Principal curvature and direction

When $\lambda_1 = \lambda_2$, for any tangent direction angle θ,

$$k_n(\theta) = \lambda_1 = \lambda_2$$

Then we cannot determine the principal direction.
We call this kind of points as **umbilical point** (脐点).

The following equations stand up for umbilical point

$$\frac{L}{E} = \frac{M}{F} = \frac{N}{G}.$$

If this ratio is zero, this umbilical point is called as **planar point** (平点),
otherwise it's called as **circular point** (圆点).

Curvature line

Proposition:
A surface S is a plane,
if and only if all points of S are planar points;
A surface S is a sphere,
if and only if all points of S are spherical points.

Definition:
Assume C is a curve on regular surface S.
If the tangent vector of C at any point is the principal direction of S at that point,
then C is a **curvature line** (曲率线) of S.

Curvature line

Assume the parametric function of curve C on surface S: $\boldsymbol{r} = \boldsymbol{r}(u, v)$ is

$$u = u(t), v = v(t)$$

According to the definition of curvature line, curve C is a curvature line if

$$W\left(\frac{d\boldsymbol{r}(u(t), v(t))}{dt}\right) = \lambda \frac{d\boldsymbol{r}(u(t), v(t))}{dt}$$

In another way, according to the definition of Weingarten map, we have

$$W\left(\frac{d\boldsymbol{r}(u(t), v(t))}{dt}\right) = -\frac{d\boldsymbol{n}(u(t), v(t))}{dt}$$

Therefore, we get the following criterion for curvature line:

Curvature line

Theorem (Rodriques theorem):

A curve $C: u = u(t), v = v(t)$ on surface $S: \boldsymbol{r} = \boldsymbol{r}(u, v)$ is a curvature line if and only if
the derivative of the normal vector field $\boldsymbol{n}(u(t), v(t))$of S along C is tangent to C, that is

$$\frac{d\boldsymbol{n}(u(t), v(t))}{dt} \parallel \frac{d\boldsymbol{r}(u(t), v(t))}{dt}$$

Curvature line

Theorem:

A curve $C: u = u(t), v = v(t)$ on surface $S: \boldsymbol{r} = \boldsymbol{r}(u, v)$ is a curvature line if and only if the normal lines (法线) of S along C form a developable surface.

Proof: (Sufficiency)

Assume the parametric function of curve C on surface $S: \boldsymbol{r} = \boldsymbol{r}(u, v)$ is

$$u = u(s), v = v(s)$$

where s is arc length parameter.

The unit normal vector field of S along C is $\boldsymbol{n}(s) = \boldsymbol{n}\big(u(s), v(s)\big)$.

So, the ruled surface formed by normal line is

$$\boldsymbol{r} = \boldsymbol{r}(s) + t\boldsymbol{n}(s)$$

where $\boldsymbol{r}(s) = \boldsymbol{r}\big(u(s), v(s)\big)$.

Curvature line

As we know, a ruled surface is a developable surface if and only if

$$\left(\boldsymbol{r}'(s), \boldsymbol{n}(s), \boldsymbol{n}'(s)\right) = 0$$

Since $\boldsymbol{n}(s)$ is the unit normal vector of S, so we have

$$\boldsymbol{n}(s).\boldsymbol{r}'(s) = 0$$

$$\boldsymbol{n}(s).\boldsymbol{n}'(s) = 0$$

That means

$$\boldsymbol{n}(s) \parallel \boldsymbol{r}'(s) \times \boldsymbol{n}'(s)$$

Let

$$\boldsymbol{r}'(s) \times \boldsymbol{n}'(s) = \lambda \boldsymbol{n}(s)$$

Then

$$\begin{aligned} 0 &= -\left(\boldsymbol{r}'(s) \times \boldsymbol{n}'(s)\right).\boldsymbol{n}(s) \\ &= \lambda \boldsymbol{n}(s).\boldsymbol{n}(s) \\ &= \lambda \end{aligned}$$

Curvature line

That means

$$\boldsymbol{r}'(s) \times \boldsymbol{n}'(s) = 0$$
$$\Leftrightarrow \boldsymbol{r}'(s) \parallel \boldsymbol{n}'(s)$$

According to Rodriques theorem, C is the curvature line of S.

Necessity

$$\Leftrightarrow \left(\boldsymbol{r}'(s), \boldsymbol{n}(s), \boldsymbol{n}'(s)\right) = 0$$

That is the ruled surface formed by normal lines of C on S is a developable surface.

Therefore, the theorem holds.

