# Principal curvatures and directions

Assume surface $S: r = r(u,v)$ is orthogonal at point $r(u_0, v_0)$, so the first and second fundamental forms of S at this point are

$$I = E(du)^2 + G(dv)^2$$

$$\Pi = L(du)^2 + 2Mdudv + N(dv)^2$$

The normal curvature of S at this point is

$$k_n = \frac{\Pi}{I} = \frac{L(du)^2 + 2Mdudv + N(dv)^2}{E(du)^2 + G(dv)^2}$$

$$= \frac{L}{E}\left(\frac{\sqrt{E}du}{\sqrt{E(du)^2 + G(dv)^2}}\right)^2$$

$$+ \frac{2M}{\sqrt{EG}}\frac{\sqrt{E}du}{\sqrt{E(du)^2 + G(dv)^2}}\frac{\sqrt{G}dv}{\sqrt{E(du)^2 + G(dv)^2}}$$

$$+ \frac{N}{G}\left(\frac{\sqrt{G}dv}{\sqrt{E(du)^2 + G(dv)^2}}\right)^2$$

# Principal curvatures and directions

Let $\theta$ be the angle between tangent direction $(du, dv)$ and then tangent direction of u-curve, So

$$\cos(\theta) = \frac{\sqrt{E}du}{\sqrt{E(du)^2 + G(dv)^2}}$$

$$\sin(\theta) = \frac{\sqrt{G}dv}{\sqrt{E(du)^2 + G(dv)^2}}$$

$$k_n = \frac{\Pi}{I} = \frac{L(du)^2 + 2Mdudv + N(dv)^2}{E(du)^2 + G(dv)^2}$$

$$= \frac{L}{E}\left(\frac{\sqrt{E}du}{\sqrt{E(du)^2 + G(dv)^2}}\right)^2$$

$$+ \frac{2M}{\sqrt{EG}} \frac{\sqrt{E}du}{\sqrt{E(du)^2 + G(dv)^2}} \frac{\sqrt{G}du}{\sqrt{E(du)^2 + G(dv)^2}}$$

$$+ \frac{N}{G}\left(\frac{\sqrt{G}du}{\sqrt{E(du)^2 + G(dv)^2}}\right)^2$$

Then

$$k_n = \frac{L}{E}\cos^2(\theta) + \frac{2M}{\sqrt{EG}}\cos(\theta)\sin(\theta) + \frac{N}{G}\sin^2(\theta)$$

$$= \frac{L}{E}\frac{1 + \cos(2\theta)}{2} + \frac{M}{\sqrt{EG}}\sin(2\theta) + \frac{N}{G}\frac{1 - \cos(2\theta)}{2}$$

# Principal curvatures and directions

$$k_n = \frac{1}{2}\left(\frac{L}{E} + \frac{N}{G}\right) + \frac{1}{2}\left(\frac{L}{E} - \frac{N}{G}\right)\cos(2\theta) + \frac{M}{\sqrt{EG}}\sin(2\theta)$$

Let

$$A = \sqrt{\left(\frac{1}{2}\left(\frac{L}{E} - \frac{N}{G}\right)\right)^2 + \left(\frac{M}{\sqrt{EG}}\right)^2}$$

When $A \neq 0$, we introduce an angle $\theta_0$ so that

$$\cos(2\theta_0) = \frac{1}{2A}\left(\frac{L}{E} - \frac{N}{G}\right)$$

$$\sin(2\theta_0) = \frac{M}{A\sqrt{EG}}$$

So

$$k_n = \frac{1}{2}\left(\frac{L}{E} + \frac{N}{G}\right) + A\left(\cos(2\theta)\cos(2\theta_0) + \sin(2\theta)\sin(2\theta_0)\right)$$

# Principal curvatures and directions

$$= \frac{1}{2}\left(\frac{L}{E} + \frac{N}{G}\right) + Acos2(\theta - \theta_0)$$

$$= \frac{1}{2}\left(\frac{L}{E} + \frac{N}{G}\right) + \sqrt{\left(\frac{1}{2}\left(\frac{L}{E} - \frac{N}{G}\right)\right)^2 + \left(\frac{M}{\sqrt{EG}}\right)^2}\, cos2(\theta - \theta_0)$$

So we know that when $\theta = \theta_0, k_n(\theta\ )$ can get its maximal value:

$$k_1 = \frac{1}{2}\left(\frac{L}{E} + \frac{N}{G}\right) + \sqrt{\left(\frac{1}{2}\left(\frac{L}{E} - \frac{N}{G}\right)\right)^2 + \left(\frac{M}{\sqrt{EG}}\right)^2}$$

when $\theta = \theta_0 + \frac{\pi}{2}$, $k_n(\theta\ )$ gets its minimal value:

$$k_2 = \frac{1}{2}\left(\frac{L}{E} + \frac{N}{G}\right) - \sqrt{\left(\frac{1}{2}\left(\frac{L}{E} - \frac{N}{G}\right)\right)^2 + \left(\frac{M}{\sqrt{EG}}\right)^2}$$

