# Set Theory

## Relations II

# Content

- Equivalence relation (等价关系)
- Order relation (序关系)

# Equivalence Relations

# Equivalence Relations

- Equivalence relations are used to relate objects that are similar in some way.
- **Definition**: A relation on a set $A$ is called an equivalence relation if it is **reflexive**, **symmetric**, and **transitive**.
- Two elements that are related by an equivalence relation R are called equivalent.

# Equivalence Relations

- Since R is symmetric, a is equivalent to b whenever b is equivalent to a.
- Since R is reflexive, every element is equivalent to itself.
- Since R is transitive, if a and b are equivalent and b and c are equivalent, then a and c are equivalent.
- Obviously, these three properties are necessary for a reasonable definition of equivalence.

# Equivalence Relations

- **Example**: Suppose that R is the relation on the set of strings that consist of English letters such that aRb if and only if l(a)=l(b),where l(x) is the length of the string x. Is R an equivalence relation?
- **Solution**:
  - R is reflexive, because l(a) =l(a) and therefore aRa for any string a.
  - R is symmetric, because if l(a) = l(b) then l(b) = l(a), so if aRb then bRa.
  - R is transitive, because if l(a) = l(b) and l(b) = l(c), then l(a) = l(c), so aRb and bRc implies aRc.
- R is an equivalence relation.

# Equivalence Classes

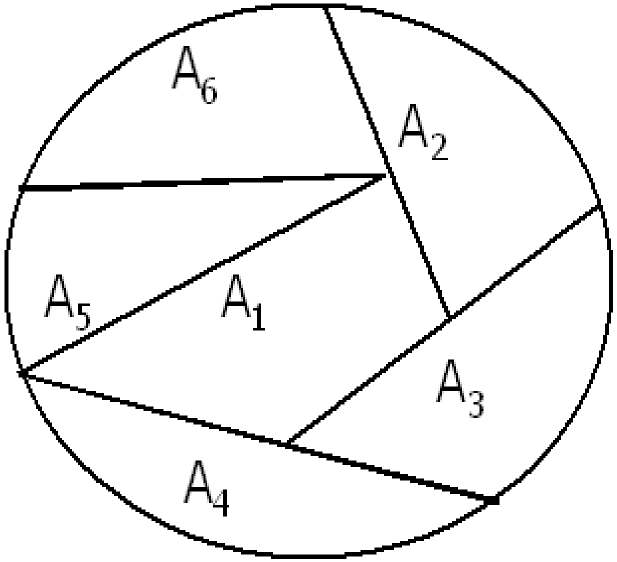
- **Definition**: Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the **equivalence class** of a.
- The equivalence class of a with respect to R is denoted by $[a]_R$.
- When only one relation is under consideration, we will delete the subscript R and write [a] for this equivalence class.
- If $b \in [a]_R$, b is called a representative of this equivalence class.

# Equivalence Classes

- **Example**: In the previous example (strings of identical length), what is the equivalence class of the word mouse, denoted by [mouse] ?
- **Solution**:
  - [mouse] is the set of all English words containing five letters.
  - For example, 'horse' would be a representative of this equivalence class.

# Equivalence Classes

- **Theorem**: Let R be an equivalence relation on a set A. The following statements are equivalent:
  - 1. aRb
  - 2. [a] = [b]
  - 3. $[a] \cap [b] \neq \emptyset$
- **Definition**: A **partition** of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets Ai, $i \in I$, forms a partition of S iff
  - 1. $A_i \neq \emptyset$ for $i \in I$
  - 2. $A_i \cap A_j = \emptyset$, if $i \neq j$
  - 3. $\cup_{i \in I} A_i = S$



# Equivalence Classes

**Examples**: Let S be the set {u, m, b, r, o, c, k, s}. Do the following collections of sets partition S ?

- {{m, o, c, k}, {r, u, b, s}} yes.
- {{c, o, m, b}, {u, s}, {r}} no (k is missing).
- {{b, r, o, c, k}, {m, u, s, t}} no (t is not in S).
- {{u, m, b, r, o, c, k, s}} yes.
- {{b, o, o, k}, {r, u, m}, {c, s}} yes ({b,o,o,k} ={b,o,k}).
- {{u, m, b}, {r, o, c, k, s}, ∅} no (∅ not allowed).

