Set Theory

Relations I

Content

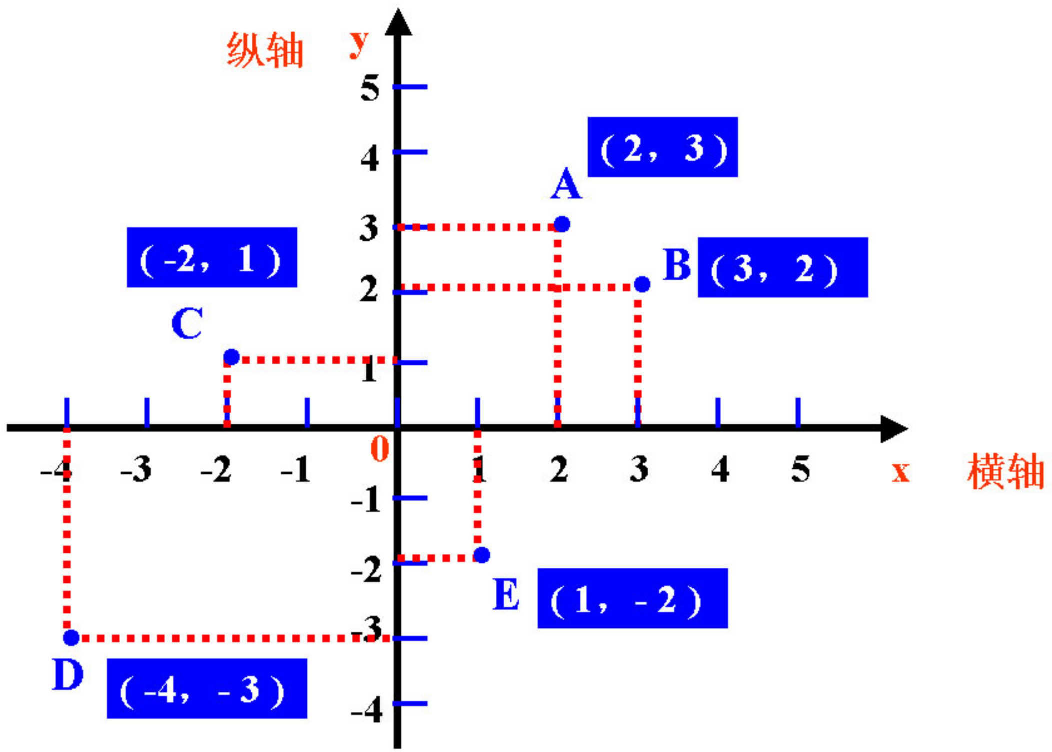
- Cartesian Product, Relations and Binary Relations
- Properties of relations
  - reflexive (自反) , irreflexive (反自反), symmetric (对称), antisymmetric (反对称)，transitive (传递)
- Representing Binary Relations
- Operations of relations
- Closure (闭包)

Cartesian Product, Relations and Binary Relations

Cartesian product (笛卡尔积)

- If $A_1, A_2, \ldots, A_m$ are nonempty sets, then the **Cartesian Product** of them is the set of all ordered m-tuples $(a_1, a_2, \ldots, a_m)$, where $a_i \in A_i$, $i = 1, 2, \ldots m$.
- Denoted $A_1 \times A_2 \times \ldots \times A_m = \{(a_1, a_2, \ldots, a_m) \mid a_i \in A_i, i = 1, 2, \ldots m\}$

Cartesian Plane



{2,3}={3,2}

$(2,3) \neq (3,2)$

Cartesian product example

- If $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$, find $A \times B$
- $A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c), (3,a), (3,b), (3,c)\}$

Using matrices to denote Cartesian product

- For Cartesian Product of two sets, you can use a matrix to find the sets.
- Example: Assume $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$. The table below represents $A \times B$.

	a	b	c
1	(1, a)	(1, b)	(1, c)
2	(2, a)	(2, b)	(2, c)
3	(3, a)	(3, b)	(3, c)

Cardinality of Cartesian product

The cardinality of the Cartesian Product equals the product of the cardinality of all of the sets:

$$| A_1 \times A_2 \times \ldots \times A_m | = | A_1 | \cdot | A_2 | \cdot \ldots \cdot | A_m |$$

Subsets of the Cartesian product

- Many of the results of operations on sets produce subsets of the Cartesian product set
- Relational database
  - Each column in a database table can be considered as a set
  - Each row is an m-tuple of the elements from each column or set
  - No two rows should be alike

	a	b	c
1	(1, a)	(1, b)	(1, c)
2	(2, a)	(2, b)	(2, c)
3	(3, a)	(3, b)	(3, c)

Property of Cartesian product

- $A=\emptyset$ or $B = \emptyset$, $A \times B = \emptyset$

- $A \times B \neq B \times A$
- $(A \times B) \times C \neq A \times (B \times C)$

- $A \times (B \cup C) = (A \times B) \cup (A \times C)$
- $A \times (B \cap C) = (A \times B) \cap (A \times C)$
- $(A \cup B) \times C = (A \times C) \cup (B \times C)$
- $(A \cap B) \times C = (A \times C) \cap (B \times C)$

- $A, B, C, D \neq \emptyset$, then
  - $A \times B \subseteq C \times D$ iff $A \subseteq C \wedge B \subseteq D$
- $C \neq \emptyset$
  - $A \subseteq B$ iff $A \times C \subseteq B \times C$ iff $C \times A \subseteq C \times B$

Example 1

A relational database with *schema* (*图表*):

1	*Name*
2	*Favorite Food*
3	*Favorite Color*
4	*Occupation*

1	Kate Winslet	Leonardo DiCaprio
2	Apple	Pear
3	Purple	Green
4	Movie star	Movie star

…etc.

Relations: Subsets of Cartesian products

A:

1. Database $\subseteq$ {Names}$\times${Foods}$\times${Colors}$\times${Jobs}

Relations: Subsets of Cartesian products

Definition: Let A_1, A_2, … , A_n be sets. An n-**ary relation** on these sets (in this order) is a subset of $A_1 \times A_2 \times \ldots \times A_n$.

Most of the time we consider n = 2 in which case have a **binary relation** and also say the relation is "**from** A_1 to A_2".

With this terminology, all functions are relations, but not vice versa.

Q: What additional property ensures that a relation is a function?

