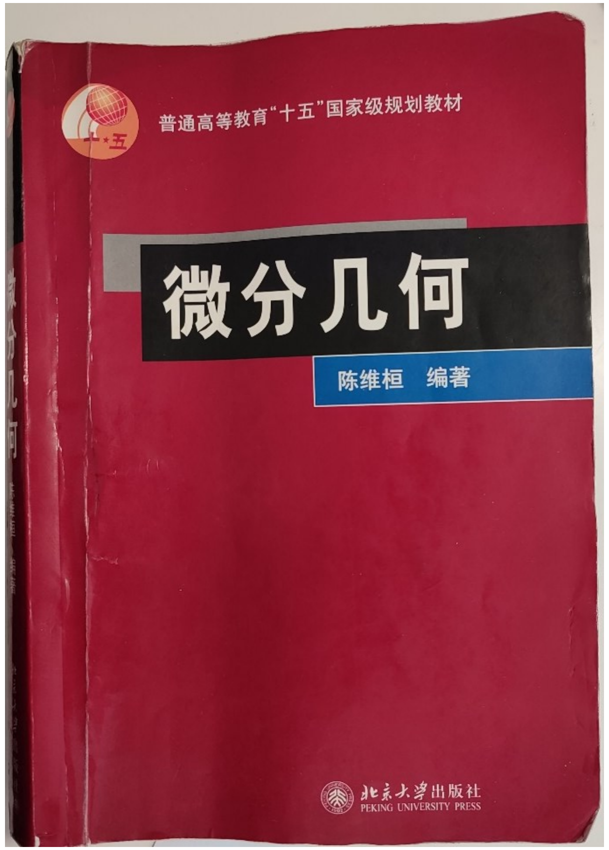

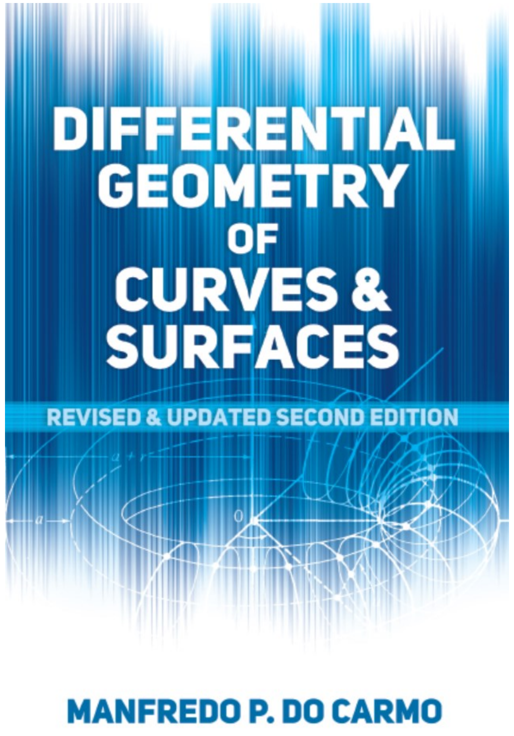

Frenet 标架的定义,

$$\boldsymbol{r}'(s) = \boldsymbol{\alpha}(s),$$
$$\boldsymbol{\alpha}'(s) = \kappa(s)\boldsymbol{\beta}(s),$$
$$\boldsymbol{\gamma}'(s) = -\tau(s)\boldsymbol{\beta}(s)$$

$$b'(s) = \tau(s)n(s)$$

for some function $\tau(s)$. (*Warning*: Many authors write $-\tau(s)$ instead of our $\tau(s)$.)

For later use, we shall call the equations

$$t' = kn,$$
$$n' = -kt - \tau b,$$
$$b' = \tau n.$$

the *Frenet formulas* (we have omitted the s, for convenience).

