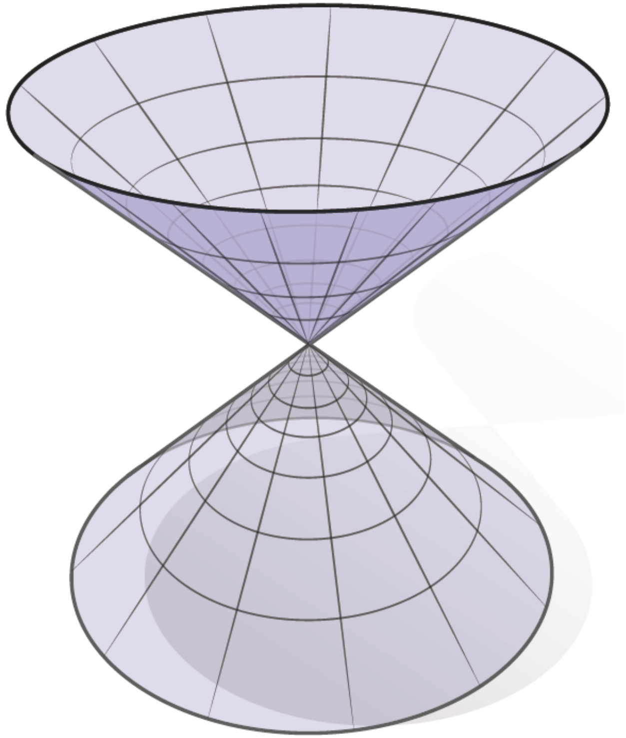
Surface



Revisit first fundamental form

$$\mathrm{I} = Edu^2 + 2Fdudv + Gdv^2 = (du, dv)\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix}$$

Length of a curve

$$l = \int_a^b \left(E\big(\alpha(t)\big)\left(\frac{du}{dt}\right)^2 + 2F\big(\alpha(t)\big)\frac{du}{dt}\frac{dv}{dt} + G\big(\alpha(t)\big)\left(\frac{dv}{dt}\right)^2 \right)^2 dt$$

Area of a surface

$$A = \iint_D \sqrt{EG - F^2}dudv$$

Angle between tangent vectors

$$cos\angle(d\boldsymbol{r}, \delta\boldsymbol{r}) = \frac{Edu\delta u + F(du\delta v + dv\delta u) + Gdv\delta v}{\sqrt{Edu^2 + 2Fdudv + Gdv^2}\sqrt{E\delta u^2 + 2F\delta u\delta v + G\delta v^2}}$$

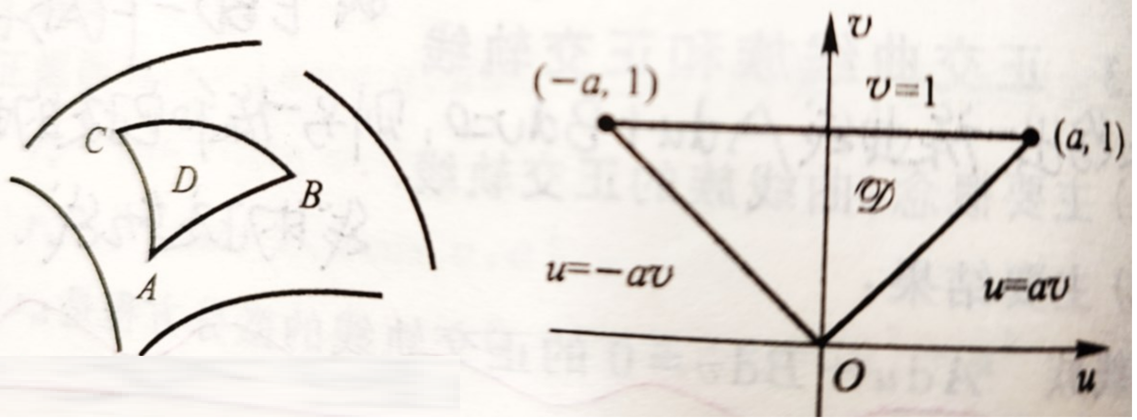
Example

Assume the 1st fundamental form of surface S is
$I = ds^2 = du^2 + (u^2 + a^2)dv^2$, try to calculate the area of the surface triangle formed by these three curves: $u = \pm av, v = 1$.

