Graph Theory

Basics for Graphs

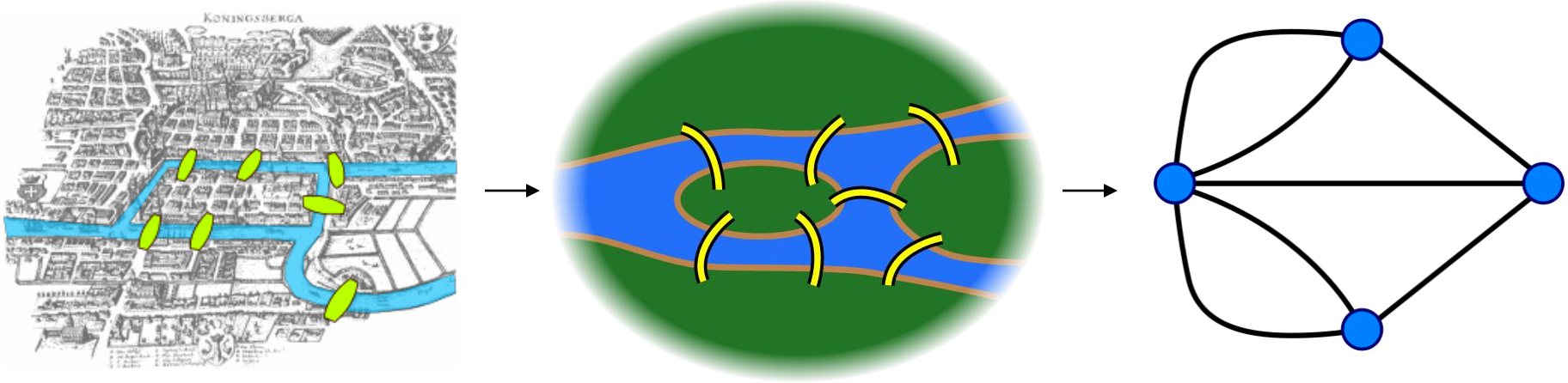
This Lecture

In this part we will study some basic graph theory.

Graph is a useful concept to model many problems in computer science.

- Seven bridges of Konigsberg
- **Basics for Graphs (图)**
- Path (路径), cycle (回路), connectedness (连通性)

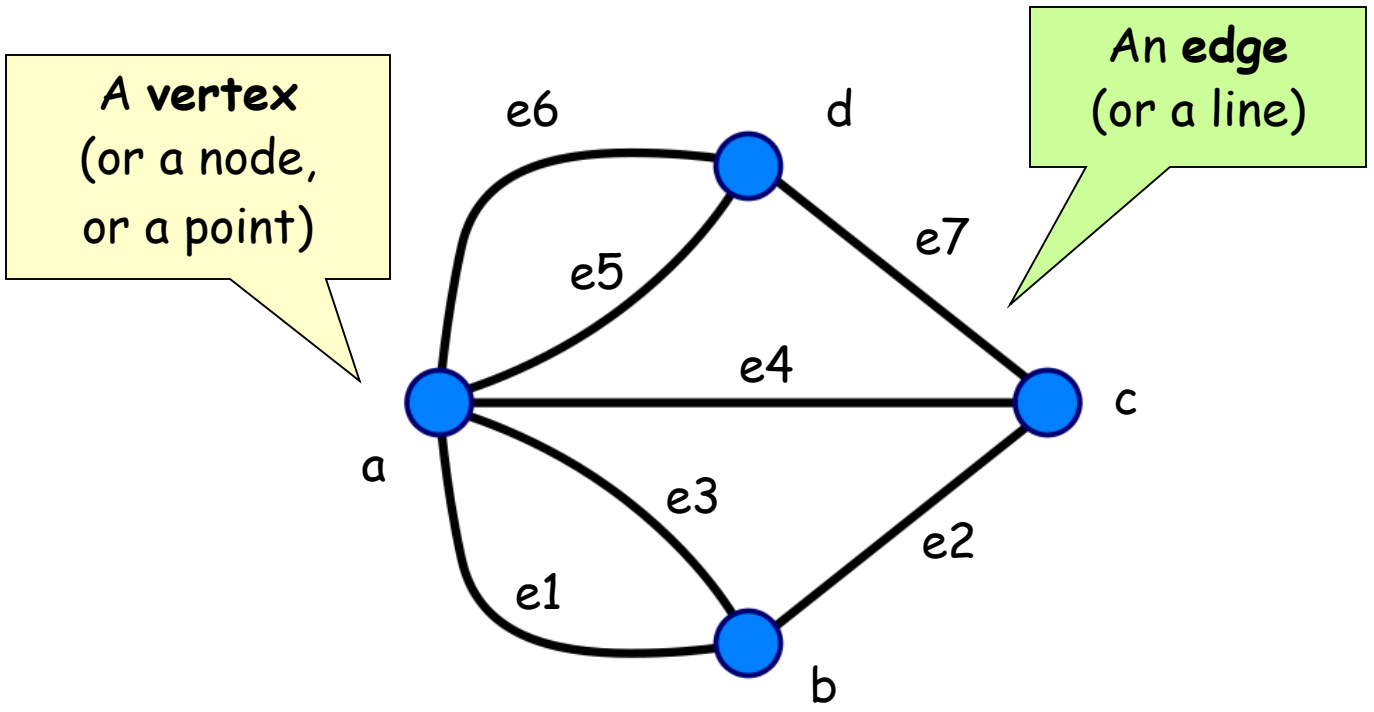
Seven Bridges of Königsberg



Forget unimportant details.

Forget even more.

A Graph

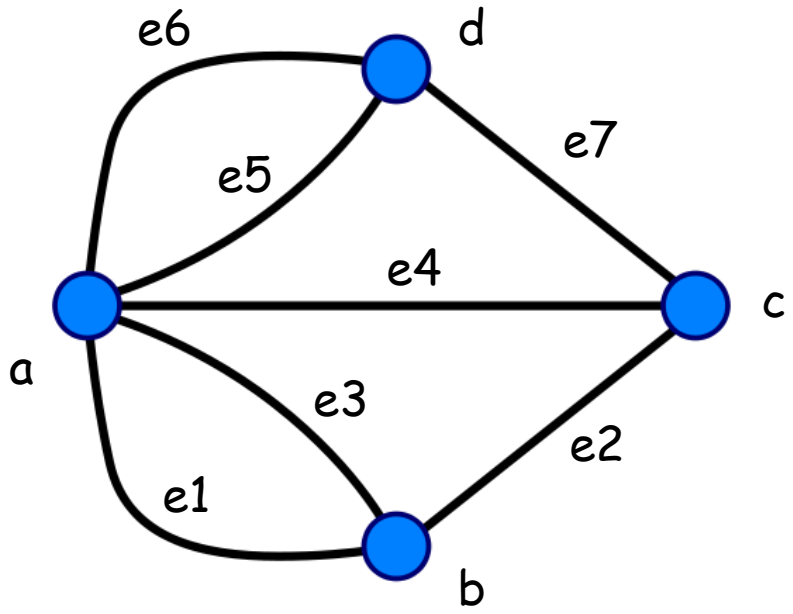


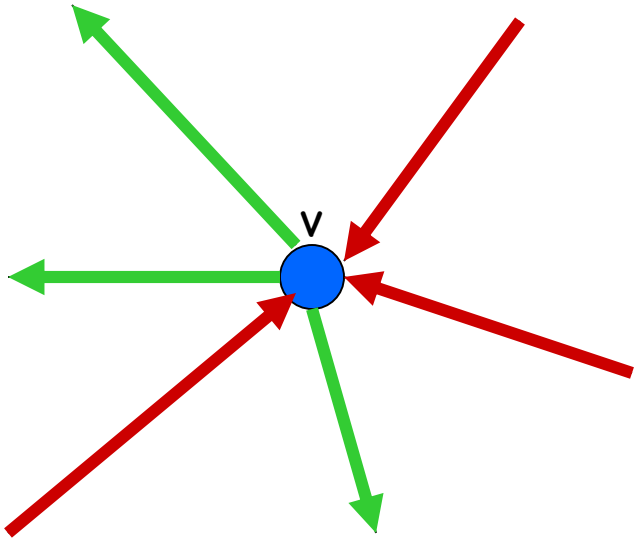
So, what is the "**Seven Bridges of Königsberg**" problem now?

To find a walk that visits each edge exactly once.

Euler's Solution

Question: Is it possible to find a walk that visits each edge exactly once.



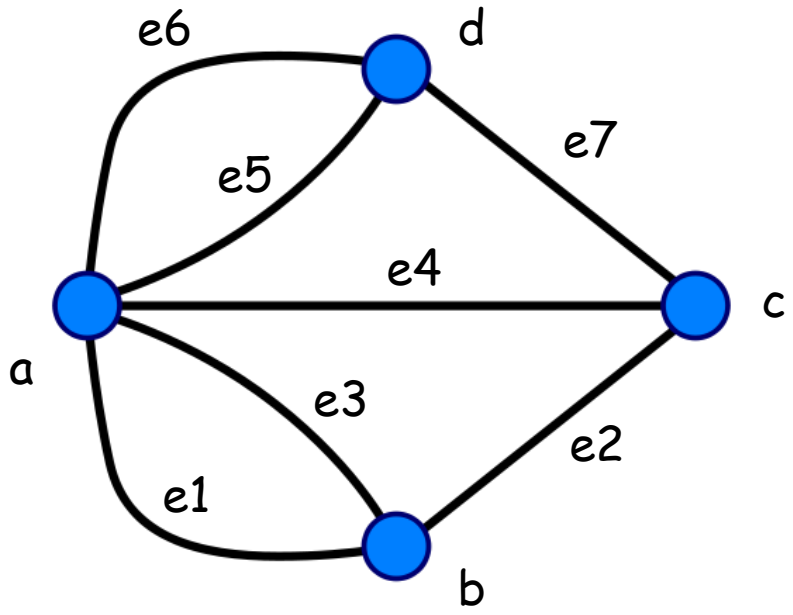


Suppose there is such a walk, there is a starting point and an endpoint point.

For every "intermediate" point v, there must be the same number of incoming and outgoing edges, and so v must have an **even number of edges**.

Euler's Solution

Question: Is it possible to find a walk that visits each edge exactly once.



So, at most **two** vertices can have odd number of edges.

In this graph, every vertex has only an odd number of edges, and so there is no walk which visits each edge exactly one.

Suppose there is such a walk, there is a starting point and an endpoint point.

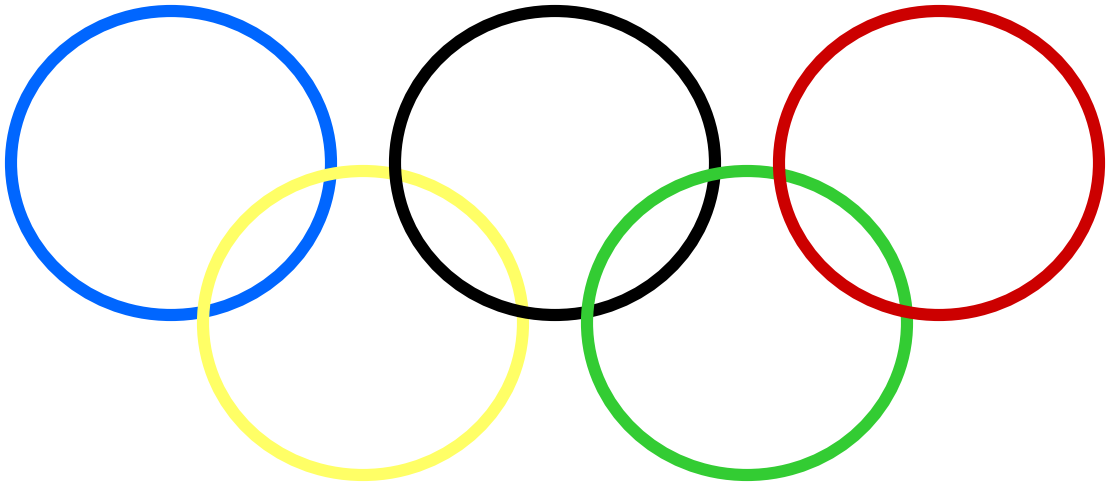
For every "intermediate" point v, there must be the same number of incoming and outgoing edges, and so v must have an **even number of edges**.

Euler's Solution

So Euler showed that the "**Seven Bridges of Königsberg**" is unsolvable.

When is it possible to have a walk that visits every edge exactly once?

一笔画问题

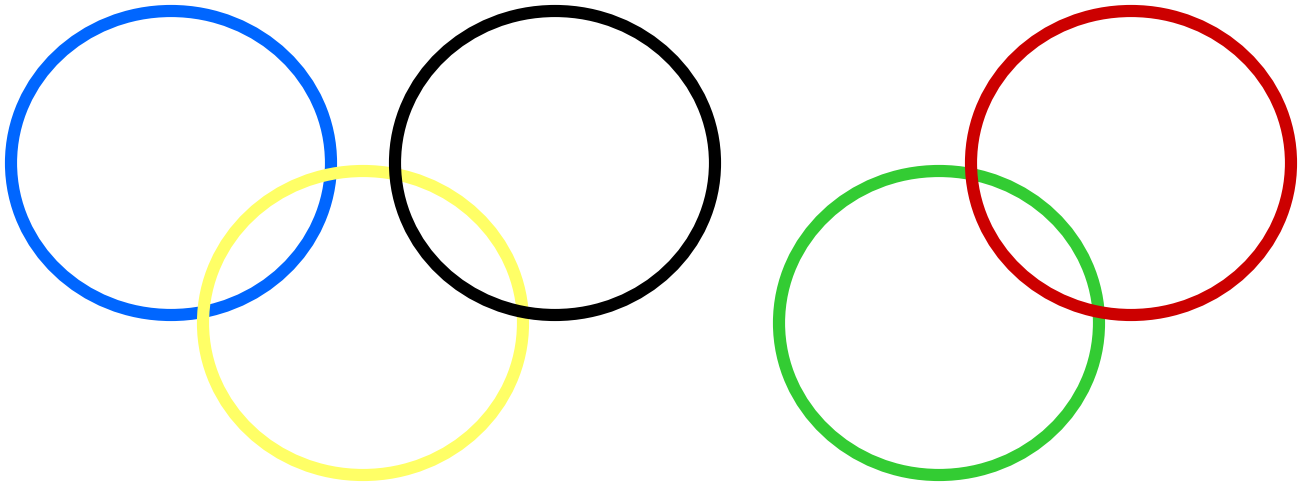


Is it always possible to find such a walk if there is **at most two** vertices with odd number of edges?