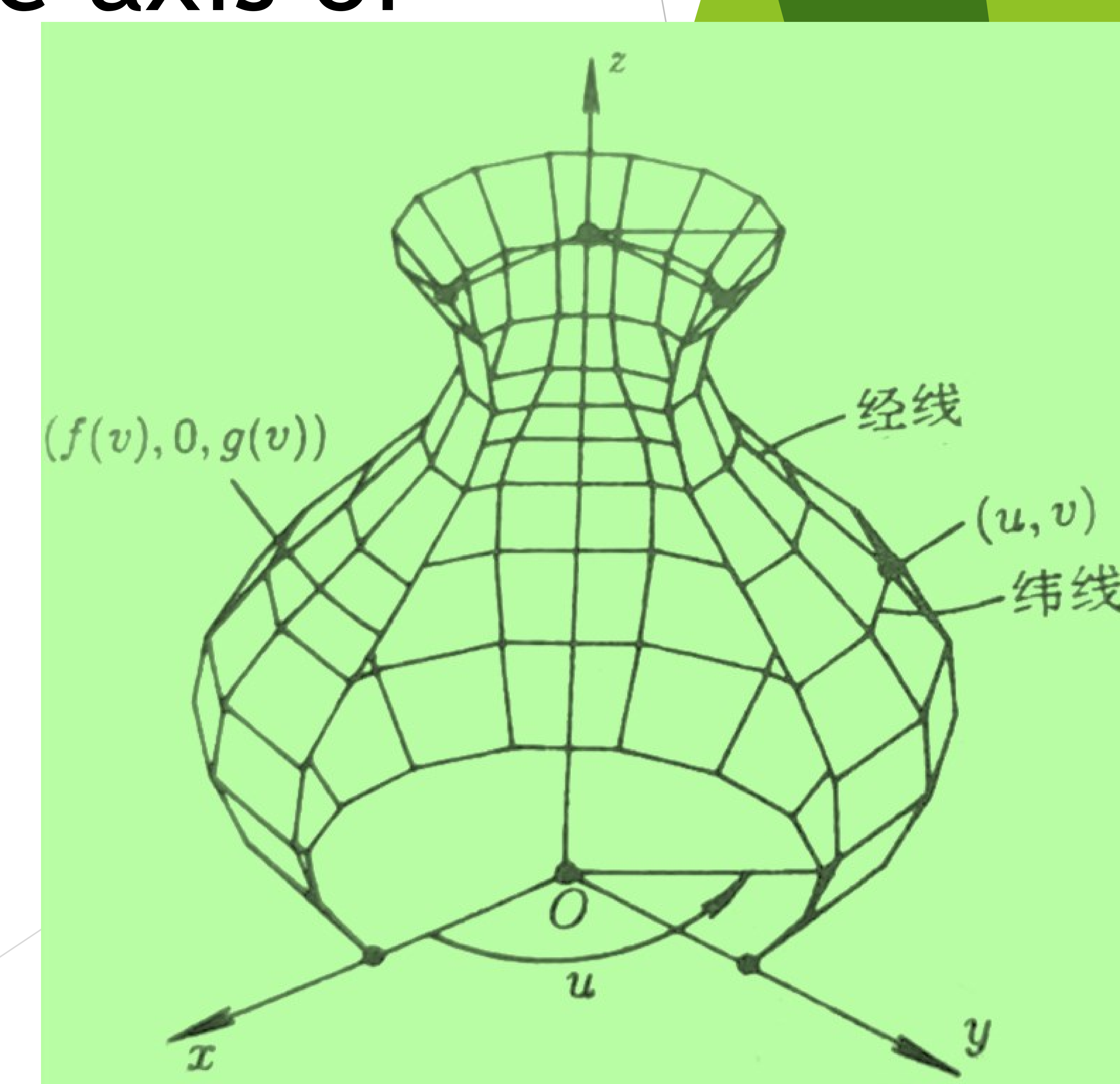
Example

Calculate the curvature line of revolution surface.

Answer:

The longitude line (经线) of revolution surface is the intersection of surface and the plane passing through the axis of revolution (旋转轴).

So the normal lines of surface along longitude line fall in the plane formed by longitude line and the axis of revolution. That is the surface formed by normal lines is developable surface.



The normal lines of surface along latitude line (纬线) pass through a fixed point on the axis of revolution.

So they form a developable surface.