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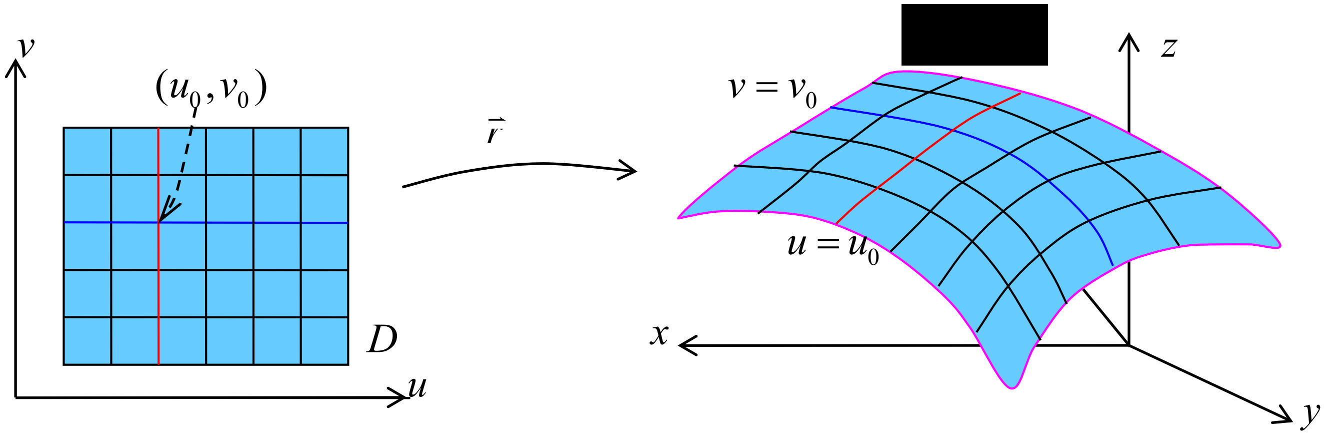
Tangent space consists of tangent vectors of curves on M, cont.

Corollary

Let $\mathbf{X} : U \to \mathbb{R}^3$ be a regular surface patch, and let $M = \mathbf{X}(U)$. Let $p \in M$ be a point in the surface. Then $T_p(M)$ consists of the tangent vectors of smooth curves on M passing through p.



Normals and unit normals

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Examples:

(i) Consider the sphere $\mathbb{S}^2(r) = \{x^2 + y^2 + z^2 = r^2\}$ which is the level set of $f(x, y, z) = x^2 + y^2 + z^2$ at the regular value r^2. Then

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$$|\mathbf{X}_u \times \mathbf{X}_v|^2 = f^2\big((g')^2 + (f')^2\big)$$

Mobiüs strip

(莫比乌斯带)

$$\mathbf{X}(\theta, v) = (\cos\theta, \sin\theta, 0) + v(\sin\frac{1}{2}\theta\cos\theta, \sin\frac{1}{2}\theta\sin\theta, \cos\frac{1}{2}\theta)$$
$$= \mathbf{a}(\theta) + v\mathbf{w}(\theta)$$

$$-\pi < \theta < \pi, \quad -\tfrac{1}{2} < v < \tfrac{1}{2}.$$

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$$\mathbf{X}_v = (\sin\frac{1}{2}\theta\cos\theta, \sin\frac{1}{2}\theta\sin\theta, \cos\frac{1}{2}\theta)$$

