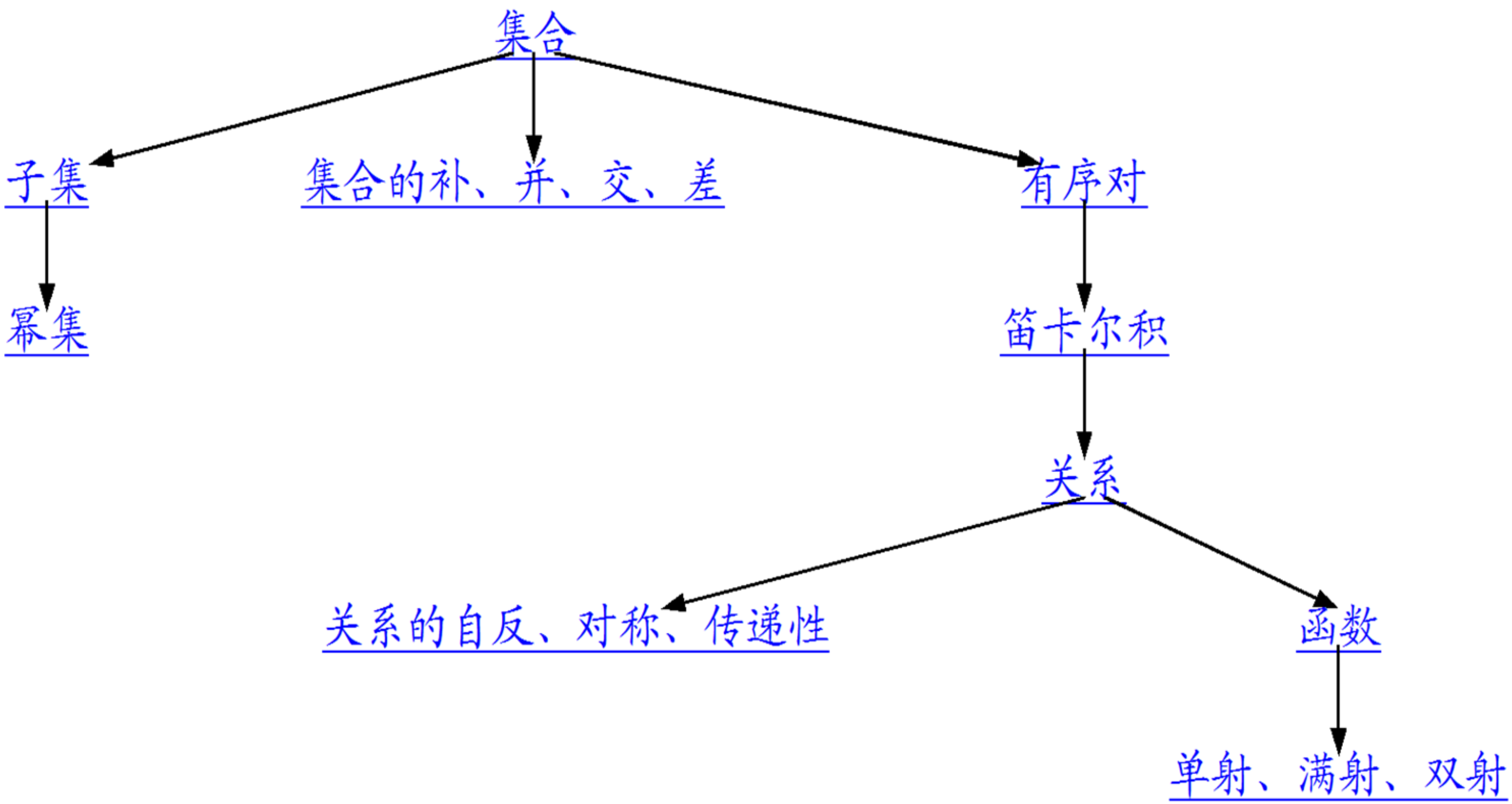
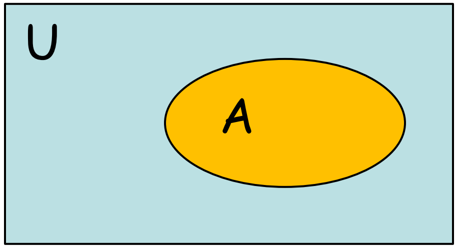
Set Theory