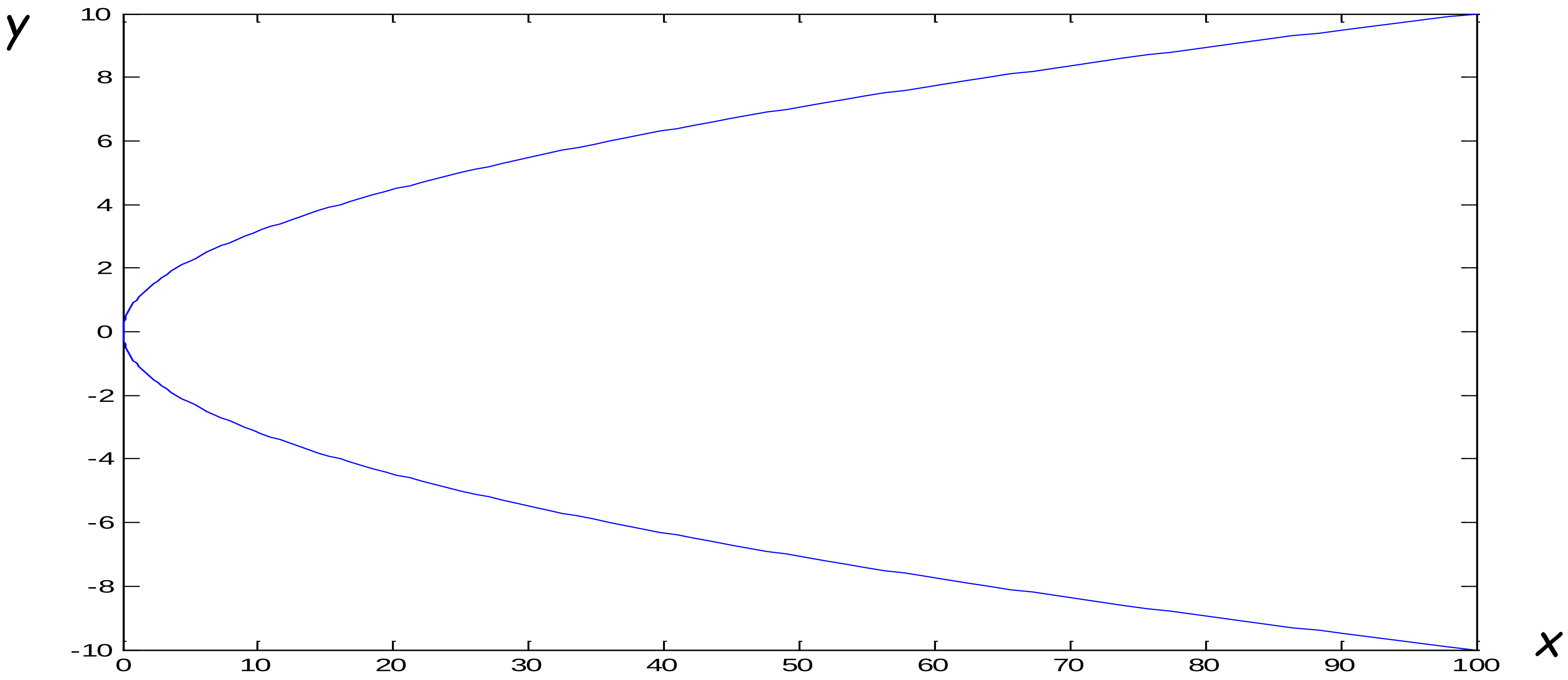
Relations vs Functions

A: Vertical line test : For every a in A_1 there is a unique b in A_2 for which (a,b) is in the relation. Here A_1 is thought of as the x-axis, A_2 is the y-axis and the relation is represented by a graph.

Q: How can this help us visualize the square root function:

Graph illustration

A: Visualize both branches of solution to $x = y^2$ as the graph of a relation:



Functions as specific relations

- Recall that a function f from a set A to a set B assigns **exactly one** element of B to each element of A
- The graph of f is the set of ordered pairs (a, b) such that b=f(a)
- Because the graph of f is a subset of A x B, it is a relation from A to B
- Furthermore, the graph of a function has the property that every element of A is the first element of exactly one ordered pair of the graph

Functions as specific relations

- Conversely, if R is a relation from A to B such that every element in A is the first element of exactly one ordered pair of R, then a function can be defined with R as its graph
- A relation can be used to express one-to-many relationship between the elements of the sets A and B where an element of A may be related to more than one element of B
- A function represents a relation where exactly one element of B is related to each element of A
- Relations are a generalization of functions

Binary relations

- Given two sets A and B, its Cartesian product $A \times B$ is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$
  - In symbols $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$
- **Definition**: Let A and B be sets. A **binary relation** R from a set A to a set B is a subset of the Cartesian product $A \times B$.
- In other words, for a binary relation R we have $R \subseteq A \times B$. We use the notation aRb to denote that $(a, b) \in R$ and $a \not R b$ to denote $(a, b) \notin R$.

Binary relations

- When (a, b) belongs to R, a is said to be related to b by R.
- **Example**: A = {1, 2, 3} and B = {a, b}
  - R = {(1, a), (1, b), (2, b), (3, a)} is a relation between A and B. 3 is related to a by R.
- **Example**: Let P be a set of people, C be a set of cars, and D be the relation describing which person drives which car(s).

Binary relations

- P = {Carl, Suzanne, Peter, Carla},
- C = {Mercedes, BMW, tricycle}
- D = {(Carl, Mercedes), (Suzanne, Mercedes), (Suzanne, BMW), (Peter, tricycle)}
- This means that Carl drives a Mercedes, Suzanne drives a Mercedes and a BMW, Peter drives a tricycle, and Carla does not drive any of these vehicles.

Domain and range

- Given a relation R from X to Y,
- The **domain** of R is the set
  - $Dom(R) = \{ x \in X \mid (x, y) \in R \text{ for some } y \in Y\}$
- The **range** of R is the set
  - $Rng(R) = \{ y \in Y \mid (x, y) \in R \text{ for some } x \in X\}$
- The **field** of R is the set
  - $FLD(R) = Dom(R) \cup Rng(R)$
- **Example**:
  - if $X = \{1, 2, 3\}$ and $Y = \{a, b\}$
  - $R = \{(1,a), (1,b), (2,b)\}$
  - Then: $Dom(R)= \{1, 2\}$, $Rng(R) = \{a, b\}$

Domain and range

- **Example**:
  - Let $X = \{1, 3, 4, 7, 9, 12, 16\}$ and
  - $Y = \{1, 2, 4, 8, 9\}$
  - Define $R_1 = \{(x, y) \mid x \in X, y \in Y, \text{and } x = y^2\}$
  - Then $R_1 = \{(?, ?), \dots\}$
  - Define $R_2 = \{(x, y) \mid x \in X, y \in Y, \text{and } x^2 = y\}$
  - Then $R_2 = ?$

Domain and range

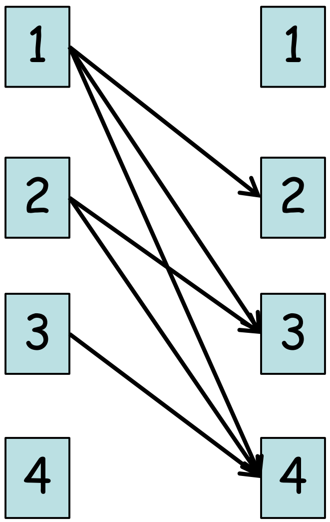
- **Example**:
  - Let $X = \{1, 3, 4, 7, 9, 12, 16\}$ and
  - $Y = \{1, 2, 4, 8, 9\}$
  - Define $R_1 = \{(x, y) \mid x \in X, y \in Y, \text{ and } x = y^2\}$
  - Then $R_1 = \{(1, 1), (4, 2), (16, 4)\}$
  - Define $R_2 = \{(x, y) \mid x \in X, y \in Y, \text{ and } x^2 = y\}$
  - Then $R_2 = \{(1, 1), (3, 9)\}$

Binary relations on a set

- **Definition**: A relation on the set A is a relation from A to A.
- In other words, a relation on the set A is a subset of $A \times A$.
- Example: Let $A = \{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a < b\}$?

Binary relations on a set

- Solution: R={(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)}



R	1	2	3	4
1		x	x	x
2			x	x
3				x
4				

Binary relations on a set

Siblinghood. A = {people}

- $\emptyset$ is the empty relation on A
- $E_X = A \times A$ is the universal relation on A
- $I_x = \{(a, a) \mid a \in A\}$ is the identity relation on A

Cardinality of binary relations on a set

- How many different relations can we define on a set A with n elements?

How many elements are in $A \times A$?
Answer: There are n^2 elements in $A \times A$.

So, how many subsets (= relations on A) does $A \times A$ have?

Answer:The number of subsets that we can form out of a set with m elements is 2^m.
Therefore, 2^{n^2} subsets can be formed out of $A \times A$.

Answer: We can define 2^{n^2} different relations on A.