$$\mathbf{X}_\theta = (-\sin\theta, \cos\theta, 0) + v\mathbf{w}'(\theta)$$

$$\therefore \mathbf{X}_\theta(\theta, 0) = (-\sin\theta, \cos\theta, 0)$$

At $(\theta, 0)$

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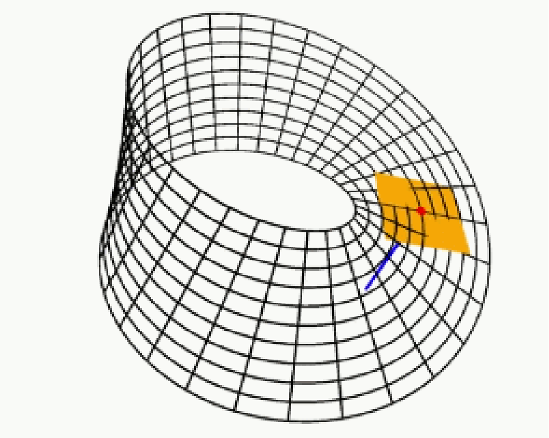
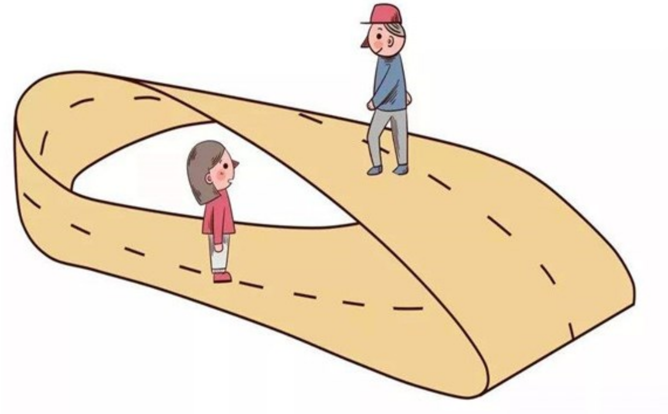
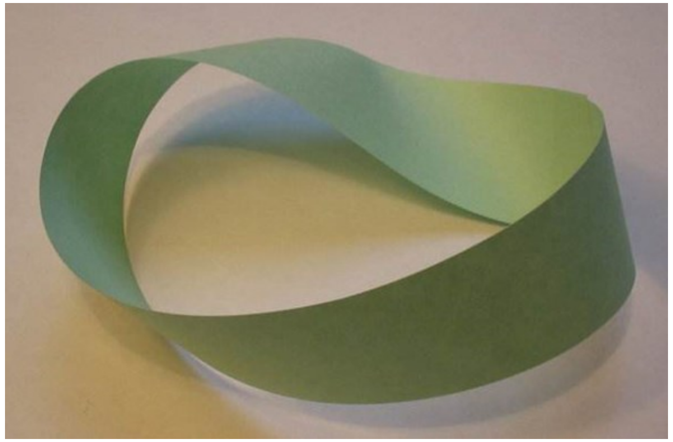
Hence the Mobiüs strip has no continuously defined unit normal vector field.

Mobiüs strip

(莫比乌斯带)







First fundamental form

(第一基本形式)

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$$I_p(w) = \langle w, w\rangle_p = |w|^2 \geq 0. \tag{1}$$

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DEFINITION 1. *The quadratic form* $\mathrm{I_p}$ *on* $\mathrm{T_p(S)}$, *defined by Eq. (1), is called the* first fundamental form *of the regular surface* $S \subset R^3$ *at* $\mathrm{p} \in \mathrm{S}$.