Sets I


集合
子集
集合的补、并、交、差
有序对
幂集
笛卡尔积
关系
关系的自反、对称、传递性
函数
单射、满射、双射

Sets

- Set = a collection of **distinct unordered** objects
- Members of a set are called **elements**
- How to determine a set
  - **Listing**
    - Example: $A = \{1,3,5,7\} = \{7, 5, 3, 1, 3\}$
  - **Description**
    - Example: $B = \{x \mid x=2k+1, 0 < k < 30\}$
  - **Venn Diagrams**
- A Venn diagram provides a graphic view of sets
- Venn diagrams are useful in representing sets and set operations which can be easily and visually identified.
- Various sets are represented by circles inside a big rectangle representing the universal set.



Examples for sets

- $A = \emptyset$ "empty set/null set"
- $A = \{z\}$ Note: $z \in A$, but $z \neq \{z\}$
- $A = \{\{b, c\}, \{c, x, d\}\}$
- $A = \{\{x, y\}\}$

  Note: $\{x, y\} \in A$, but $\{x, y\} \neq \{\{x, y\}\}$
- $A = \{x \mid P(x)\}$

  "set of all x such that P(x)"
- $A = \{x \mid x \in \mathbf{N} \land x > 7\} = \{8, 9, 10, \ldots\}$

  "set builder notation"

Defining sets

Definition: A **set** is an **unordered** collection of objects.

The objects in a set are called the **elements** or **members** of the set S, and we say S **contains** its elements.

We can define a set by directly listing all its elements.

e.g. $S = \{2, 3, 5, 7, 11, 13, 17, 19\}$,

$S = \{CSC1130, CSC2110, ERG2020, MAT2510\}$

After we define a set, the set is a single mathematical object, and it can be an element of another set.

e.g. $S = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$

Defining sets by properties

It is inconvenient, and sometimes impossible, to define a set by listing all its elements.

Alternatively, we can define a set by describing the *properties* that its elements should satisfy.

We use the notation $\{x \in A \mid P(x)\}$

to define the set as the set of elements, *x*, in *A* such that *x* satisfies property *P*.

e.g. $\{x \mid x \text{ is a prime number and } x < 1000\}$

$\{x \mid x \text{ is a real number and } -2 < x < 5\}$

Sets

Curly braces "{" and "}" are used to denote sets.

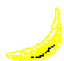
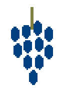
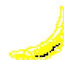
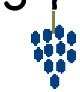
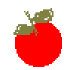
Java note: In Java curly braces denote arrays, a data-structure with inherent ordering. Mathematical sets are ***unordered*** so different from Java arrays. Java arrays require that all elements be of the same type. Mathematical sets don't require this, however. EG:

- { 11, 12, 13 }
- {  ,  ,  }
- {  ,  ,  , 11, Leo }

Sets

A set is defined only by the elements which it contains. Thus repeating an element, or changing the ordering of elements in the description of the set, does not change the set itself:

{ 11, 11, 11, 12, 13 } = { 11, 12, 13 }

{  ,  ,  } = {  ,  ,  }

Examples of sets

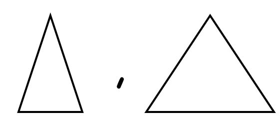
Well known sets:

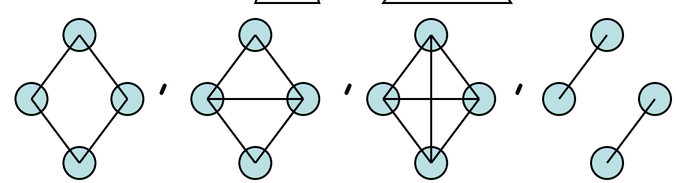
- the set of all real numbers, $\mathbb{R}$
- the set of all complex numbers, $\mathbb{C}$
- the set of all integers, $\mathbb{Z}$
- the set of all positive integers $\mathbb{Z}^{+}$
- the set of natural numbers **N**
- **empty set**, $\emptyset = \{\}$, the set with no elements.

Other examples:

The set of all polynomials with degree at most three: $\{1, x, x^2, x^3, 2x+3x^2, \ldots\}$.

