Methods of Mathematical Physics

—Lecture 3 Complex Integrations—

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April, 2022

Contents

Introduction

In the theory of real variables, the integration is considered from two perspectives: the indefinite integration as an operation inverse to that of differentiation and the definite integration as the limit of a sum. The concept of the indefinite integral as the process of inverse differentiation in case a function of a real variable is extended to a function of a complex variable if the complex function $f(z)$ is analytic. It means that, if $f(z)$ is an analytic function of a complex variable z and

$$\int f(z)dz = F(z),$$

then the differential of $F(z)$ is equal to $f(z)$, i.e., $F'(z) = f(z)$.

However, the concept of the definite integral of a function of a real variable does not extend out, rightly to the domain of complex variables. For example, in the case of real variable, the path of integration of $\int_a^b f(x)dx$ is always along the real axis from $x = a$ to $x = b$. But, in the case of a complex function $f(z)$, the path of the definite integral

$$\int_a^b f(z)dz,$$

may be along any curve joining the points $z = a$ and $z = b$ and so its value depends upon the path (curve) of integration.

Some Definitions

- Let $[a, b]$ be a closed interval where a and b are real numbers. Subdivide the interval $[a, b]$ into n sub-intervals:

$$[t_0, t_1], \quad [t_1, t_2], \quad [t_2, t_3], \quad \cdots, \quad [t_{n-1}, t_n]$$

  by inserting $n-1$ intermediate points $t_1, t_2, \cdots, t_{n-1}$ satisfying the inequalities:

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b.$$

  Then the set

$$P = \{t_0, t_1, t_2, \cdots .t_n\}$$

  is called a partition of the interval $[a, b]$ and the greatest of the numbers

$$t_1 - t_0, t_2 - t_1, \cdots, t_n - t_{n-1}$$

  is called the norm of the partition P, which is denoted by $|P|$.

- Suppose that a point z lies on an arc L is defined by

$$z = z(t) = x(t) + iy(t),$$

  where t runs through the interval $a \leq t \leq b$ and $x(t), y(t)$ are continuous functions of t. Then the arc L is said to be a continuous arc.

Some Definitions

- Arc L is said to be continuously differentiable or simply differentiable if $z'(t)$ exists and is continuous. If, in addition to the existence of $z'(t)$, we also have $z'(t) \neq 0$, then we say that L is a regular arc (or a smooth arc). Thus a regular arc is characterized by the property that it has, at every point, a tangent whose direction is determined by $\arg z'(t)$. In fact, as t increases from a to b, z continuously traces out the arc L and, at the same time, $\arg z'(t)$ varies continuously since $z'(t)$ changes continuously without vanishing.
- An arc L is said to be simple or a Jordan arc if $z(t_1) = z(t_2)$ only when $t_1 = t_2$. If $z(a) = z(b)$, then the arc L is said to be a closed curve. If L is the arc defined by $z = z(t)(a \leq t \leq b)$, then the arc defined by

$$z = z(-t) \quad (-b \leq t \leq -a)$$

is said to be the opposite arc of L and is defined by $-L$.

Complex Integrals

Let L be a Jordan arc defined by

$$z = z(t) = x(t) + iy(t) \quad (a \leq t \leq b)$$

and let $f(z)$ be a function of a complex variable z which has a definite value at each point of a rectifiable arc L. Consider an arbitrary partition

$$P = \{a = t_0, t_1, t_2, \cdots, t_{n-1}, t_n = b\}$$

of $[a, b]$. We divide the arc L into a small arcs by means of the points $z_0, z_1, z_2, \cdots, z_{n-1}, z_n$, which correspond to the values

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$$

of the parameter t, and form the sum

$$\sum = \sum_{k=1}^{n} f(\zeta_k)\,(z_k - z_{k-1}) = \sum_{k=1}^{n} f(\zeta_k)\,\Delta z_k,$$

where $z_k = z(t_k), \zeta_k = z(\alpha_k)$ and $t_{k-1} \leq \alpha_k \leq t_k$, is a point of L between z_{k-1} and z_k, $\Delta z_k = z_k - z_{k-1}$.