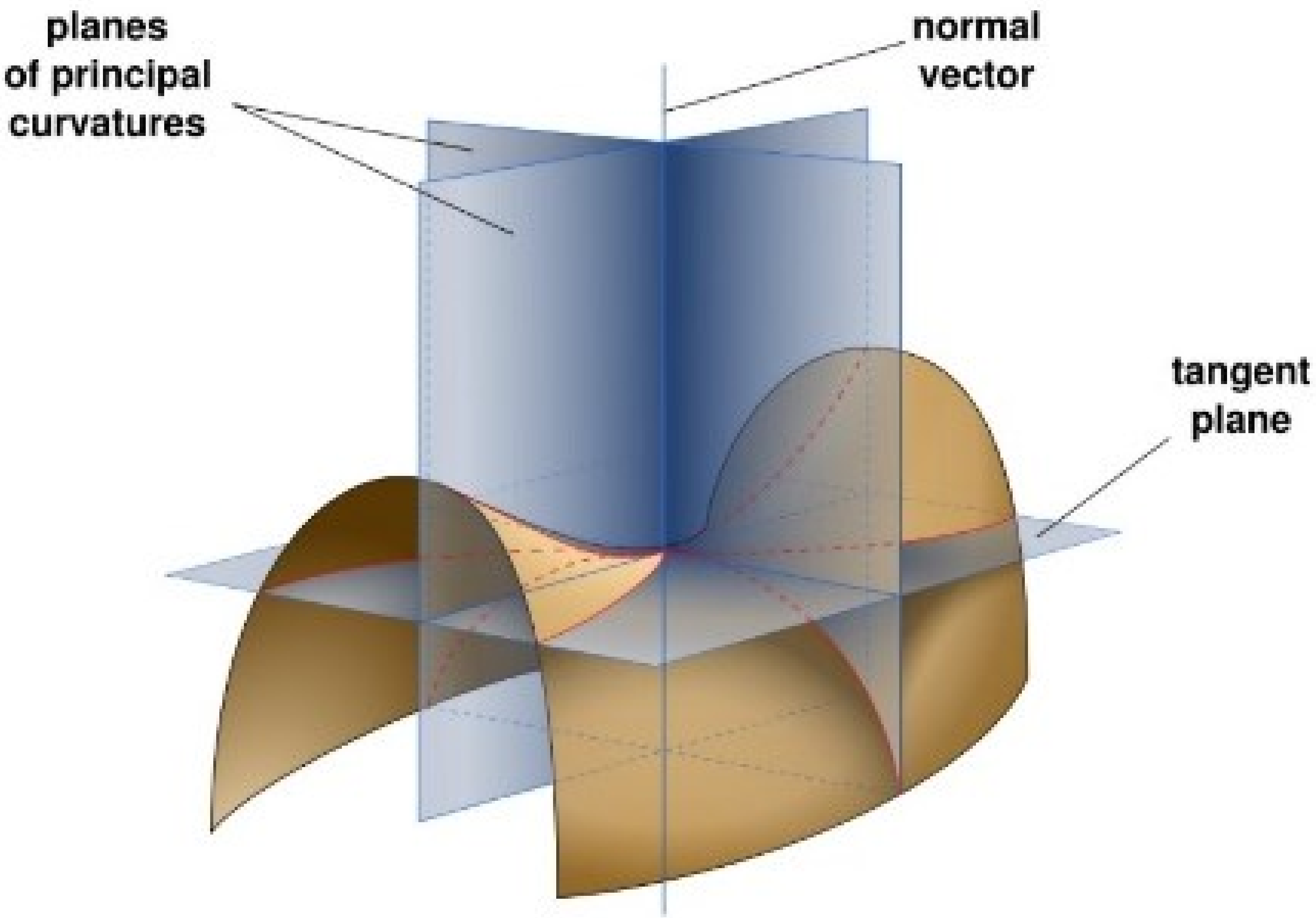
# Principal curvatures and directions

When $A = 0$, then $k_n(\theta)$ is independent of $\theta$.

$$k_n = \frac{1}{2}\left(\frac{L}{E} + \frac{N}{G}\right) + A(\cos(2\theta)\cos(2\theta_0) + \sin(2\theta)\sin(2\theta_0))$$