Euler's Solution

So Euler showed that the "**Seven Bridges of Königsberg**" is unsolvable.

When is it possible to have a walk that visits every edge exactly once?



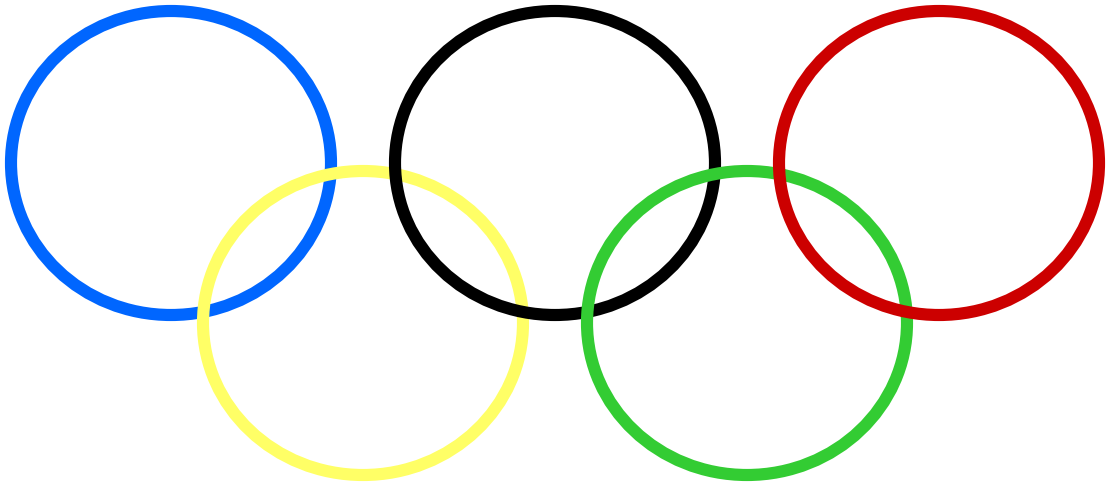
Is it always possible to find such a walk if there is **at most two** vertices with odd number of edges?

NO!

Euler's Solution

So Euler showed that the "**Seven Bridges of Königsberg**" is unsolvable.

When is it possible to have a walk that visits every edge exactly once?



Is it always possible to find such a walk if the graph is "**connected**" and there are **at most two** vertices with odd number of edges?

YES!

Euler's Solution

So Euler showed that the **"Seven Bridges of Königsberg"** is unsolvable.

When is it possible to have a walk that visits every edge exactly once?

Eulerian path

Euler's theorem: *A graph has an Eulerian path if and only if it is "connected" and has at most two vertices with an odd number of edges.*

This theorem was proved in 1736,
and was regarded as the starting point of graph theory.

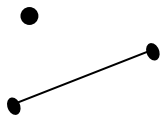
Basics for Graphs

Content

- Graph basics and definitions
  - Vertices/nodes, edges, adjacency(邻域), incidence
  - Degree, in-degree, out-degree
  - Subgraphs, unions, isomorphism
  - Adjacency matrices
- Types of Graphs
  - Undirected graphs
    - Simple graphs, Multigraphs (多重边图), Pseudographs (伪图)
  - Digraphs, Directed multigraph
  - Bipartite (二部图)
  - Complete graphs, cycles (圈图), wheels (轮图), cubes, complete bipartite

Graphs – Intuitive Notion

A graph is a bunch of vertices (or nodes) represented by circles which are connected by edges, represented by line segments



Mathematically, graphs are binary-relations on their vertex set (except for multigraphs).

In Data Structures one often starts with trees and generalizes to graphs. In this course, opposite approach: We start with graphs and restrict to get trees.

Simple Graphs

Different purposes require different types of graphs.

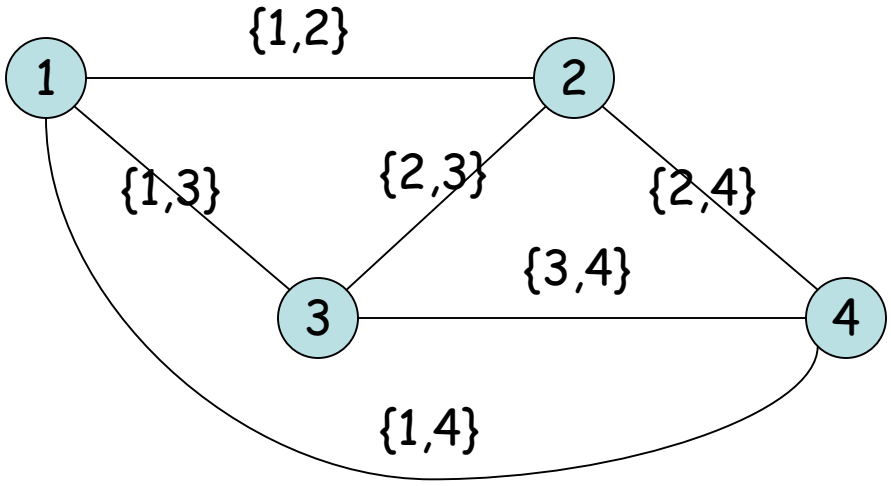
Exempli Gratia (EG):

Suppose a local computer network

- Is bidirectional (undirected)
- Has no loops (no “self-communication”)
- Has unique connections between computers

Sensible to represent as follows:

Simple Graphs



- Vertices are labeled to associate with particular computers
- Each edge can be viewed as the set of its two endpoints

Simple Graphs

Definition: A **simple graph** $G = (V,E)$ consists of a non-empty set V of **vertices** (or **nodes**) and a set E (possibly empty) of **edges** where each edge is a subset of V with cardinality 2 (an **unordered pair**).

Q: For a set V with n elements, how many possible edges there?

A: The number of pairs in V

$= C(n,2) = n \cdot (n-1) / 2$

Simple Graphs

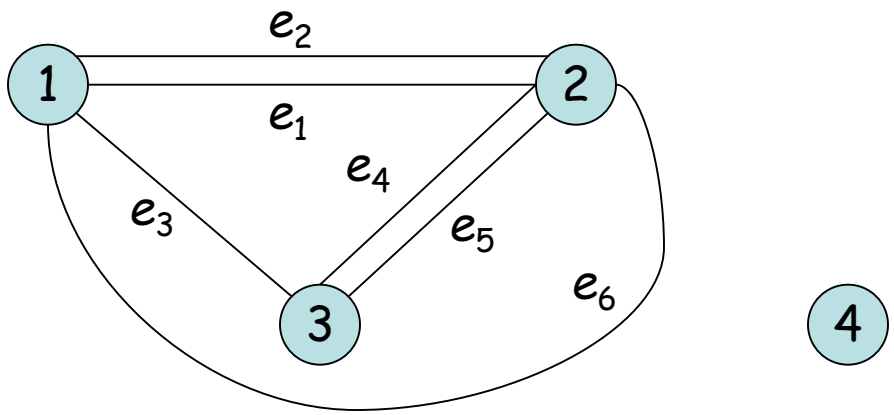
Q: How many possible graphs are there for the same set of vertices V ?

A: The number of subsets in the set of possible edges. There are $n \cdot (n-1) / 2$ possible edges, therefore the number of graphs on V is $2^{n(n-1)/2}$

Multigraphs

If computers are connected via internet instead of directly, there may be several routes to choose from for each connection. Depending on traffic, one route could be better than another. Makes sense to allow multiple edges, but still no self-loops:

Multigraphs



Edge-labels distinguish between edges sharing same endpoints. Labeling can be thought of as function:

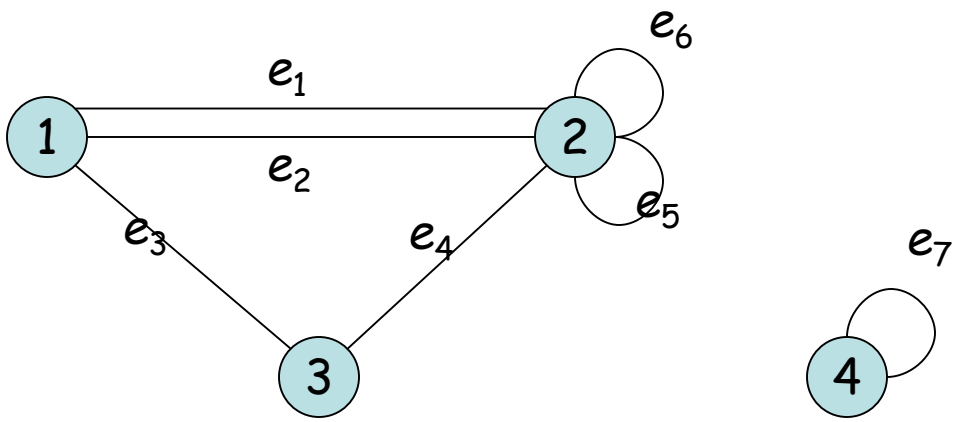
$e_1 \rightarrow \{1,2\}$, $e_2 \rightarrow \{1,2\}$, $e_3 \rightarrow \{1,3\}$,
$e_4 \rightarrow \{2,3\}$, $e_5 \rightarrow \{2,3\}$, $e_6 \rightarrow \{1,2\}$

Multigraphs

Definition: A **multigraph** $G = (V,E,f)$ consists of a non-empty set V of **vertices** (or **nodes**), a set E (possibly empty) of **edges** and a function f with domain E and codomain the set of pairs in V.

Pseudographs

If self-loops are allowed we get a pseudograph:



Now edges may be associated with a single vertex, when the edge is a loop

$e_1 \rightarrow \{1,2\}$, $e_2 \rightarrow \{1,2\}$, $e_3 \rightarrow \{1,3\}$,

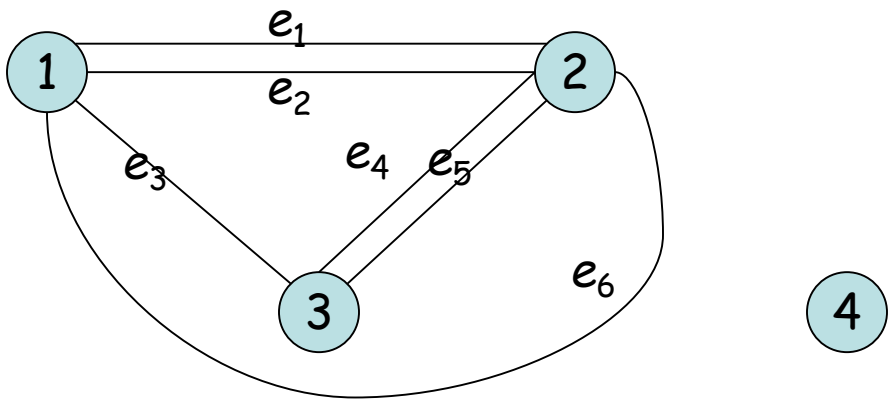
$e_4 \rightarrow \{2,3\}$, $e_5 \rightarrow \{2\}$, $e_6 \rightarrow \{2\}$, $e_7 \rightarrow \{4\}$

Pseudographs

Definition: A **pseudograph** $G = (V,E,f)$ consists of a non-empty set V of **vertices** (or **nodes**), a set E (possibly empty) of **edges** and a function f with domain E and codomain the set of pairs and singletons in V.

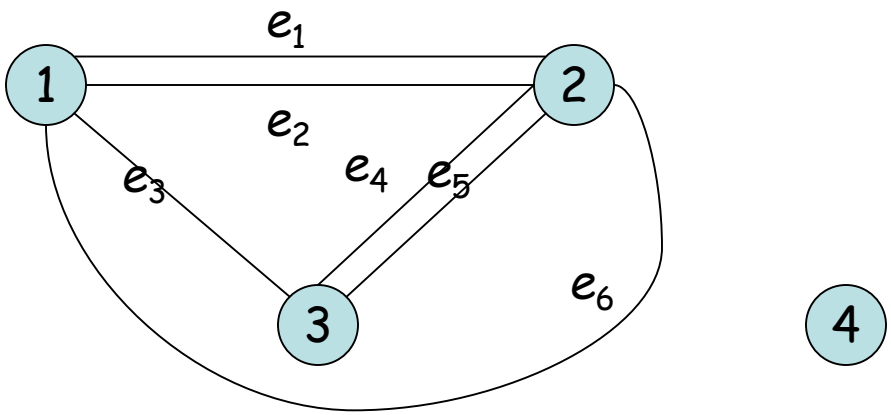
Undirected Graphs Terminology

Vertices are **adjacent** if they are the endpoints of the same edge.



Q: Which vertices are adjacent to 1? How about adjacent to 2, 3, and 4?

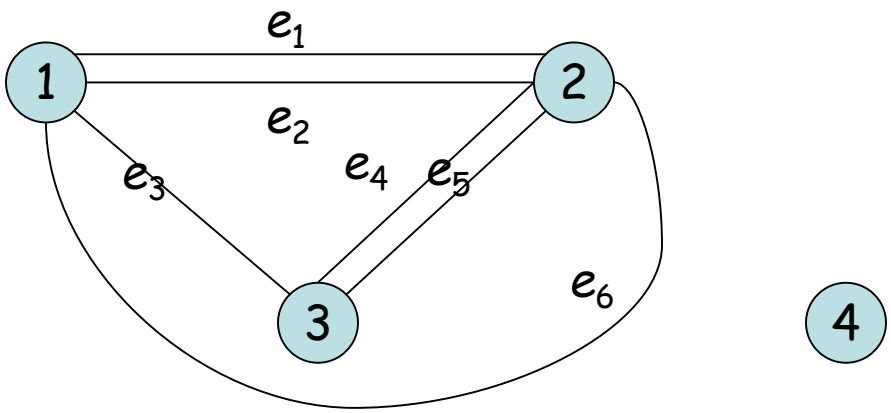
Undirected Graphs Terminology



A: 1 is adjacent to 2 and 3
2 is adjacent to 1 and 3
3 is adjacent to 1 and 2
4 is not adjacent to any vertex

Undirected Graphs Terminology

A vertex is **incident** with an edge (and the edge is incident with the vertex) if it is the endpoint of the edge.