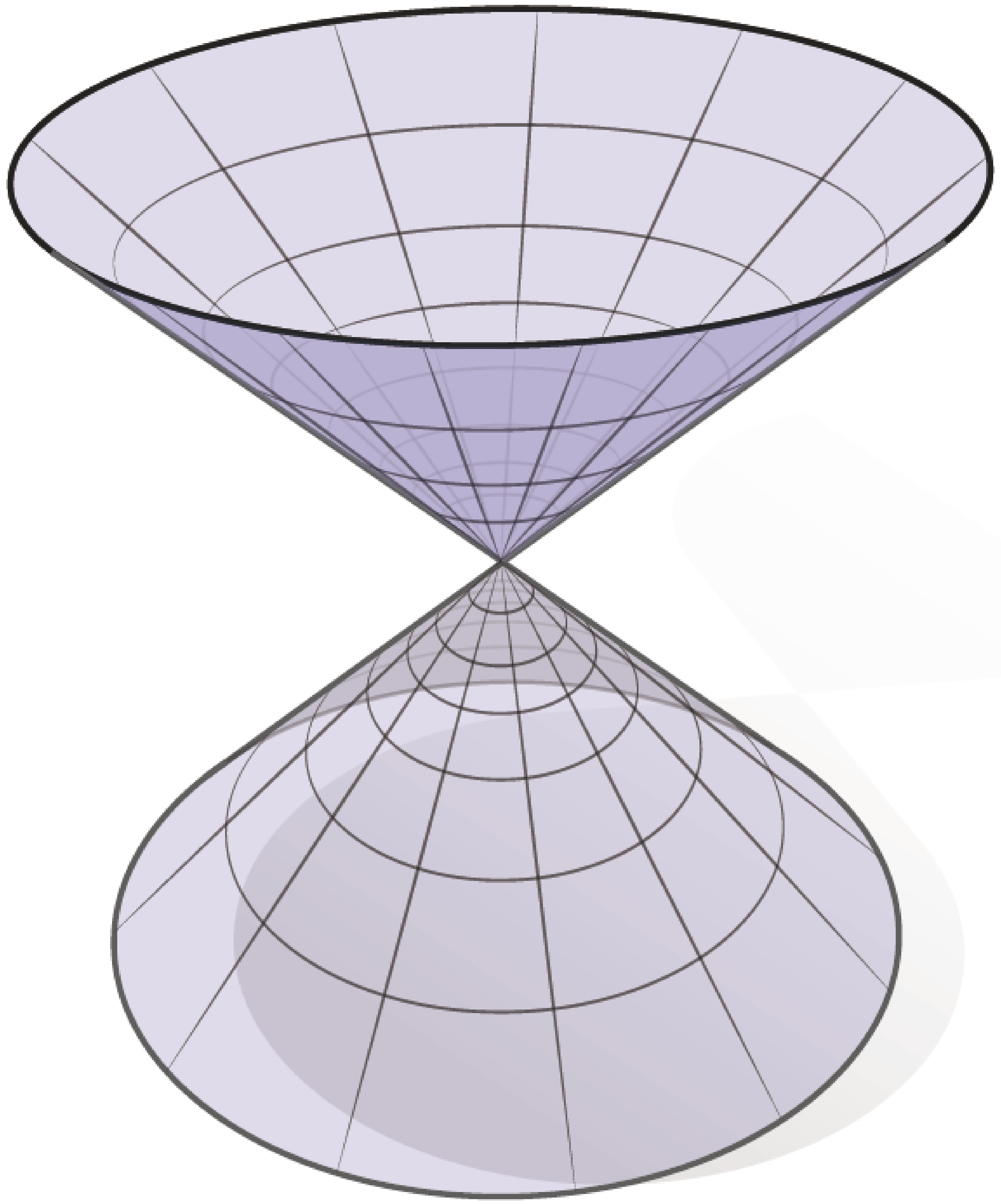
Example

Calculate the curvature line of developable surface.

Answer:

Developable surface is ruled surface.
The tangent vectors along generating line keep unchanged.
That is to say, the unit normal vector field of surface along generating line is a constant vector field.
So the derivative of the unit normal vector along generating line is zero.
Then $\frac{d\boldsymbol{n}(u(t),v(t))}{dt} \parallel \frac{d\boldsymbol{r}(u(t),v(t))}{dt}$ is satisfied.
The curves orthogonal to the generating lines are the other family of curvature lines.

Calculation of principal curvature and principle direction



Mean curvature and Gauss curvature

Assume the parametric function of surface S is $\boldsymbol{r} = \boldsymbol{r}(u, v)$.
Assume $\delta\boldsymbol{r} = \boldsymbol{r}_u\delta u + \boldsymbol{r}_v\delta v$ is a principal direction of S at point (u, v), that is $(\delta u, \delta v) \neq 0$, and there is real number λ so that

$$W(\delta\boldsymbol{r}) = \lambda\delta\boldsymbol{r} \longleftarrow$$

$$W = -g_*: T_pS \to T_pS$$

$$g_*(\boldsymbol{r}_u) = \boldsymbol{n}_u, \qquad g_*(\boldsymbol{r}_v) = \boldsymbol{n}_v$$

Then

$$-(\boldsymbol{n}_u\delta u + \boldsymbol{n}_v\delta v) = \lambda(\boldsymbol{r}_u\delta u + \boldsymbol{r}_v\delta v)$$

$$\Rightarrow \begin{cases} -(\boldsymbol{n}_u\delta u + \boldsymbol{n}_v\delta v).\boldsymbol{r}_u = \lambda(\boldsymbol{r}_u\delta u + \boldsymbol{r}_v\delta v).\boldsymbol{r}_u \\ -(\boldsymbol{n}_u\delta u + \boldsymbol{n}_v\delta v).\boldsymbol{r}_v = \lambda(\boldsymbol{r}_u\delta u + \boldsymbol{r}_v\delta v).\boldsymbol{r}_v \end{cases}$$

$$\Rightarrow \begin{cases} L\delta u + M\delta v = \lambda(E\delta u + F\delta v) \\ (M\delta u + N\delta v) = \lambda(F\delta u + G\delta v) \end{cases}$$