The set of all n-bit strings: {000…0, 000…1, …, 111…1}

The set of all triangles without an obtuse angle: {  ,… }

The set of all graphs with four nodes: {  ,…}

Standard numerical sets

- The natural numbers:
  $\mathbf{N} = \{ 0, 1, 2, 3, 4, \ldots \}$
- The integers:
  $\mathbf{Z} = \{ \ldots -3, -2, -1, 0, 1, 2, 3, \ldots \}$
- The positive integers:
  $\mathbf{Z}^+ = \{1, 2, 3, 4, 5, \ldots \}$
- The real numbers: **R** --contains any decimal number of arbitrary precision
- The rational numbers: **Q** --these are decimal numbers whose decimal expansion repeats

Q: Give examples of numbers in **R** but not **Q**.

Examples for sets

- We are now able to define the set of rational numbers **Q**:
  - **Q** = $\{a/b \mid a \in \mathbf{Z} \wedge b \in \mathbf{Z}^+\}$
  - or
  - **Q** = $\{a/b \mid a \in \mathbf{Z} \wedge b \in \mathbf{Z} \wedge b \neq 0\}$
- And how about the set of real numbers R?
  - **R** = $\{r \mid r \text{ is a real number}\}$
    - That is the best we can do.

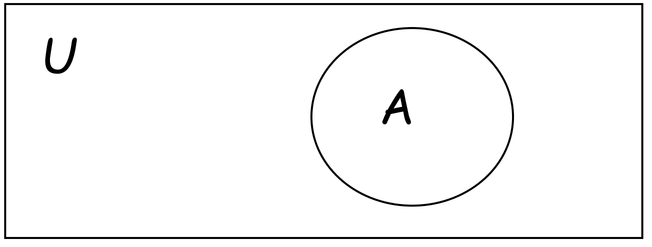
Some important sets

- **Empty set** ∅ = { } has no elements.
  also called null set or void set, and is denoted by:
  - {}
  - $\varnothing$

- **Universal set**: the set of all elements about which we make assertions.
  Examples:
  - U = {all natural numbers}
  - U = {all real numbers}
  - U = {x| x is a natural number and $1 \leq x \leq 10$}

Universal set (continued)

- The ***Universal Set U*** is the set containing all objects for which the discussion is meaningful. (e.g., examining whether “a sock is an integer” is a meaningless exercise.)
- Any set is a subset of the universal set from which it derives its meaning
- In Venn diagrams, the universal set is denoted with a rectangle.



Set builder notation

Up to now sets have been defined using the curly brace notation "{ … }" or descriptively "the set of all natural numbers".

The set builder notation allows for concise definition of new sets. For example

- $\{ x \mid x \text{ is an even integer} \}$
- $\{ 2x \mid x \text{ is an integer} \}$

are equivalent ways of specifying the set of all even integers.

Set builder notation

In general, one specifies a set by writing

$$\{ f(x) \mid P(x) \}$$

where $f(x)$ is a function of x (okay we haven't really gotten to functions yet…) and $P(x)$ is a propositional function of x. The notation is read as

"the set of all elements $f(x)$ such that $P(x)$ holds"

- Stuff between "{" and "|"
  - specifies how elements look
- Stuff between the "|" and "}"
  - gives properties elements satisfy
- Pipe symbol "|" is
  - short-hand for "such that".

Set builder notation. Shortcuts.

- To specify a subset of a pre-defined set, $f(x)$ takes the form $x \in S$.
  For example
  $$\{x \in \mathbf{N} \mid \exists y\,(x = 2y)\}$$
  defines the set of all even natural numbers (assuming universe of reference **Z**).
- When universe of reference is understood, don't need to specify propositional function
- EG: $\{x^3 \mid \}$ or simply $\{x^3\}$ specifies the set of perfect cubes
  $$\{0,1,8,27,64,125, \ldots\}$$
  assuming U is the set of natural numbers.

Set builder notation.
Examples.

Q1: $U = \mathbf{N}$. $\{ x \mid \forall y\, (y \geq x) \} = ?$

Q2: $U = \mathbf{Z}$. $\{ x \mid \forall y\, (y \geq x) \} = ?$

Q3: $U = \mathbf{Z}$. $\{ x \mid \exists y\, (y \in \mathbf{R} \wedge y^2 = x)\} = ?$

Q4: $U = \mathbf{Z}$. $\{ x \mid \exists y\, (y \in \mathbf{R} \wedge y^3 = x)\} = ?$

Q5: $U = \mathbf{R}$. $\{ |x| \mid x \in \mathbf{Z} \} = ?$

Q6: $U = \mathbf{R}$. $\{ |x| \} = ?$

Set builder notation.
Examples.