Answer: from $I = ds^2 = du^2 + (u^2 + a^2)dv^2$ we know that

$$E = 1, F = 0, G = u^2 + a^2$$

The intersection points of these lines can be solved with
$\begin{cases} u = av \\ u = -av \end{cases}, \begin{cases} u = av \\ v = 1 \end{cases}, \begin{cases} u = -av \\ v = 1 \end{cases}$, that is
A: (u=0, v=0), B:(u=a, v=1), C: (u=-a, v=1)

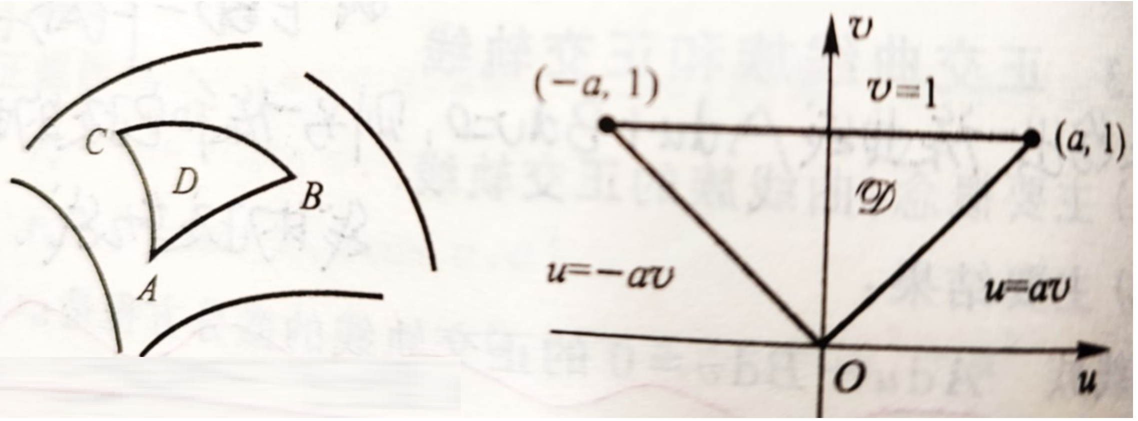


Example

Assume the 1st fundamental form a the surface S is $I = ds^2 = du^2 + (u^2 + a^2)dv^2$, try to calculate the area of the surface triangle formed by these three curves:$u = \pm av, v = 1$.

Answer: Then the area of the surface triangle is

$$\sigma = \iint_{\mathfrak{D}} \sqrt{u^2 + a^2}\,\mathrm{dudv} = \cdots = \mathrm{a}^2\left[\frac{2}{3} - \frac{\sqrt{2}}{3} + \ln\left(\sqrt{2} + 1\right)\right]$$



Example

Find the differential equation determined by the trajectory of bisecting angle of the parametric curve network.

Answer: assume the regular parametric surface $S: \boldsymbol{r} = \boldsymbol{r}(u, v)$. Its first fundamental form is

$$I = Edu^2 + 2Fdudv + Gdv^2$$

Let the tangent vector of the bisecting angle trajectory is

$$\boldsymbol{dr} = \boldsymbol{r}_u du + \boldsymbol{r}_v dv$$

As we know, the angle between $\boldsymbol{dr}$ and u-curve ($\boldsymbol{r}_u$) should be equal to the angle between $\boldsymbol{dr}$ and v-curve ($\boldsymbol{r}_v$), that is

$$\frac{\boldsymbol{dr}.\boldsymbol{r}_u}{|\boldsymbol{dr}||\boldsymbol{r}_u|} = \pm\frac{\boldsymbol{dr}.\boldsymbol{r}_v}{|\boldsymbol{dr}||\boldsymbol{r}_v|}$$

Since $|\boldsymbol{r}_u| = \sqrt{E}, \boldsymbol{r}_v = \sqrt{G}$, we get

Example

$$\sqrt{G}(Edu + Fdv) = \pm\sqrt{E}(Fdu + Gdv)$$

That is

$$\sqrt{E}\left(\sqrt{EG} - F\right)du = \pm\sqrt{G}\left(\sqrt{EG} - F\right)dv$$