So $(\delta u, \delta v)$ should be the nonzero solution of the following linear equation system

$$\begin{cases} (L - \lambda E)\delta u + (M - \lambda F)\delta v = 0 \\ (M - \lambda F)\delta u + (N - \lambda G)\delta v = 0 \end{cases}$$

Mean curvature and Gauss curvature

According to linear algebra, the above linear equation system has nonzero solution if and only if
λ satisfied the following quadric function

$$\begin{vmatrix} L-\lambda E & M-\lambda F \\ M-\lambda F & N-\lambda G \end{vmatrix} = 0$$

That is

$$\lambda^2(EG-F^2) - \lambda(LG-2MF+NE) + (LN-M^2) = 0$$

$$\Longrightarrow \lambda^2 - \frac{LG-2MF+NE}{EG-F^2}\lambda + \frac{LN-M^2}{EG-F^2} = 0$$

As we know the above quadric function must have two real roots k_1, k_2 (Weingarten map of regular surface must have two eigenvalues.)
Then we have

$$k_1 + k_2 = \frac{LG-2MF+NE}{EG-F^2} = 2H$$

$$k_1 k_2 = \frac{LN-M^2}{EG-F^2} = K$$

Mean curvature and Gauss curvature

So we call

$$\lambda^2 - \frac{LG-2MF+NE}{EG-F^2}\lambda + \frac{LN-M^2}{EG-F^2} = 0$$

$$H = \frac{1}{2}(k_1 + k_2)$$

as the **Mean curvature** (平均曲率) of S,

And

$$K = k_1 k_2$$

as the **Guass (Total) curvature** (高斯曲率) of S.

Therefore, principal curvatures satisfy the following equation:

$$\lambda^2 - 2H\lambda + K = 0$$

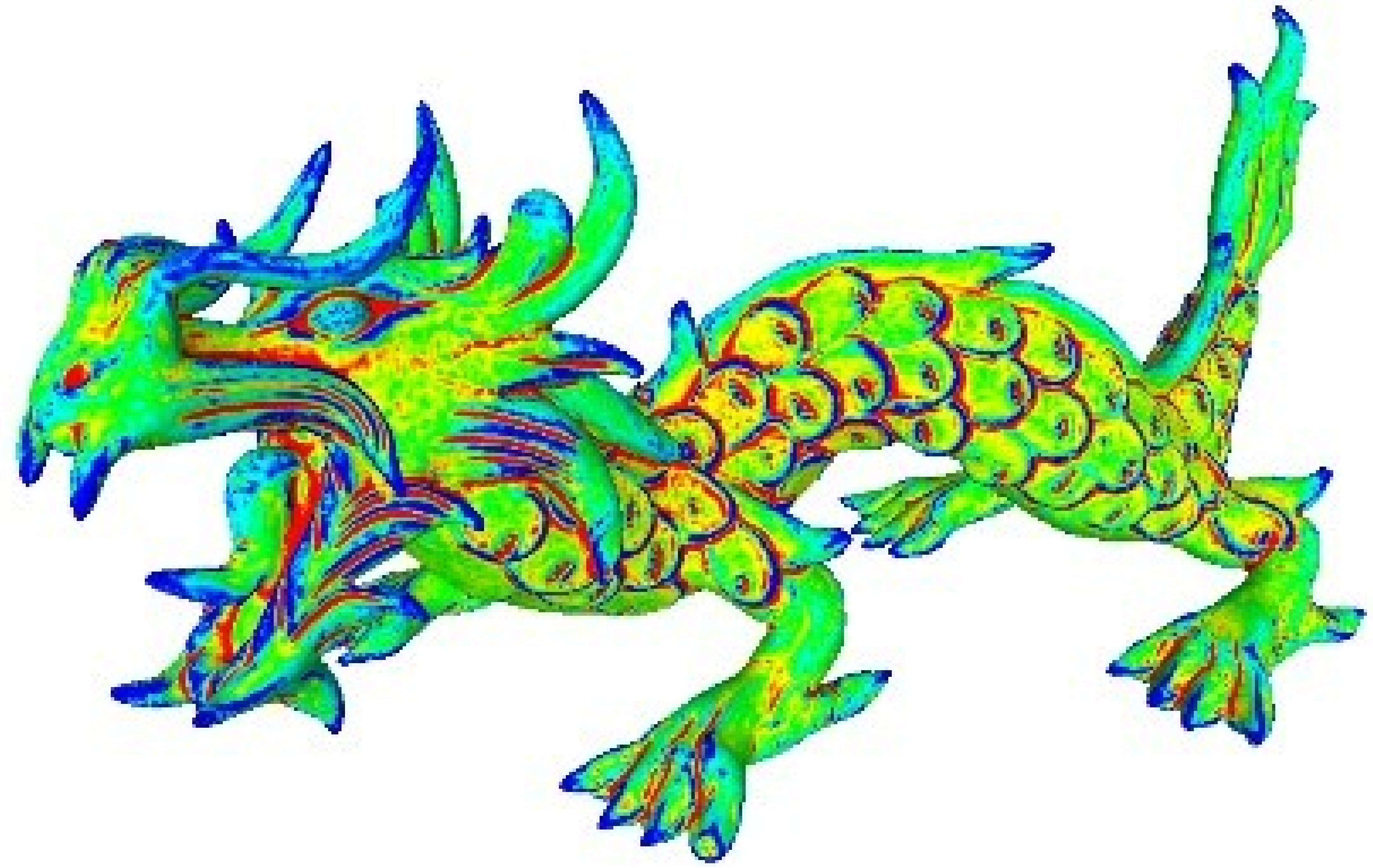
So

$$k_1 = H + \sqrt{H^2 - K}$$

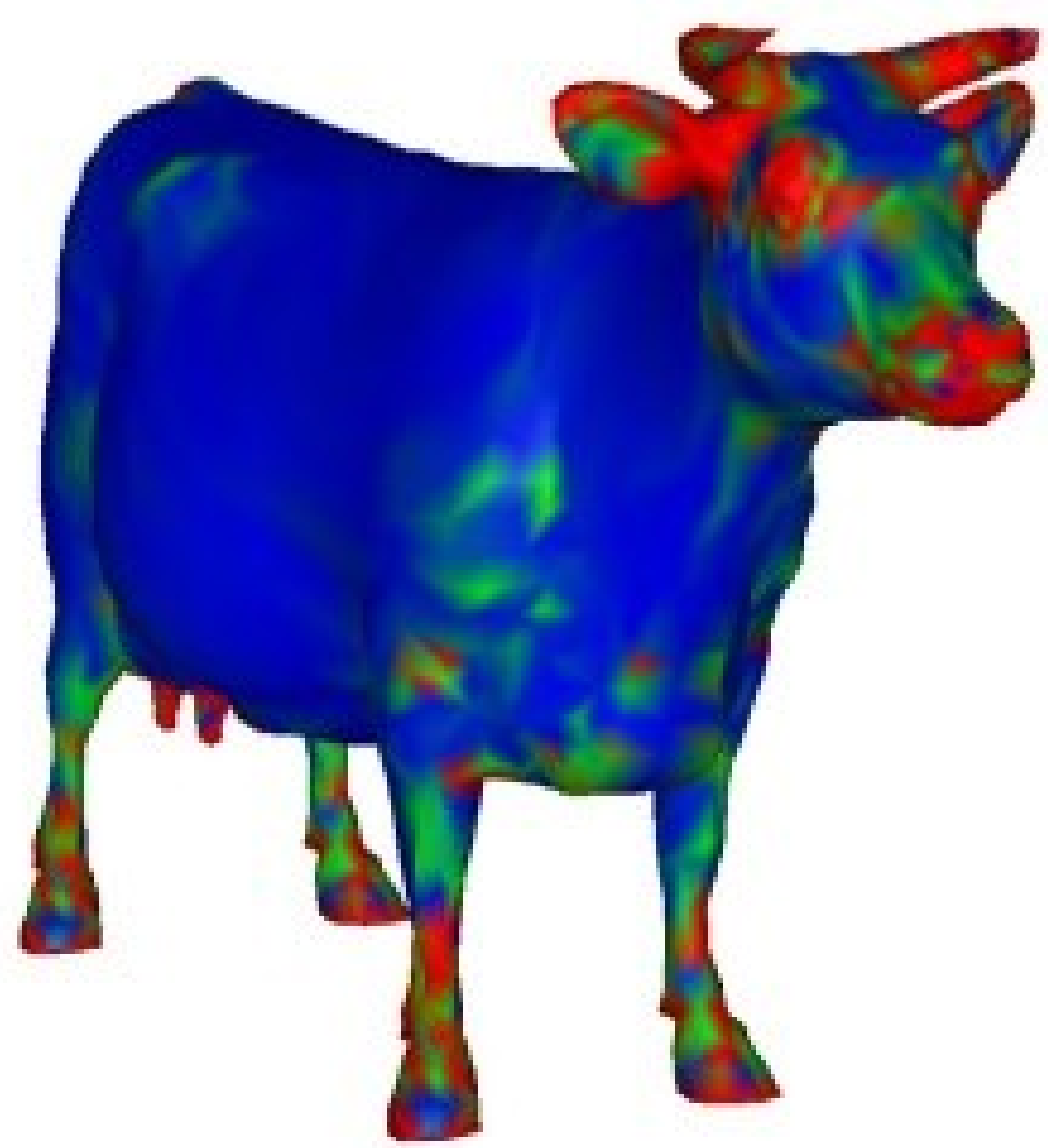
$$k_2 = H - \sqrt{H^2 - K}$$

Note: Guass curvature and Mean curvature keep unchanged under the parameter transformation that keeps orientation