A1: $U = \mathbf{N}$. $\{ x \mid \forall y\ (y \geq x) \} = \{ 0 \}$

A2: $U = \mathbf{Z}$. $\{ x \mid \forall y\ (y \geq x) \} = \{ \}$

A3: $U = \mathbf{Z}$. $\{ x \mid \exists y\ (y \in \mathbf{R} \ \wedge\ y^2 = x)\}$
$= \{ 0, 1, 2, 3, 4, \ldots \} = \mathbf{N}$

A4: $U = \mathbf{Z}$. $\{ x \mid \exists y\ (y \in \mathbf{R} \ \wedge\ y^3 = x)\} = \mathbf{Z}$

A5: $U = \mathbf{R}$. $\{ |x| \mid x \in \mathbf{Z} \} = \mathbf{N}$

A6: $U = \mathbf{R}$. $\{ |x| \}$ = non-negative reals.

Membership

The order and number of occurrence are not important.

e.g. {a,b,c} = {c,b,a} = {a,a,b,c,b}

The most basic question in set theory is whether an element is in a set.

$x \in A$ — *x* is an element of *A*; *x* is in *A*

$x \notin A$ — *x* is not an element of *A*; *x* is not in *A*

e.g. Recall that Z is the set of all integers. So $7 \in \mathbb{Z}$ and $2/3 \notin \mathbb{Z}$.

Let P be the set of all prime numbers. Then $97 \in P$ and $321 \notin P$

Let Q be the set of all rational numbers. Then $0.5 \in Q$ and $\sqrt{2} \notin Q$

$\in$-Notation

The Greek letter "$\in$" (epsilon) is used to denote that an object is an element of a set. When crossed out "$\notin$" denotes that the object is not an element."

EG: $3 \in S$ reads:

"3 is an element of the set S ".

Q: Which of the following are true:

1. $3 \in \mathbf{R}$
2. $-3 \in \mathbf{N}$
3. $-3 \in \mathbf{R}$
4. $0 \notin \mathbf{Z}^+$
5. $\exists x \quad x \in \mathbf{R} \;\wedge\; x^2 = -5$

$\in$-Notation

A: 1, 3 and 4

1. $3 \in \mathbf{R}$. True: 3 is a real number.
2. $-3 \in \mathbf{N}$. False: natural numbers don't contain negatives.
3. $-3 \in \mathbf{R}$. True: -3 is a real number.
4. $0 \notin \mathbf{Z}^+$. True: 0 isn't positive.
5. $\exists x \quad x \in \mathbf{R} \wedge x^2 = -5$. False: square of a real number is non-neg., so can't be -5.

$\in$-Notation

- $x \in A$ "x is an element of A"
  "x is a member of A"
- $x \notin A$ "x is not an element of A"
- $A = \{x_1, x_2, \ldots, x_n\}$ "A contains…"
- Order of elements is meaningless.
- It does not matter how often the same element is listed.

Cardinality (基数：Size of a Set)

In this course we mostly focus on finite sets.

> Definition: The **size** of a set S, denoted by **|S|**, is defined as the number of elements contained in S.

e.g. if $S = \{2, 3, 5, 7, 11, 13, 17, 19\}$, then $|S|=8$.

if $S = \{CSC1130, CSC2110, ERG2020, MAT2510\}$, then $|S|=4$.

if $S = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$, then $|S|=6$.

Cardinality

The **cardinality** of a set is the number of distinct elements in the set. $|S|$ denotes the cardinality of S.

Q: Compute each cardinality.

1. $|\{1, -13, 4, -13, 1\}|$
2. $|\{3, \{1,2,3,4\}, \emptyset\}|$
3. $|\{\}|$
4. $|\{ \{\}, \{\{\}\}, \{\{\{\}\}\} \}|$

Cardinality

Hint: After eliminating the redundancies just look at the number of top level commas and add 1 (except for the empty set).