Since $EG - F^2 = |\boldsymbol{r}_u \times \boldsymbol{r}_v|^2 > 0$,

then $\left(\sqrt{EG} - F\right)\left(\sqrt{EG} + F\right) > 0$

so we get

$$\sqrt{E}du \pm \sqrt{G}dv = 0$$

Equivalently,

$$Edu^2 + Gdv^2 = 0$$

or

$$\frac{du}{dv} = \pm\frac{\sqrt{G}}{\sqrt{E}}$$

Existence of orthogonal parametric curve networks (正交参数曲线网) on surfaces

Theorem:

Assume there are two linearly independent tangent vectors $\boldsymbol{a}(u,v)$ and $\boldsymbol{b}(u,v)$ of regular parametric surface $S: \boldsymbol{r} = \boldsymbol{r}(u,v)$. Then for any point $\boldsymbol{p} \in S$, there must be a neighborhood $\boldsymbol{p} \in U \subset S$, so that there is a new parameter $(\tilde{u}, \tilde{v})$ on U satisfying

$$\boldsymbol{r}_{\tilde{u}} \parallel \boldsymbol{a}, \boldsymbol{r}_{\tilde{v}} \parallel \boldsymbol{b}$$

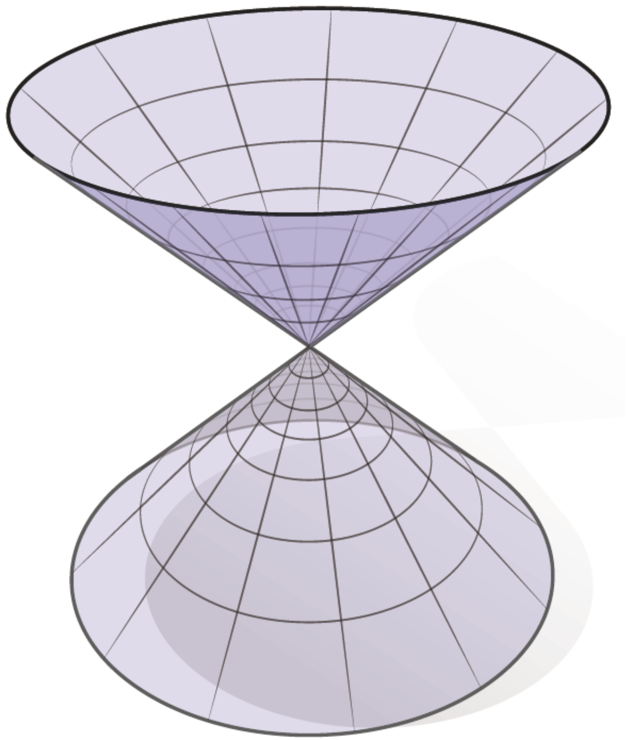
Theorem:

For any point on regular parametric $\boldsymbol{p} \in S: \boldsymbol{r} = \boldsymbol{r}(u,v)$, there must be a neighborhood of $\boldsymbol{p} \in U \subset S$, so that there is a new parameter $(\tilde{u}, \tilde{v})$ in U satisfying

$$\tilde{F} = \boldsymbol{r}_{\tilde{u}} . \boldsymbol{r}_{\tilde{v}} = 0$$

Then $(\tilde{u}, \tilde{v})$ is an orthogonal parametric system of S.

Isometric and conformal parameterization



Mapping between surfaces

Assume S_1 and S_2 are two regular parametric surfaces.
Since any point p on surface correspondences to its parameter (Grain coordinate 纹理/曲纹坐标) one to one, the map $\varphi: S_1 \rightarrow S_2$ can be represented with their parameters.
So there is a map $\psi: D_1 \rightarrow D_2$, so that $\varphi = \boldsymbol{r}_2 \circ \psi \circ \boldsymbol{r}_1^{-1}$ or $\psi = \boldsymbol{r}_2^{-1} \circ \varphi \circ \boldsymbol{r}_1$

$$\begin{tikzcd} E^3 \supset S_1 \arrow[r, "\varphi"] & S_2 \subset E^3 \\ D_1 \arrow[u, "\vec{r}_1"] \arrow[r, "\psi"] & D_2 \arrow[u, "\vec{r}_2"'] \end{tikzcd}$$

Mapping between surfaces

Their parametric functions are $\boldsymbol{r}_1 = \boldsymbol{r}_1(u, v), (u, v) \in D_1, and\ \boldsymbol{r}_2 = \boldsymbol{r}_2(\bar{u}, \bar{v}), (\bar{u}, \bar{v}) \in D_2$.