Representing Binary Relations

Representing binary relations

- We have many ways of representing binary relations. We now take a closer look at two ways of representation: **Boolean (zero-one) matrices** and **directed graphs**.

Representing binary relations - Boolean matrices

- If R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$, then R can be represented by the Boolean matrix $M_R = [m_{ij}]$ with $m_{ij} = 1$, if $(a_i, b_j) \in R$, and $m_{ij} = 0$, if $(a_i, b_j) \notin R$.
- **Boolean matrices** are 2 dimensional tables consisting of 0's and 1's.
- Note that for creating this matrix we first need to list the elements in A and B in a particular, but arbitrary order.

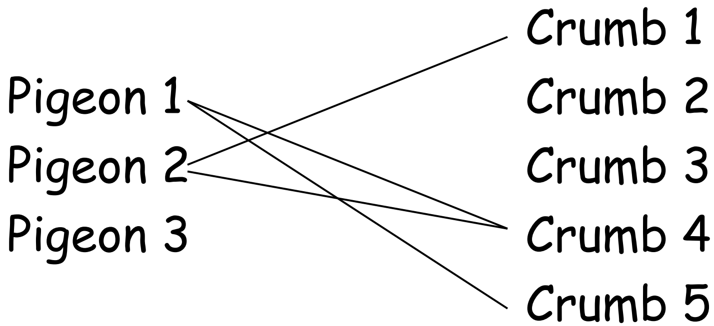
For a relation R from A to B define matrix M_R by:

- Rows – one for each element of A
- Columns – one for each element of B
- Value at i th row and j th column is
  - 1 if i th element of A is related to j
  - 0 otherwise

	b 1	b 2	b 3	b 4	b 5
a1	0	0	0	1	1
a2	1	0	0	1	0
a3	0	0	0	0	0

The matrix of a relation

Q: How is the pigeon-crumb relation represented?



M_R =

	C1	C2	C3	C4	C5
P1	0	0	0	1	1
P2	1	0	0	1	0
P3	0	0	0	0	0

Q: What's M_R's shape for a relation *on A*?

A: Square.

The matrix of a relation

- Example:
  - Let $X = \{1, 2, 3\}$, $Y = \{a, b, c, d\}$
  - Let $R = \{(1,a), (1,d), (2,a), (2,b), (2,c)\}$
  - The matrix M_R of the relation R is

$M_R =$

	a	b	c	d
1	1	0	0	1
2	1	1	1	0
3	0	0	0	0

The matrix of a relation

- If R is a relation from a set X to itself and M_R is the matrix of R, then M_R is a square matrix.
- Example: Let X = {2,3,4,5} and R = {(x,y) | x+y divides by 3}. Then :

M_R =

	2	3	4	5
2				
3				
4				
5				

fill it

The matrix of a relation

- Example: Let X = {2,3,4,5} and R = {(x,y) | x+y divides by 3}. Then :

M_R =

	2	3	4	5
2	0	0	1	0
3	0	1	0	0
4	1	0	0	1
5	0	0	1	0

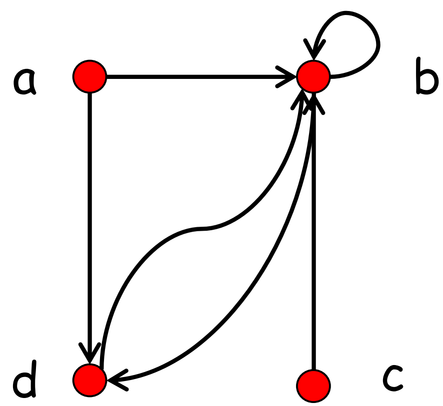
Representing relations - Digraphs

Another way of representing a relation R on a set A is with a **digraph** which stands for "**di**rected **graph**". The set A is represented by **nodes** (or **vertices**) and whenever aRb occurs, a **directed edge** (or **arrow**) a→b is created. Self pointing edges (or **loops**) are used to represent aRa.

- **Definition**: A directed graph, or digraph, consists of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs).
- The vertex a is called the initial vertex of the edge (a, b), and the vertex b is called the terminal vertex of this edge.
- We can use arrows to display graphs.

Representing relations - Digraphs

- **Example**:
  - Display the digraph with V = {a, b, c, d}, E = {(a, b), (a, d), (b, b), (b, d), (c, a), (c, b), (d, b)}.



- An edge of the b called a loop form (b, b) is loop.

Representing relations - Digraphs

- Obviously, we can represent any relation R on a set A by the digraph with A as its vertices and all pairs $(a, b) \in R$ as its edges.
- Vice versa, any digraph with vertices V and edges E can be represented by a relation on V containing all the pairs in E.
- This one-to-one correspondence between relations and digraphs means that any statement about relations also applies to digraphs, and vice versa.

Properties of Relations

Properties of relations

Special properties for relation on a set A:

- **Reflexive** : every element is self-related. I.e. aRa for all a $\in A$
- **Symmetric** : order is irrelevant. I.e. for all a,b $\in A$ aRb iff bRa
- **Transitive** : when a is related to b and b is related to c, it follows that a is related to c. I.e. for all a,b,c $\in A$ aRb and bRc implies aRc

Q: Which of these properties hold for:

1) "Siblinghood" 2) "<" 3) "$\leq$"

Properties of relations

A:

1) "Siblinghood": not reflexive (I'm not my brother), is symmetric, is transitive.
2) "<": not reflexive, not symmetric, is transitive
3) "≤": is reflexive, not symmetric, is transitive

Definition: An **equivalence relation** is a relation on *A* which is **reflexive**, **symmetric** and **transitive**.

Generalizes the notion of "equals".

(Will be discussed in next slide)

Properties of relations - Warnings

- Warnings: there are additional concepts with confusing names
  - **Antisymmetric**: not equivalent to “not symmetric”. Meaning: it’s never the case for $a \neq b$ that both aRb and bRa hold.
  - **Asymmetric**: also not equivalent to “not symmetric”. Meaning: it’s never the case that both aRb and bRa hold.
  - **Irreflexive**: not equivalent to “not reflexive”. Meaning: it’s never the case that aRa holds.

Definition: *A* **partial order relation** on *A* which is **reflexive**, **antisymmetric** and **transitive**.

(Also will be discussed in next slide)

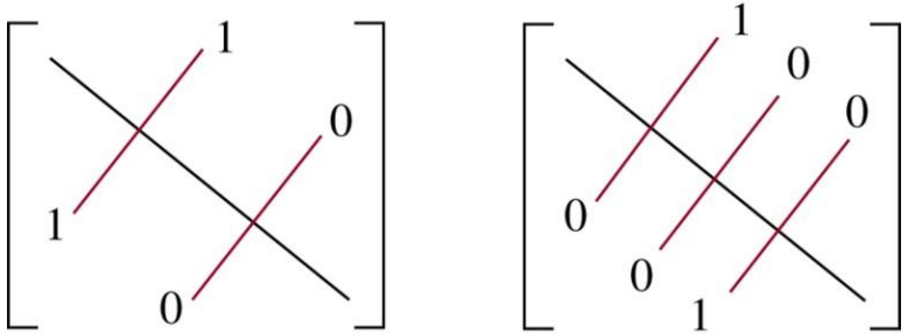
Reflexive

- The matrix of a relation on a set, which is a square matrix, can be used to determine whether the relation has certain properties
- Recall that a relation R on A is reflexive if $(a,a) \in R$. Thus R is reflexive if and only if $(a_i,a_i) \in R$ for $i=1,2,\ldots,n$
- Hence R is reflexive iff $m_{ii}=1$, for $i=1,2,\ldots, n$.
- R is reflexive if all the elements on the main diagonal of M_R are 1

$$\begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & \cdot & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix}$$