**Theorem:**

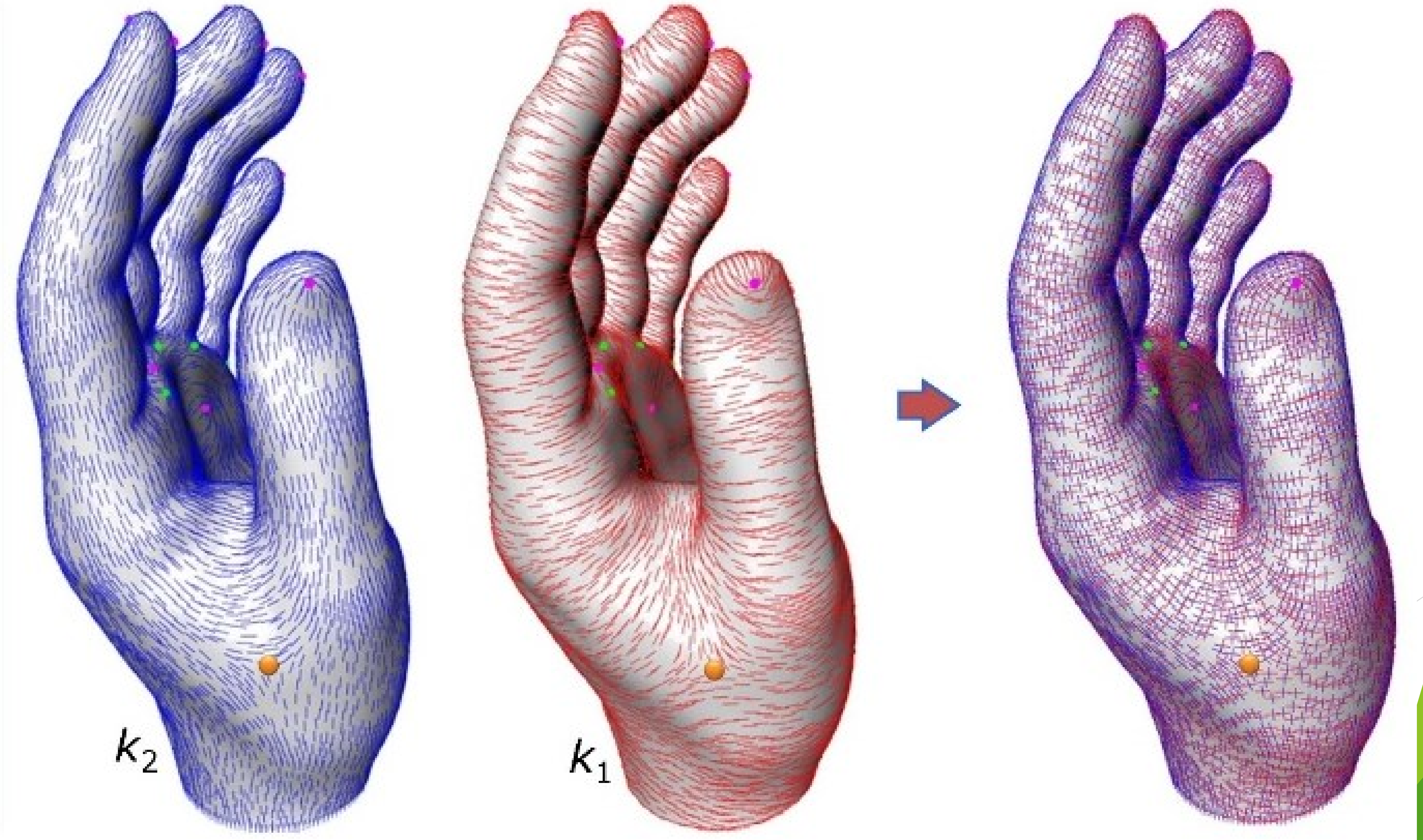
For any point on a regular parametric surface, its maximal and minimal normal curvatures must be obtained in two orthogonal directions.



# Principal curvatures and directions

**Definition**

For any point on a regular parametric surface, its maximal and minimal normal curvatures are called as the **principal curvature** (主曲率) of this point on the surface, and the corresponding directions are the **principal direction** (主方向).



# Asymptotic direction and curve (渐近方向/曲线)

Then the normal curvature of tangent vector along direction angle $\theta$ is

$$k_n(\theta) = k_1 \cos^2(\theta - \theta_0) + k_2 \sin^2(\theta - \theta_0)$$

This equation is called as the **Euler's formula** (欧拉公式) of normal curvature.

**Definition**

For a point on surface S, the tangent vector corresponds to zero normal curvature is called as the **asymptotic direction** (渐近方向).

If there is a curve whose tangent vector at any point is the asymptotic direction of S at that point, then this curve is the **asymptotic curve** (渐近曲线) of S.

# Asymptotic direction and curve (渐近方向/曲线)

For a fixed point $(u, v)$, asymptotic direction satisfies:

$$k_n = \frac{L(du)^2 + 2Mdudv + N(dv)^2}{E(du)^2 + 2Fdudv + G(dv)^2} = 0$$

$$\Leftrightarrow L(du)^2 + 2Mdudv + N(dv)^2 = 0$$

So, point $(u, v)$ has asymptotic direction if and only if

$$LN - M^2 \leq 0$$

If $LN - M^2 < 0$, there are two different asymptotic directions:

$$\frac{du}{dv} = -\frac{M \pm \sqrt{M^2 - LN}}{L}$$

If $LN - M^2 = 0$, there is one asymptotic direction:

$$\frac{du}{dv} = -\frac{M}{L} = -\frac{N}{M}$$

# Asymptotic direction and curve (渐近方向/曲线)

If $LN - M^2 < 0$ is satisfied for any point of surface S, then there are two linearly independent asymptotic direction fields, and there is a parametric curve network formed by asymptotic curves.