A:

1. $|\{1, -13, 4, -13, 1\}| = |\{1, -13, 4\}| = 3$
2. $|\{3, \{1,2,3,4\}, \varnothing\}| = 3$. To see this, set $S = \{1,2,3,4\}$. Compute the cardinality of $\{3, S, \varnothing\}$
3. $|\{\}| = |\varnothing| = 0$
4. $|\{ \{\}, \{\{\}\}, \{\{\{\}\}\} \}| = |\{ \varnothing, \{\varnothing\}, \{\{\varnothing\}\}| = 3$

Cardinality

Definition: The set S is said to be **finite** if its cardinality is a nonnegative integer. Otherwise, S is said to be infinite.

EG: **N**, **Z**, **Z**$^+$, **R**, **Q** are each infinite.

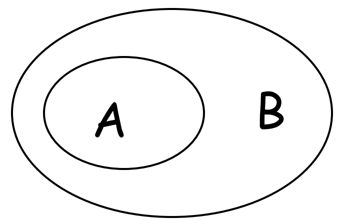
Note: Not all infinities are the same. In fact, **R** will end up having a bigger infinity-type than **N**, but surprisingly, **N** has same infinity-type as **Z**, **Z**$^+$, and **Q**.

Finite and infinite sets

- Finite sets
  - Examples:
    - $A = \{1, 2, 3, 4\}$
    - $B = \{x \mid x \text{ is an integer}, 1 < x < 4\}$
- Infinite sets
  - Examples:
    - $Z = \{\text{integers}\} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
    - $S=\{x \mid x \text{ is a real number and } 1 \leq x \leq 4] = [0\ 4]$

Subset

Definition: Given two sets A and B, we say A is a **subset** of B, denoted by $A \subseteq B$, if every element of A is also an element of B.



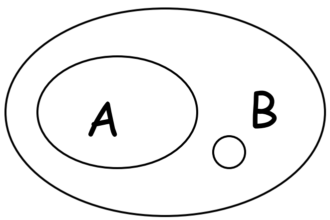
not a subset

- If A={4, 8, 12, 16} and B={2, 4, 6, 8, 10, 12, 14, 16}, then $A \subseteq B$ but $B \not\subseteq A$
- $A \subseteq A$ because every element in A is an element of A.
- $\emptyset \subseteq A$ for any A because the empty set has no elements.
- If A is the set of prime numbers and B is the set of odd numbers, then $A \not\subseteq B$

Fact: If $A \subseteq B$, then |A| <= |B|.

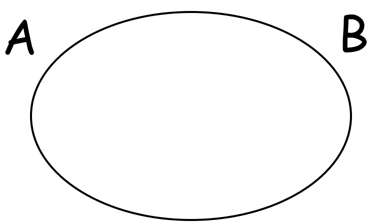
Proper Subset, Equality

Definition: Given two sets A and B, we say A is a **proper subset** of B, denoted by $A \subset B$, if every element of A is an element of B, But there is an element in B that is not contained in A.



Fact: If $A \subset B$, then |A| < |B|.

Definition: Given two sets A and B, we say A = B if $A \subseteq B$ and $B \subseteq A$.



Fact: If A = B, then |A| = |B|.

$\subseteq$-Notation

Definition: A set S is said to be a **subset** of the set T iff every element of S is also an element of T. This situation is denoted by

$$S \subseteq T$$

A synonym of "subset" is "contained by".

Definitions are often just a means of establishing a logical equivalence which aids in notation. The definition above says that:

$$S \subseteq T \qquad \Leftrightarrow \qquad \forall x \ (x \in S) \rightarrow (x \in T)$$

We already had all the necessary concepts, but the "$\subseteq$" notation saves work.

$\subset$-Notation

When "$\subset$" is used instead of "$\subseteq$", proper containment is meant. A subset S of T is said to be a **proper subset** if S is not equal to T.

Notationally:

$$S \subset T \quad \Leftrightarrow \quad S \subseteq T \wedge \exists x \ (x \notin S \wedge x \in T)$$

Q: What algebraic symbol is $\subset$ reminiscent of?