Symmetric

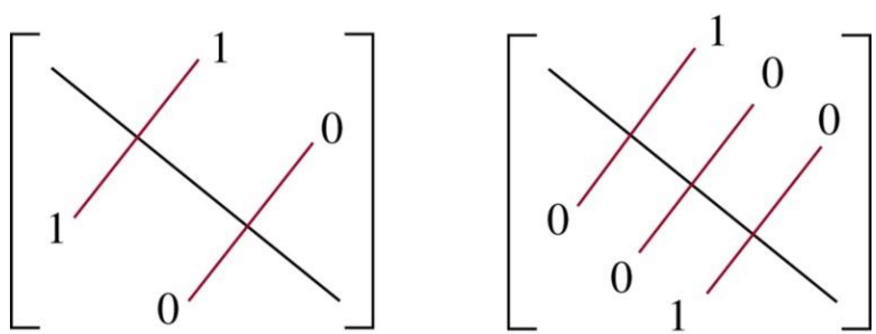
- The relation R is symmetric if $(a,b) \in R$ implies that $(b,a) \in R$
- In terms of matrix, R is symmetric if and only $m_{ji}=1$ whenever $m_{ij}=1$, i.e., $M_R=(M_R)^T$
- R is symmetric iff M_R is a symmetric matrix



(a) Symmetric (b) Antisymmetric

Antisymmetric

- The relation R is antisymmetric if (a,b)∈R and (b,a)∈R imply a=b
- The matrix of an antisymmetric relation has the property that if m_{ij}=1 with i≠j, then m_{ji}=0
- In other words, either m_{ij}=0 or m_{ji}=0 when i≠j



(a) Symmetric (b) Antisymmetric

Symmetric, Asymmetric and Antisymmetric

- 对称的(symmetric)：对所有的aRb,都有bRa
- 非对称的(not symmetric)：存在一些aRb,满足$b \not R a$
- 不对称的(asymmetric)：对所有的aRb,都有$b \not R a$
- 非不对称的(not asymmetric)：存在一些aRb,满足bRa
- 反对称的(antisymmetric)：对所有的aRb和bRa,都有a=b
- 非反对称的(not antisymmetric)：存在一些a≠b,满足aRb和bRa

可见：

- (1) asymmetric→not symmetric,而not symmetric不能得出asymmetric
- (2) asymmetric→antisymmetric,而antisymmetric不能得出asymmetric

举例1：A={1,2,3,4},R={(1,2),(2,2),(3,4),(4,1)},则：

- R是非对称的(not symmetric),因为(1,2)属于R,而(2,1)不属于R;
- R是非不对称的(not asymmetric),因为(2,2)属于R
- R是反对称的(antisymmetric),因为对于任意a≠b,不存在(a,b)和(b,a)都属于R

Useful summary

- Let R be a relation on a set *A*, i.e. R is a subset of the Cartesian product $A \times A$
  - R is **reflexive** if $(x, x) \in R$ for every $x \in A$
  - R is **irreflexive** if $(x, x) \notin R$ for every element $x \in A$.
  - R is **symmetric** if for all $x, y \in A$ such that $(x,y) \in R$ then $(y, x) \in R$
  - R is **antisymmetric** if for all $x, y \in A$ such that $x \neq y$, if $(x, y) \in R$ then $(y, x) \notin R$
  - R is **transitive** if $(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$

Example

- Suppose that the relation R on a set is represented by the matrix

Is R reflexive, symmetric or antisymmetric?

- As all the diagonal elements are 1, R is reflexive. As M_R is symmetric, R is symmetric. It is also easy to see R is not antisymmetric

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Cardinality of reflexive relations

- **Example**:
  - How many different **reflexive** relations can be defined on a set A containing n elements?
- **Solution**:
  - Relations on R are subsets of $A \times A$, which contains n^2 elements.
  - Therefore, different relations on A can be generated by choosing different subsets out of these n^2 elements, so there are 2^{n^2} relations.

Cardinality of reflexive relations

- A **reflexive** relation, however, must contain the n elements (a, a) for every $a \in A$.
- Consequently, we can only choose among $n^2 - n = n(n - 1)$ elements to generate reflexive relations, so there are $2^{n(n-1)}$ of them.

Properties of relations

- Through these properties, we can classify relations.
- Example:
  - Are the following relations on {1,2,3,4} reflexive?
  - R = {(1, 1), (1, 2), (2, 3), (3, 3), (4, 4)} NO
  - R = {(1, 1), (2, 2), (2, 3), (3, 3), (4, 4)} YES
  - R = {(1, 1), (2, 2), (3, 3)} NO

Properties of relations

- Are the following relations on {1, 2, 3, 4} symmetric or antisymmetric?
  - R = {(1, 2), (2, 1), (3, 3), (4, 4)} symmetric
  - R = {(1, 1)} sym. and antisym.
  - R = {(1, 3), (3, 2), (2, 1)} antisym.
  - R = {(4, 4), (3, 3), (1, 4)} antisym.

Properties of relations

- Are the following relations on {1, 2, 3, 4} transitive?

- R = {(1, 1), (1, 2), (2, 2), (2, 1), (3, 3)} Yes
- R = {(1, 3), (3, 2), (2, 1)} NO
- R = {(2, 4), (4, 3), (2, 3), (4, 1)} NO

Visualization

For relations R on a set A.

Q: What does M_R look like when R is reflexive?

Visualization

A: Reflexive. Upper-Left corner to Lower-Right corner diagonal is all 1's. EG:

$$M_R = \begin{pmatrix} 1 & * & * & * \\ * & 1 & * & * \\ * & * & 1 & * \\ * & * & * & 1 \end{pmatrix}$$

Q: How about if R is symmetric?

Visualization

A: A **symmetric matrix**. I.e., flipping across diagonal does not change matrix. EG:

$$M_R = \begin{pmatrix} * & 0 & 1 & 1 \\ 0 & * & 0 & 0 \\ 1 & 0 & * & 1 \\ 1 & 0 & 1 & * \end{pmatrix}$$

Visualization

- What do we know about the matrices representing symmetric relations?
  - These matrices are symmetric, that is, $M_R=(M_R)^T$.

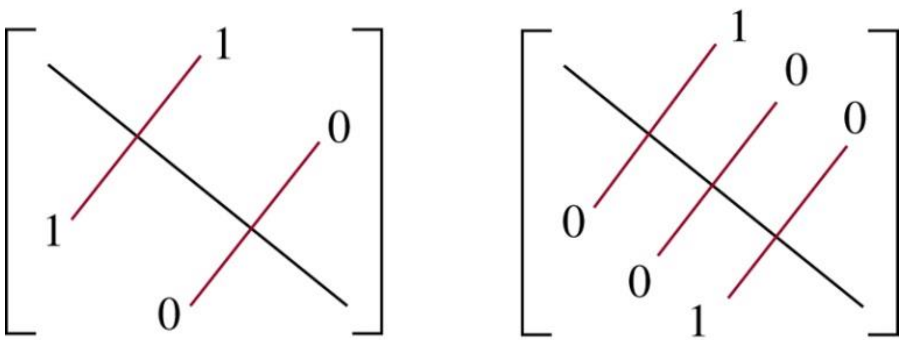
$$M_R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

symmetric matrix,
symmetric relation.