# Equivalence Classes

- **Theorem**: Let R be an equivalence relation on a set S. Then the **equivalence classes** of R form a **partition** of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets $A_i$, $i \in I$, as its equivalence classes.
- **Example**:
  - Let us assume that Frank, Suzanne and George live in Boston, Stephanie and Max live in Lübeck, and Jennifer lives in Sydney.

# Equivalence Classes

- Let R be the equivalence relation {(a, b) | a and b live in the same city} on the set P = {Frank, Suzanne, George, Stephanie, Max, Jennifer}.
- Then R = { (Frank, Frank), (Frank, Suzanne), (Frank,George), (Suzanne,Frank), (Suzanne,Suzanne), (Suzanne,George), (George,Frank), (George, Suzanne), ( George,George), (Stephanie, Stephanie), ( Stephanie, Max), ( Max, Stephanie), (Max Max), (Jennifer, Jennifer) }.

# Equivalence Classes

- Then the equivalence classes of R are:
  - {{Frank, Suzanne, George}, {Stephanie, Max}, {Jennifer}}.
  - This is a partition of P.
- The equivalence classes of any equivalence relation R defined on a set S constitute a partition of S, because every element in S is assigned to exactly one of the equivalence classes.

# Equivalence Classes

- **Example**:
  - Let R be the relation $\{(a,b) \mid a \equiv b \pmod 3\}$ on the set of integers.
  - Is R an equivalence relation?
    - Yes, R is reflexive, symmetric, and transitive.
  - What are the equivalence classes of R ?
    - $\{ \{\ldots, -6, -3, 0, 3, 6, \ldots\}, \{\ldots, -5, -2, 1, 4, 7, \ldots\}, \{\ldots, 4, 1, 2, 5, 8, \ldots\} \}$

# Equivalence relations

- **Example**:
  - Consider set $X = \{1,2,\ldots,13\}$. Define $xRy$ as 5 divides $x - y$ (i.e., $x - y = 5k$, for some int $k$). We can verify that $R$ is reflexive, symmetric, and transitive. Here is how.
  - The equivalence class $[1]$ consists of all $x$ with $xR1$. Thus:
    - $[1] = \{x \in X \mid 5 \text{ divides } x - 1\} = \{1, 6, 11\}$
  - Similarly:
    - $[2] = \{2, 7, 12\}$
    - $[3] = \{3, 8, 13\}$
    - $[4] = \{4, 9\}$
    - $[5] = \{5, 10\}$

# Equivalence relations

- These 5 sets partition X. Note that:
- [1] = [6] = [11]
- [2] = [7] = [12]
- [3] = [8] = [13]
- [4] = [9]
- [5] = [10]
- For this relation, equivalence is “has the same reminder when divided by 5”.

# Partial Orders Relations

# Order relations

- **Definition**: Let X be a set and R a relation on X, R is a partial order on X if R is **reflexive**, **antisymmetric** and **transitive**. A set X together with a partial ordering R is called a ***partially ordered set***, or ***poset***, or ***PO***, and is denoted by (X, R).
- **Example**: Is $(x, y) \in R$ in partial order if $x >= y$?
  - Yes, since:
    - Reflexive: $(x, x) \in R$
    - Anti-symmetric: If $(x, y) \in R$ and $x \neq y$, then $(y, x) \notin R$
    - Transitive: If $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$

# Order relations

- **Example**: Is the "inclusion relation" $\subseteq$ a partial ordering on the power set of a set $S$?
  - $\subseteq$ is reflexive, because $A \subseteq A$ for every set $A \in S$.
  - $\subseteq$ is antisymmetric, because if $A \neq B$, then $A \subseteq B \wedge B \subseteq A$ is false.
  - $\subseteq$ is transitive, because if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
- Consequently, $(P(S), \subseteq)$ is a partially ordered set or poset.

