デジタル信号処理
中間演習解説

立命館大学　情報理工学部
岩居　健太

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フーリエ級数

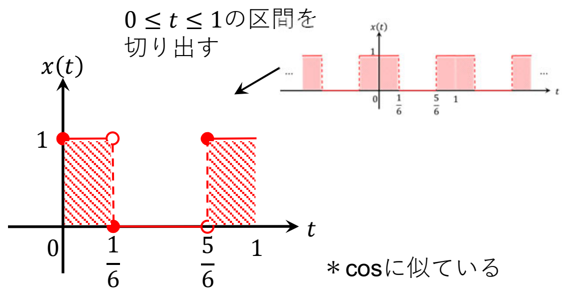
周期$T=1$の信号$x(t)$のフーリエ級数

$$x(t) = a_0 + \sum_{m=1}^{\infty} a_m \cos(2\pi m t) + \sum_{m=1}^{\infty} b_m \sin(2\pi m t)$$

$$a_0 = \int_0^1 x(t)dt \qquad a_m = 2\int_0^1 x(t)\cos(2\pi m t)\,dt \qquad b_m = 2\int_0^1 x(t)\sin(2\pi m t)\,dt$$

信号$x(t)$

$$x(t) = \begin{cases} 1, & \left(0 \le t \le \frac{1}{6}\right) \\ 0, & \left(\frac{1}{6} \le t \le \frac{5}{6}\right) \\ 1, & \left(\frac{5}{6} \le t \le 1\right) \end{cases}$$



＊cosに似ている

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(1) フーリエ係数a_0を求める

$$
\begin{aligned}
a_0 &= \int_0^1 x(t)dt \\
&= \int_0^{\frac{1}{6}} 1dt + \int_{\frac{1}{6}}^{\frac{5}{6}} 0dt + \int_{\frac{5}{6}}^{1} 1dt \\
&= [t]_0^{\frac{1}{6}} + [t]_{\frac{5}{6}}^{1} \\
&= \frac{1}{6} - 0 + 1 - \frac{5}{6} \\
&= \frac{1}{3}
\end{aligned}
$$

＊信号の周期性を仮定するため,
$0 \le t \le T, -\frac{\mathrm{T}}{2} \le t \le \frac{T}{2}$
のように1周期分の信号が入る区間を指定すればOK.

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(2) フーリエ係数a_mを求める (1/2)

$$
\begin{aligned}
a_m &= 2\int_0^1 x(t)\cos(2\pi mt)\,dt \\
&= 2\int_0^{\frac{1}{6}} \cos(2\pi mt)\,dt + 2\int_{\frac{5}{6}}^{1} \cos(2\pi mt)\,dt \\
&= 2\left[\frac{\sin(2\pi mt)}{2\pi m}\right]_0^{\frac{1}{6}} + 2\left[\frac{\sin(2\pi mt)}{2\pi m}\right]_{\frac{5}{6}}^{1} \\
&= 2\left(\frac{\sin(\pi m/3)}{2\pi m} - \frac{\sin(5\pi m/3)}{2\pi m}\right) \\
&= 2\left(\frac{\sin(\pi m/3)}{2\pi m} + \frac{\sin(\pi m/3)}{2\pi m}\right) \\
&= \frac{2\sin(\pi m/3)}{\pi m}
\end{aligned}
$$

$$\int \cos(x)\,dt = \sin(x) + C$$

$$\sin(0) = \sin(2\pi) = 0$$

$$\sin(2\pi m - \theta) = -\sin(\theta)$$

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(2) フーリエ係数$\boldsymbol{a_m}$を求める (2/2)

$$a_m = \frac{2\sin(\pi m/3)}{\pi m}$$

$$a_1 = \frac{2\sin(\pi/3)}{\pi} = \frac{\sqrt{3}}{\pi} \approx 0.5513$$

$$a_2 = \frac{2\sin(2\pi/3)}{2\pi} = \frac{\sqrt{3}}{2\pi} \approx 0.2757$$

$$a_3 = \frac{2\sin(3\pi/3)}{3\pi} = 0$$

$$a_4 = \frac{2\sin(4\pi/3)}{4\pi} = -\frac{\sqrt{3}}{4\pi} \approx 0.1378$$

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(3) フーリエ係数$\boldsymbol{b_m}$を求める