**Theorem:**

The parametric curve network is **asymptotic curve network** (渐近曲线网) if and only if $L = N = 0$.

**Proof (Necessity):**

If the parametric curve network is the asymptotic curve network, then $(du, dv) = (1,0)$ and $(du, dv) = (0, 1)$ are asymptotic directions.

Since asymptotic direction satisfies

$$L(du)^2 + 2Mdudv + N(dv)^2 = 0$$

so we have $L = N = 0$.

# Asymptotic direction and curve (渐近方向/曲线)

**Theorem:**

The parametric curve network is the asymptotic curve network if and only if $L = N = 0$.

**Proof (Sufficiency):**

If $L = N = 0$,

$$L(du)^2 + 2Mdudv + N(dv)^2 = 0$$
$$\Leftrightarrow 2Mdudv = 0$$

The solution are $du = 0$ and $dv = 0$.

That is the parametric curve network are the asymptotic curve network.

# Asymptotic direction and curve (渐近方向/曲线)

**Theorem:**

A curve of S is an asymptotic curve if and only if it is a straight line (直线) or it osculating plane (密切平面) is the tangent plane of S.

Proof:

We know the normal curvature satisfies

$$k_n = kcos(\theta)$$

where $\theta$ is the angle between normal vector $\boldsymbol{N}$ of curve and normal vector $\boldsymbol{n}$ of surface S. So

$$k_n = 0$$

$$\Leftrightarrow k = 0 \; or \cos(\theta\ ) = 0$$

If $k = 0$ for any point of the curve, then this curve will be straight line.

# Asymptotic direction and curve (渐近方向/曲线)

If for a point,

$$\cos(\theta\ ) = 0$$

then

$$\theta = \frac{\pi}{2},$$

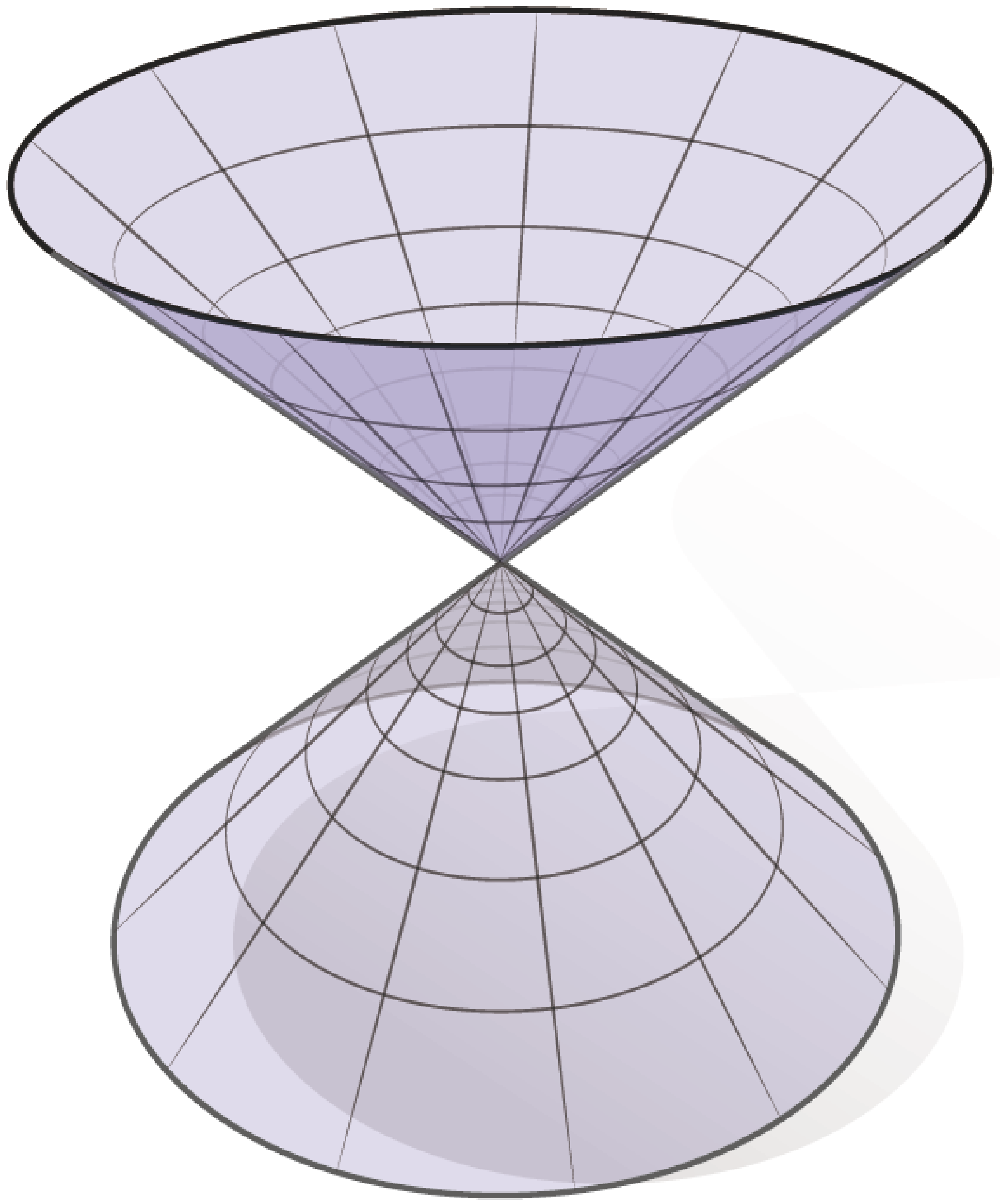
that is $\boldsymbol{N}$ is perpendicular to $\boldsymbol{n}$,