# Order relations

- Let $x,y \in X$,
  - If $(x,y)$ or $(y,x)$ are in R, then x and y are **comparable**.
  - If $(x,y) \notin R$ and $(y,x) \notin R$, then x and y are **incomparable**.
  - **Definition**: If every pair of elements in X are comparable, then R is a **total order** on X.
    - In this case, X is called a totally ordered or linearly ordered set, and $\leq$ is called a total order or linear order. A totally ordered set is also called a **chain**.

# Order relations

- **Example**: Is $(Z, \leq)$ a totally ordered poset?
  - Yes, because $a \leq b$ or $b \leq a$ for all integers a and b.
- **Example**: Is $(Z^+$, division) a totally ordered poset?
  - No, because it contains incomparable elements such as 5 and 7.

# Order relations

- In a poset the notation $a \leq b$ denotes that $(a, b) \in R$.
- Note that the symbol $\leq$ is used to denote the relation in any poset, not just the "less than or equal" relation.
- The notation $a < b$ denotes that $a \leq b$, but $a \neq b$.
- If $a < b$ we say "a is less than b" or "b is greater than a".

# Lexicographic Order

- How can we define a lexicographic ordering on the set of English words?
- This is a **special case** of an ordering of strings on a set constructed from a partial ordering on the set.
- We already have an ordering of letters (such as $a < b$, $b < c$, …), and from that we want to derive an ordering of strings.
- Let us take a look at the *general* case, that is, how the construction works in any poset.

# Lexicographic Order

- **First step**: Construct a partial ordering on the Cartesian product of two posets,$(A_1, \preceq_1)$ and $(A_2, \preceq_2)$:
- $(a_1,a_2) \prec (b_1,b_2)$ if $(a_1 \prec_1 b_1) \lor [(a_1=b_1) \land (a_2 \prec_2 b_2)]$
- $(a_1,a_2) \preceq (b_1, b_2)$ if $(a_1 \prec_1 b_1) \lor [(a_1 = b_1) \land (a_2 \preceq_2 b_2)]$
- **Examples**:
  - In the poset $(Z \times Z, \preceq)$, …
    - is $(5, 5) \prec (6, 4)$ ? YES
    - is $(6, 5) \prec (6, 4)$ ? NO
    - is $(3, 3) \prec (3, 3)$ ? NO

# Lexicographic Order

- **Second step**: Extend the previous definition to the Cartesian product of n posets $(A_1, \preceq_1), (A_2, \preceq_2), \ldots, (A_n, \preceq_n)$:
- $(a_1, a_2, \ldots, a_n) \prec (b_1, b_2, \ldots, b_n)$ if $(a_1 \prec_1 b_1) \vee \exists i>0\ (a_1 = b_1, a_2 = b_2, \ldots, a_i = b_i, a_{i+1} \prec_{i+1} b_{i+1})$
- Examples:
  - Is $(1,1,1,2,1) \prec (1,1,1,1,2)$? No
  - Is $(1,1,1,1,1) \prec (1,1,1,1,2)$? Yes

# Lexicographic Order

We can now define lexicographic ordering of strings. Consider the strings $a_1a_2 \ldots a_m$ and $b_1b_2 \ldots b_n$ on a partially ordered set $S$.

Suppose these strings are not equal. Let $t$ be the minimum of $m$ and $n$. The definition of lexicographic ordering is that the string $a_1a_2 \ldots a_m$ is less than $b_1b_2 \ldots b_n$ if and only if

- $(a_1, a_2, \ldots, a_t) < (b_1, b_2, \ldots, b_t)$, or
- $(a_1, a_2, \ldots, a_t) = (b_1, b_2, \ldots, b_t)$ and $m < n$,

where $<$ in this inequality represents the lexicographic ordering of $S^t$.

# Lexicographic Order

- In other words, to determine the ordering of two different strings, the longer string is truncated to the length of the shorter string, namely, to $t = \min(m, n)$ terms.
- Then the t-tuples made up of the first t terms of each string are compared using the lexicographic ordering on $S^t$ .
- One string is less than another string if the t-tuple corresponding to the first string is less than the t-tuple of the second string, or if these two t-tuples are the same, but the second string is longer.