$\subset$-Notation

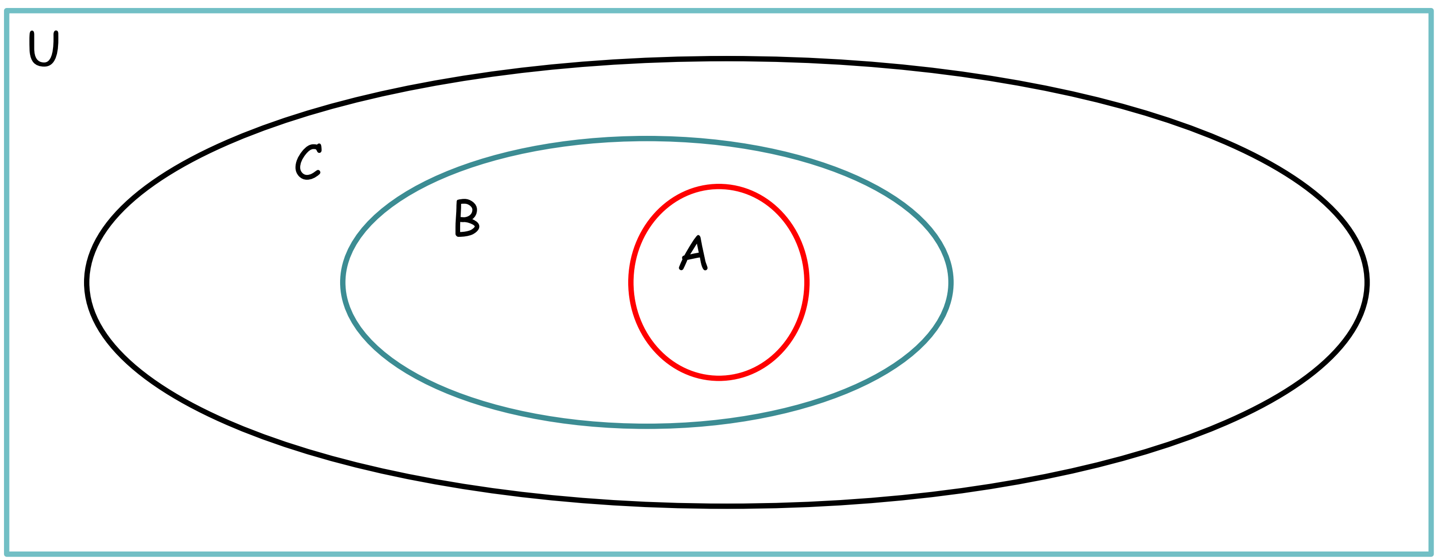
A: $\subset$ is to $\subseteq$, as $<$ is to $\leq$.

Subsets

- $A \subseteq B$
  - "A is a subset of B"
- $A \subseteq B$
  - iff every element of A is also an element of B.
- We can completely formalize this:
  - $A \subseteq B \Leftrightarrow \forall x\, (x \in A \rightarrow x \in B)$
- **Examples**:
  - $A = \{3, 9\}$, $B = \{5, 9, 1, 3\}$, $A \subseteq B$? True
  - $A = \{3, 3, 3, 9\}$, $B = \{5, 9, 1, 3\}$, $A \subseteq B$? True
  - $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, $A \subseteq B$? False

Subsets

- Useful rules:
- $A = B \Leftrightarrow (A \subseteq B) \land (B \subseteq A)$
- $(A \subseteq B) \land (B \subseteq C) \Rightarrow A \subseteq C$ (next Venn Diagram)



Subsets

- Useful rules:
  - $\emptyset \subseteq A$ for any set A
  - $A \subseteq A$ for any set A
- Proper subsets:
  - $A \subset B$ "A is a proper subset of B"
  - $A \subset B \Leftrightarrow \forall x\, (x \in A \rightarrow x \in B) \wedge \exists x\, (x \in B \wedge x \notin A)$
  - or
  - $A \subset B \Leftrightarrow \forall x\, (x \in A \rightarrow x \in B) \wedge \neg\forall x\, (x \in B \rightarrow x \in A)$

Subset examples

Q: Which of the following are true:

1. $\mathbf{N} \subset \mathbf{R}$
2. $\mathbf{Z} \subseteq \mathbf{N}$
3. $-3 \subseteq \mathbf{R}$
4. $\{1,2\} \not\in \mathbf{Z}^+$
5. $\varnothing \subseteq \varnothing$
6. $\varnothing \subset \varnothing$

