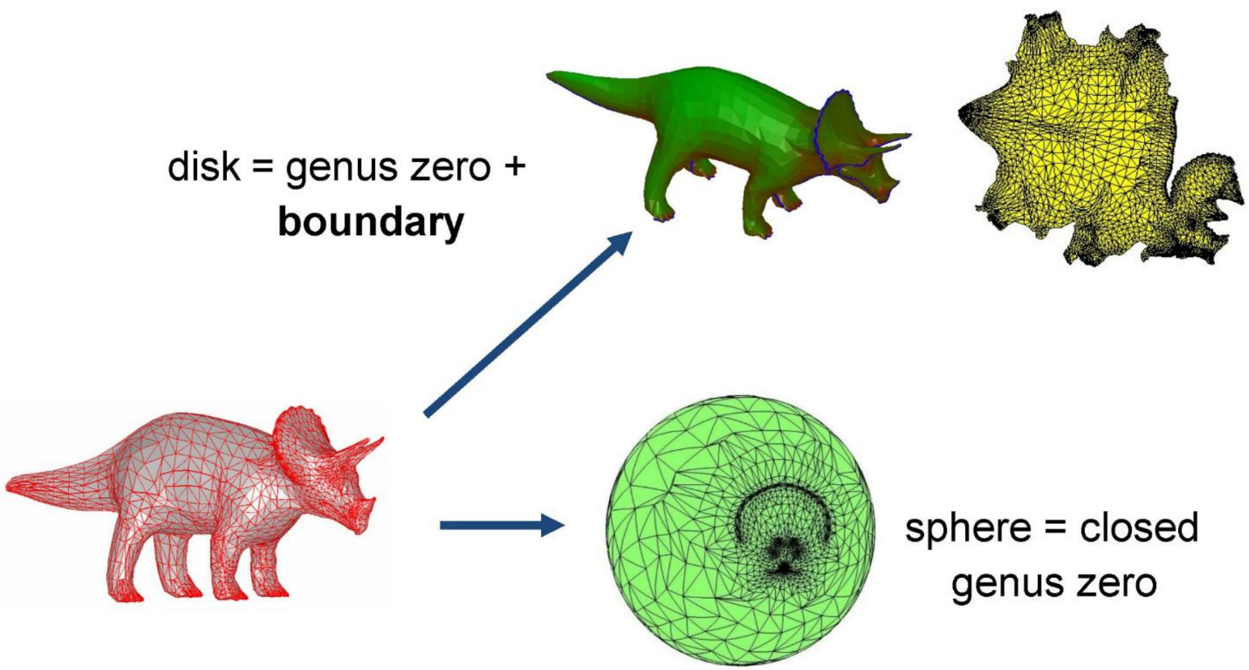
Constrained Parameterizations



Shape parameterization







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