- $(a_1a_2 \ldots a_{m)}$ R $(b_1b_2 \ldots b_n)$ if
  - $(a_1, a_2, \ldots, a_t) = (b_1, b_2, \ldots, b_t)$ and $t= m = n$,
  - $(a_1, a_2, \ldots, a_t) < (b_1, b_2, \ldots, b_t)$, or $(a_1, a_2, \ldots, a_t) = (b_1, b_2, \ldots, b_t)$ and $m < n$,
- R is a partial ordering

# Hasse Diagram (哈斯图)

- Hasse diagram is a graphical display of a poset.
- A point is drawn for each element of the poset, and line segments are drawn between these points according to the following two rules:
  - 1. If $x < y$ in the poset, then the point corresponding to $x$ appears lower in the drawing than the point corresponding to $y$.
  - 2. The line segment between the points corresponding to any two elements $x$ and $y$ of the poset is included in the drawing iff $x$ covers $y$ or $y$ covers $x$.

# Cover Relation

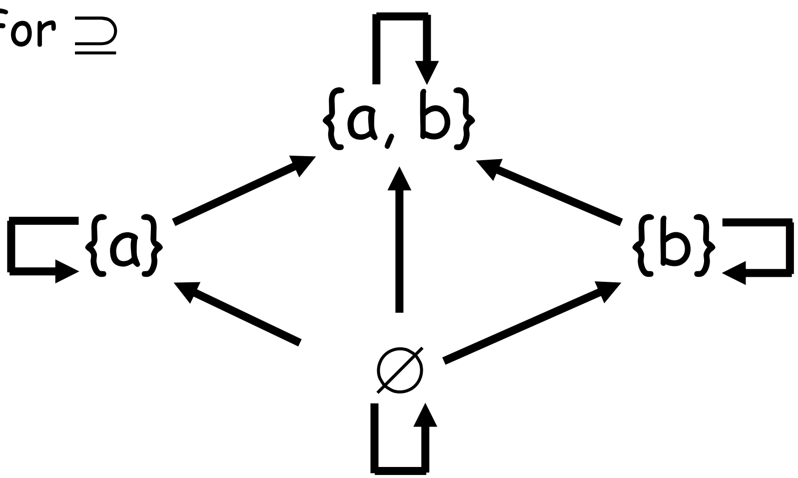
- Let $(S, \leq)$ be a poset. We say that an element $y \in S$ **covers** an element $x \in S$ if $x < y$ and there is no element $z \in S$ such that $x < z < y$. The set of pairs $(x, y)$ such that $y$ covers $x$ is called **the covering relation** of $(S, \leq)$.

# Hasse Diagrams

We produce Hasse Diagrams from directed graphs of relations by doing a **transitive reduction** plus a **reflexive reduction** (if weak) and (usually) **dropping arrowheads** (using, instead, "above" to give direction)

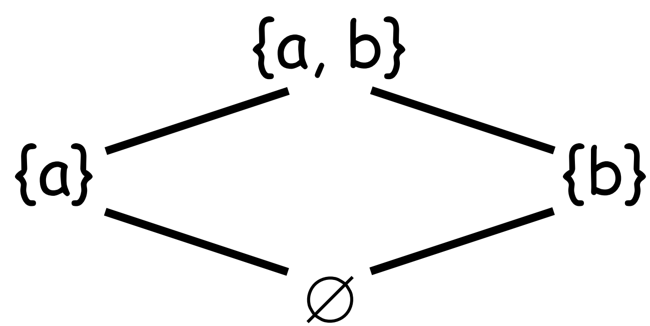
1) Transitive reduction — discard all arcs except those that "directly cover" an element.
2) Reflexive reduction — discard all self loops.