so osculating plane of the curve is the tangent plane of S.

# Homework

1. Calculate the normal curvature of paraboloid surface $z = \frac{1}{2}(ax^2 + by^2)$ at point $(0,0)$, along direction $(dx{:}\,dy)$.
2. Assume the distance from a plane $\pi$ to the center of unit sphere S is $d$ $(0 < d < 1)$, calculate the curvature and normal curvature of the intersection curve of S and $\pi$.
3. Assume the parametric function of catenoid (悬链面) is $\boldsymbol{r} = (\sqrt{u^2 + a^2}\cos(v), \sqrt{u^2 + a^2}\sin(v), a\log(u + \sqrt{u^2 + a^2})$, calculate its first and second fundamental forms, and the normal curvature at point (0, 0), along tangent vector $d\boldsymbol{r} = 2\boldsymbol{r}_u + \boldsymbol{r}_v$.

# Weingarten map and principal curvature



# Gauss map

Assume $S: \boldsymbol{r} = \boldsymbol{r}(u, v)$ is a regular parametric surface.
For each point of S, there is a certain unit normal vector $\boldsymbol{n}(u, v)$.
We move the start node of $\boldsymbol{n}(u, v)$ to the origin of the coordinate system,
then the ending node will fall at a unite sphere $\Sigma$ in $E^3$.
Therefore, we can define a differentiable map from S to $\Sigma$