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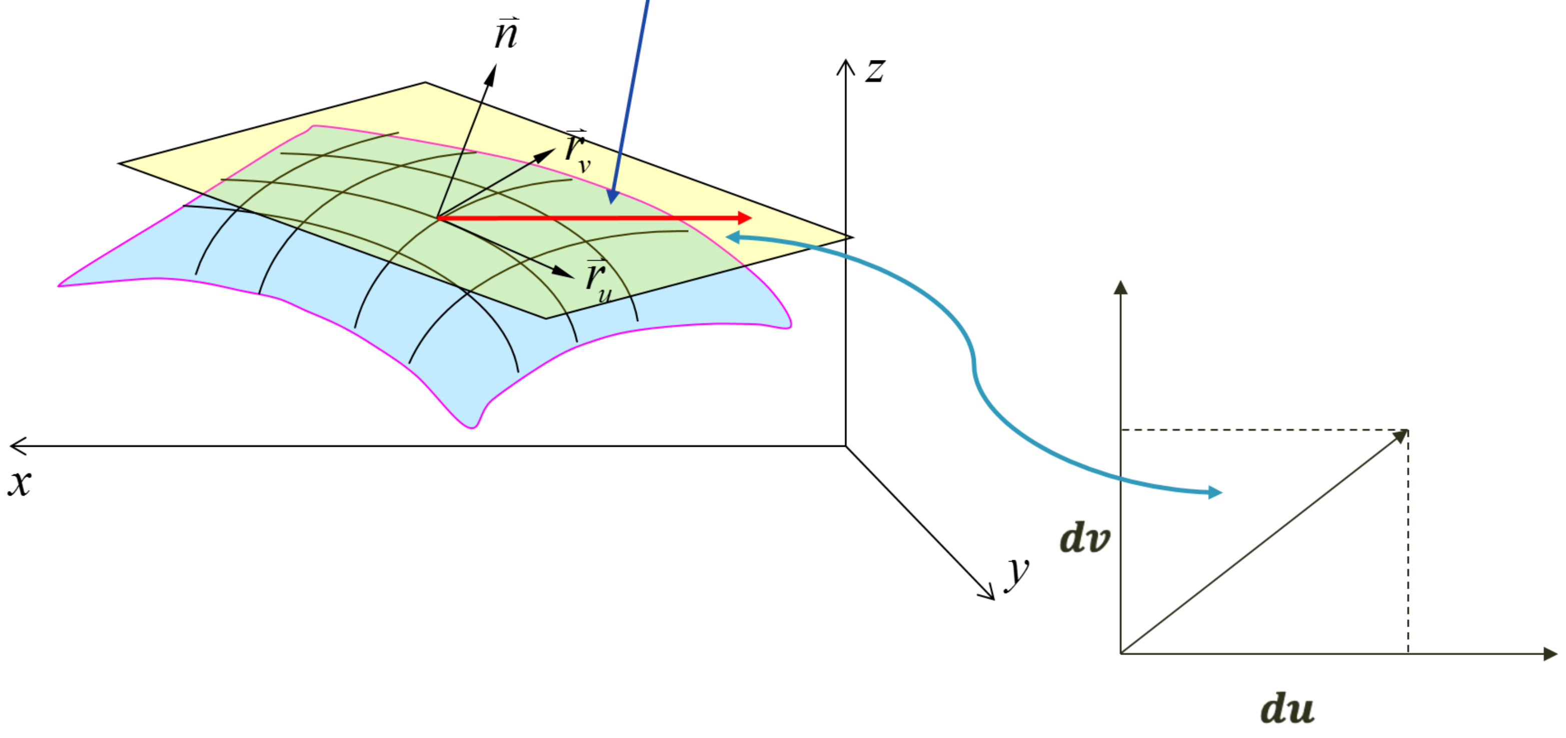
Therefore, the first fundamental form is merely the expression of how the surface S inherits the natural inner product of R^3. Geometrically, as we shall see in a while, the first fundamental form allows us to make measurements on the surface (lengths of curves, angles of tangent vectors, areas of regions) without referring back to the ambient space R^3 where the surface lies.

Geometric meaning of $d\boldsymbol{r}$

$$d\vec{r}(u,v) = \vec{r}_u(u,v)du + \vec{r}_v(u,v)dv$$



The first fundamental form

$$\mathrm{I} =< d\boldsymbol{r}, d\boldsymbol{r} > = d\boldsymbol{r}.d\boldsymbol{r}$$
$$= (\boldsymbol{r}_u du + \boldsymbol{r}_v dv)^2$$
$$= \boldsymbol{r}_u.\boldsymbol{r}_u du^2 + \boldsymbol{r}_u.\boldsymbol{r}_v dudv + \boldsymbol{r}_v.\boldsymbol{r}_u dudv + \boldsymbol{r}_v.\boldsymbol{r}_v dv^2$$

Let

$$E(u,v) = \boldsymbol{r}_u(u,v).\boldsymbol{r}_u(u,v)$$
$$F(u,v) = \boldsymbol{r}_u(u,v).\boldsymbol{r}_v(u,v) = \boldsymbol{r}_u(u,v).\boldsymbol{r}_v(u,v)$$
$$G(u,v) = \boldsymbol{r}_v(u,v).\boldsymbol{r}_v(u,v)$$