For $\supseteq$




$\equiv$

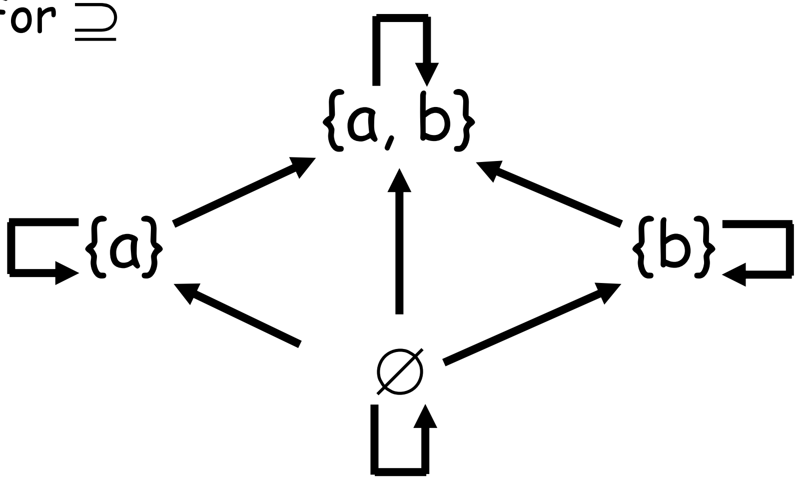
we write:



# The Procedure Summary

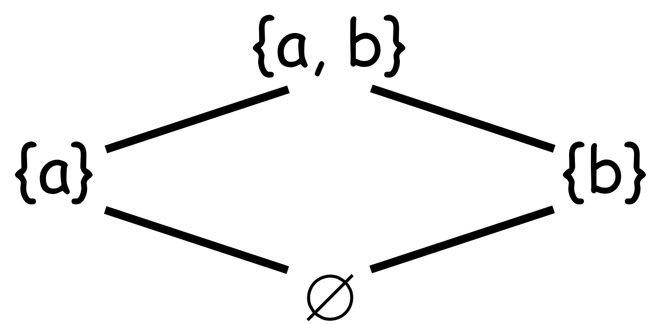
- Start with the directed graph for this relation.
- Because a partial ordering is reflexive, a loop (a, a) is present at every vertex a. Remove these loops.
- Next, remove all edges that must be in the partial ordering because of the presence of other edges and transitivity. That is, remove all edges $(x, y)$ for which there is an element $z \in S$ such that $x < z$ and $z < x$.
- Finally, arrange each edge so that its initial vertex is below its terminal vertex (as it is drawn on paper). Remove all the arrows on the directed edges, because all edges point "upward" toward their terminal vertex.