Mapping $\varphi: S_1 \to S_2$ is continuous and differentiable, and its parametric function is

$$\varphi = \boldsymbol{r}_2 \circ \psi \circ \boldsymbol{r}_1^{-1}$$

where

$$\psi: D_1 \to D_2: (u, v) \to \psi(u, v) = (\bar{u}, \bar{v}) = \big(\bar{u}(u, v), \bar{v}(u, v)\big)$$

$$\begin{tikzcd}
E^3 \supset S_1 \arrow[r, "\varphi"] & S_2 \subset E^3 \\
D_1 \arrow[u, "\vec{r}_1"] \arrow[r, "\psi"] & D_2 \arrow[u, "\vec{r}_2"']
\end{tikzcd}$$

Tangential map (切映射)

For any point $\mathrm{p} \in S_1$, we can define a linear map as

$$\varphi_*: T_pS_1 \to T_{\varphi(p)}S_2: X = a(\boldsymbol{r}_1)_u + b(\boldsymbol{r}_1)_v \to \varphi_* X$$

The above map φ_* is called as **tangential map** induced by the continuous and differential map φ.

$$\begin{tikzcd}
T_pS_1 \arrow[r, "\varphi_*"] & T_{\varphi(p)} \\
E^3 \supset S_1 \arrow[u] \arrow[r, "\varphi"] & S_2 \subset E^3 \arrow[u] \\
D_1 \arrow[u, "\vec{r}_1"] \arrow[r, "\psi"] & D_2 \arrow[u, "\vec{r}_2"']
\end{tikzcd}$$

Tangential map(切映射)

Tangential map can be defined in another way:
φ maps a curve C on S_1 into a curve $\bar{C}$ on S_2, that is

$$\bar{C}: \left(\bar{u}(t), \bar{v}(t)\right) = \left(\bar{u}\left(u(t), v(t)\right), \bar{v}\left(u(t), v(t)\right)\right)$$

$\varphi_*(X)$ is defined as the tangent vector of $\bar{C}$ at t, that is

$$\varphi_*(X) = \varphi_*\left(\frac{d\boldsymbol{r}_1\left(u(t), v(t)\right)}{dt}\right) = \frac{d}{dt}\left[\boldsymbol{r}_2\left(\bar{u}(t), \bar{v}(t)\right)\right]$$

$$= \frac{d}{dt}\left[\boldsymbol{r}_2\left(\bar{u}\left(u(t), v(t)\right), \bar{v}\left(u(t), v(t)\right)\right)\right]$$

$$= \frac{\partial \boldsymbol{r}_2}{\partial \bar{u}}\left[\frac{\partial \bar{u}}{\partial u}\frac{du}{dt} + \frac{\partial \bar{u}}{\partial v}\frac{dv}{dt}\right] + \frac{\partial \boldsymbol{r}_2}{\partial \bar{v}}\left[\frac{\partial \bar{v}}{\partial u}\frac{du}{dt} + \frac{\partial \bar{v}}{\partial v}\frac{dv}{dt}\right]$$

$$= \left(\frac{\partial \boldsymbol{r}_2}{\partial \bar{u}}\frac{\partial \bar{u}}{\partial u} + \frac{\partial \boldsymbol{r}_2}{\partial \bar{v}}\frac{\partial \bar{v}}{\partial u}\right)\frac{du}{dt} + \left(\frac{\partial \boldsymbol{r}_2}{\partial \bar{u}}\frac{\partial \bar{u}}{\partial v} + \frac{\partial \boldsymbol{r}_2}{\partial \bar{v}}\frac{\partial \bar{v}}{\partial v}\right)\frac{dv}{dt}$$

Tangential map(切映射)

Assume $p = \boldsymbol{r}_1\big(u(t_0), v(t_0)\big)$, let

$$\varphi_{*p}\left(\frac{d\boldsymbol{r}_1\big(u(t), v(t)\big)}{dt}\Big|_{t=t_0}\right) = \frac{d}{dt}\left[\boldsymbol{r}_2\big(\bar{u}(t), \bar{v}(t)\big)\right]\Big|_{t=t_0}$$