Then,

$$\mathrm{I} =< d\boldsymbol{r}, d\boldsymbol{r} > = d\boldsymbol{r}.d\boldsymbol{r}$$
$$= Edu^2 + 2Fdudv + Gdv^2$$
$$= (du, dv)\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix}$$
$$= I$$

Independent of the selection of parametrization

The first fundamental form

Let $u = u(\tilde{u}, \tilde{v}), v = v(\tilde{u}, \tilde{v})$, the Jacobi matrix of parameter transformation is

$$J = \begin{pmatrix} \dfrac{\partial u}{\partial \tilde{u}} & \dfrac{\partial v}{\partial \tilde{u}} \\ \dfrac{\partial u}{\partial \tilde{v}} & \dfrac{\partial v}{\partial \tilde{v}} \end{pmatrix}$$

Then

$$\begin{pmatrix} \boldsymbol{r}_{\tilde{u}} \\ \boldsymbol{r}_{\tilde{v}} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u}{\partial \tilde{u}} & \dfrac{\partial v}{\partial \tilde{u}} \\ \dfrac{\partial u}{\partial \tilde{v}} & \dfrac{\partial v}{\partial \tilde{v}} \end{pmatrix} \begin{pmatrix} \boldsymbol{r}_u \\ \boldsymbol{r}_v \end{pmatrix} = J \begin{pmatrix} \boldsymbol{r}_u \\ \boldsymbol{r}_v \end{pmatrix}$$

$$(du, dv) = (d\tilde{u}, d\tilde{v})J$$

The first fundamental form

Since

$$\begin{pmatrix} \boldsymbol{r}_{\tilde{u}} \\ \boldsymbol{r}_{\tilde{v}} \end{pmatrix} = J \begin{pmatrix} \boldsymbol{r}_u \\ \boldsymbol{r}_v \end{pmatrix}$$

we get

$$\begin{aligned} \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} &= \begin{pmatrix} r_{\tilde{u}} \\ r_{\tilde{v}} \end{pmatrix} \begin{pmatrix} r_{\tilde{u}} & r_{\tilde{v}} \end{pmatrix} \\ &= J \begin{pmatrix} \boldsymbol{r}_u \\ \boldsymbol{r}_v \end{pmatrix} \begin{pmatrix} r_u & r_v \end{pmatrix} J^T \\ &= J \begin{pmatrix} E & F \\ F & G \end{pmatrix} J^T \end{aligned}$$

The first fundamental form

The relationship between the coefficients of the 1st fundamental form:

$$\begin{cases} \tilde{E} = \boldsymbol{r}_{\tilde{u}}^2 = E\left(\dfrac{\partial u}{\partial \tilde{u}}\right)^2 + 2F\dfrac{\partial u}{\partial \tilde{u}}\dfrac{\partial v}{\partial \tilde{u}} + G\left(\dfrac{\partial v}{\partial \tilde{u}}\right)^2 \\ \tilde{F} = \boldsymbol{r}_{\tilde{u}}.\boldsymbol{r}_{\tilde{v}} = E\dfrac{\partial u}{\partial \tilde{u}}\dfrac{\partial u}{\partial \tilde{v}} + F\left(\dfrac{\partial u}{\partial \tilde{u}}\dfrac{\partial v}{\partial \tilde{v}} + \dfrac{\partial u}{\partial \tilde{v}}\dfrac{\partial v}{\partial \tilde{u}}\right) + G\dfrac{\partial v}{\partial \tilde{u}}\dfrac{\partial v}{\partial \tilde{v}} \\ \tilde{G} = \boldsymbol{r}_{\tilde{v}}^2 = E\left(\dfrac{\partial u}{\partial \tilde{v}}\right)^2 + 2F\dfrac{\partial u}{\partial \tilde{v}}\dfrac{\partial v}{\partial \tilde{v}} + G\left(\dfrac{\partial v}{\partial \tilde{v}}\right)^2 \end{cases}$$