For $\supseteq$



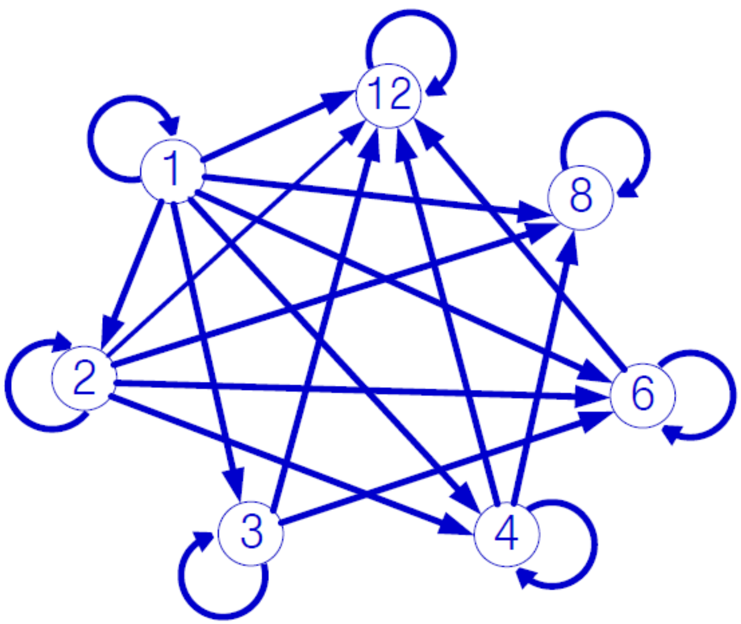

$\equiv$

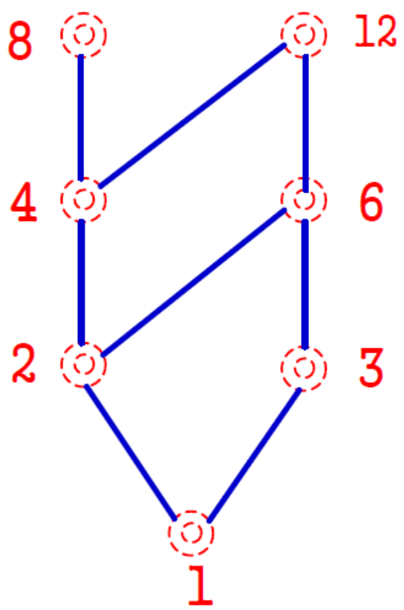
we write:



# Hasse Diagram

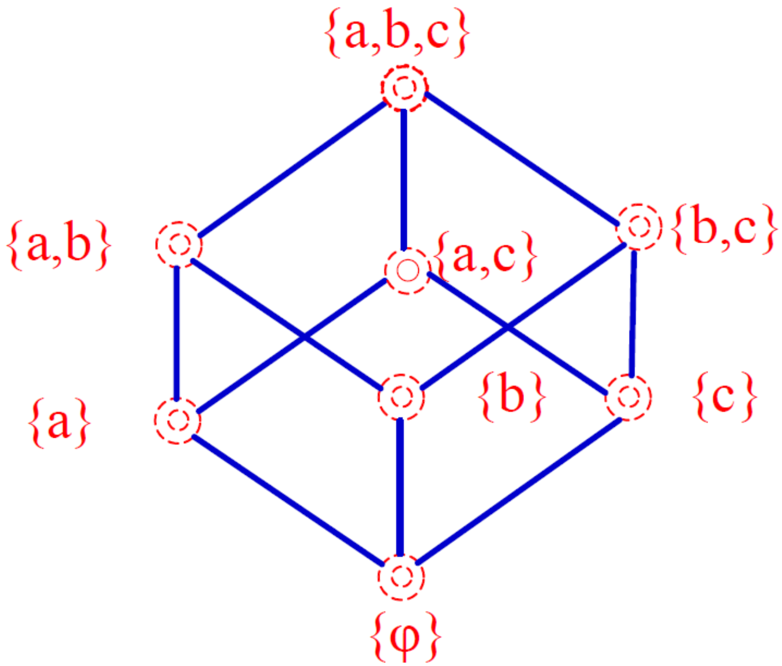
- **Example**: $A=\{1,2,3,4,6,8,12\}$, integral division relation.



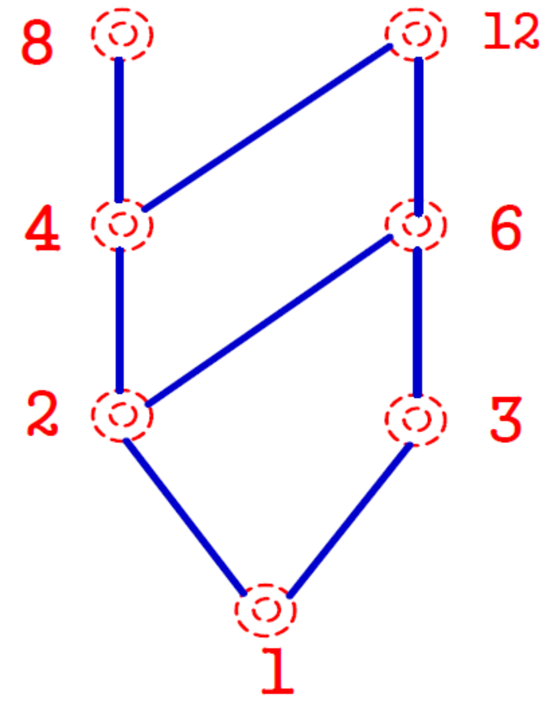
# Hasse Diagram

- **Example**: S={a, b, c}, (P(S), ⊆)