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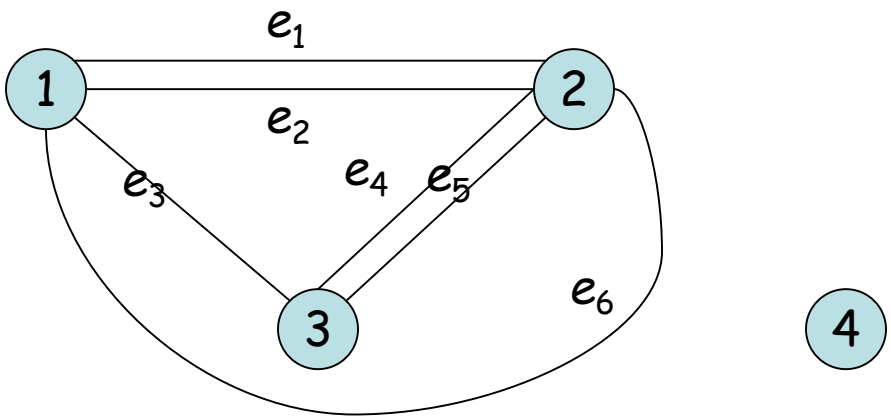
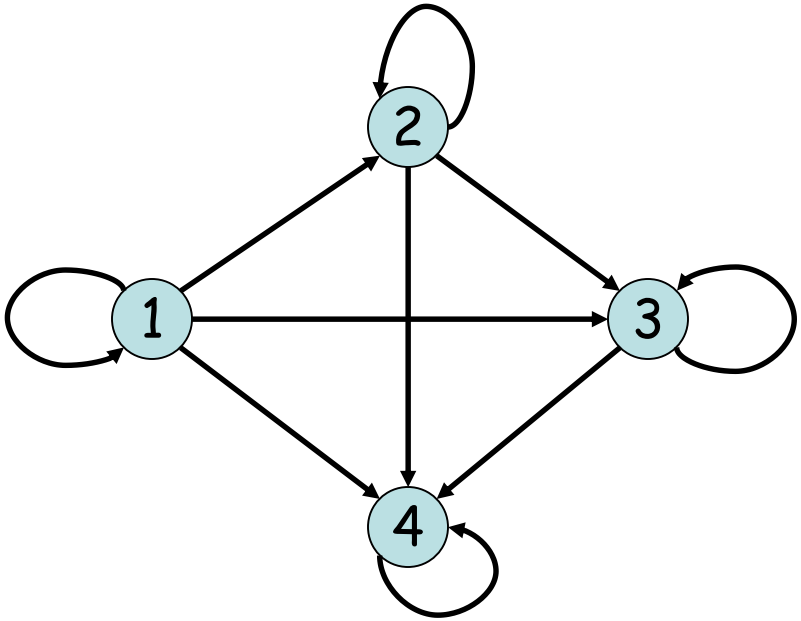
Undirected Graphs Terminology



A: e_1, e_2, e_3, e_6 are incident with 1
2 is incident with e_1, e_2, e_4, e_5, e_6
3 is incident with e_3, e_4, e_5
4 is not incident with any edge

Digraphs

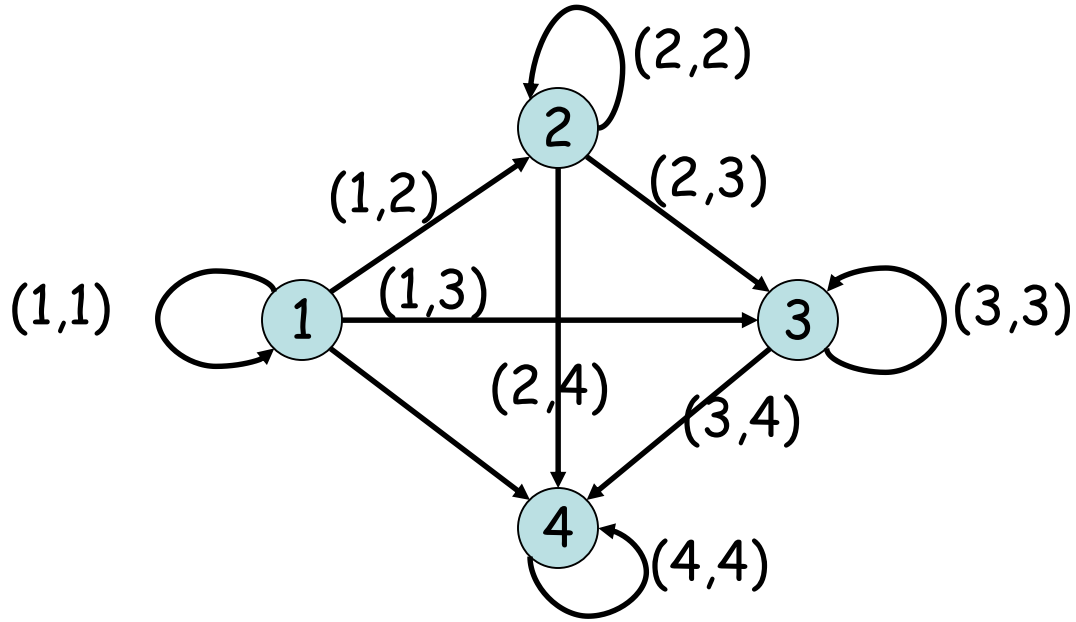
Last time introduced digraphs as a way of representing relations:



Q: What type of pair should each edge be (multiple edges not allowed)?

Digraphs

A: Each edge is directed so an ordered pair (or tuple) rather than unordered pair.



Thus the set of edges E is just the represented relation on V.

Digraphs

Definition: A **directed graph** (or **digraph**) $G = (V,E)$ consists of a non-empty set V of **vertices** (or **nodes**) and a set E of **edges** with $E \subseteq V \times V$.

The edge (a,b) is also denoted by $a \rightarrow b$ and a is called the **source** of the edge while b is called the **target** of the edge.

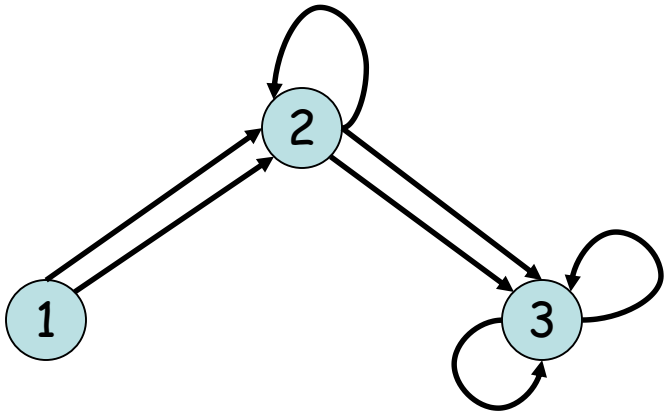
Q: For a set V with n elements, how many possible digraphs are there?

Digraphs

A: The same as the number of relations on V, which is the number of subsets of V×V so $2^{(n^2)}$

Directed Multigraphs

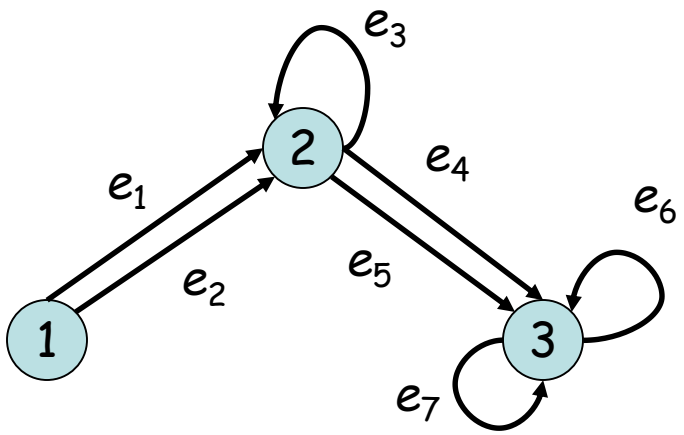
If also want to allow multiple edges in a digraph, get a **directed multigraph** (or **multi-digraph**).



Q: How to use sets and functions to deal with multiple directed edges, loops?

Directed Multigraphs

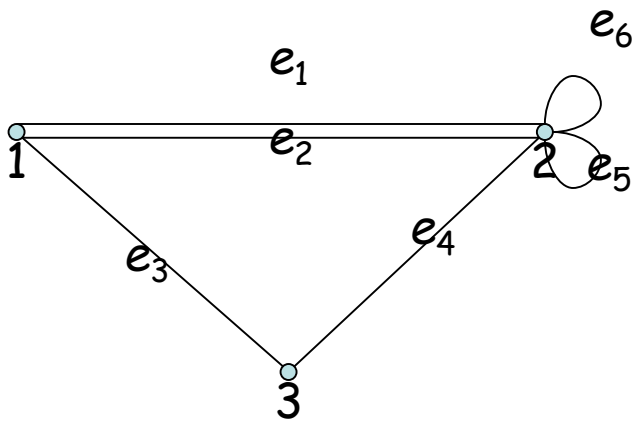
A: Have function with domain the edge set and codomain V ×V .



$e_1 \rightarrow (1,2)$, $e_2 \rightarrow (1,2)$, $e_3 \rightarrow (2,2)$, $e_4 \rightarrow (2,3)$,
$e_5 \rightarrow (2,3)$, $e_6 \rightarrow (3,3)$, $e_7 \rightarrow (3,3)$

Degree

The **degree** of a vertex counts the number of edges that seem to be sticking out if you looked under a magnifying glass:



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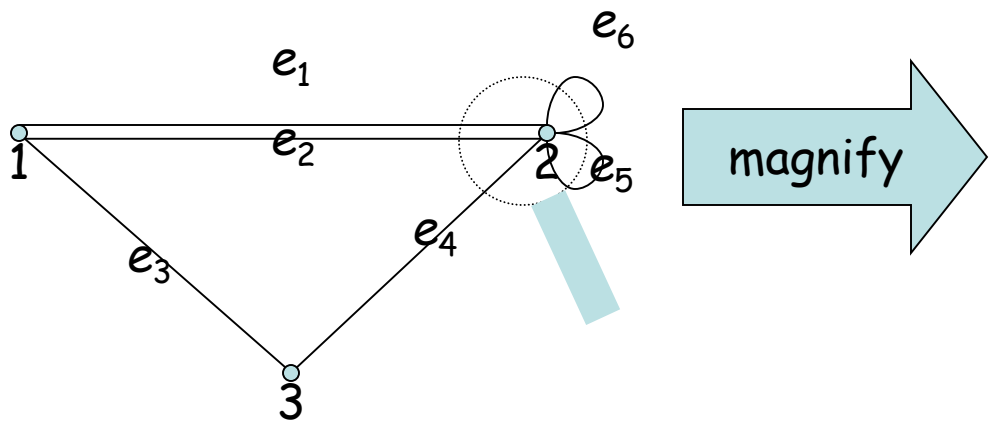
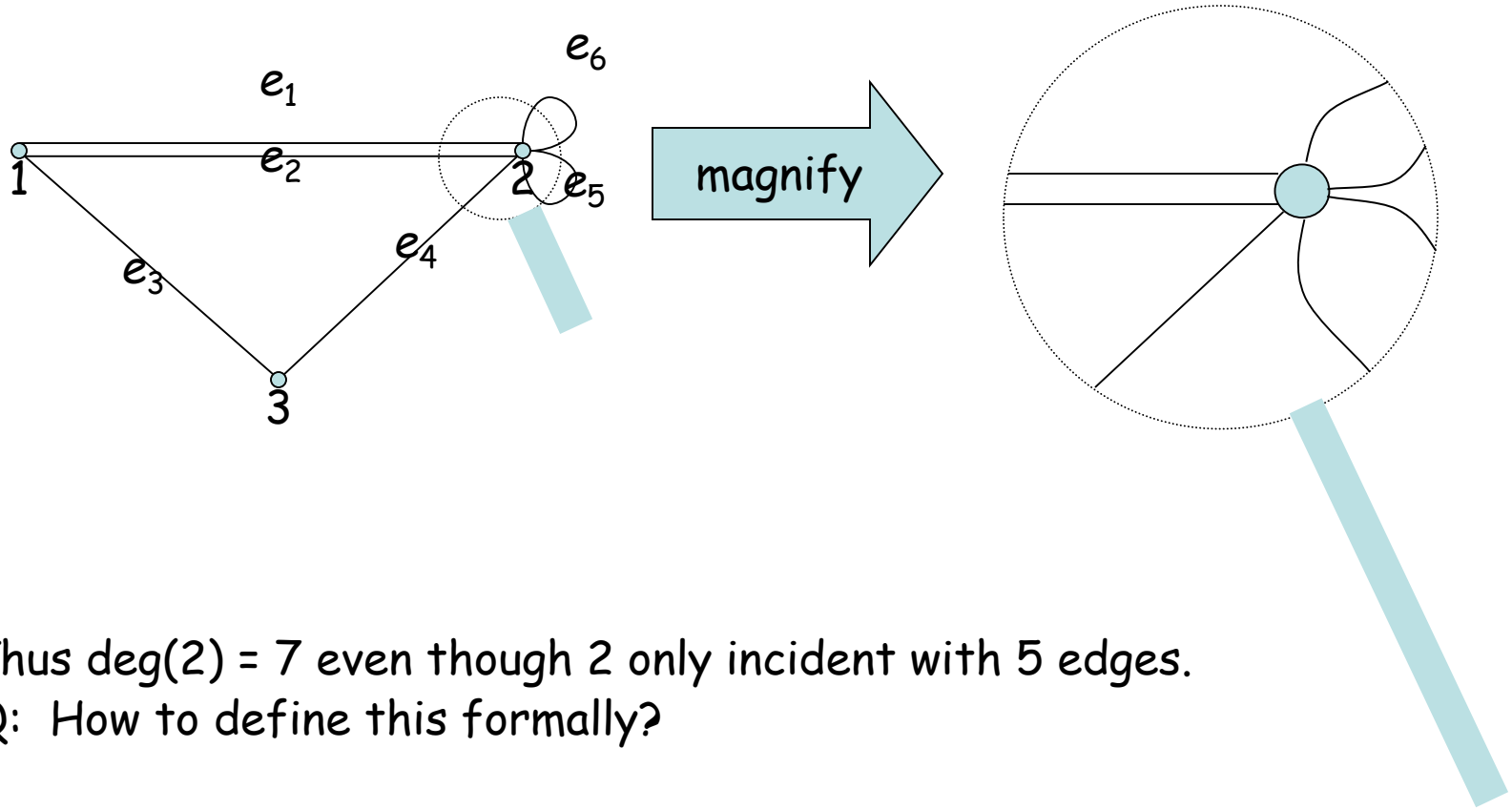