Then we have

$$\Rightarrow \varphi_{*p}\left(\frac{\partial \boldsymbol{r}_1}{\partial u}\frac{du}{dt} + \frac{\partial \boldsymbol{r}_1}{\partial v}\frac{dv}{dt}\right)$$

$$= \left(\frac{\partial \boldsymbol{r}_2}{\partial \bar{u}}\frac{\partial \bar{u}}{\partial u} + \frac{\partial \boldsymbol{r}_2}{\partial \bar{v}}\frac{\partial \bar{v}}{\partial u}\right)\frac{du}{dt} + \left(\frac{\partial \boldsymbol{r}_2}{\partial \bar{u}}\frac{\partial \bar{u}}{\partial v} + \frac{\partial r_2}{\partial \bar{v}}\frac{\partial \bar{v}}{\partial v}\right)\frac{dv}{dt}$$

Since tangential map is linear, then we get

$$\varphi_{*p}\left(\frac{\partial \boldsymbol{r}_1}{du}\right) = \frac{\partial \boldsymbol{r}_2}{\partial \bar{u}}\frac{\partial \bar{u}}{\partial u} + \frac{\partial \boldsymbol{r}_2}{\partial \bar{v}}\frac{\partial \bar{v}}{\partial u}$$

$$\varphi_{*p}\left(\frac{\partial \boldsymbol{r}_1}{dv}\right) = \frac{\partial \boldsymbol{r}_2}{\partial \bar{u}}\frac{\partial \bar{u}}{\partial v} + \frac{\partial \boldsymbol{r}_2}{\partial \bar{v}}\frac{\partial \bar{v}}{\partial v}$$

Tangential map(切映射)

Write in matrix

$$\varphi_{*p}\begin{pmatrix}\dfrac{\partial \boldsymbol{r}_1}{du}\\ \dfrac{\partial \boldsymbol{r}_1}{dv}\end{pmatrix}=\begin{pmatrix}\dfrac{\partial \bar{u}}{\partial u} & \dfrac{\partial \bar{v}}{\partial u}\\ \dfrac{\partial \bar{u}}{\partial v} & \dfrac{\partial \bar{v}}{\partial v}\end{pmatrix}\begin{pmatrix}\dfrac{\partial \boldsymbol{r}_2}{\partial \bar{u}}\\ \dfrac{\partial \boldsymbol{r}_2}{\partial \bar{v}}\end{pmatrix}$$

$$=J\begin{pmatrix}\dfrac{\partial \boldsymbol{r}_2}{\partial \bar{u}}\\ \dfrac{\partial r_2}{\partial \bar{v}}\end{pmatrix}$$

where

$$J=\begin{pmatrix}\dfrac{\partial \bar{u}}{\partial u} & \dfrac{\partial \bar{v}}{\partial u}\\ \dfrac{\partial \bar{u}}{\partial v} & \dfrac{\partial \bar{v}}{\partial v}\end{pmatrix}$$

is the Jacobi matrix of parameter transformation.

Tangential map(切映射)

For any tangent vector $X = a\frac{\partial \boldsymbol{r}_1}{\partial u} + b\frac{\partial \boldsymbol{r}_1}{\partial v}$ (a and b are constants) in T_pS_1, we have

$$\varphi_{*p}(X) = \varphi_{*p}\left(a\frac{\partial \boldsymbol{r}_1}{\partial u} + b\frac{\partial \boldsymbol{r}_1}{\partial v}\right)$$

$$= \mathrm{a}\varphi_{*p}\left(\frac{\partial \boldsymbol{r}_1}{\partial u}\right) + \mathrm{b}\varphi_{*p}\left(\frac{\partial \boldsymbol{r}_1}{\partial v}\right)$$

$$= (a, b)J_p\begin{pmatrix}\frac{\partial \boldsymbol{r}_2}{\partial \bar{u}} \\ \frac{\partial \boldsymbol{r}_2}{\partial \bar{v}}\end{pmatrix}$$

Tangential map $\varphi_{*p}: T_pS_1 \to T_{\varphi(p)}S_2$ is isomorphic(同构) if and only if