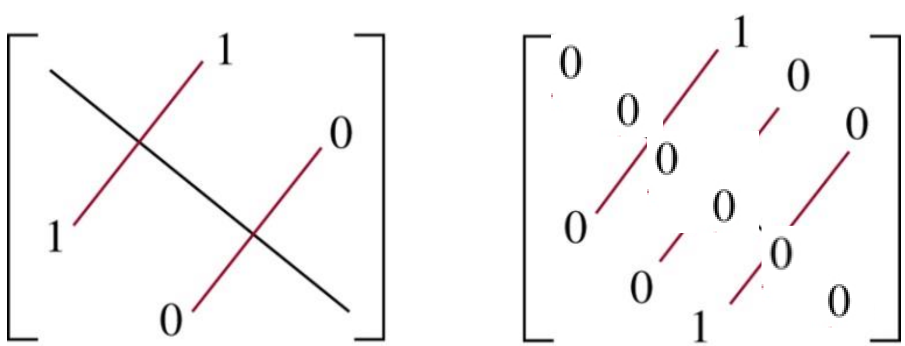
$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

nonsymmetric matrix,
nonsymmetric relation.

Visualization



(a) Symmetric (b) Antisymmetric



(a) Symmetric (b) Asymmetric

Operations on Relations

Operators on Relations

- Operators on Sets
- Inversion
- Composite

Operations on relations - Set operations

- Since a relation is also a set, we can apply union and intersection of two relations, respectively.

- Suppose R and S are relations on a set A represented by M_R and M_S
- The matrices representing the union and intersection of these relations are

$M_{R \cup S} = M_R \vee M_S$

$M_{R \cap S} = M_R \wedge M_S$

Relations as subsets: $\cup$, $\cap$, $\oplus$, $-$, ^{c}

Because relations are just subsets, all the usual set theoretic operations are defined between relations which belong to the same Cartesian product.

Q: Suppose we have relations on {1,2} given by R = {(1,1), (2,2)}, S = {(1,1),(1,2)}. Find:

1. The union $R \cup S$
2. The intersection $R \cap S$
3. The symmetric difference $R \oplus S$
4. The difference $R - S$
5. The complement R^c

Relations as subsets:
$\cup$, $\cap$, $\oplus$, $-$, ^{c}

A: $R = \{(1,1),(2,2)\}$, $S = \{(1,1),(1,2)\}$

1. $R \cup S = \{(1,1),(1,2),(2,2)\}$
2. $R \cap S = \{(1,1)\}$
3. $R \oplus S = \{(1,2),(2,2)\}$.
4. $R - S = \{(2,2)\}$.
5. $R^c = \{(1,2),(2,1)\}$

Examples

- Example 1:
  - Let $R_1 = \{(1, 2), (3, 6)\}$
  - Let $R_2 = \{(2, u), (6, v)\}$
  - Then $R_2 \bullet R_1 = \{(1, u), (3, v)\}$
  - Relations are sets, and therefore, we can apply the usual set operations to them.
  - If we have two relations R_1 and R_2, and both of them are from a set A to a set B, then we can combine them to $R_1 \cup R_2$, $R_1 \cap R_2$, or $R_1 - R$
  - In each case, the result will be another relation from A to B.

Examples

- Let the relations R and S be represented by the matrices

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad M_s = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

- What are the matrices representing R $\cup$ S and R $\cap$ S?
- **Solution**: These matrices are given by

$$M_{R\cup S} = M_R \vee M_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad M_{R\cap S} = M_R \wedge M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Inverting Relations

Relational inversion amounts to just reversing all the tuples of a binary relation.

Definition: If R is a relation from A to B, the ***inversion*** of R is the relation R^{-1} from B to A defined by setting $bR^{-1}a$ if and only aRb.

Q: Suppose R is defined on **N** by: xRy iff $y = x^2$.

What is the inverse R^{-1}?

Inverting Relations

A: xRy iff $y = x^2$.

R is the square function so R^{-1} is sqaure root: i.e. the union of the two square-root branches. I.e:

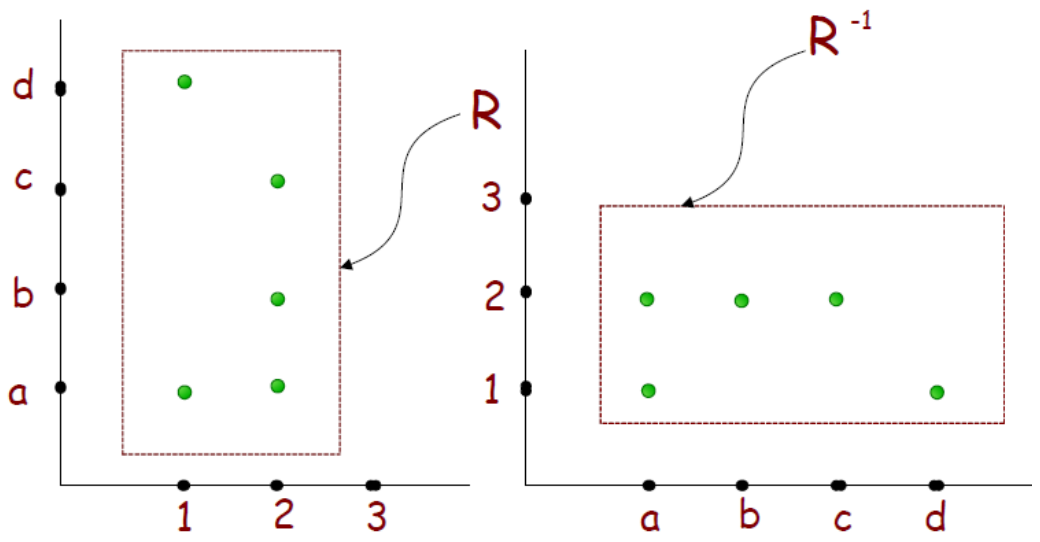
$yR^{-1}x$ iff $y = x^2$

or in terms of square root:

$xR^{-1}y$ iff $y = \pm\sqrt{x}$ where x is non-negative

Inverse of a relation

- Given a relation R from A to B, its inverse R^{-1} is the relation from B to A defined by $R^{-1} = \{ (b, a) \mid (a, b) \in R \}$
- Example: if $R = \{(1,a), (1,d), (2,a), (2,b), (2,c)\}$ then $R^{-1} = \{(a,1), (d,1), (a,2), (b,2), (c,2)\}$



Combining Relations

- Let R_1 be a relation from X to Y
- Let R_2 be a relation from Y to Z
- **Definition**: The composition of R_1 and R_2, denoted $R_2 \bullet R_1$ (or $R_2 \odot R_1$), is a relation from X to Z defined by $\{(x, z) \mid (x, y) \in R_1 \text{ and } (y, z) \in R_2 \text{ for some } y \in Y\}$
- In other words, if relation R_1 contains a pair (x, y) and relation R_2 contains a pair (y, z), then $R_2 \bullet R_1$ contains a pair (x, z).
  - Note how y plays the role of range and then domain in this transitive relationship

Combining Relations

Just as functions may be composed, so can binary relations:

Definition: If R is a relation from A to B, and S is a relation from B to C then the **composite** of R and S is the relation $S \bullet R$ (or just SR) from A to C defined by setting $a\,(S \bullet R)\,c$ if and only if there is some b such that aRb and bSc.

Notation is weird because generalizing functional composition: $f \bullet g\,(x) = f\,(g\,(x))$.

Combining Relations

Composite relation: $S \circ R$

$$(a, b) \in S \circ R \leftrightarrow \exists x : (a, x) \in R \wedge (x, b) \in S$$

Note:

$$(a, b) \in R \wedge (b, c) \in S \rightarrow (a, c) \in S \circ R$$

Example:

$$R = \{(1,1), (1,4), (2,3), (3,1), (3,4)\}$$

$$S = \{(1,0), (2,0), (3,1), (3,2), (4,1)\}$$

$$S \circ R = \{(1,0), (1,1), (2,1), (2,2), (3,0), (3,1)\}$$

Examples

- Let D and S be relations on $A=\{1,2,3,4\}$.
- $D = \{(a, b) \mid b = 5 - a\}$ "b equals (5 – a)"
- $S = \{(a, b) \mid a < b\}$ "a is smaller than b"
- $D = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$
- $S = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$
- $S \bullet D = \{ (2, 4), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4) \}$

D maps an element a to the element $(5 - a)$, and afterwards S maps $(5 - a)$ to all elements larger than $(5 - a)$, resulting in $S \bullet D = \{(a,b) \mid b > 5 - a\}$ or $S \bullet D = \{(a,b) \mid a + b > 5\}$.