The first fundamental form

Under the new parameter $(\tilde{u}, \tilde{v})$, the 1st fundamental form keeps unchanged, that is

$$\begin{aligned}\tilde{I} &= (d\tilde{u}, d\tilde{v})\begin{pmatrix}\tilde{E} & \tilde{F}\\ \tilde{F} & \tilde{G}\end{pmatrix}\begin{pmatrix}d\tilde{u}\\ d\tilde{v}\end{pmatrix}\\ &= (d\tilde{u}, d\tilde{v})J\begin{pmatrix}E & F\\ F & G\end{pmatrix}J^T\begin{pmatrix}d\tilde{u}\\ d\tilde{v}\end{pmatrix}\\ &= (du, dv)\begin{pmatrix}E & F\\ F & G\end{pmatrix}\begin{pmatrix}du\\ dv\end{pmatrix}\\ &= I\end{aligned}$$

Therefore, the first fundamental form is independent of the parameterization.

Example

Calculate the 1st fundamental form of the revolution surface
$\boldsymbol{r}(u, v) = \big(f(v) con(u), f(v)\sin(u), g(v)\big)$

Angle between two tangent vectors

For two tangent vectors $\boldsymbol{dr} = \boldsymbol{r}_u du + \boldsymbol{r}_v dv$ and $\boldsymbol{\delta r} = \boldsymbol{r}_u \delta u + \boldsymbol{r}_v \delta v$, their angle $\angle(\boldsymbol{dr}, \boldsymbol{\delta r})$ satisfies

$$cos\angle(d\boldsymbol{r}, \delta\boldsymbol{r}) = \frac{\boldsymbol{dr}.\boldsymbol{\delta r}}{|\boldsymbol{dr}||\boldsymbol{\delta r}|}$$

$$= \frac{Edu\delta u + F(du\delta v + dv\delta u) + Gdv\delta v}{\sqrt{Edu^2 + 2Fdudv + Gdv^2}\sqrt{E\delta u^2 + 2F\delta u\delta v + G\delta v^2}}$$

Then, $\boldsymbol{dr}$ and $\boldsymbol{\delta r}$ are perpendicular to each other if and only if

$$Edu\delta u + F(du\delta v + dv\delta u) + Gdv\delta v = 0$$

Parametric curve network(参数曲线网)

The parametric curves network is a orthogonal curves network**, if and only if** $F \equiv 0$.
That is $\boldsymbol{r}_u \perp \boldsymbol{r}_v$.

Length of a curve

Suppose $\alpha(t) = (x(t), y(t), z(t))$ is a smooth curve on M, $a \leq t \leq b$ such that $\alpha(t) = \mathbf{X}((u(t), v(t))$ in local coordinates.

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Then the length of α is given by

$$\ell = \int_a^b |\alpha'|(t)dt$$

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If we use (u^1, u^2) instead of (u, v) and $\mathbf{X}_i = \frac{\partial \mathbf{X}}{\partial u^i}$,

$$\ell = \int_a^b \left(\sum_{i,j=1}^{2} g_{ij} \frac{du^i}{dt}\frac{du^j}{dt} \right)^{\frac{1}{2}} dt. \quad g_{ij} =< X_i, X_j >$$

Length of a curve, cont.