Degree

The **degree** of a vertex counts the number of edges that *seem* to be sticking out if you looked under a magnifying glass:

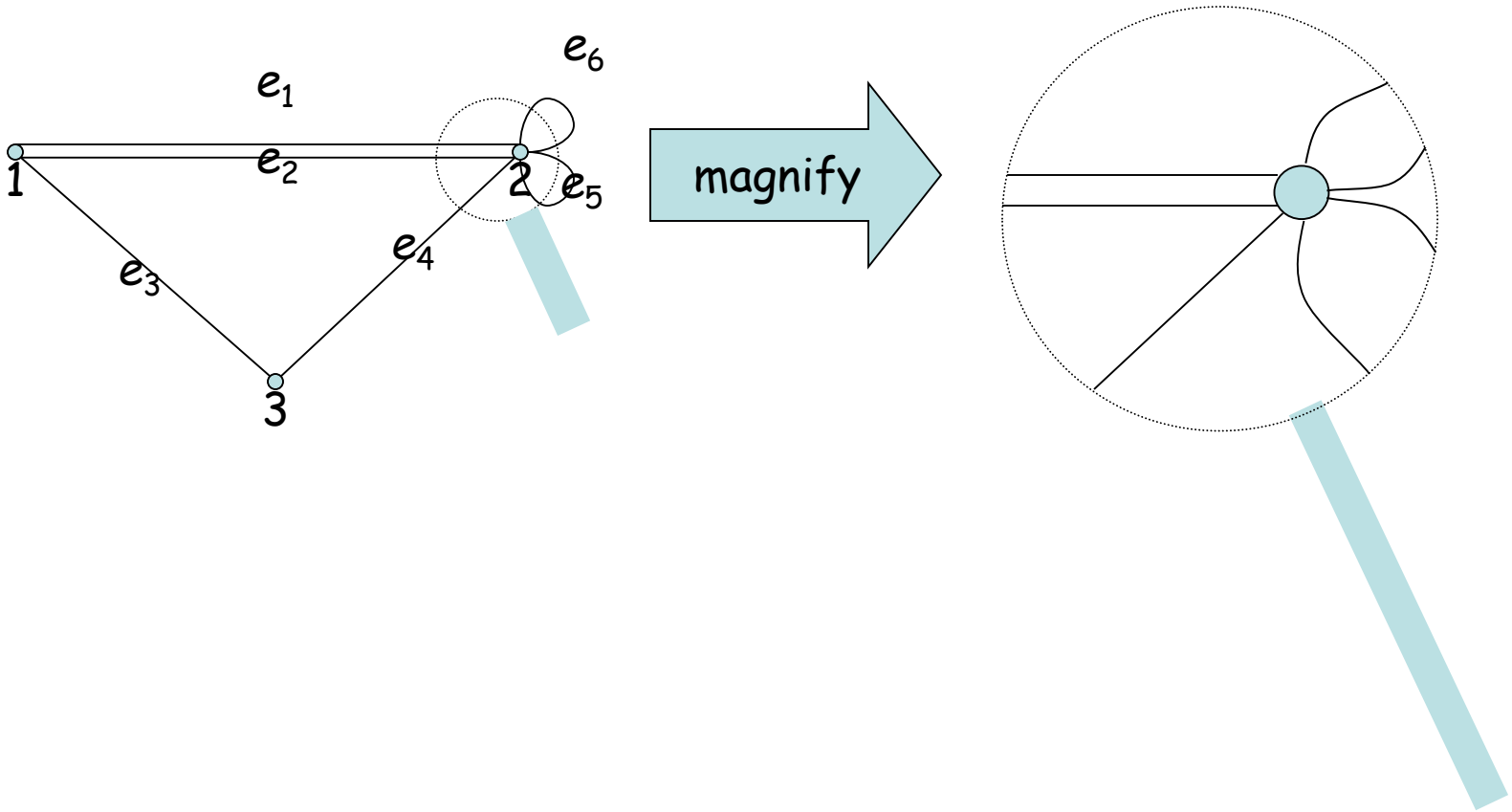


Thus deg(2) = 7 even though 2 only incident with 5 edges.

Q: How to define this formally?

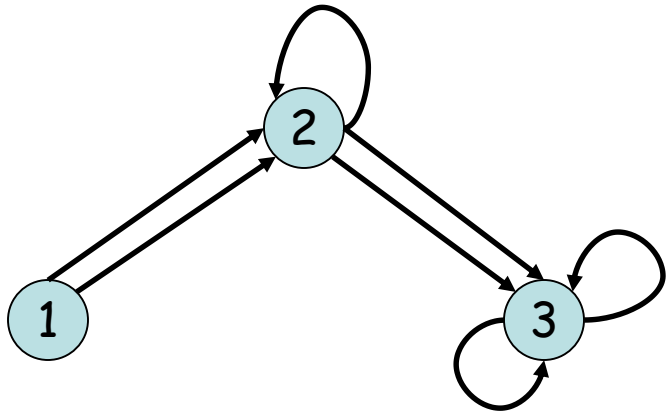
Degree

A: Add 1 for every regular edge incident with vertex and 2 for every loop. Thus deg(2) = 1 + 1 + 1 + 2 + 2 = 7



Oriented Degree when Edges Directed

The **in-degree** of a vertex ($\deg^-$) counts the number of edges that stick in to the vertex. The **out-degree** ($\deg^+$) counts the number sticking out.



Q: What are in-degrees and out-degrees of all the vertices?

Oriented Degree when Edges Directed

A: $\deg^-(1) = 0$

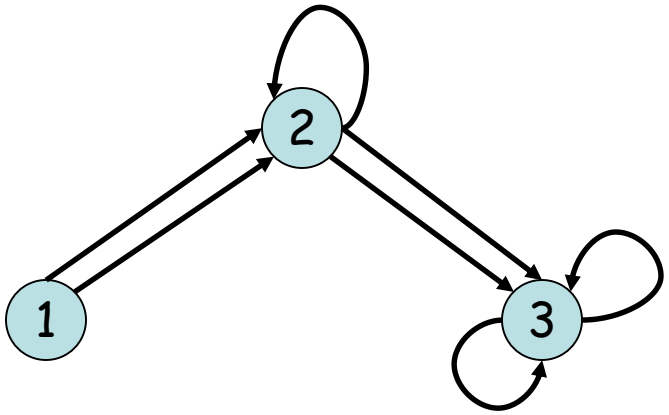
$\deg^-(2) = 3$

$\deg^-(3) = 4$

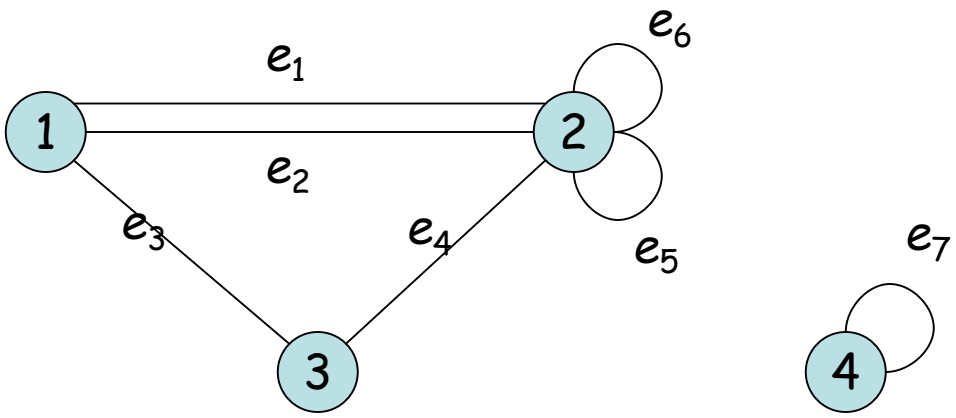
$\deg^+(1) = 2$

$\deg^+(2) = 3$

$\deg^+(3) = 2$



Handshaking Theorem



There are two ways to count the number of edges in the above graph:

1. Just count the set of edges: 7
2. Count seeming edges vertex by vertex and divide by 2 because double-counted edges: (deg(1)+deg(2)+deg(3)+deg(4))/2 = (3+7+2+2)/2 = 14/2 = 7

Handshaking Theorem

Theorem: In an undirected graph $|E| = \frac{1}{2}\sum_{v \in V} \deg(v)$

In a directed graph $|E| = \sum_{v \in V} \deg^{+}(v) = \sum_{v \in V} \deg^{-}(v)$

Q: In a party of 5 people can each person be friends(symmetric) with exactly three others?

Handshaking Theorem

A: Imagine a simple graph with 5 people as *vertices* and edges being undirected edges between friends (simple graph assuming friendship is symmetric and irreflexive). Number of friends each person has is the degree of the person.

Handshaking would imply that

$|E|$ = (sum of degrees)/2 or

$2|E|$ = (sum of degrees) = $(5 \cdot 3) = 15$.

Impossible as 15 is not even. In general:

Handshaking Theorem

Lemma: The number of vertices of odd degree must be even in an undirected graph.

Proof : Otherwise would have

$2|E|$ = Sum of even no.'s

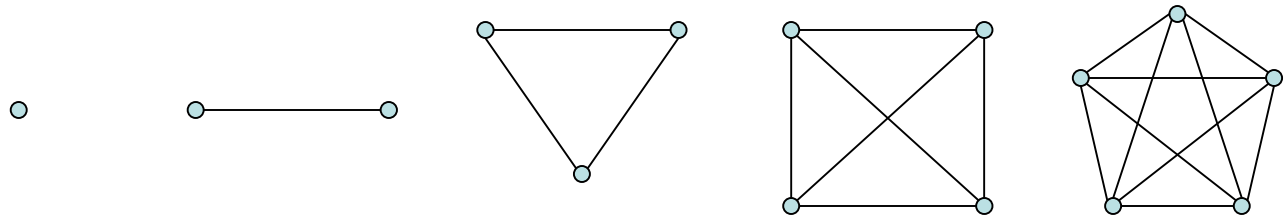
+ an odd number of odd no.'s

➔ even = even + odd

–this is impossible. •

Graph Patterns Complete Graphs - K_n

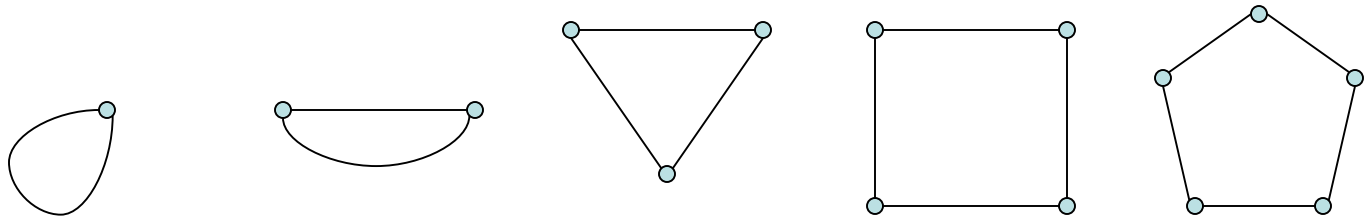
A simple graph is **complete** if every pair of distinct vertices share an edge. The notation K_n denotes the complete graph on n vertices.



K_1 K_2 K_3 K_4 K_5

Graph Patterns Cycles - C_n

The ***cycle graph*** C_n is a circular graph with $V = \{0,1,2,\ldots,n-1\}$ where vertex i is connected to $i+1$ **mod** n and to $i-1$ **mod** n. They look like polygons:



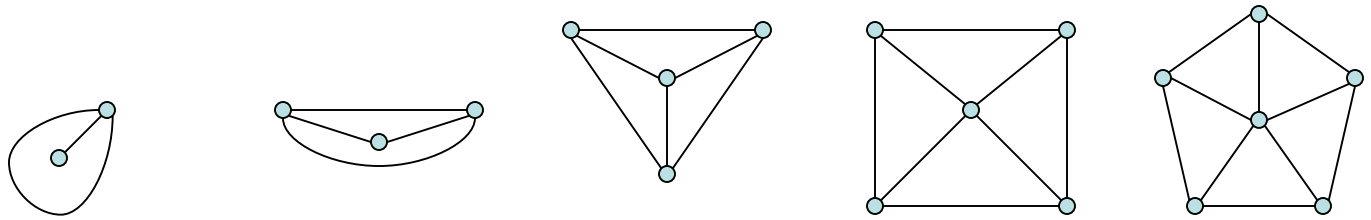
C_1 C_2 C_3 C_4 C_5

Q: What type of graph are C_1 and C_2 ?