Examples

- Let X and Y be relations on $A = \{1, 2, 3, \ldots\}$.
- $X = \{(a, b) \mid b = a + 1\}$ "b equals a plus 1"
- $Y = \{(a, b) \mid b = 3a\}$ "b equals 3 times a"
- $X = \{(1, 2), (2, 3), (3, 4), (4, 5), \ldots\}$
- $Y = \{(1, 3), (2, 6), (3, 9), (4, 12), \ldots\}$
- $X \bullet Y = \{(1,4),(2,7),(3,10),(4,13),\ldots\}$

Y maps an element a to the element 3a, and afterwards X maps 3a to 3a + 1), resulting in $X \bullet Y = \{(a,b) \mid b = 3a + 1\}$

$Y \bullet X = \{(a,b) \mid b = 3a + 3\}$

Examples

- Let X and Y be relations on $A = \{1, 2, 3, \ldots\}$.
- $X = \{(a, b) \mid b = a + 1\}$ "b equals a plus 1"
  ($X = \{(b, c) \mid c = b + 1\}$ "c equals b plus 1")
- $Y = \{(a, b) \mid b = 3a\}$ "b equals 3 times a"
  ($Y = \{(b, c) \mid c = 3b\}$ "c equals 3 times b")

Y maps an element a to the element 3a, and afterwards X maps 3a to 3a + 1), resulting in $X \bullet Y = \{(a,b) \mid b = 3a + 1\}$
$X \bullet Y = \{(a,c) \mid c = 3a + 1\} = \{(a,b) \mid b = 3a + 1\}$

$Y \bullet X = \{(a, b) \mid b = 3a + 3\}$

Property of Relation Composite (optional)

$$(a)\ \ R_1 \circ (R_2 \bigcup R_3) = (R_1 \circ R_2) \bigcup (R_1 \circ R_3)$$

$$(b)\ \ R_1 \circ (R_2 \bigcap R_3) \subseteq (R_1 \circ R_2) \bigcap (R_1 \circ R_3)$$

$$(c)\ \ (R_2 \bigcup R_3) \circ R_4 = (R_2 \circ R_4) \bigcup (R_3 \circ R_4)$$

$$(d)\ \ (R_2 \bigcap R_3) \circ R_4 \subseteq (R_2 \circ R_4) \bigcap (R_3 \circ R_4)$$

Power of relations

- **Definition**: Let R be a relation on the set *A*. The powers R^n, n = 1, 2, 3, …, are defined inductively by:
  - $R^1 = R$
  - $R^{n+1} = R^n \bullet R$
- In other words:
- $R^n = R \bullet R \bullet \ldots \bullet R$ (n times the letter R)

Power of relations

Power of relation: R^n

$$R^0 = I_A \quad R^1 = R \quad R^{n+1} = R^n \circ R$$

Example: $R = \{ (1,1), (2,1), (3,2), (4,3) \}$

$$R^2 = R \circ R = \{ (1,1), (2,1), (3,1)\ (4,2) \}$$

$$R^3 = R^2 \circ R = \{ (1,1), (2,1), (3,1)\ (4,1) \}$$

$$R^4 = R^3 \circ R = R^3$$

Power of relations

Theorem: Suppose $|A|=n$, R is a relation on A. Then there exists s and t, such that $R^s = R^t$, $0 \le s,t \le 2^{n^2}$

Proof:
For any k, R^k is the subset of $A \times A$, as $|A \times A| = n^2$, $|P(A \times A)|=2^{n^2}$. Then we can list all the relations as $R^0, R^1, \ldots, R^{2^{n^2}}$. It is easy to check that there are $2^{n^2}+1$ relations. So there must exist $R^s = R^t$, $0 \le s,t \le 2^{n^2}$

有穷集合上关系的幂序列式一个周期变化的序列!

Examples

X={a,b,c}， R_1, R_2, R_3, R_4, Show the power of these relations

$$R_1 = \{(a,b),(a,c),(c,b)\}$$

$$R_2 = \{(a,b),(b,c),(c,a)\}$$

$$R_3 = \{(a,b),(b,c),(c,c)\}$$

$$R_4 = \{(a,b),(b,a),(c,c)\}$$

$$R_1 = \{(a,b),(a,c),(c,b)\}$$
$$R_2 = \{(a,b),(b,c),(c,a)\}$$
$$R_3 = \{(a,b),(b,c),(c,c)\}$$
$$R_4 = \{(a,b),(b,a),(c,c)\}$$

$$R_1^2 = \{(a,b)\}, R_1^3 = \varnothing, R_1^4 = \varnothing, \cdots$$

$$R_2^2 = \{(a,c),(b,a),(c,b)\},$$

$$R_2^3 = \{(a,a),(b,b),(c,c)\} = R_2^0$$

$$R_2^4 = R_2, R_2^5 = R_2^2$$

$$R_2^6 = R_2^3, \cdots$$

$$R_3^2 = \{(a,c),(b,c),(c,c)\} = R_3^3 = R_3^4 = R_3^5 \cdots$$

$$R_4^2 = \{(a,a),(b,b),(c,c)\} = R_4^0$$

$$R_4^3 = R_4, R_4^5 = R_4^3 \cdots$$

Combining Relations

- **Theorem**: The relation R on a set A is transitive if and only if $R^n \subseteq R$ for all positive integers n.
- Recall the definition of transitivity:
  - **Definition**: A relation R on a set A is called **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for $a, b, c \in A$.

Theorem: A relation R is transitive if an only if $R^n \subseteq R$ for all $n = 1,2,3,\ldots$

Proof: 1. If part: $R^2 \subseteq R$

2. Only if part: use induction

1. If part: We will show that if $R^2 \subseteq R$ then R is transitive

---

$$\left.\begin{array}{r} \text{Assumption: } R^2 \subseteq R \\ \text{Definition of power: } R^2 = R \circ R \\ \text{Definition of composition:} \\ (a,b) \in R \wedge (b,c) \in R \rightarrow (a,c) \in R \circ R \end{array}\right\} \Rightarrow (a,c) \in R$$

Therefore, R is transitive

2. Only if part:

We will show that if R is transitive then $R^n \subseteq R$ for all $n \geq 1$

Proof by induction on n

Inductive basis: $n = 1$

It trivially holds $R^1 = R \subseteq R$

Inductive hypothesis:

Assume that $R^k \subseteq R$

for all $1 \leq k \leq n$

Inductive step: We will prove $R^{k+1} \subseteq R$

Take arbitrary $(a, b) \in R^{k+1}$

We will show $(a, b) \in R$

$$(a, b) \in R^{k+1}$$

$\Downarrow$ definition of power

$$(a, b) \in R^{k} \circ R$$

$\Downarrow$ definition of composition

$$\exists x : (a, x) \in R \wedge (x, b) \in R^{k}$$

$\Downarrow$ inductive hypothesis $R^{k} \subseteq R$

$$\exists x : (a, x) \in R \wedge (x, b) \in R$$

$\Downarrow$ R is transitive

$$(a, b) \in R$$

End of Proof

Summary

Theorem: Let R be the relation on a set A. Then we have

R is reflexive iff $I_A \subseteq R$

R is irreflexive iff $I_A \cap R = \emptyset$

R is symmetric iff $R = R^{-1}$

R is antisymmetric iff $R \cap R^{-1} \subseteq I_A$

R is transitive iff $R \bullet R \subseteq R$

Theorem: A relation R is transitive if an only if $R^n \subseteq R$ for all $n = 1,2,3,\ldots$

Closures of Relations

Closures of Relations

- **Definition**:
  - Let R be a relation on a set A. R may or may not have some property P, such as reflexivity, symmetry, or transitivity.
  - If there is a relation S with property P containing R such that S is a subset of every relation with property P containing R, then S is called the closure of R with respect to P.
- Note that the closure of a relation with respect to a property may not exist.