So sometimes, the first fundamental form is written symbolically as

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2,$$

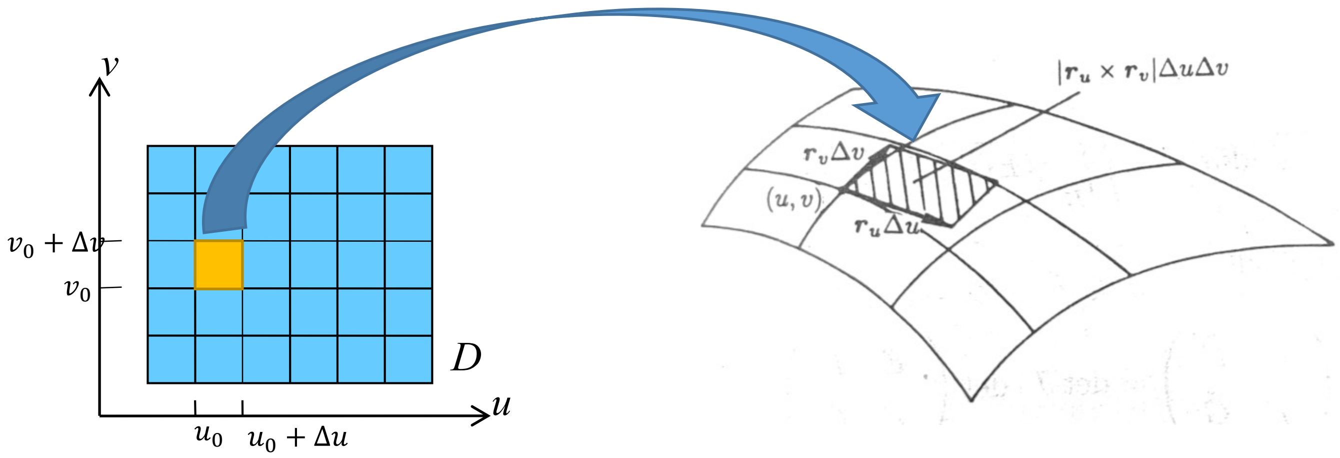
or

$$g = \sum_{i,j=1}^{2} g_{ij} du^i du^j.$$

Area of a region

Assume the domain of definition of a regular parametric surface $S\colon r = r(u,v)$ is $D \subset E^2$.
Consider a small patch formed by parametric curves

$$u = u_0, u = u_0 + \Delta u, v = v_0, v = v_0 + \Delta v$$



Area of a region

The area of the surface region is approximated as

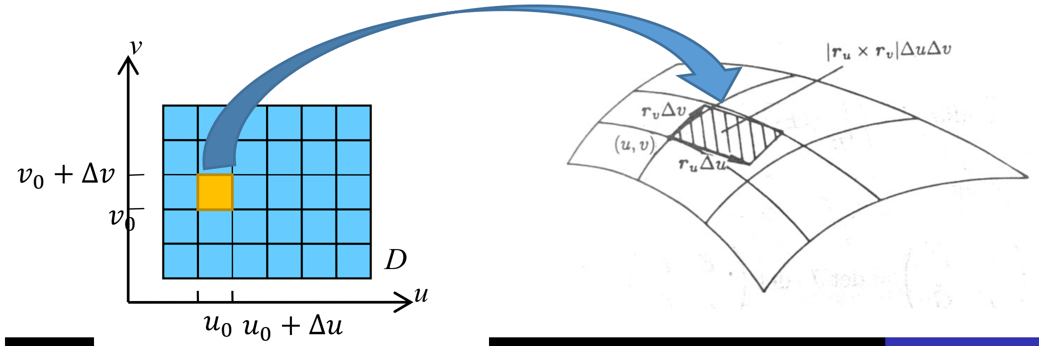
$$\begin{aligned} A &\approx |(r_u \Delta u) \times (r_v \Delta v)| = |r_u \times r_v| \Delta u \Delta v \\ &= |r_u||r_v| \sin\angle(r_u, r_v) \Delta u \Delta v \\ &= \sqrt{EG - F^2} \Delta u \Delta v \end{aligned}$$