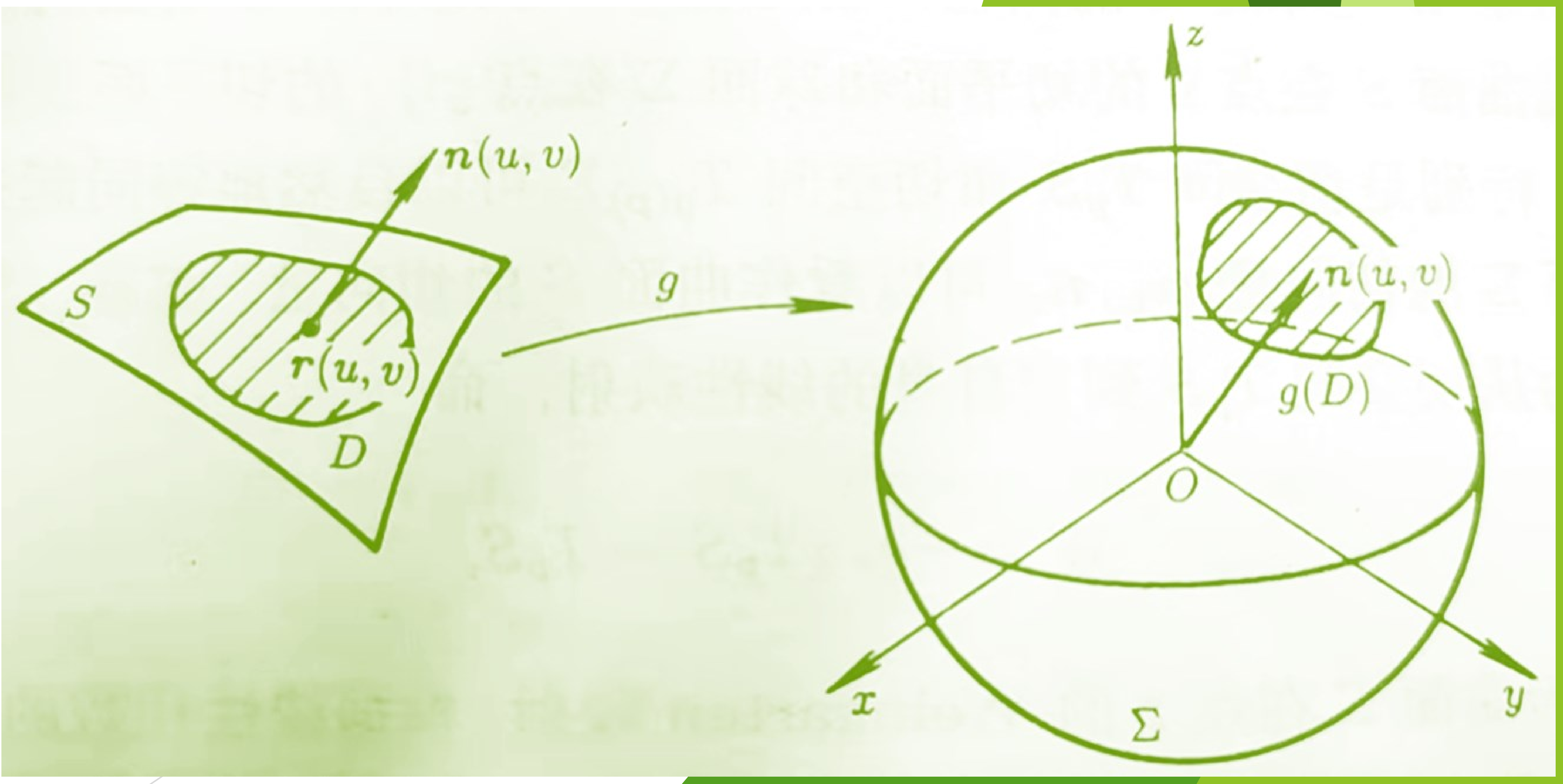
$$g: S \to \Sigma$$

That is

$$g(\boldsymbol{r}(u, v)) = \boldsymbol{n}(u, v)$$

This map is called as **Guass map** (高斯映射).

Obviously, for a surface S has severe bending,
when change a point on S,
its normal will have large change.



# Gauss map

From Guass map $g: S \to \Sigma$, a tangent map $g_*$ from the tangent space $T_pS$ of $S$ at point $p$ to the tangent space $T_{g(p)}\Sigma$ of $\Sigma$ at point $g(p)$ can be induced:

$$g_*: T_pS \to T_{g(p)}\Sigma$$

Next we will introduce the expression of $g_*$.

Assume the parametric function of a curve on S is

$$u = u(t), v = v(t)$$

Its image (像) under Gauss map is

$$g\left(\boldsymbol{r}\big(u(t), v(t)\big)\right) = \boldsymbol{n}\big(u(t), v(t)\big)$$

According the definition of induced tangent map we know

$$g_*\left(\frac{d\boldsymbol{r}}{dt}\right) = \frac{d\boldsymbol{n}\big(u(t), v(t)\big)}{dt}$$

# Gauss map

$$g_*\left(\frac{d\boldsymbol{r}}{dt}\right) = \frac{d\boldsymbol{n}\big(u(t), v(t)\big)}{dt}$$

$$= \boldsymbol{n}_u \frac{du(t)}{dt} + \boldsymbol{n}_v \frac{dv(t)}{dt}$$

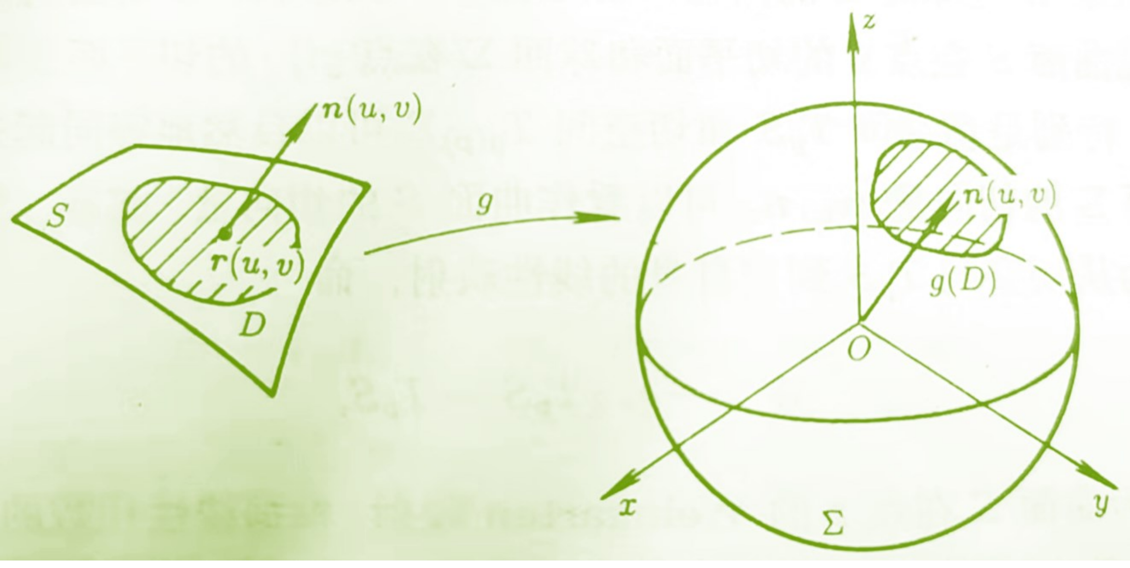
Since tangent map is linear, so we get

$$g_*\left(\frac{d\boldsymbol{r}}{dt}\right) = g_*\left(\boldsymbol{r}_u \frac{du(t)}{dt} + \boldsymbol{r}_v \frac{dv(t)}{dt}\right)$$