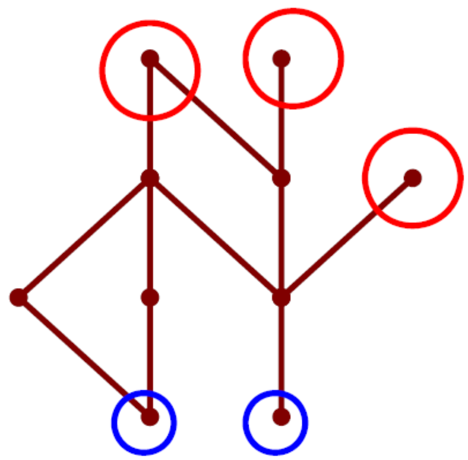
# Maximum/Minimum/Greatest/Least

- Maximum/Minimum element
- 极大、极小
- Greatest/Least element
- 最大、最小
- Upper/Lower bound
- 上界、下界
- Least upper/Greatest lower bound
- 最小上界、最大下界



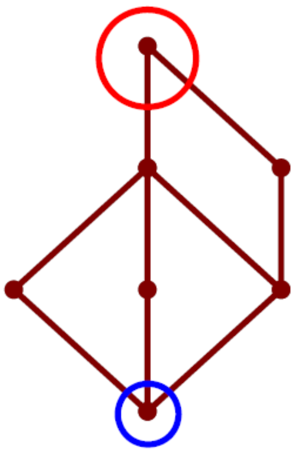
# Minimum and Maximum

- **Definition**: In a poset S, an element z is a **minimum** element if there is no element $b \in S$, thus $b \leq z$ and $b \neq z$.
- How about definition for **maximum** element?
- **Example**:
  - Reds are maximal.
  - Blues are minimal.



# Least and Greatest

- **Definition**: In a poset S, an element z is a **Least** element if $\forall b \in S$, $z \leq b$.
- How about definition for **Greatest** element.
- **Example**:
  - Reds are greatest.
  - Blues are least.
- Greatest/Least may not exist.

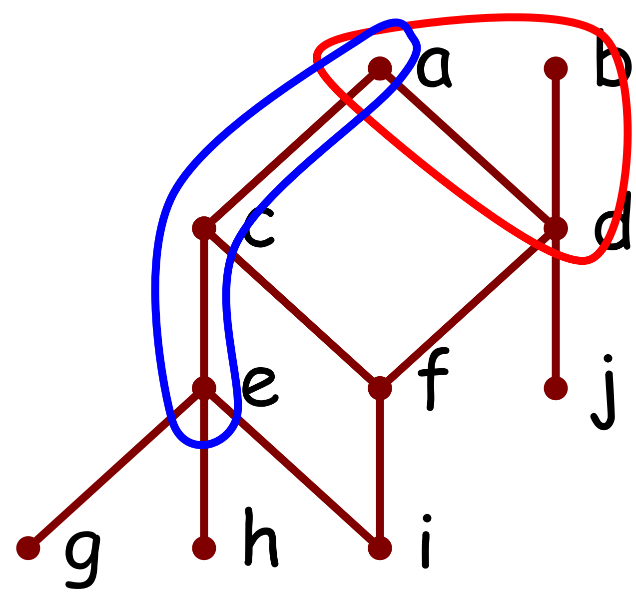


# Least and Greatest

- **Theorem**: In every poset, if the **greatest** element exists, then it is **unique**. Similarly for the **least**.
- **Proof**:
  - Suppose there are two greatest elements, $a_1$ and $a_2$, with $a_1 \neq a_2$. Then $a_1 \leq a_2$, and $a_2 \leq a_1$, by defn of greatest. So $a_1 = a_2$, a contradiction. Thus, our assumption was incorrect, and the greatest element, if it exists, is unique.
  - Similar proof for least.