Closures of Relations

R: X→X

关系**S**满足

(1) S 是自 可传递的

(2)R⊆ S

(3)对任何 可传递 关系**S′**

R ⊆ S′ ⇒ **S⊆S′**

R的 可传递 闭包

r(R)

s

t(R)

Reflexive Closures

- **Example**: Find the reflexive closure of relation R = {(1, 1),(1, 2), (2, 1), (3, 2)} on the set A={1,2,3}.
- **Solution**:
  - We know that any reflexive relation on A must contain the elements (1,1), (2,2),and (3, 3).
  - By adding (2, 2) and (3, 3) to R, we obtain the reflexive relation S, which is given by S = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 2), (3, 3)}.
  - S is reflexive, contains R, and is contained within every reflexive relation that contains R.
  - Therefore S is reflexive closure of R.

Symmetric Closures

- **Example**: Find the symmetric closure of the relation $R = \{(a, b) \mid a > b\}$ on the set of positive integers.
- **Solution**:
  - The symmetric closure of R is given by $R \cup R^{-1} = \{(a, b) \mid a > b\} \cup \{(b, a) \mid a > b\} = \{(a, b) \mid a \neq b\}$

Transitive Closures

- **Example:** Find the transitive closure of the relation R = {(1, 3), (1, 4), (2, 1), (3, 2)} on the set A = {1, 2, 3, 4}.
- **Solution**:
  - R would be transitive, if for all pairs (a, b) and (b, c) in R there were also a pair (a, c) in R.
  - If we add the missing pairs (1, 2), (2, 3), (2, 4), and (3, 1), will R be transitive?

Transitive Closures

- No, because the extended relation R contains (3, 1) and (1, 4), but does not contain (3, 4).
- By adding new elements to R, we also add new requirements for its transitivity. We need to look at paths in digraphs to solve this problem.
- Imagine that we have a relation R that represents all train connections in the US.

Transitive Closures

- For example, if (Boston, Philadelphia) is in R, then there is a direct train connection from Boston to Philadelphia.
- If R contains (Boston, Philadelphia) and (Philadelphia, Washington), there is an indirect connection from Boston to Washington.

- The transitive closure of R contains exactly those pairs of cities that are connected, either directly or indirectly.
- For getting transitive closure, we need next definition.

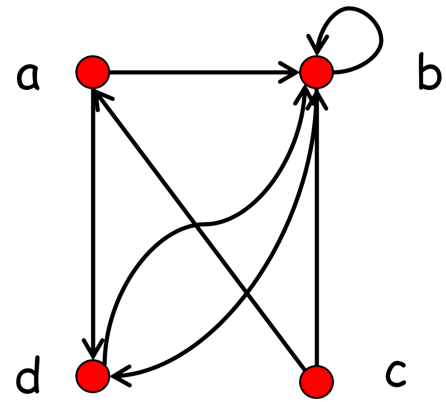
- **Definition**: *A* **path** from a to b in the directed graph *G* is a sequence of one or more edges (x_0, x_1), (x_1, x_2), (x_2, x_3), …, (x_{n-1}, x_n) in *G*, where x_0 = a and x_n = b.

Transitive Closures

- In other words, a path is a sequence of edges where the **terminal** vertex of an edge is the same as the **initial** vertex of the next edge in the path.
- The path mentioned above is denoted by $x_0, x_1, x_2, \ldots, x_n$ and has **length** n.
- A path that begins and ends at the same vertex is called a **circuit** or **cycle**.

Transitive Closures

- **Example**: Let us take a look at the following graph:



- Is **c**,**a**,**b**,**d**,**b** a path in this graph? YES
- Is **d**,**b**,**b**,**b**,**d**,**b**,**d** a circuit in this graph? YES
- Is there any circuit including c in this graph? NO

$$R = \{(a, b), (a, d), (b, b), (b, d), (c, a), (c, b), (d, b)\}$$

Transitive Closures

- Due to the one-to-one correspondence between graphs and relations, we can transfer the definition of path from graphs to relations:
- **Definition:** There is a path from a to b in a relation R, if there is a sequence of elements a, x_1, x_2, …, x_{n-1}, b with (a, x_1)$\in$R, (x_1, x_2)$\in$R, …, and (x_{n-1}, b)$\in$R.
- **Theorem:** Let R be a relation on a set A. There is a path (of length n) from a to b if and only if (a, b)$\in R^n$.

Transitive Closures

- According to the train example, the transitive closure of a relation consists of the pairs of vertices in the associated directed graph that are connected by a path.
- **Definition:** Let R be a relation on a set A. The **connectivity relation** R^* consists of the pairs (a, b) such that there is a path between a and b in R.
- We know that R^n consists of the pairs (a, b) such that a and b are connected by a path of length n.

Transitive Closures

- Therefore, R^* is the union of R^n across all positive integers n:

$$R^* = \bigcup_{n=1}^{\infty} R^{\mathrm{n}} = R^1 \bigcup R^2 \bigcup R^3 \cdots$$

Transitive Closures

- **Theorem**: The transitive closure of a relation R equals to the connectivity relation R^*.

- But how can we compute R^* ?

- **Lemma**: Let A be a set with n elements, and let R be a relation on A. If there is a path in R from a to b, then there is such a path with length not exceeding n. Moreover, if $a \neq b$ and there is a path in R from a to b, then there is such a path with length not exceeding $(n - 1)$.

Transitive Closures

- This lemma is based on the observation that if a path from a to b visits any vertex more than once, it must include at least one circuit.
- These circuits can be eliminated from the path, and the reduced path will still connect a and b.
- **Theorem**: For a relation R on a set A with n elements, the transitive closure R* is given by:

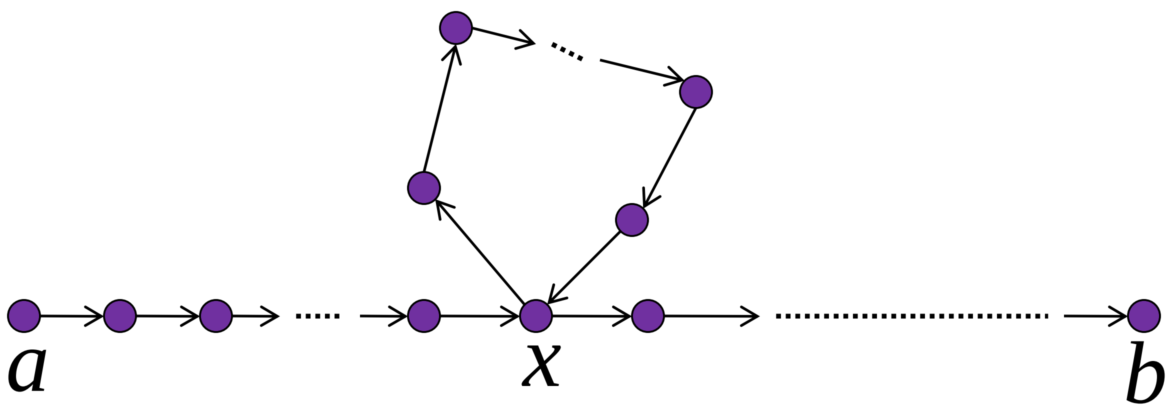
$$R^* = R \cup R^2 \cup R^3 \cup \ldots \cup R^n$$