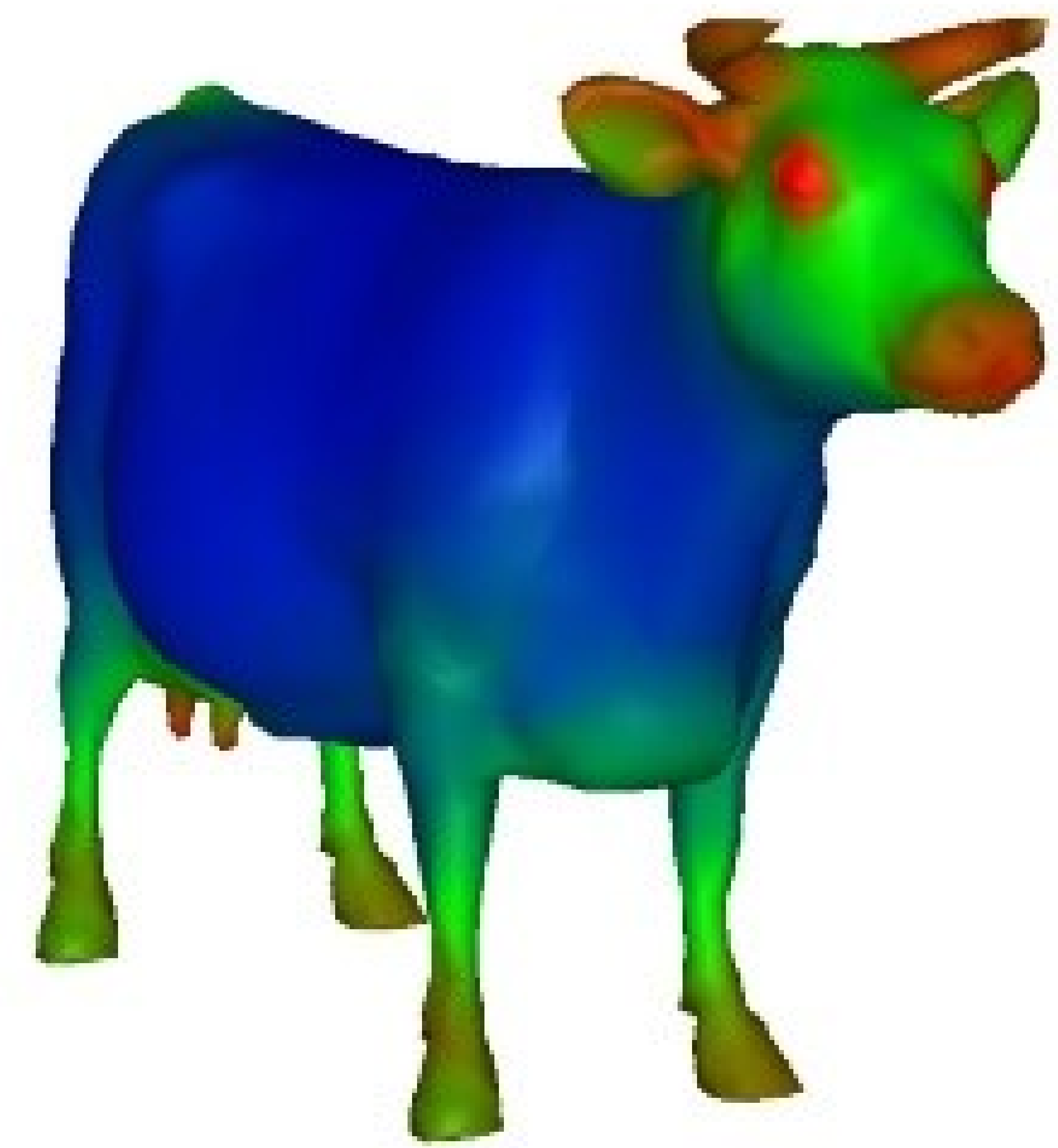
Example: Mean Curvature



Example: Gaussian Curvature



original



smoothed

Discrete Gauss-Bonnet (Descartes) theorem:

$$\sum_v K_v = \sum_v \left[2\pi - \sum_i \theta_i \right] = 2\pi\chi$$

Principal curvature and direction

Theorem:

Principal curvatures k_1, k_2 are continuous functions on surface, and they are continuous differentiable functions in the neighborhood of non-umbilical point.