Subset examples

A: 1, 4 and 5

1. $\mathbf{N} \subset \mathbf{R}$. All natural numbers are real.
2. $\mathbf{Z} \subseteq \mathbf{N}$. Negative numbers aren't natural.
3. $-3 \subseteq \mathbf{R}$. Nonsensical. -3 is not a subset but an element! (This could have made sense if we viewed -3 as a set –which in principle is the case– in this case the proposition is **false**).
4. $\{1,2\} \notin \mathbf{Z}^+$. This actually makes sense. The set {1,2} is an object in its own right, so could be an element of some set; however, {1,2} is not a number, therefore is not an element of **Z**.
5. $\varnothing \subseteq \varnothing$. Any set contains itself.
6. $\varnothing \subset \varnothing$. No set can contain itself properly.

Set equality

- Sets A and B are equal if and only if they contain exactly the same elements.
- Examples:
  - $A = \{9, 2, 7, -3\}$, $B = \{7, 9, -3, 2\}$: $A = B$
  - $A = \{dog, cat, horse\}$, $B = \{cat, horse,$
  - $squirrel, dog\}$: $A \neq B$
  - $A = \{dog, cat, horse\}$, $B = \{cat, horse, dog, dog\}$: $A = B$

Power set （幂集）

Definition: The **power set** of S is the set of all subsets of S.
Denote the power set by P(S) or by 2^s .
The latter weird notation comes from the following.
Lemma: $| 2^s | = 2^{|s|}$

Power set

To understand the previous fact consider

$$S = \{1,2,3\}$$

Enumerate all the subsets of S :

0-element sets: $\{\}$	1
1-element sets: $\{1\}, \{2\}, \{3\}$	+3
2-element sets: $\{1,2\}, \{1,3\}, \{2,3\}$	+3
3-element sets: $\{1,2,3\}$	+1

Therefore: $|\, 2^S \,| = 8 = 2^3 = 2^{|S|}$

Power set

- Cardinality of power sets: $| 2^S | = 2^{|S|}$
- Imagine each element in S has an "on/off" switch
- Each possible switch configuration in A corresponds to one element in 2^S
- For 3 elements in S, there are $2 \times 2 \times 2 = 8$ elements in 2^S

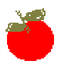
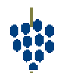
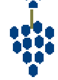
S	1	2	3	4	5	6	7	8
x	x	x	x	x	x	x	x	x
y	y	y	y	y	y	y	y	y
z	z	z	z	z	z	z	z	z

Power set

- 2^S or $P(S)$ "power set of S"
- $2^S = \{X \mid X \subseteq S\}$ (contains all subsets of S)
- Examples:
  - $S = \{x, y, z\}$
  - $2^S = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x,y\}, \{x,z\}, \{y,z\}, \{x,y,z\}\}$
  - $S = \emptyset$
  - $2^S = \{\emptyset\}$
- Note: $|S| = 0$, $|2^S| = 1$

Ordered *n*-tuples (n元有序组)

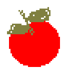
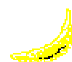
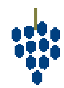
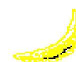
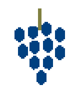
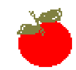
Notationally, *n*-tuples look like sets except that curly braces are replaced by parentheses:

- (11, 12) –a 2-tuple aka *ordered pair*
- ( ,  , ) –a 3-tuple
- ( ,  ,  , 11, Leo) –a 5-tuple

Java: *n* -tuples are similar to Java arrays "{…}", except that type-mixing isn't allowed in Java.

Ordered *n*-tuples

As opposed to sets, **repetition** and **ordering** do matter with *n*-tuples.

- $(11, 11, 11, 12, 13) \neq (11, 12, 13)$
- ( ,  , ) ≠ ( ,  , )

Ordered *n*-tuples

- The ordered n-tuple $(a_1, a_2, a_3, \ldots, a_n)$ is an ordered collection of objects.
- Two ordered n-tuples $(a_1, a_2, a_3, \ldots, a_n)$ and $(b_1, b_2, b_3, \ldots, b_n)$ are equal if and only if they contain exactly the same elements in the same order
  - i.e. $a_i = b_i$ for $1 \le i \le n$