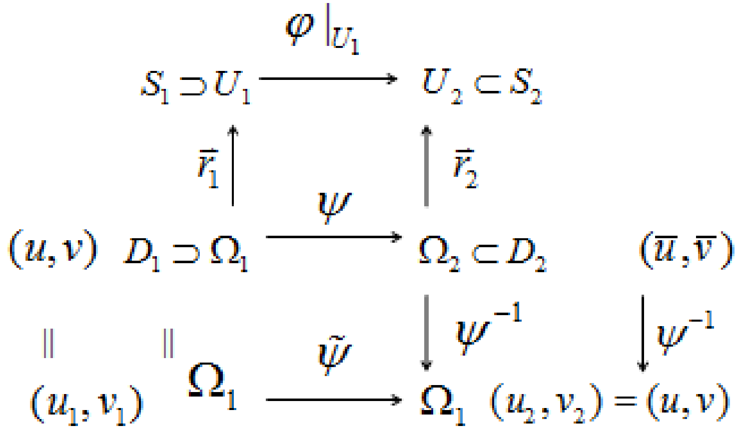
$$|J_p| \neq 0$$

Theorem

Assume $\varphi: S_1 \to S_2$ is a more than 3-times continuous and differentiable map.
Assume tangential map $\varphi_{*p}: T_pS_1 \to T_{\varphi(p)}S_2$ is linearly isomorphic, then there are neighborhoods of $p \in U_1 \subset S_1, \varphi(p) \in U_2 \subset S_2$, and the parameter systems of U_1, U_2 are $(u_1, v_1), (u_2, v_2)$,
so that $\varphi(U_1) \subset U_2$, and the parametric expression of map $\varphi|_{U_1}$

$$\tilde{\psi} = id: \Omega_1 \to \Omega_2: (u_1, v_1) \to (u_2, v_2) = (u_1, v_1)$$

where $\Omega_1 = \boldsymbol{r}_1^{-1}(U_1), \Omega_2 = \boldsymbol{r}_2^{-1}(U_2)$.
These parameter systems are called applicable parameter systems (适用参数系).



φ^* mapping

Assume $\varphi: S_1 \to S_2$ is an continuous and differentiable map, $(u_1, v_1)\ and\ (\bar{u}, \bar{v})$ are grain coordinates (曲纹坐标) of $S_1\ and\ S_2$. The parametric expression of φ is $\bar{u} = \bar{u}(u, v), \bar{v} = \bar{v}(u, v)$. Since

$$(d\bar{u}, d\bar{v}) = (du, dv)\begin{pmatrix} \frac{\partial \bar{u}}{\partial u} & \frac{\partial \bar{v}}{\partial u} \\ \frac{\partial \bar{u}}{\partial v} & \frac{\partial \bar{v}}{\partial v} \end{pmatrix} = (du, dv)J$$

For any quadric differential form of S_2

$$w = \bar{A}(\bar{u}, \bar{v})d\bar{u}^2 + 2\bar{B}(\bar{u}, \bar{v})d\bar{u}d\bar{v} + \bar{C}(\bar{u}, \bar{v})d\bar{v}^2$$
$$= (d\bar{u}, d\bar{v})\begin{pmatrix} \bar{A} & \bar{B} \\ \bar{B} & \bar{C} \end{pmatrix}\begin{pmatrix} d\bar{u} \\ d\bar{v} \end{pmatrix}$$

φ^* mapping

We can define a quadric differential form(二次微分式) on S_1 as

$$\varphi^* w = A(u,v)du^2 + 2B(u,v)dudv + C(u,v)dv^2$$
$$= (du, dv) J \begin{pmatrix} \bar{A} & \bar{B} \\ \bar{B} & \bar{C} \end{pmatrix} J^T \begin{pmatrix} du \\ dv \end{pmatrix}$$

Quadric differential form $\varphi^* w$ is determined by pulling a quadric differential form w on S_2 back to the quadric differential form on S_1.

Isometric correspondence (等距/保长对应)

Assume $\varphi: S_1 \to S_2$ is a more than 3-times continuous and differentiable map.
If for any point $p \in S_1$, tangential map φ_* can keep the length of tangent vectors, that is

$$\forall X \in T_p S_1, |\varphi_* X| = |X|,$$

then φ is an isometric correspondence from S_1 to S_2

Isometric correspondence (等距/保长对应)

Note 1: if a linear map is able to keep the vector length, it can keep inner product.

So if $\varphi\colon S_1 \to S_2$ is an isometric correspondence, then

$$\varphi_*(X).\varphi_*(Y) = X.Y, \forall X, Y \in T_pS_1, \forall p \in S_1$$

Conversely, if a linear map can keep the inner product, it can also keep the vector length.