Calculate principal curvature and direction

1) Calculate Mean curvature H and Gauss curvature G with the coefficients of first and second fundamental, that is L, M, N, E, F, G.

$$k_1 + k_2 = 2H = \frac{LG - 2MF + NE}{EG - F^2}$$
$$k_1 k_2 = K = \frac{LN - M^2}{EG - F^2}$$

2) Calculate principal curvatures k_1, k_2 with

$$\lambda^2 - 2H\lambda + K = 0$$

3) If $k_1 \neq k_2$, substitute the k_1, k_2 into

$$\begin{cases}(L - \lambda E)\delta u + (M - \lambda F)\delta v = 0\\(M - \lambda F)\delta u + (N - \lambda G)\delta v = 0\end{cases}$$

then we get the principal directions corresponding to k_1, k_2

$$\frac{\delta u}{\delta v} = -\frac{M - k_1 F}{L - k_1 E} = -\frac{N - k_1 G}{M - k_1 F}$$
$$\frac{\delta u}{\delta v} = -\frac{M - k_2 F}{L - k_2 E} = -\frac{N - k_2 G}{M - k_2 F}$$

4) If $k_1 = k_2$ $(umbilical\ point)$, then

$$k_1 = k_2 = \frac{L}{E} = \frac{M}{F} = \frac{N}{G}$$

Calculate principal curvature and direction

Substitute $k_1 = k_2 = \frac{L}{E} = \frac{M}{F} = \frac{N}{G}$ into

$$\begin{cases}(L - \lambda E)\delta u + (M - \lambda F)\delta v = 0\\(M - \lambda F)\delta u + (N - \lambda G)\delta v = 0\end{cases}$$

$\frac{\delta u}{\delta v}$ cannot be determined.

That is the principal direction is uncertain at umbilical points.

Inversely, we can firstly calculate the principal directions, and then calculate the principal curvatures.

1) With the following equations

$$\begin{cases}(L - \lambda E)\delta u + (M - \lambda F)\delta v = 0\\(M - \lambda F)\delta u + (N - \lambda G)\delta v = 0\end{cases}$$

we get

$$\lambda = \frac{L\delta u + M\delta v}{E\delta u + F\delta v} = \frac{M\delta u + N\delta v}{F\delta u + G\delta v}$$

Calculate principal curvature and direction

That means principal direction must satisfies

$$\begin{vmatrix} L\delta u + M\delta v & E\delta u + F\delta v \\ M\delta u + N\delta v & F\delta u + G\delta v \end{vmatrix} = 0$$

$$\Longrightarrow (LF - ME)(\delta u)^2 + (LG - NE)\delta u\delta v + (MG - NF)(\delta v)^2 = 0$$

2) After solving the above equation, we can obtain principal direction $\delta u : \delta v$.

3) Then principal curvature can be obtained by substituting the principal direction in

$$\lambda = \frac{L\delta u + M\delta v}{E\delta u + F\delta v} = \frac{M\delta u + N\delta v}{F\delta u + G\delta v}$$

4) Just for memorization, we change $(\delta u, \delta v)$ into (du, dv). Then principal direction satisfies the following equation:

$$\begin{vmatrix} (dv)^2 & -dudv & (du)^2 \\ E & F & G \\ L & M & N \end{vmatrix} = 0$$

If we consider (u, v) as moving point, curvature line of the surface S satisfies the above function.

Principal curvature and direction

Theorem:

For any fixed point (u, v) on surface $S: r = r(u, v)$, the directions of parametric curves are the orthogonal principal directions if and only if
$F = M = 0$ at this point.

Then the principal curvature along u-curve is $k_1 = \frac{L}{E}$, and the principal curvature along v-curve is $k_2 = \frac{N}{G}$.

Proof: (**Necessity**)

Since r_u and r_v are orthogonal, so $F = 0$. Let assume the principal direction is $(du, dv) = (1,0)$. With

$$\begin{vmatrix} (dv)^2 & -dudv & (du)^2 \\ E & F & G \\ L & M & N \end{vmatrix} = 0$$

we have $EM = 0$, that is $M = 0$ (r_u and r_v are orthogonal, $E = r_u . r_u \neq 0$).

Principal curvature and direction

Proof: (**Sufficiency**)

Assume we have $F = M = 0$ at fixed point (u, v). Then with equation

$$\begin{vmatrix} (dv)^2 & -dudv & (du)^2 \\ E & F & G \\ L & M & N \end{vmatrix} = 0$$

we have

$$(EN - GL)dudv = 0$$

So $(du, dv) = (1,0) and \ (du, dv) = (0,1)$ are the solutions of the above equation.

That is the directions of parametric curve are orthogonal principal directions.

After substituting $(du, dv) = (1,0) and \ (du, dv) = (0,1)$ into

$$\lambda = \frac{L\delta u + M\delta v}{E\delta u + F\delta v} = \frac{M\delta u + N\delta v}{F\delta u + G\delta v}$$

we get $k_1 = \frac{L}{E}$ and $k_2 = \frac{N}{G}$.

Principal curvature and direction

Theorem:

At a Surface S, parametric curve network is an orthogonal curvature line network if and only if $F = M = 0$, then the first and second fundamental forms can be represented as

$$I = E(du)^2 + G(dv)^2$$

$$\Pi = k_1 E(du)^2 + k_2 G(dv)^2$$

where k_1 and k_2 are principal curvatures of S.