$$= g_*(\boldsymbol{r}_u)\frac{du(t)}{dt} + g_*(\boldsymbol{r}_v)\frac{dv(t)}{dt}$$

$$\Longrightarrow g_*(\boldsymbol{r}_u) = \boldsymbol{n}_u, \qquad g_*(\boldsymbol{r}_v) = \boldsymbol{n}_v$$

# Weingarten map




Since each point on a unit sphere is the unit normal of sphere at that point, $\boldsymbol{n}\big(u(t), v(t)\big)$ is the unit normal of $\Sigma$.
That is the tangent plane of S at point p and the tangent plane of $\Sigma$ at point g(p) are parallel.
Therefore, tangent space $T_pS$ and tangent space $T_{g(p)}\Sigma$ can be identified (等同起来).
That is tangent vectors $\boldsymbol{n}_u, \boldsymbol{n}_v$ of $\Sigma$ can be considered as the tangent vectors of $S$.
Then tangent map $g_*$ can be a map from tangent space $T_pS$ to itself, that is

$$W = -g_* \colon T_pS \to T_pS$$

Let $W$ be the **Weingarten map** of $S$ at point $p$.

# Weingarten map

**Theorem**:
The second fundamental form $\Pi$ can be represented with Weingarten map, that is

$$\Pi = W(d\boldsymbol{r}).d\boldsymbol{r}$$

**Proof:**
Any tangent vector of S can be represented as

$$d\boldsymbol{r} = \boldsymbol{r}_u du + \boldsymbol{r}_v dv$$

where $du$ and $dv$ are the components of tangent vector.
According to the definition of Weingarten map we know

$$\begin{aligned} W(d\boldsymbol{r}) &= W(\boldsymbol{r}_u du + \boldsymbol{r}_v dv) \\ &= -g_*(\boldsymbol{r}_u du + \boldsymbol{r}_v dv) \\ &= -g_*(\boldsymbol{r}_u)du - g_*(\boldsymbol{r}_v)dv \\ &= -\boldsymbol{n}_u du - \boldsymbol{n}_v dv \\ &= -d\boldsymbol{n} \end{aligned}$$

So

$$W(d\boldsymbol{r}).d\boldsymbol{r} = -d\boldsymbol{n}.d\boldsymbol{r} = \Pi$$

# Weingarten map

**Theorem**:
Weingarten map W is a self-conjugate map (自共轭映射) from tangent space $T_pS$ to itself,
that is for any two tangent vectors $d\boldsymbol{r}$ and $\delta\boldsymbol{r}$ at point $(u, v)$, the following equation is satisfied

$$W(d\boldsymbol{r}).\delta\boldsymbol{r} = d\boldsymbol{r}.W(\delta\boldsymbol{r})$$

**Proof:**
Let assume

$$d\boldsymbol{r} = \boldsymbol{r}_u du + \boldsymbol{r}_v dv$$
$$\delta\boldsymbol{r} = \boldsymbol{r}_u \delta u + \boldsymbol{r}_v \delta v$$

Then we get

$$W(d\boldsymbol{r}) = -(\boldsymbol{n}_u du + \boldsymbol{n}_v dv)$$
$$W(\delta\boldsymbol{r}) = -(\boldsymbol{n}_u \delta u + \boldsymbol{n}_v \delta v)$$

# Weingarten map

Then

$$
\begin{gathered}
W(d\boldsymbol{r}).\,\delta\boldsymbol{r} \\
= -(\boldsymbol{n}_u du + \boldsymbol{n}_v dv).\,(\boldsymbol{r}_u \delta u + \boldsymbol{r}_v \delta v) \\
= L du \delta u + M(du \delta v + dv \delta u) + N dv \delta v \\
= -(\boldsymbol{r}_u du + \boldsymbol{r}_v dv).\,(\boldsymbol{n}_u \delta u + \boldsymbol{n}_v \delta v) \\
= d\boldsymbol{r}.\,W(\delta\boldsymbol{r})
\end{gathered}
$$

# Principal curvature and direction

If there is a tangent vector $d\boldsymbol{r} \neq 0$ and a real number $\lambda \neq 0$, so that

$$W(d\boldsymbol{r}) = \lambda d\boldsymbol{r}$$

Then we call $\lambda$ is the **eigenvalue** (特征值) of the Weingarten map and $d\boldsymbol{r}$ is the corresponding **eigenvector** (特征向量).