Graph Patterns Wheels - W_n

A: Pseudographs

The ***wheel graph*** W_n is just a cycle graph with an extra vertex in the middle:



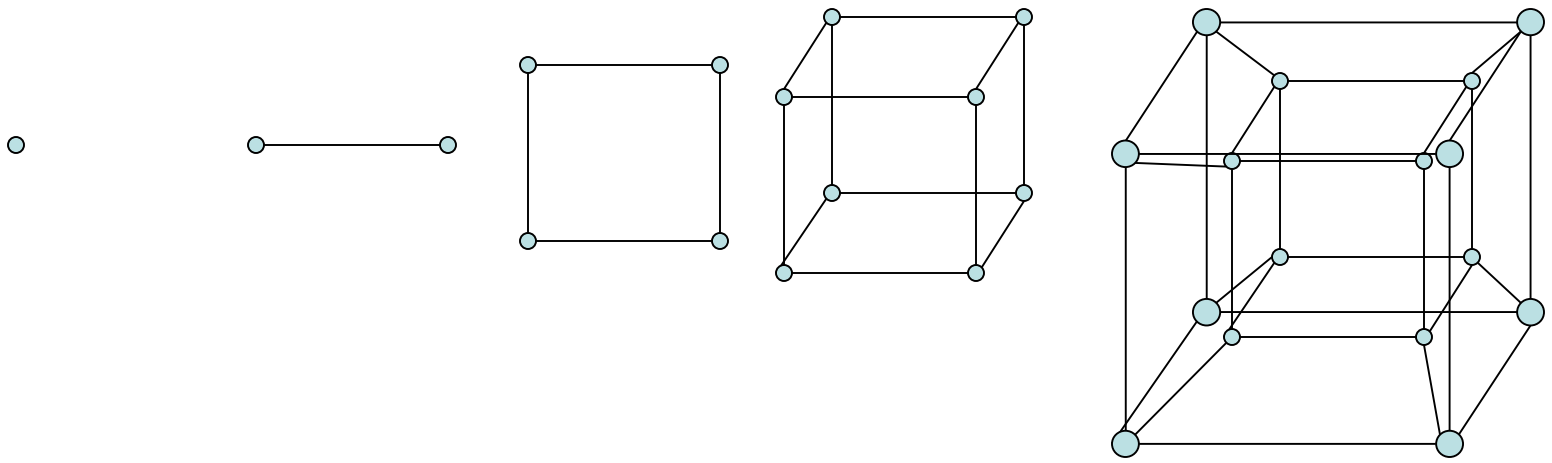
W_1 W_2 W_3 W_4 W_5

Usually consider wheels with 3 or more spokes only.

Graph Patterns Cubes - Q_n

The ***n-cube*** Q_n is defined recursively. Q_0 is just a vertex. Q_{n+1} is gotten by taking 2 copies of Q_n and joining each vertex v of Q_n with its copy v' :

Q_0 Q_1 Q_2 Q_3 Q_4 (hypercube)

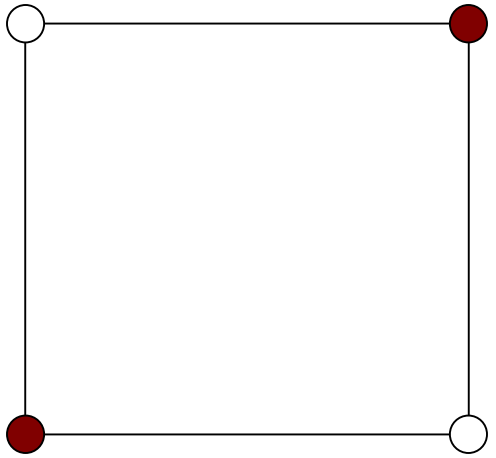


Bipartite Graphs

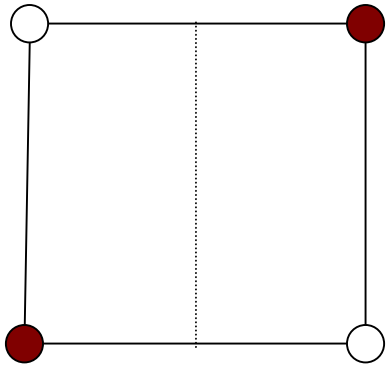
A simple graph is ***bipartite*** if V can be partitioned into $V = V_1 \cup V_2$ so that any two adjacent vertices are in different parts of the partition. Another way of expressing the same idea is ***bichromatic*** : vertices can be colored using two colors so that no two vertices of the same color are adjacent.

Bipartite Graphs

EG: C_4 is a bichromatic:

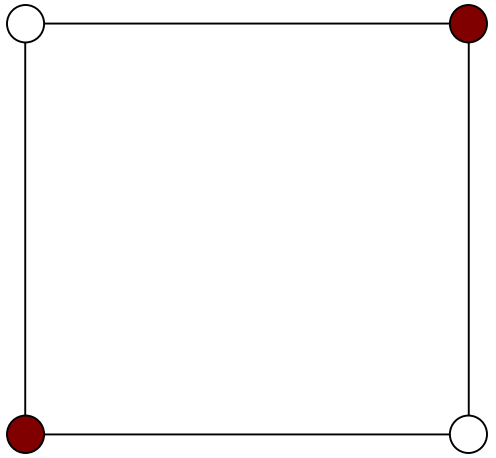


And so is bipartite, if we redraw it:

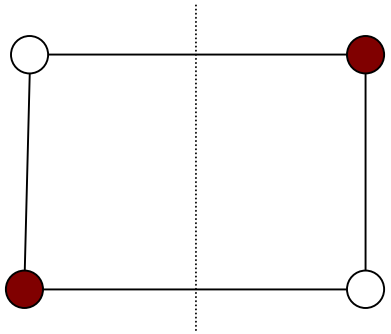


Bipartite Graphs

EG: C_4 is a bichromatic:



And so is bipartite, if we redraw it:



Bipartite Graphs

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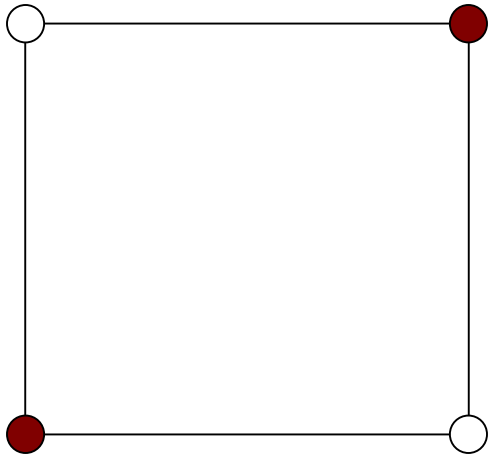

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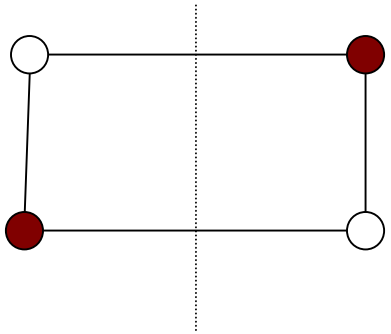

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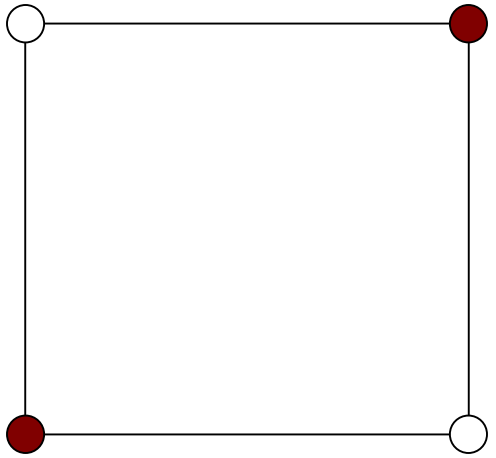

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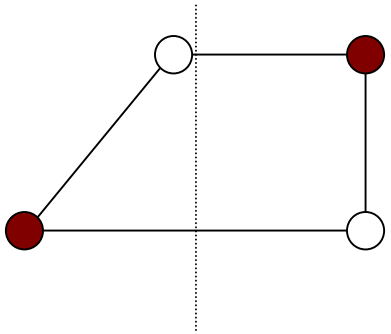

Bipartite Graphs

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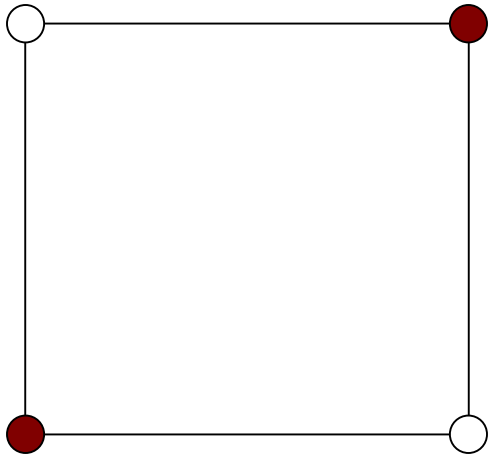

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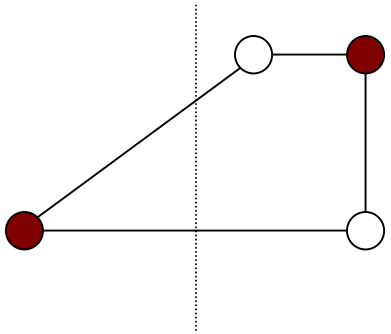

Bipartite Graphs

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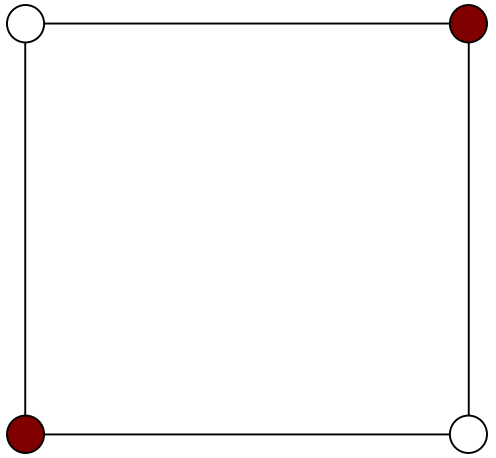

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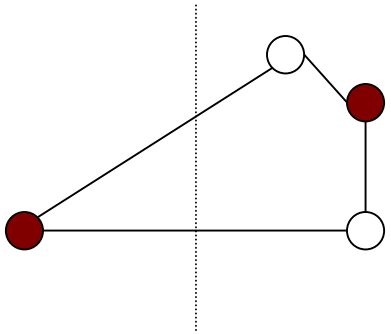

Bipartite Graphs

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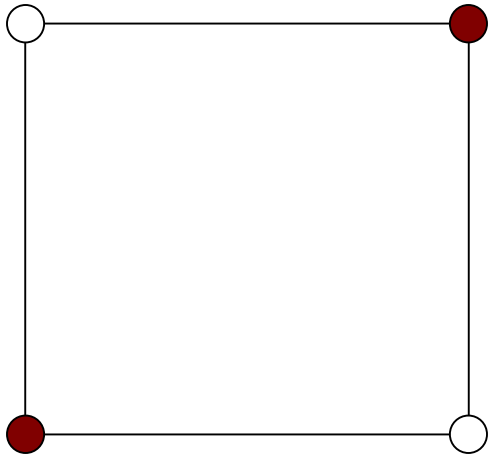

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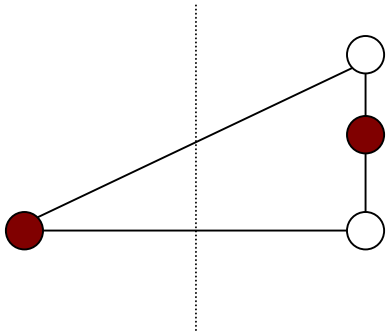

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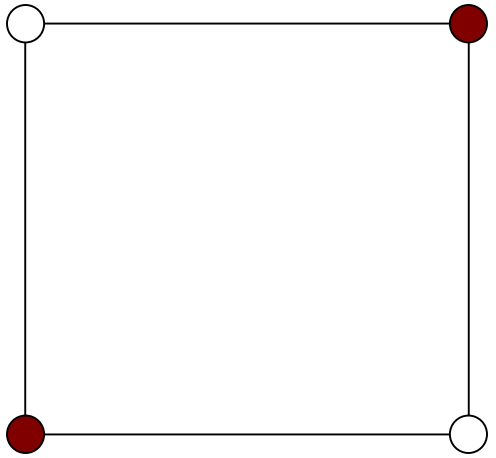

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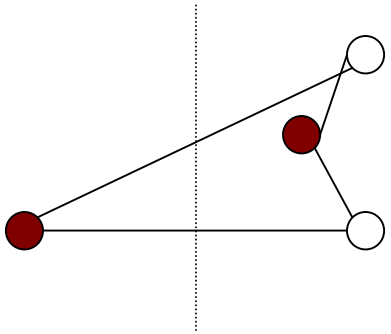

Bipartite Graphs

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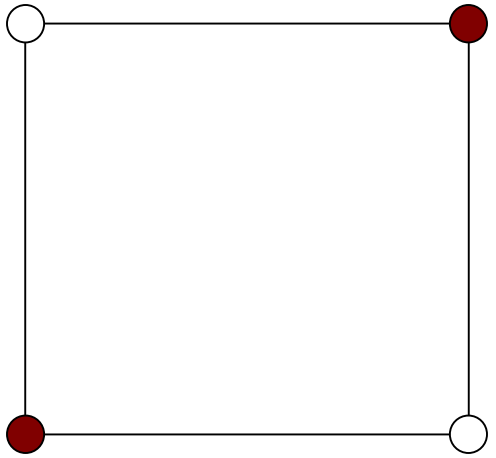

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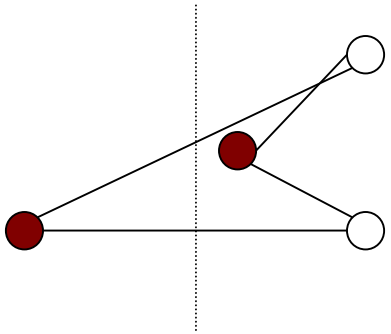

Bipartite Graphs

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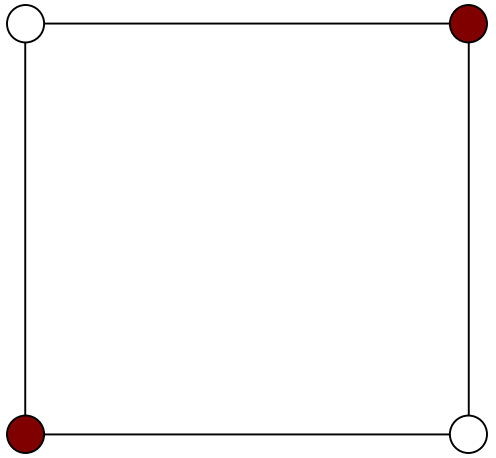

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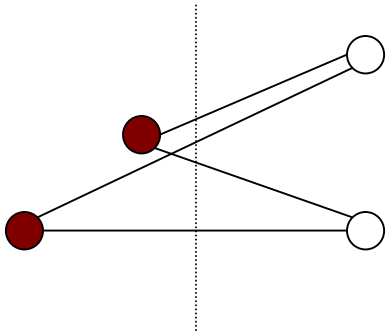