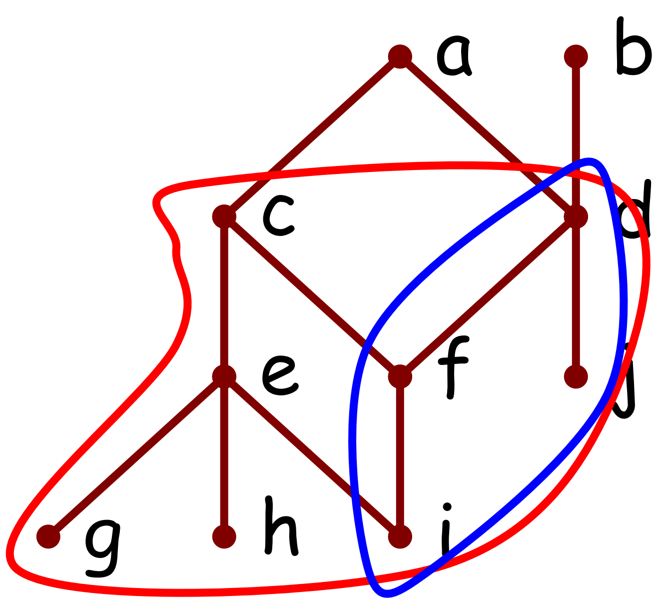
# Upper and Lower Bound

- **Definition**: Let $(S, \leq)$ be a partial order. If $A \subseteq S$, then
  - an **upper bound** (or UB) for $A$ is any element $x \in S$ (**perhaps in $A$ also**) such that $\forall a \in A$, $a \leq x$.
  - a **lower bound** (or LB) for $A$ is any $x \in S$ such that $\forall a \in A$, $a \geq x$.
- **Example**: a b
  - The UB of {g, j} is a.
  - Why not b?
  - What is/are UB of {g, i}?
  - Does {a, b} have UB?



# Upper and Lower Bound

- **Example**:
  - The LBs of {a, b} are d, f, i, and j.
  - What is/are the LB of {c, d}?
  - Does {g, h, i, j} have LB?



# Least Upper and Greatest Lower Bounds

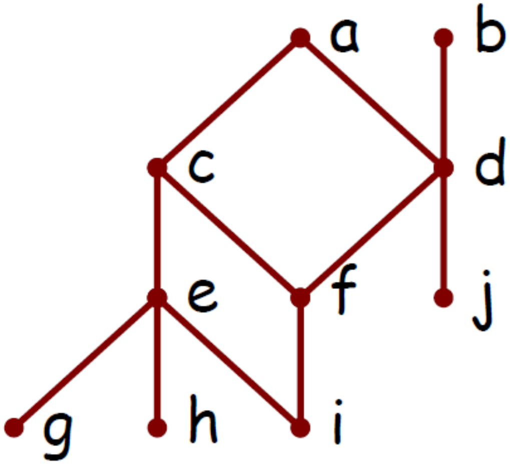
- **Definition**: Given poset $(S, \leq)$ and $A \subseteq S$,
  - $x \in S$ is a least upper bound (or LUB) for $A$ if $x$ is an UB and for every UB $y$ of $A$, $y \geq x$.
  - $x$ is a greatest lower bound (or GUB) for $A$ if $x$ is a LB and if $x \leq y$ for every LB $y$ of $A$.
- **Example**: LUB of {i,j} is d.
- **Example**: GLB of {g j} is

  A. I have no clue.

  B. a

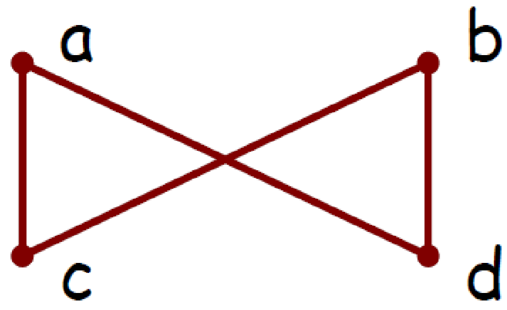
  C. non-existent

  D. e, f, j



# Least Upper and Greatest Lower Bounds

- **Example**:
  - In the following poset, c and d are lower bounds for {a, b}, but there is no GLB.
  - Similarly, a and b are upper bounds for {c,d}, but there is no LUB.



# The End