Let

$$d\sigma = \sqrt{EG - F^2}\, du dv$$

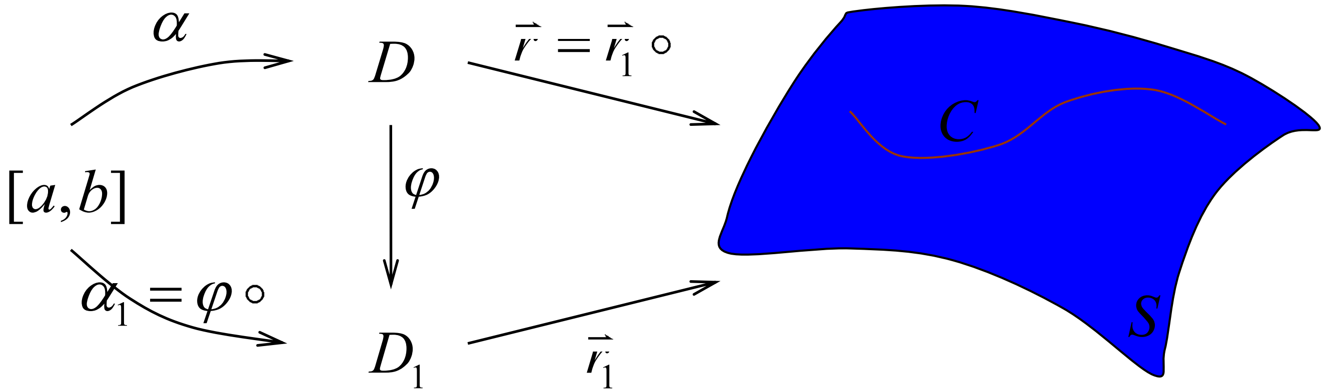
$d\sigma$ is called as the **area element (面积元素)** of surface S.
The area of S is

$$A = \iint_D \sqrt{EG - F^2}\, du dv$$



Geometric quantities(几何量) of surface

The arc length ds, area element(面积元) $d\sigma$ and area of a surface A are geometric quantities of this surface



Examples

Graphs: Let $M = \{(x, y, z)|\ z = f(x, y), (x, y) \in U \subset \mathbb{R}^2\}$. It is parametrized by $\mathbf{X}(u, v) = (u, v, f(u, v))$. Hence

$$E = 1 + f_u^2, F = f_u f_v, G = 1 + f_v^2.$$

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$$E = 1 + f_u^2, F = f_u f_v, G = 1 + f_v^2.$$

The surface area of $\mathbf{X}(U)$ is given by

$$\begin{aligned} A &= \iint_U \sqrt{(1 + f_u^2)(1 + f_v^2) - f_u^2 f_v^2}\, dudv \\ &= \iint_U \sqrt{1 + f_u^2 + f_v^2}\, dudv \end{aligned}$$

Sphere: $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 | \ x^2 + y^2 + z^2 = 1\}$.
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(i) One of them is $\mathbf{X}(x, y) = (x, y, \sqrt{1 - (x^2 + y^2)})$, $(x, y) \in D$ which is the unit disk in $\mathbb{R}^2$. This is graph. So the coefficients of the first fundamental form can be computed as before.

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(ii) (Spherical coordinates) One of them is:

$$\mathbf{X}(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

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$$\mathbf{X}_\theta = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)$$

$$\mathbf{X}_\varphi = (-\cos\theta\sin\varphi, \cos\theta\cos\varphi, 0)$$

So $E = 1$; $F = 0$; $G = \cos^2\theta$.

(iii) (Stereographic projection) The unit sphere M is considered as the set $\{x^2 + y^2 + (z-1)^2 = 1\}$, parametrized by

$$\mathbf{X}(u, v) = \left(\frac{4u}{u^2 + v^2 + 4}, \frac{4v}{u^2 + v^2 + 4}, \frac{2(u^2 + v^2)}{u^2 + v^2 + 4}\right)$$

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The first fundamental form is:

$$E = G = \frac{1}{1+\frac{1}{4}(u^2+v^2)^2}; F = 0.$$

Homework

1. Calculate the first fundamental form of the following parametric functions:
   a) $r = (a(u+v), b(u-v), 2uv)$, where a and b are constants
   b) $r = (ucos(v), usin(v), bv)$, where b is a constant.

2. Parametrized the torus by:

$$\mathbf{X}(u,v) = ((a + r\cos u)\cos v, (a + r\cos u)\sin v, r\sin u).$$

$0 < u, v < 2\pi$. Find the coefficients of the first fundamental form and find the area of the torus.

3. The 1^{st} fundamental form of surface S is $I = du^2 + \sinh^2(u)dv^2$, try to calculate the arc length of the curve $u = v$ under S.