Principal curvature and direction

Theorem:

At a regular surface S, there is a parameter system (u, v) in a neighborhood of a non-umbilical point, so that the parametric curves form an orthogonal curvature line network.

For umbilical point, the point whose $\frac{\Pi}{I} = c_0$, so it must be planar point or spherical point.
If this umbilical point is not isolated, its neighborhood form an open set which will be a plane or sphere.
Then its orthogonal parametric curve network will be orthogonal curvature line network.
For isolated umbilical point, its neighborhood is complex.
We cannot guarantee the existence of the orthogonal curvature line network.

Matrix of Weingarten map

Assume the parametric function of surface S is $\boldsymbol{r} = \boldsymbol{r}(u, v)$.
According the definition of Weingarten map, we have

$$W(\boldsymbol{r}_u) = -\boldsymbol{n}_u, W(\boldsymbol{r}_v) = -\boldsymbol{n}_v$$

Since $\boldsymbol{n}_u, \boldsymbol{n}_v$ are tangent vectors of S, so

$$\begin{pmatrix} -\boldsymbol{n}_u \\ -\boldsymbol{n}_v \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{r}_u \\ \boldsymbol{r}_v \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} -\boldsymbol{n}_u \\ -\boldsymbol{n}_v \end{pmatrix} . \begin{pmatrix} \boldsymbol{r}_u & \boldsymbol{r}_v \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{r}_u \\ \boldsymbol{r}_v \end{pmatrix} . \begin{pmatrix} \boldsymbol{r}_u & \boldsymbol{r}_v \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$$

Since we know

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix}$$

Matrix of Weingarten map

That is

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{1}{EG - F^2} \begin{pmatrix} LG - MF & -LF + ME \\ MG - NF & -MF + NE \end{pmatrix}$$

So we get

$$W\begin{pmatrix} \boldsymbol{r}_u \\ \boldsymbol{r}_v \end{pmatrix} = \frac{1}{EG - F^2} \begin{pmatrix} LG - MF & -LF + ME \\ MG - NF & -MF + NE \end{pmatrix} \begin{pmatrix} \boldsymbol{r}_u \\ \boldsymbol{r}_v \end{pmatrix}$$

For a linear transformation, the **trace** (迹) and **determinant** (行列式) of its transformation matrix under a basis keep unchanged.

Therefore, the trace and determinant of the Weingarten transformation matrix under natural basis $\{\boldsymbol{r}_u, \boldsymbol{r}_v\}$ are

$$\begin{aligned} Trace: \quad & a_{11} + a_{22} \\ &= \frac{LG - 2MF + NE}{EG - F^2} \\ &= 2H \end{aligned}$$

Matrix of Weingarten map

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{1}{EG-F^2}\begin{pmatrix} LG-MF & -LF+ME \\ MG-NF & -MF+NE \end{pmatrix}$$

Determinant:

$$\begin{aligned} & a_{11}a_{22} - a_{12}a_{21} \\ &= \frac{LN-M^2}{EG-F^2} \\ &= K \end{aligned}$$

$$\begin{pmatrix} -\boldsymbol{n}_u \\ -\boldsymbol{n}_v \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} \boldsymbol{r}_u \\ \boldsymbol{r}_v \end{pmatrix}$$

The geometric meaning of Gauss curvature

$$\begin{aligned} & \boldsymbol{n}_u \times \boldsymbol{n}_v \\ &= (a_{11}a_{22} - a_{12}a_{21})\boldsymbol{r}_u \times \boldsymbol{r}_v \\ &= K(\boldsymbol{r}_u \times \boldsymbol{r}_v) \\ \Longrightarrow & |\boldsymbol{n}_u \times \boldsymbol{n}_v| = |K|.|\boldsymbol{r}_u \times \boldsymbol{r}_v| \end{aligned}$$

We know area element $d\sigma = |\boldsymbol{r}_u \times \boldsymbol{r}_v|dudv$ is the area of a region on S ($\boldsymbol{r} = \boldsymbol{r}(u,v)$) bounded by lines:

$$u = u_0, u = u_0 + du, v = v_0, v = v_0 + dv$$

The area of the above region after Gauss map is

$$\begin{aligned} d\sigma_0 &= |\boldsymbol{n}_u \times \boldsymbol{n}_v|dudv \\ &= |K|d\sigma \\ \Longrightarrow |K| &= \frac{d\sigma_0}{d\sigma} \end{aligned}$$

Matrix of Weingarten map

Assume *D* is a neighborhood around point p on surface S. Let $g(D)$ represent the image of D under Guass map. Then the area of $g(D)$ is

$$A\big(g(D)\big) = \int_{g(D)} d\sigma_0$$

$$= \int_D |K| d\sigma$$

With mean value theorem of integrals (积分中值定理), and calculate the limit when region *D* contract to point p, we get

$$|K(p)| = \lim_{D \to p} \frac{A\big(g(D)\big)}{A(D)}$$