$$\begin{aligned}
b_m &= 2\int_0^1 x(t)\sin(2\pi mt)\,dt \\
&= 2\int_0^{\frac{1}{6}} \sin(2\pi mt)\,dt + 2\int_{\frac{5}{6}}^1 \sin(2\pi mt)\,dt \\
&= 2\left[-\frac{\cos(2\pi mt)}{2\pi m}\right]_0^{\frac{1}{6}} + 2\left[\frac{\cos(2\pi mt)}{2\pi m}\right]_{\frac{5}{6}}^1 \\
&= 2\left(\frac{1-\cos(\pi m/3)}{2\pi m} - \frac{1-\cos(5\pi m/3)}{2\pi m}\right) \\
&= 2\left(\frac{1-\cos(\pi m/3)}{2\pi m} - \frac{1-\cos(\pi m/3)}{2\pi m}\right) \\
&= 0
\end{aligned}$$

$$\int \sin(x)\,dt = -\cos(x) + C$$

$$\sin(0) = \sin(2\pi) = 0$$

$$\cos(2\pi m - \theta) = \cos(\theta)$$

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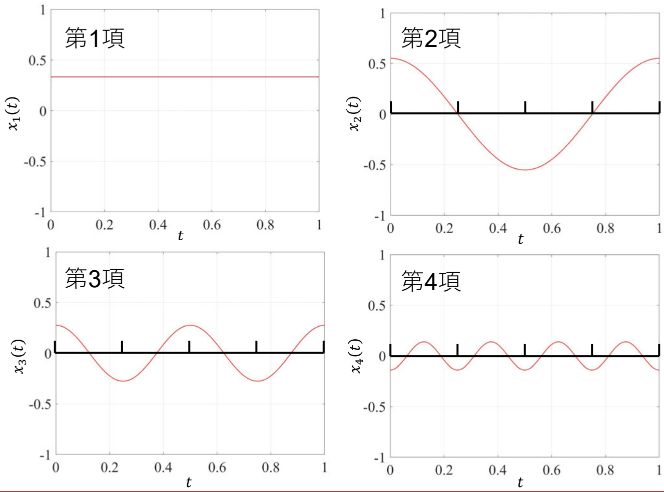
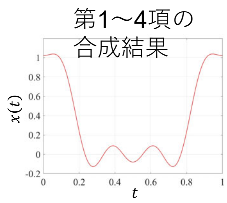
(4) 式 (1) のフーリエ級数展開

$$x(t) \approx \underbrace{\frac{1}{3}}_{=a_0} + \underbrace{0.5513}_{=a_1}\cos(2\pi t) + \underbrace{0.2757}_{=a_2}\cos(4\pi t) + \underbrace{(-0.1378)}_{=a_4}\cos(8\pi m t) + \cdots$$

＊$a_3 = 0$のため，$\cos(6\pi t)$成分は存在しない．





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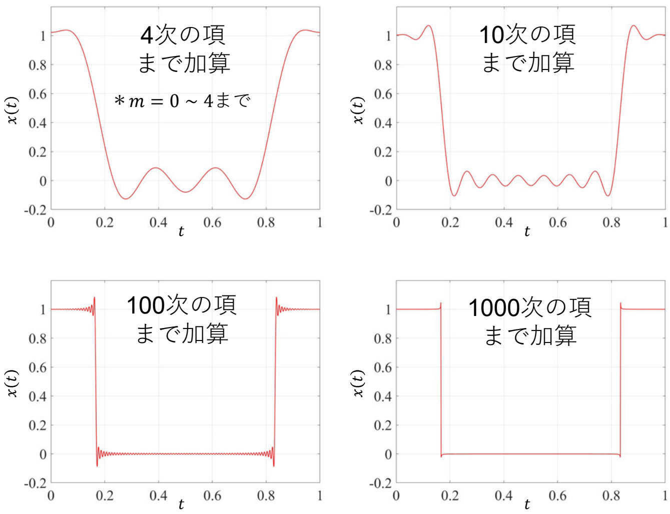
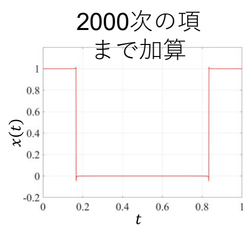
補足: フーリエ級数展開による合成結果





次数を高く設定すれば，
限りなく矩形波に近づく

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