For matrices representing relations we have:

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee \ldots \vee M_R^{[n]}$$

Theorem: $R^* = R^1 \cup R^2 \cup R^3 \cup \cdots \cup R^n$

Proof: if $(a, b) \in R^{n+1}$ then $(a, b) \in R^i$ for some $i \in \{1, \ldots, n\}$



Repeated node

Transitive Closures

- Let us finally find the transitive closure of the relation R = {(1, 3), (1, 4), (2, 1), (3, 2)} on the set A = {1, 2, 3, 4} (discussed in previous slide).
- R can be represented by the following matrix M_R:

$$M_R = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Transitive Closures

$$M_R = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad M_R^{[2]} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_R^{[3]} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad M_R^{[4]} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee \ldots \vee M_R^{[n]} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Transitive Closures

- **Answer**: The transitive closure of the relation R = {(1, 3), (1, 4), (2, 1), (3, 2)} on the set A = {1, 2, 3, 4} is given by the relation
- Tran(R)={(1, 1), (1, 2), (1, 3), (1, 4), (2, 1),(2, 2), (2, 3), (2, 4),(3, 1), (3, 2), (3, 3), (3, 4)}

Equivalence Relations

Equivalence Relations

- Equivalence relations are used to relate objects that are similar in some way.
- **Definition**: A relation on a set *A* is called an equivalence relation if it is **reflexive**, **symmetric**, and **transitive**.
- Two elements that are related by an equivalence relation R are called equivalent.

Equivalence Relations

- Since R is symmetric, a is equivalent to b whenever b is equivalent to a.
- Since R is reflexive, every element is equivalent to itself.
- Since R is transitive, if a and b are equivalent and b and c are equivalent, then a and c are equivalent.
- Obviously, these three properties are necessary for a reasonable definition of equivalence.

Equivalence Relations

- **Example**: Suppose that R is the relation on the set of strings that consist of English letters such that aRb if and only if l(a)=l(b),where l(x) is the length of the string x. Is R an equivalence relation?
- **Solution**:
  – R is reflexive, because l(a) =l(a) and therefore aRa for any string a.
  – R is symmetric, because if l(a) = l(b) then l(b) = l(a), so if aRb then bRa.
  – R is transitive, because if l(a) = l(b) and l(b) = l(c), then l(a) = l(c), so aRb and bRc implies aRc.
- R is an equivalence relation.

Equivalence Classes

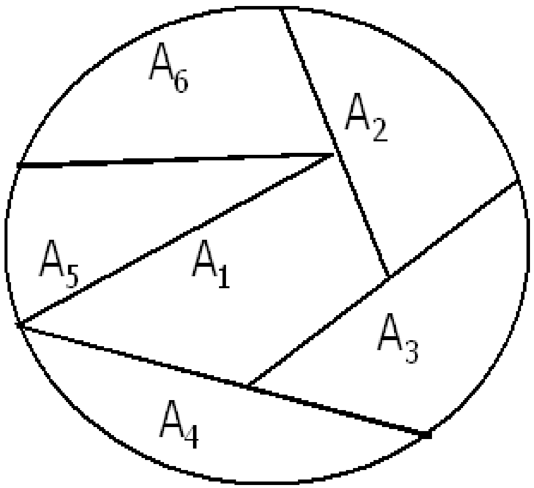
- **Definition**: Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the **equivalence class** of a.
- The equivalence class of a with respect to R is denoted by $[a]_R$.
- When only one relation is under consideration, we will delete the subscript R and write [a] for this equivalence class.
- If $b \in [a]_R$, b is called a representative of this equivalence class.

Equivalence Classes

- **Example**: In the previous example (strings of identical length), what is the equivalence class of the word mouse, denoted by [mouse] ?
- **Solution**:
  - [mouse] is the set of all English words containing five letters.
  - For example, ‘horse’ would be a representative of this equivalence class.

Equivalence Classes

- **Theorem**: Let R be an equivalence relation on a set A. The following statements are equivalent:
  - 1. aRb
  - 2. [a] = [b]
  - 3. $[a] \cap [b] \neq \emptyset$
- **Definition**: A **partition** of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets Ai, $i \in I$, forms a partition of S iff
  - 1. $A_i \neq \emptyset$ for $i \in I$
  - 2. $A_i \cap A_j = \emptyset$, if $i \neq j$
  - 3. $\bigcup_{i \in I} A_i = S$



划分的图形表示

Equivalence Classes

Examples: Let S be the set {u, m, b, r, o, c, k, s}. Do the following collections of sets partition S ?

- {{m, o, c, k}, {r, u, b, s}} yes.
- {{c, o, m, b}, {u, s}, {r}} no (k is missing).
- {{b, r, o, c, k}, {m, u, s, t}} no (t is not in S).
- {{u, m, b, r, o, c, k, s}} yes.
- {{b, o, o, k}, {r, u, m}, {c, s}} yes ({b,o,o,k} ={b,o,k}).
- {{u, m, b}, {r, o, c, k, s}, ∅} no (∅ not allowed).

Equivalence Classes

- **Theorem**: Let R be an equivalence relation on a set S. Then the **equivalence classes** of R form a **partition** of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.
- **Example**:
  - Let us assume that Frank, Suzanne and George live in ***Boston***, Stephanie and Max live in ***Lübeck***, and Jennifer lives in ***Sydney***.

  - Let R be the equivalence relation {(a, b) | a and b live in the same city} on the set P = {Frank, Suzanne, George, Stephanie, Max, Jennifer}.
  - Then R = { (Frank, Frank), (Frank, Suzanne), (Frank,George), (Suzanne,Frank), (Suzanne,Suzanne), (Suzanne,George), (George,Frank), (George, Suzanne), (George,George), (Stephanie, Stephanie), (Stephanie, Max), (Max, Stephanie), (Max Max), (Jennifer, Jennifer) }.

Then the equivalence classes of R are:

{{Frank, Suzanne, George}, {Stephanie, Max}, {Jennifer}}.

This is a partition of P.

Equivalence Classes

- **Example**:
  - Let R be the relation $\{(a,b) \mid a \equiv b \pmod 3\}$ on the set of integers.
  - Is R an equivalence relation?

    Yes, R is reflexive, symmetric, and transitive.
  - What are the equivalence classes of R ?

    { {…, -6, -3, 0, 3, 6, …}, {…, -5, -2, 1, 4, 7, …}, {…, 4, 1, 2, 5, 8, …} }

Equivalence relations

- **Example**:
  - Consider set $X = \{1,2,\ldots,13\}$. Define xRy as 5 divides $x - y$ (i.e., $x - y = 5k$, for some int k). We can verify that R is reflexive, symmetric, and transitive. Here is how.
  - The equivalence class $[1]$ consists of all x with $xR1$. Thus:
    - $[1] = \{x \in X \mid 5 \text{ divides } x - 1\} = \{1, 6, 11\}$
  - Similarly:
    - $[2] = \{2, 7, 12\}$
    - $[3] = \{3, 8, 13\}$
    - $[4] = \{4, 9\}$
    - $[5] = \{5, 10\}$

Equivalence relations

- These 5 sets partition X. Note that:
- [1] = [6] = [11]
- [2] = [7] = [12]
- [3] = [8] = [13]
- [4] = [9]
- [5] = [10]
- For this relation, equivalence is “has the same reminder when divided by 5”.

The End