Bipartite Graphs

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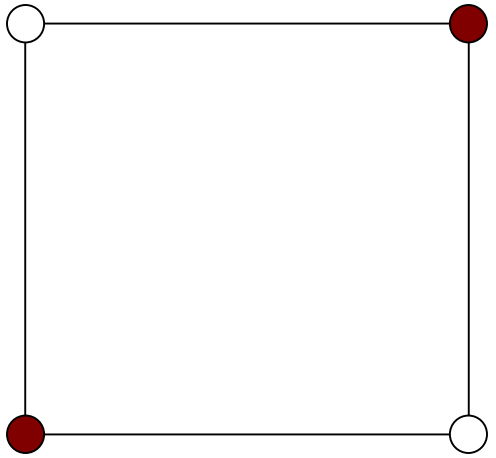
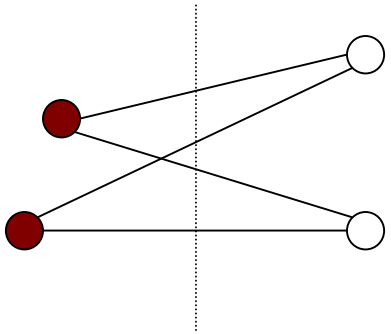


And so is bipartite, if we redraw it:



Bipartite Graphs

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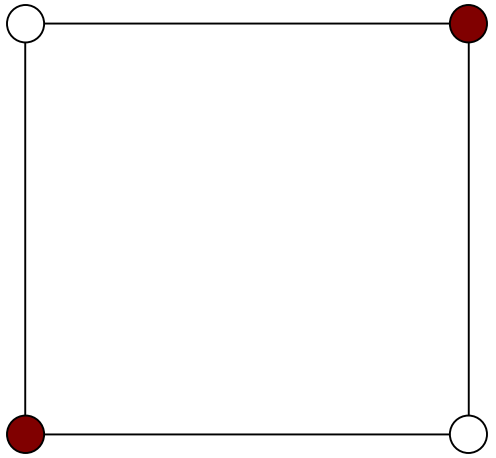
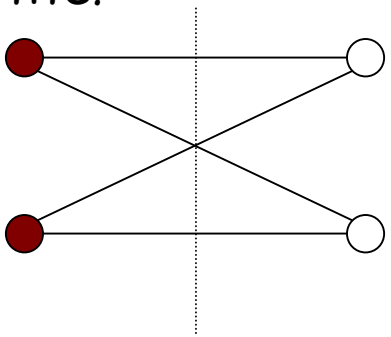


And so is bipartite, if we redraw it:

Q: For which n is C_n bipartite?



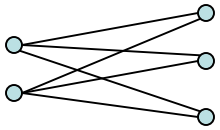
Bipartite Graphs

A: C_n is bipartite when n is even. For even n color all odd numbers red and all even numbers green so that vertices are only adjacent to opposite color.

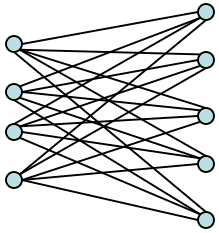
If n is odd, C_n is not bipartite.

Graph Patterns Complete Bipartite - $K_{m,n}$

When all possible edges exist in a simple bipartite graph with *m* red vertices and *n* green vertices, the graph is called ***complete bipartite*** and the notation $K_{m,n}$ is used. EG:



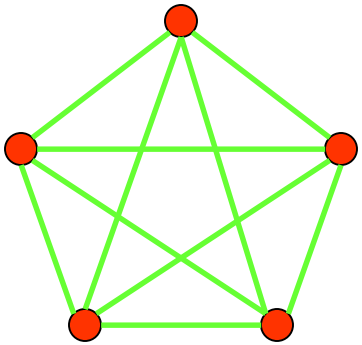
$K_{2,3}$



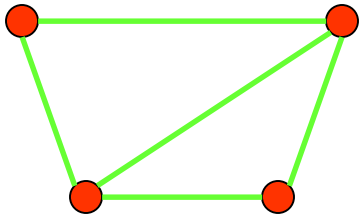
$K_{4,5}$

Operations on Graphs

- **Definition:** *A* **subgraph** of a graph $G = (V, E)$ is a graph $H = (W, F)$ where $W \subseteq V$ and $F \subseteq E$.
- **Example:**

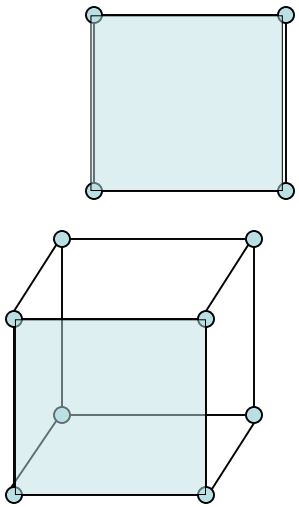


K_5



subgraph of K_5

Subgraphs



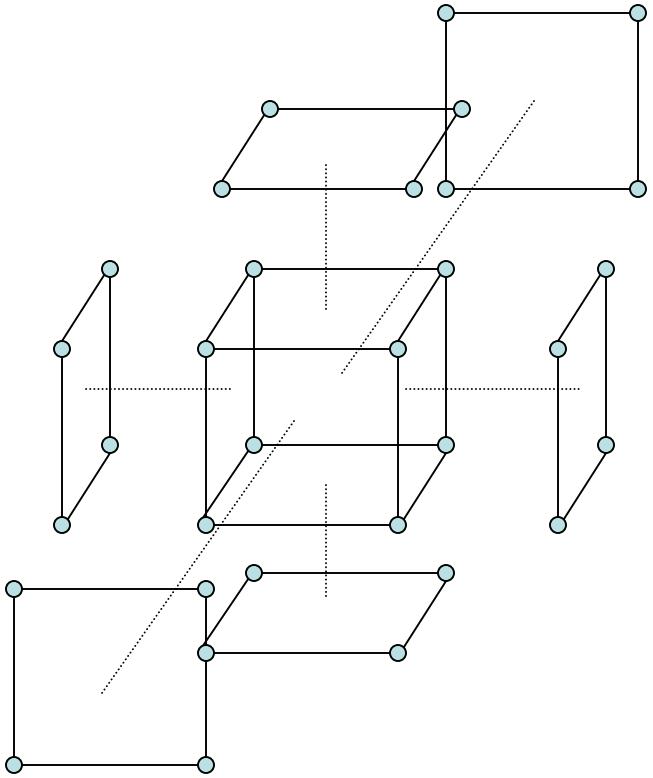
Notice that the 2-cube occurs inside the 3-cube. In other words, Q_2 is a subgraph of Q_3 :

Definition: Let $G = (V,E)$ and $H = (W,F)$ be graphs. H is said to be a **subgraph** of G, if $W \subseteq V$ and $F \subseteq E$.

Q: How many Q_2 subgraphs does Q_3 have?

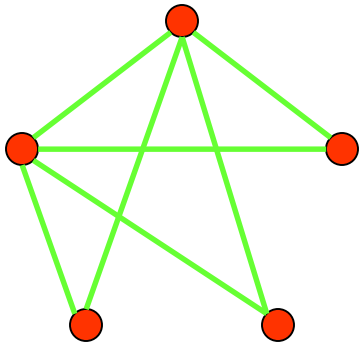
Subgraphs

A: Each face of Q_3 is a Q_2 subgraph so the answer is 6, as this is the number of faces on a 3-cube:

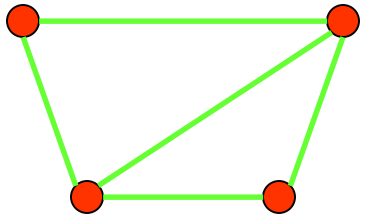


Operations on Graphs

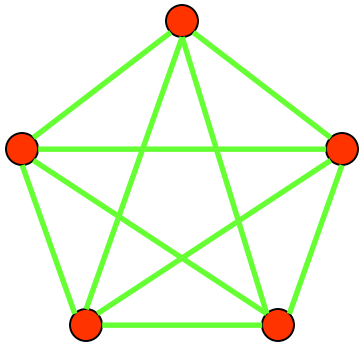
- **Definition:** The **union** of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$.
- The union of G_1 and G_2 is denoted by ***G1 ∪ G2***.



G_1



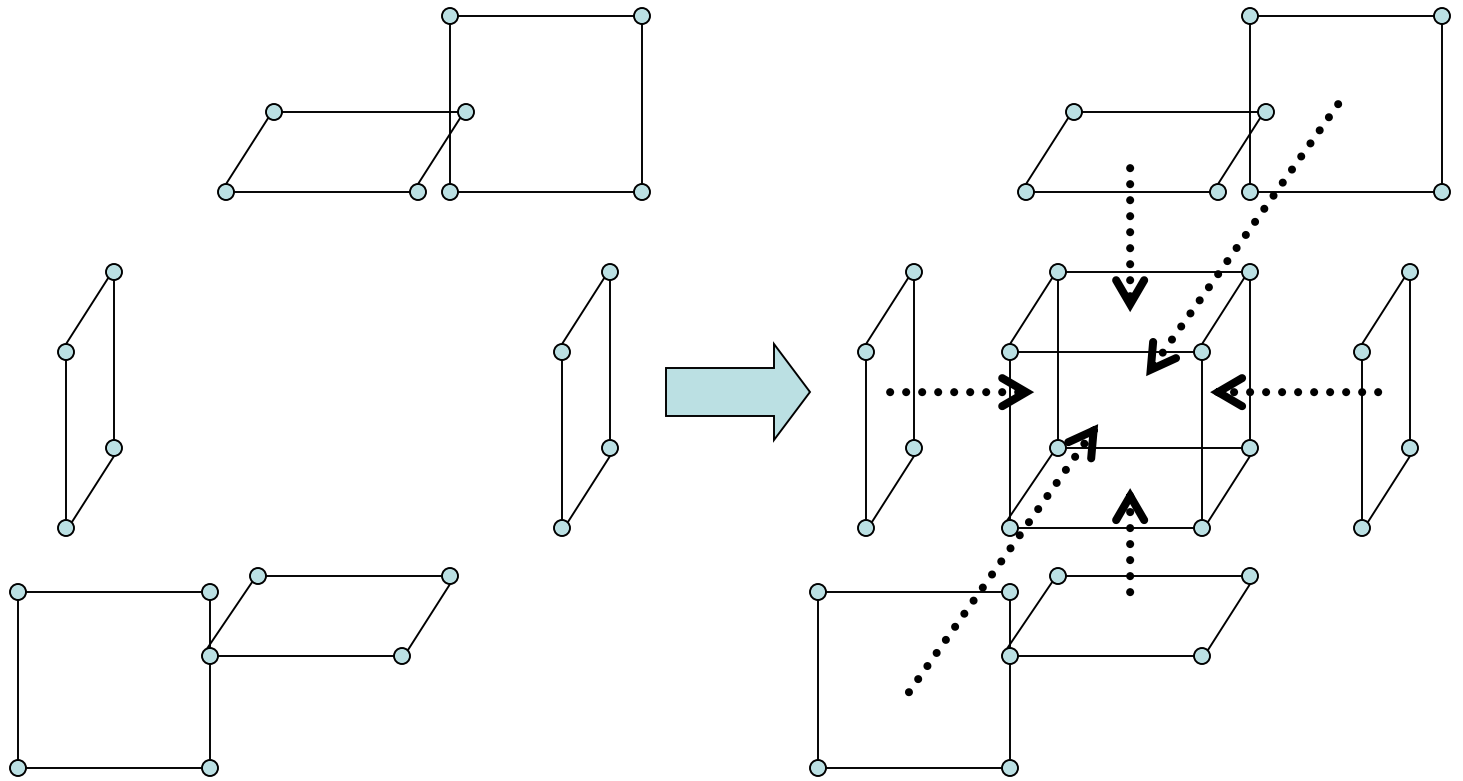
G_2



$G_1 \cup G_2 = K_5$

Unions

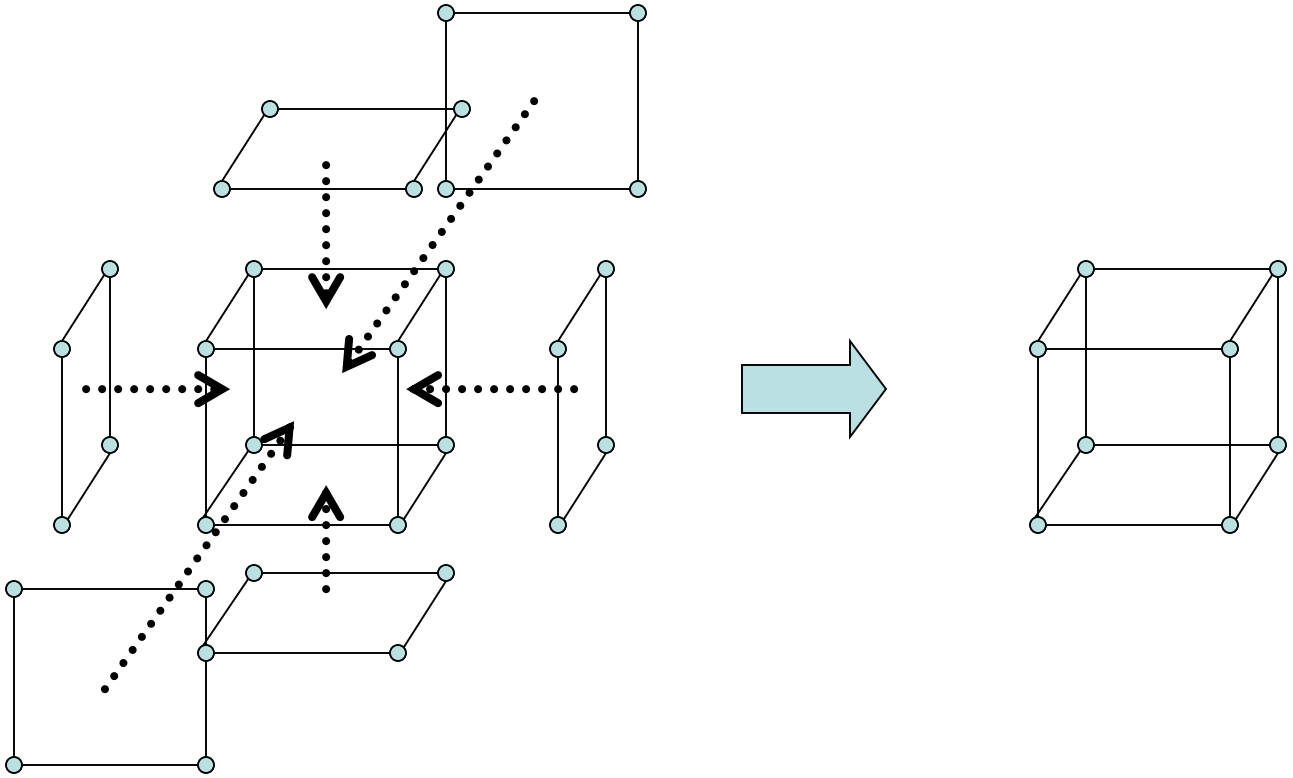
In previous example can actually reconstruct the 3-cube from its 6 2-cube faces:



Unions

If we assign the 2-cube faces (aka *S*quares) the names S_1, S_2, S_3, S_4, S_5, S_6 then Q_3 is the union of its faces:

$$Q_3 = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6$$



Unions

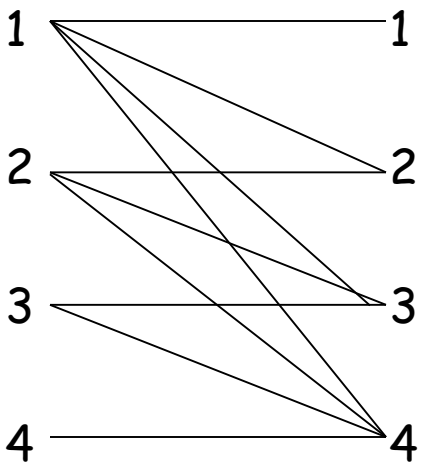
Definition: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs (and V_1, V_2 may or may not be disjoint). The **union** of G_1, G_2 is formed by taking the union of the vertices and edges. I.E: $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$.