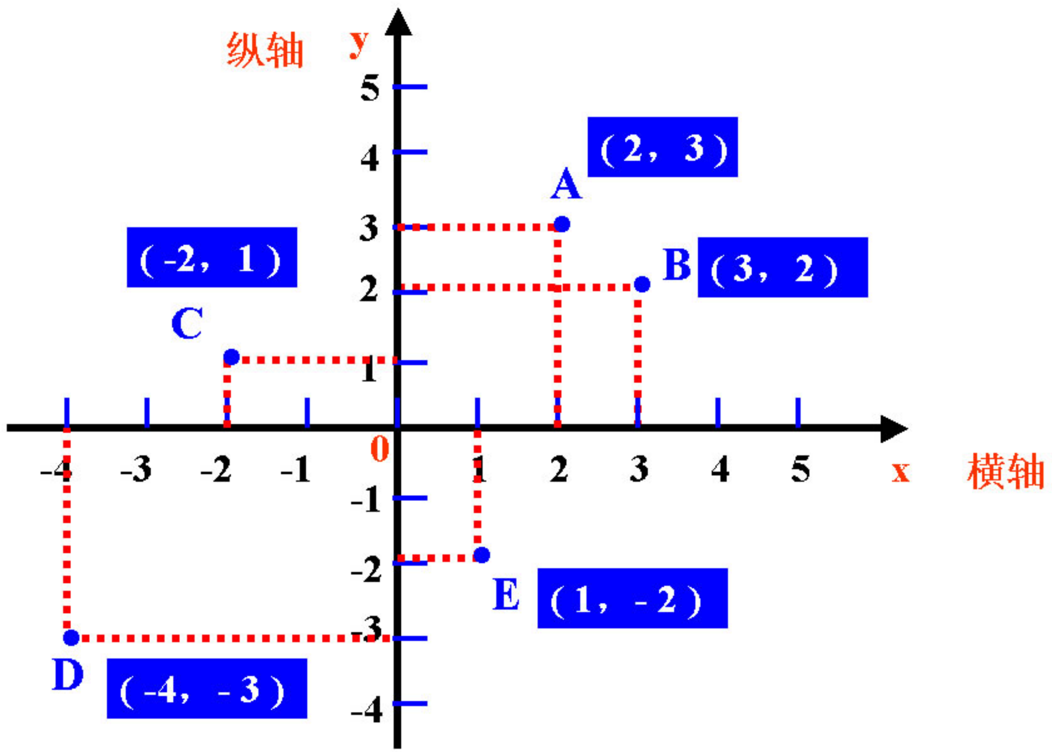
Cartesian product (笛卡儿积)

The most famous example of 2-tuples are points in the **Cartesian** plane $\mathbf{R}^2$. Here ordered pairs (x,y) of elements of $\mathbf{R}$ describe the coordinates of each point. We can think of the first coordinate as the value on the x-axis and the second coordinate as the value on the y-axis.

Definition: The **Cartesian product** of two sets A and B –denoted by $A \times B$ – is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.

Q: Describe $\mathbf{R}^2$ as the Cartesian product of two sets.

Cartesian plane



$\{2,3\}=\{3,2\}$

$(2,3) \neq (3,2)$

Cartesian product

A: $\mathbf{R}^2 = \mathbf{R}\times\mathbf{R}$. I.e., the Cartesian plane is formed by taking the Cartesian product of the x-axis with the y-axis.

One can generalize the Cartesian product to several sets simultaneously.

Q: If $A = \{1,2\}$, $B = \{3,4\}$, $C = \{5,6,7\}$

what is $A \times B \times C$?

Cartesian product

A:

$A = \{1,2\}$, $B = \{3,4\}$, $C = \{5,6,7\}$

$A \times B \times C =$

$\{ (1,3,5), (1,3,6), (1,3,7),$
$(1,4,5), (1,4,6), (1,4,7),$
$(2,3,5), (2,3,6), (2,3,7),$
$(2,4,5), (2,4,6), (2,4,7) \}$

Lemma: The cardinality of the Cartesian product is the product of the cardinalities:

$$| A_1 \times A_2 \times \ldots \times A_n | = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_n|$$

Q: What does $\emptyset \times S$ equal?

Cartesian product

A: From the lemma:

$$|\emptyset\times S| = |\emptyset|\cdot|S| = 0\cdot|S| = 0$$

There is only one set with no elements –the empty set– therefore, $\emptyset\times S$ must be the empty set $\emptyset$.

One can also check this directly from the definition of the Cartesian product.

The End