So we have

$$\Pi = W(d\boldsymbol{r}).d\boldsymbol{r} = \lambda d\boldsymbol{r}.d\boldsymbol{r}$$

Then the normal curvature $k_n$ along eigenvector $d\boldsymbol{r}$ is written as

$$k_n = \frac{\Pi}{I} = \frac{W(d\boldsymbol{r}).d\boldsymbol{r}}{d\boldsymbol{r}.d\boldsymbol{r}} = \lambda$$

It proves that the eigenvalue $\lambda$ of the Weingarten map W is the normal curvature of surface at that point along the corresponding eigenvector $d\boldsymbol{r}$.

# Principal curvature and direction

In 2D vector space, Weingarten map has two eigenvalues $\lambda_1$ and $\lambda_2$ which correspond to two linearly independent orthogonal eigenvectors.

If $\lambda_1 \neq \lambda_2$, then their corresponding eigenvectors are uniquely determined.

Otherwise, the corresponding eigenvectors cannot be determined,

that is any tangent vector at that point is the eigenvector.

# Principal curvature and direction

**Theorem**:
The two eigenvalues of the Weingarten map for any point on a regular parametric surface are the principal curvatures of the surface at that point,
and the corresponding eigenvectors are the corresponding principal directions.

**Proof:**
Choose an orthogonal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ for tangent space $T_pS$, so that $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are eigenvectors of $S$ at point $p$.
The corresponding eigenvalues $\lambda_1 \geq \lambda_2$ satisfy:

$$W(\boldsymbol{e}_1) = \lambda_1 \boldsymbol{e}_1$$
$$W(\boldsymbol{e}_2) = \lambda_2 \boldsymbol{e}_2$$

Let assume $\boldsymbol{e}$ is an arbitrary tangent vector of $S$ at point $p$.

# Principal curvature and direction

Then it can be represented as

$$\boldsymbol{e} = \cos(\theta)\,\boldsymbol{e}_1 + \sin(\theta)\,\boldsymbol{e}_2$$

So

$$\begin{aligned} W(\boldsymbol{e}) &= \cos(\theta)\,W(\boldsymbol{e}_1) + sin(\theta)\,W(\boldsymbol{e}_2) \\ &= \lambda_1 \cos(\theta)\,\boldsymbol{e}_1 + \lambda_2 \sin(\theta)\,\boldsymbol{e}_2 \end{aligned}$$

The normal curvature along tangent vector $\boldsymbol{e}$ is

$$k_n(\theta) = \frac{W(\boldsymbol{e}).\boldsymbol{e}}{\boldsymbol{e}.\boldsymbol{e}}$$

$$\begin{aligned} &= (\lambda_1 \cos(\theta)\,\boldsymbol{e}_1 + \lambda_2 \sin(\theta)\,\boldsymbol{e}_2).(\cos(\theta)\,\boldsymbol{e}_1 + \sin(\theta)\,\boldsymbol{e}_2) \\ &= \lambda_1 \cos^2(\theta) + \lambda_2 \sin^2(\theta) \\ &= \lambda_1 - (\lambda_1 - \lambda_2)\sin^2(\theta) \\ &= \lambda_2 + (\lambda_1 - \lambda_2)\cos^2(\theta) \end{aligned}$$

Then, the maximal normal curvature $\lambda_1$ is obtained when $\theta = 0$, and the minimal normal curvature $\lambda_2$ is obtained when $\theta = \frac{\pi}{2}$.

# Principal curvature and direction

**Theorem (Euler formula)**:
Assume $e_1, e_2$ are orthogonal unit principal vectors of $S$ at point $p$, the corresponding principal curvatures are $k_1, k_2$. Then the normal curvature of $S$ at point $p$ along any tangent vector $\boldsymbol{e} = \cos(\theta)\,\boldsymbol{e}_1 + \sin(\theta)\,\boldsymbol{e}_2$ is

$$k_n(\theta) = \lambda_1 \cos^2(\theta) + \lambda_2 \sin^2(\theta)$$

When $\lambda_1 = \lambda_2$, for any tangent direction angle $\theta$,

$$k_n(\theta) = \lambda_1 = \lambda_2$$

Then we cannot determine the principal direction.
We call this kind of points as **umbilical point** (脐点).
For umbilical point, its normal curvature

$$k_n = \frac{L(du)^2 + 2Mdudv + N(dv)^2}{E(du)^2 + 2Fdudv + G(dv)^2}$$

is independent of the tangent vector $(du, dv)$,

# Principal curvature and direction

*So*

$$(L - k_n E)(du)^2 + 2(M - k_n F)dudv + (N - k_n G)(dv)^2 = 0$$

is always satisfied.

Therefore, the following equations stand up at that point

$$\frac{L}{E} = \frac{M}{F} = \frac{N}{G}$$

Then we can see that umbilical point is the point where the coefficients of the first fundamental form is proportional to the coefficients of the second fundamental form of surface.

If this ratio is zero, this umbilical point is called as **planar point** (**平点**),

otherwise it’s called as **circular point** (**圆点**).