A similar definitions can be created for unions of digraphs, multigraphs, pseudographs, etc.

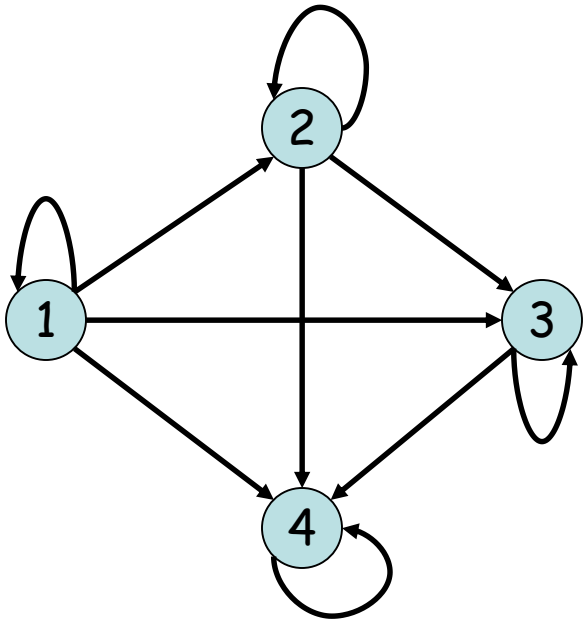
Graph Representation Adjacency Matrix

We already saw a way of representing relations on a set with a Boolean matrix:

R



digraph(*R*)



M_R

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Graph Representation Adjacency Matrix

Since digraphs are relations on their vertex sets, can adopt the concept to represent digraphs. In the context of graphs, we call the representation an **adjacency matrix**:

For a digraph $G = (V,E)$ define matrix A_G by:

- Rows, Columns –one for each vertex in V
- Value at i^{th} row and j^{th} column is
  - 1 if i^{th} vertex connects to j^{th} vertex ($i \rightarrow j$)
  - 0 otherwise

Adjacency Matrix-Directed Multigraphs

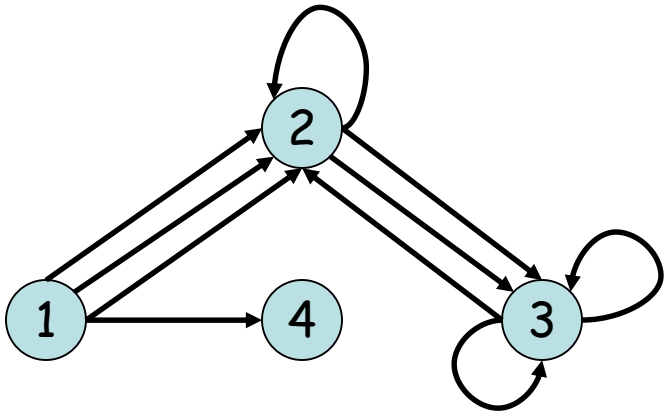
Can easily generalize to directed multigraphs by putting in the number of edges between vertices, instead of only allowing 0 and 1:

For a directed multigraph $G = (V,E)$ define the matrix A_G by:

- Rows, Columns –one for each vertex in V
- Value at i^{th} row and j^{th} column is the number of edges with source the i^{th} vertex and target the j^{th} vertex

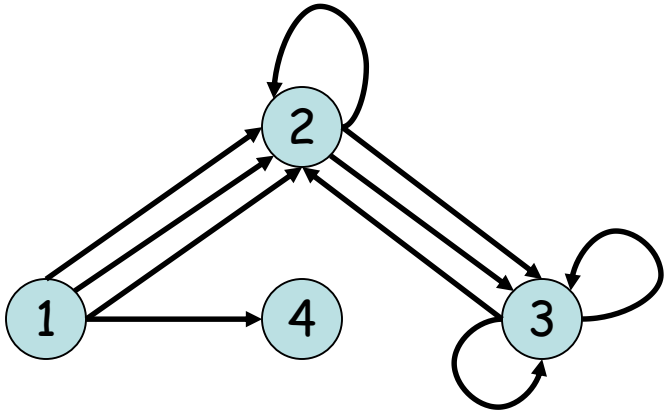
Adjacency Matrix-Directed Multigraphs

Q: What is the adjacency matrix?



Adjacency Matrix-Directed Multigraphs

A:

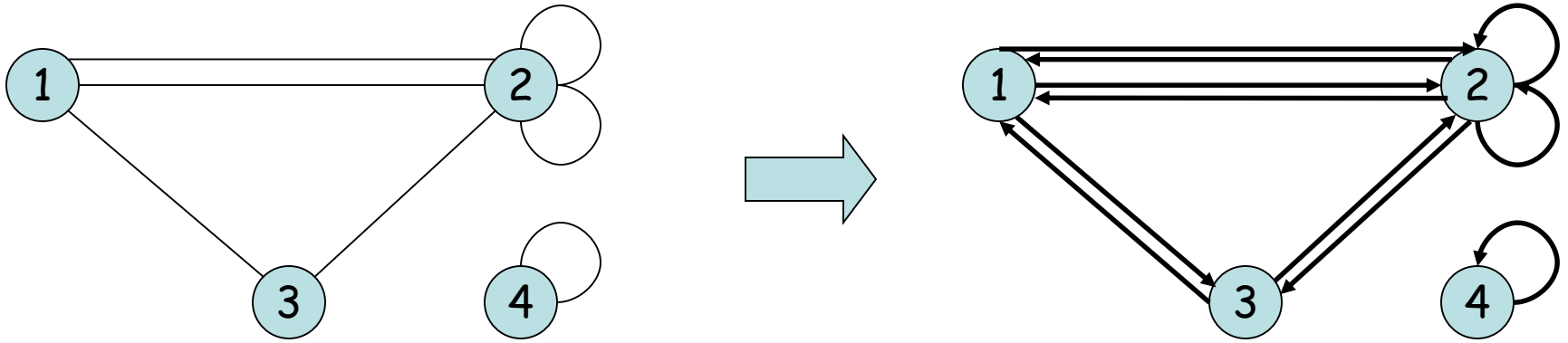


$$\begin{pmatrix} 0 & 3 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Adjacency Matrix-General

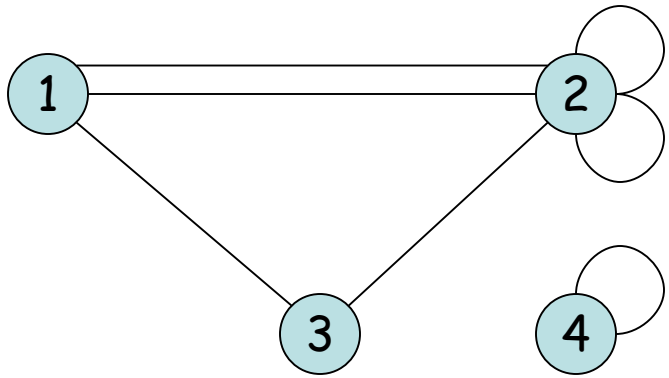
Undirected graphs can be viewed as directed graphs by turning each undirected edge into two oppositely oriented directed edges, except when the edge is a self-loop in which case only 1 directed edge is introduced.

EG:



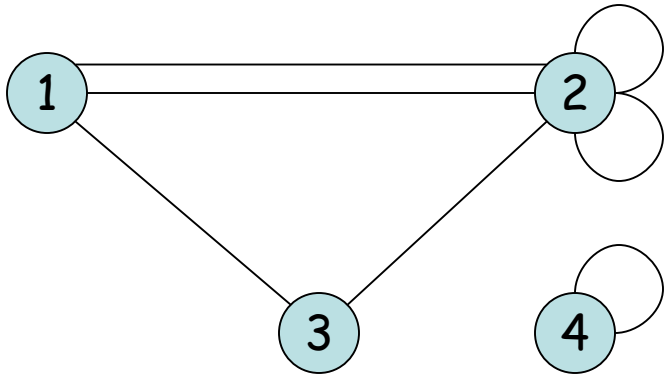
Adjacency Matrix-General

Q: What's the adjacency matrix?



Adjacency Matrix-General

A:



Notice that answer is *symmetric*.

$$\begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Adjacency Matrix-General

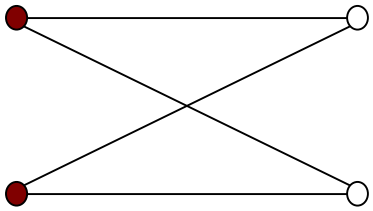
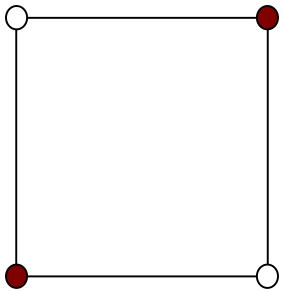
For an undirected graph G = (V,E) define the matrix A_G by:

- Rows, Columns –one for each element of V
- Value at i th row and j th column is the number of edges incident with vertices i and j.

This is equivalent to converting first to a directed graph as above. Or by allowing undirected edges to take us from i to j can simply use definition for directed graphs.

Graph Isomorphism

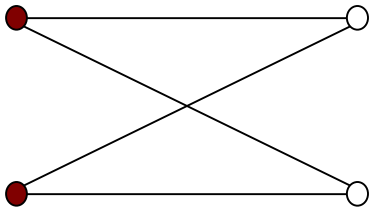
Various mathematical notions come with their own concept of *equivalence*, as opposed to equality:

- Equivalence for sets is bijectivity:
  - EG {  ,  ,  } ≈ {12, 23, 43}
- Equivalence for graphs is isomorphism:
  - EG  ≈ 

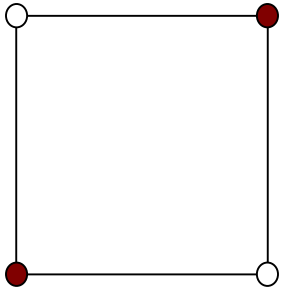
Graph Isomorphism

Intuitively, two graphs are isomorphic if can bend, stretch and reposition vertices of the first graph, until the second graph is formed. Etymologically, *isomorphic* means "same shape".

EG: Can twist or relabel:



to obtain:



Graph Isomorphism Undirected Graphs

Definition: Suppose $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are pseudographs. Let $f : V_1 \rightarrow V_2$ be a function s.t.:

1. f is bijective
2. for all vertices u,v in V_1, the number of edges **between u and v** in G_1 is the same as the number of edges between f (u) and f (v) in G_2.

Then f is called an **isomorphism** and G_1 is said to be **isomorphic** to G_2.

Graph Isomorphism Digraphs

Definition: Suppose $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are directed multigraphs. Let $f : V_1 \rightarrow V_2$ be a function s.t.:

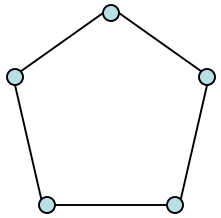
1) f is bijective

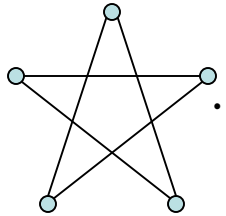
2) for all vertices u,v in V_1, the number of edges **from u to v** in G_1 is the same as the number of edges between f (u) and f (v) in G_2.

Then f is called an **isomorphism** and G_1 is said to be **isomorphic** to G_2.

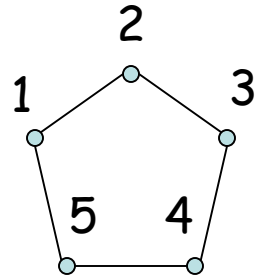
Note: Only difference between two definitions is the italicized "from" in no. 2 (was "between").

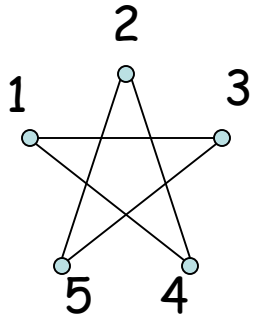
Graph Isomorphism-Example

EG: Prove that 

is isomorphic to  .