In addition, isometric correspondence can also keep the arc length of the continuous and differentiable curves.

Isometric correspondence (等距/保长对应)

Note2: If a linear map can keep inner product,
it must be a linear isomorphism (线性同构).
So for an isometric correspondence φ, for each point $p \in S_1$,
tangential map $\varphi_*: T_p S_1 \to T_{\varphi(p)} S_2$ is linear isomorphism.
Then φ is locally diffeomorphism and exists applicable parameter system.

Isometric correspondence (等距/保长对应)

Assume $\varphi: S_1 \to S_2$ is a more than 3-times continuous and differentiable map.

Then φ is a isometric correspondence **if and only if**

$$\varphi^*(I_2) = \varphi_*(d\boldsymbol{r}_1).\varphi_*(d\boldsymbol{r}_1) = (d\boldsymbol{r}_1).(d\boldsymbol{r}_1) = I_1$$

That is the following equation is valid at the corresponding points

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = J \begin{pmatrix} \bar{E} & \bar{F} \\ \bar{F} & \bar{G} \end{pmatrix} J^T$$

$$= \begin{pmatrix} \frac{\partial \bar{u}}{\partial u} & \frac{\partial \bar{v}}{\partial u} \\ \frac{\partial \bar{u}}{\partial v} & \frac{\partial \bar{v}}{\partial v} \end{pmatrix} \begin{pmatrix} \bar{E} & \bar{F} \\ \bar{F} & \bar{G} \end{pmatrix} \begin{pmatrix} \frac{\partial \bar{u}}{\partial u} & \frac{\partial \bar{v}}{\partial v} \\ \frac{\partial \bar{u}}{\partial u} & \frac{\partial \bar{v}}{\partial v} \end{pmatrix}$$

$$\varphi^* w = A(u,v)du^2 + 2B(u,v)dudv + C(u,v)dv^2$$

$$= (du, dv) J \begin{pmatrix} \bar{A} & \bar{B} \\ \bar{B} & \bar{C} \end{pmatrix} J^T \begin{pmatrix} du \\ dv \end{pmatrix}$$

Isometric correspondence (等距/保长对应)

Since φ is an isometric correspondence, so we get

$$|d\boldsymbol{r}_1(u,v)|^2$$

$$= \left|\varphi_{*p}\left(d\boldsymbol{r}_1(u,v)\right)\right|^2$$

$$= (du,dv)J\begin{pmatrix}\frac{\partial \boldsymbol{r}_2}{\partial \bar{u}}\\ \frac{\partial \boldsymbol{r}_2}{\partial \bar{v}}\end{pmatrix}\cdot\begin{pmatrix}\frac{\partial \boldsymbol{r}_2}{\partial \bar{u}} & \frac{\partial \boldsymbol{r}_2}{\partial \bar{v}}\end{pmatrix}J^T\begin{pmatrix}du\\ dv\end{pmatrix}$$

$$= (du,dv)J\begin{pmatrix}\bar{E} & \bar{F}\\ \bar{F} & \bar{G}\end{pmatrix}J^T\begin{pmatrix}du\\ dv\end{pmatrix}$$

$$= \varphi^*(|d\boldsymbol{r}_2(\bar{u},\bar{v})|^2)$$

$$\varphi_{*p}\begin{pmatrix}\frac{\partial r_1}{du}\\ \frac{\partial r_1}{dv}\end{pmatrix} = \begin{pmatrix}\frac{\partial \bar{u}}{\partial u} & \frac{\partial \bar{v}}{\partial u}\\ \frac{\partial \bar{u}}{\partial v} & \frac{\partial \bar{v}}{\partial v}\end{pmatrix}\begin{pmatrix}\frac{\partial r_2}{\partial \bar{u}}\\ \frac{\partial r_2}{\partial \bar{v}}\end{pmatrix}$$

$$\varphi^*(I_2) = \varphi_*(dr_1).\ \varphi_*(dr_1)$$

Isometric correspondence

Theorem:
There is an isometric correspondence between surfaces S_1 and S_2 **if and only if**
an applicable parameter system can be selected on S_1 and S_2 so that S_1 and S_2 have the same coefficients of the 1st fundamental form, that is

$$E_1(u,v) = E_2(u,v)$$
$$F_1(u,v) = F_2(u,v)$$
$$G_1(u,v) = G_2(u,v)$$