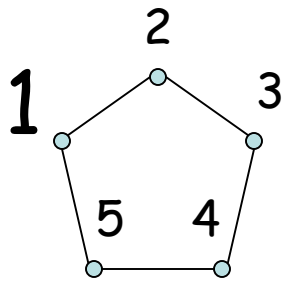
First label the vertices:

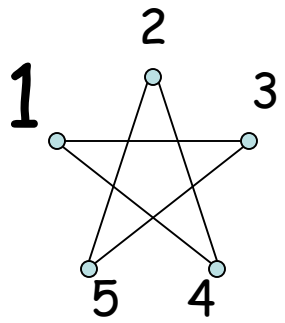




Graph Isomorphism-Example

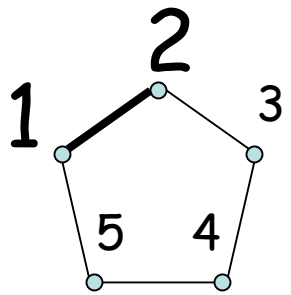
Next, set f (1) = 1 and try to walk around clockwise on the star.

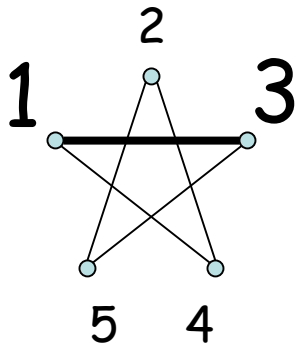




Graph Isomorphism-Example

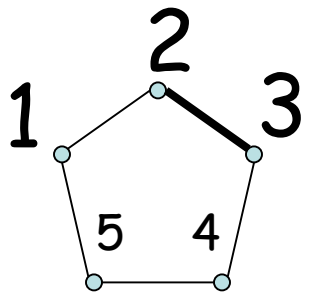
Next, set f (1) = 1 and try to walk around clockwise on the star. The next vertex seen is 3, not 2 so set f (2) = 3.

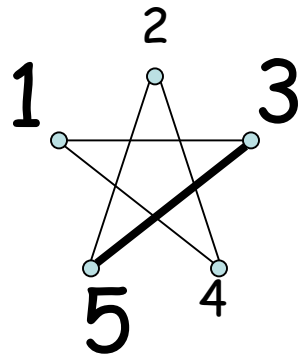




Graph Isomorphism-Example

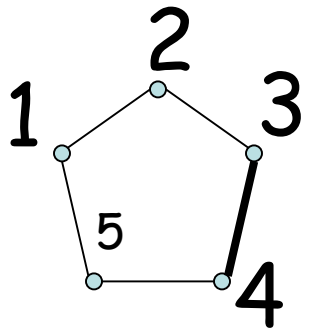
Next, set f (1) = 1 and try to walk around clockwise on the star. The next vertex seen is 3, not 2 so set f (2) = 3. Next vertex is 5 so set f (3) = 5.

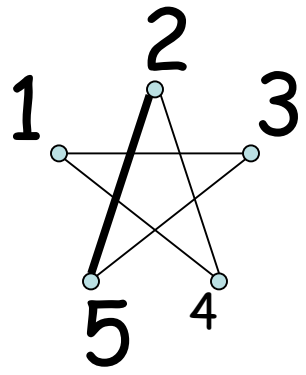




Graph Isomorphism-Example

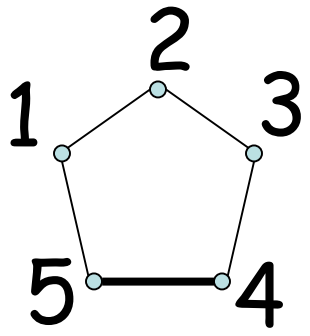
Next, set f (1) = 1 and try to walk around clockwise on the star. The next vertex seen is 3, not 2 so set f (2) = 3. Next vertex is 5 so set f (3) = 5. In this fashion we get f (4) = 2

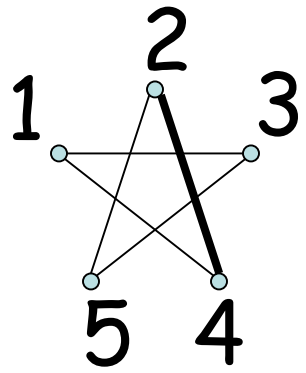




Graph Isomorphism-Example

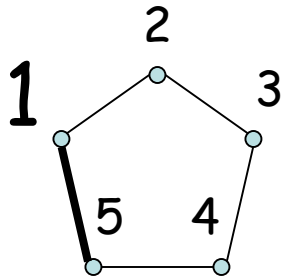
Next, set f (1) = 1 and try to walk around clockwise on the star. The next vertex seen is 3, not 2 so set f (2) = 3. Next vertex is 5 so set f (3) = 5. In this fashion we get f (4) = 2, f (5) = 4.

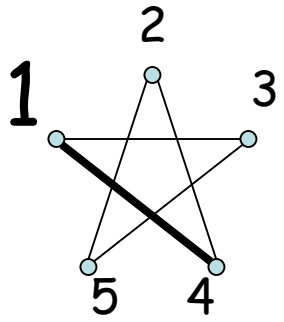




Graph Isomorphism-Example

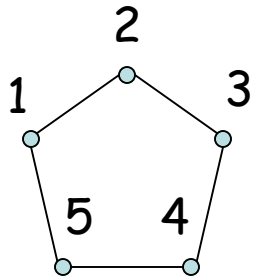
Next, set f (1) = 1 and try to walk around clockwise on the star. The next vertex seen is 3, not 2 so set f (2) = 3. Next vertex is 5 so set f (3) = 5. In this fashion we get f (4) = 2, f (5) = 4. If we would continue, we would get back to f (1) =1 so this process is well defined and f is a morphism.

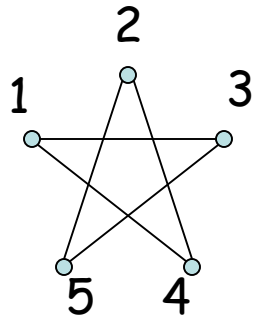




Graph Isomorphism-Example

Next, set f (1) = 1 and try to walk around clockwise on the star. The next vertex seen is 3, not 2 so set f (2) = 3. Next vertex is 5 so set f (3) = 5. In this fashion we get f (4) = 2, f (5) = 4. If we would continue, we would get back to f (1) =1 so this process is well defined and f is a morphism. Finally since f is bijective, f is an isomorphism.





Properties of Isomorphims

Since graphs are completely defined by their vertex sets and the number of edges between each pair, isomorphic graphs must have the same intrinsic properties. I.e. isomorphic graphs have the same...

...number of vertices and edges

...degrees at corresponding vertices

...types of possible subgraphs

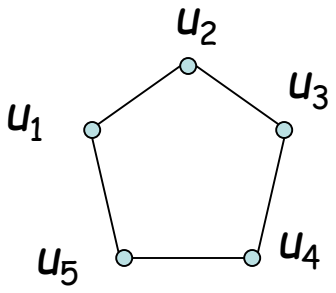
...any other property defined in terms of the basic graph theoretic building blocks!

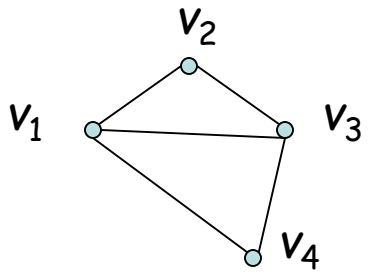
Graph Isomorphism-Negative Examples

Once you see that graphs are isomorphic, easy to prove it. Proving the opposite, is usually more difficult. To show that two graphs are non-isomorphic need to show that no function can exist that satisfies defining properties of isomorphism. In practice, you try to find some intrinsic property that differs between the 2 graphs in question.

Graph Isomorphism-Negative Examples

Q: Why are the following non-isomorphic?

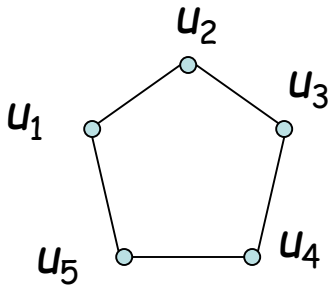


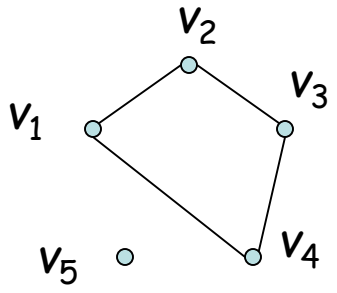


A: 1st graph has more vertices than 2nd.

Graph Isomorphism-Negative Examples

Q: Why are the following non-isomorphic?

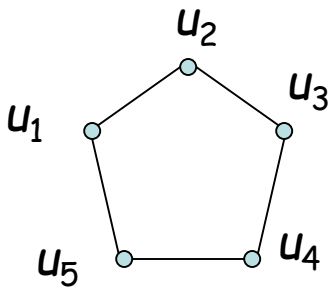


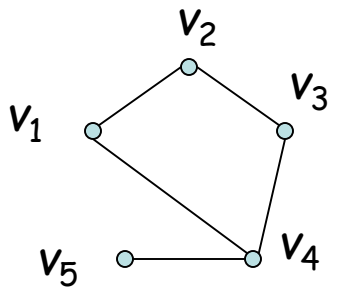


A: 1st graph has more edges than 2nd.

Graph Isomorphism-Negative Examples

Q: Why are the following non-isomorphic?

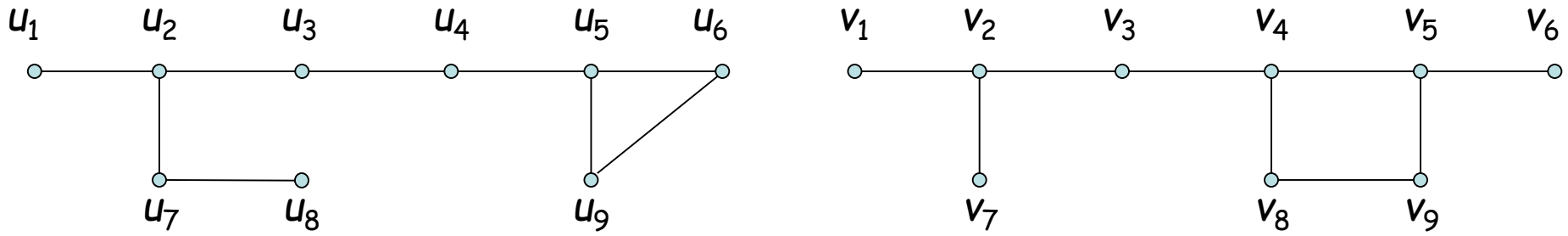




A: 2nd graph has vertex of degree 1, 1st graph doesn't.

Graph Isomorphism-Negative Examples

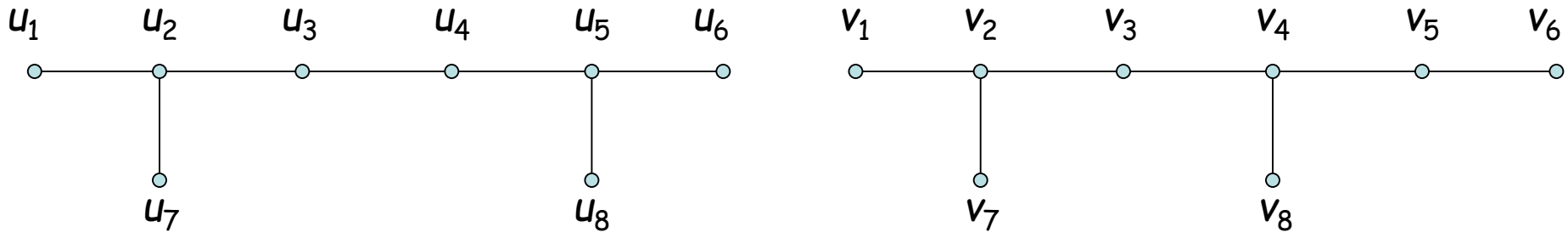
Q: Why are the following non-isomorphic?



A: 1st graph has 2 degree 1 vertices, 4 degree 2 vertex and 2 degree 3 vertices.
2nd graph has 3 degree 1 vertices, 3 degree 2 vertex and 3 degree 3 vertices.

Graph Isomorphism-Negative Examples

Q: Why are the following non-isomorphic?



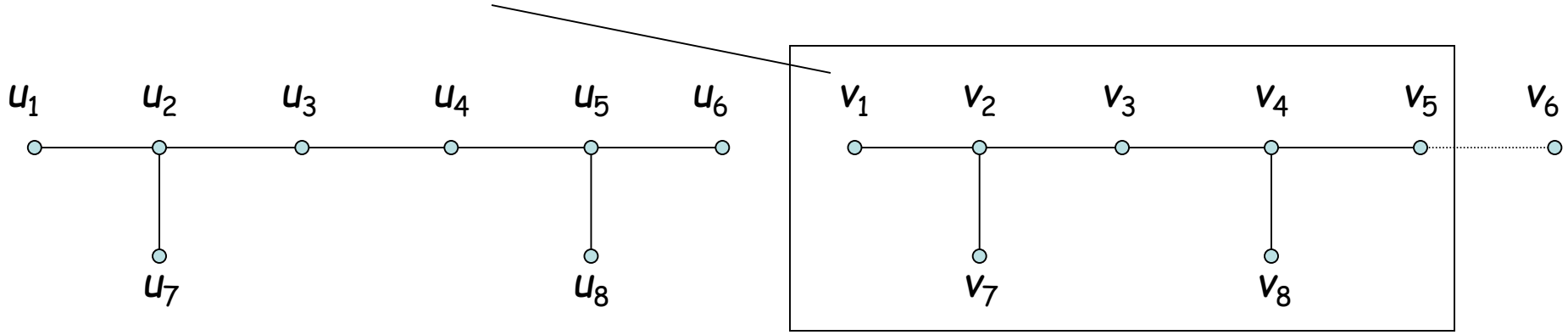
A: None of the previous approaches work as there are the same no. of vertices, edges, and same no. of vertices per degree.

Lemma: If G and H are isomorphic, then any subgraph of G will be isomorphic to some subgraph of H.

Find a subgraph of 2nd graph which isn't a subgraph of 1st graph.

Graph Isomorphism-Negative Examples

This subgraph is not a subgraph of the left graph.



Why not? Deg. 3 vertices must map to deg. 3 vertices. Since subgraph and left graph are symmetric, can assume v_2 maps to u_2. Adjacent deg. 1 vertices to v_2 must map to degree 1 vertices, forcing the deg. 2 adjacent vertex v_3 to map to u_3. This forces the other vertex adjacent to v_3, namely v_4 to map to u_4. But then a deg. 3 vertex has mapped to a deg. 2 vertex→← •