Complex Integrals

If this $\mathrm{sum} \sum$ tends to a unique limit I as $n \to \infty$ and the norm of P, i.e., $|P|$ tends to zero, then we say that $f(z)$ is integrable from a to b along the arc L and we write

$$I = \int_L f(x)dt.$$

We also call $\int_L f(z)dz$ the complex line integral or, simply, the line integral of $f(z)$ along the arc L or the definite integral of $f(z)$ from a to b along L. The sense of direction of integration is from a to b, since the points $x(t) + iy(t)$, for increasing values of t, are oriented in the very sense on the arc L. In fact, the value of t depends not only on the end points of arc L, but also on the actual form. Thus we have

$$\int_L f(z)dz = \lim_{|P|\to 0, n\to\infty} \sum_{k=1}^{n} f(\zeta_k)\left(z_k - z_{k-1}\right).$$

Note that an integral of this type exists under pretty general conditions. However, we may do without the assumption that $x'(t)$ and $y'(t)$ exist at each point of L. In fact, the continuity of $f(z)$ on L is a sufficient condition.

Complex Integrals

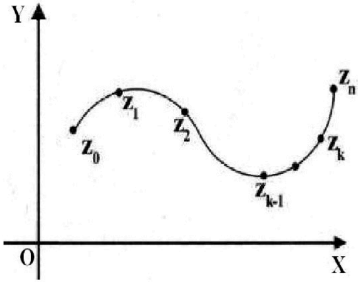

Figure: 1. The complex line integral of $f(z)$ along the arc L.

Evaluation of Integrals by the Direct Definition

Example

1. $\int_L dz$;
2. $\int_L |dz|$;
3. $\int_L zdz$, where L is any rectifiable[a] joining the points $z = \alpha$ and $z = \beta$.

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[a] The length of the polygonal arc, obtained by joining successively z_0 and z_1, z_1 and $z_2, \cdots, z_{n-1}$ and z_n, by straight line segments is given by

$$\Sigma = \sum_{k=1}^{n} L_k = \sum_{k=1}^{n} |z_k - z_{k-1}|$$

where $L_k = \text{Arc}\, z_{k-1}z_k (k = 1, 2, \cdots, n)$ and $z_l = z(t_l)\,(l = 0, 1, \cdots, n)$. If this sum $\sum$ tends to a unique limit l, say, as $n \to \infty$ and the norm of the partition P tends to zero, then we say that the arc L defined by $z = x(t) + iy(t)$ $(a \leq t \leq b)$ is rectifiable and its length is l. Rectifiable Jordan arcs with continuously turning tangents are called regular arcs.

Properties of Complex Integrals

Some elementary properties of complex integrals are as follows:

1. $\int_L [f(z) + g(z)]dz = \int_L f(z)dz + \int_L g(z)dz.$
2. $\int_L f(z)dz = -\int_{-L} f(z)dz$, where by $-L$ we mean the curve L traversed in the opposite direction.
3. $\int_{L_1+L_2} f(z)dz = \int_{L_1} f(z)dz + \int_{L_2} f(z)dz$, where the terminal point of L_1 coincides with the initial point of L_2.
4. $\int_L cf(z)dz = c\int_L f(z)dz$, where c is any complex constant.
5. We have
$$\int_L [c_1 f_1(z) + c_2 f_2(z) + \cdots + c_n f_n(z)]\, dz$$
$$= c_1 \int_L f_1(z)dz + c_2 \int_L f_2(z)dz + \cdots + c_n \int_L f_n(z)dz.$$
6. $\left|\int_L f(z)dz\right| \leq \int_L |f(z)||dz|.$

Integrations along Regular Arcs

Theorem

Let $f(z)$ be continuous on the regular arc L which is defined by

$$z = z(t) = x(t) + iy(t) \quad (z \leq t \leq b).$$

Then $f(z)$ is integrable along L and

$$\int_L f(z)dz = \int_a^b F(t)\{\dot{x}(t) + i\dot{y}(t)\}dt,$$

where $F(t)$ denotes the value of $f(z)$ at the point $z = x(t) + iy(t)$ of L corresponding to the parameter value t.

Proof.

Consider the sum

$$\sum = \sum_{k=1}^{n} f(\zeta_k)(z_k - z_{k-1})$$

where ζ_k is a point of L between z_{k-1} and z_k and $\zeta_k = z(\tau_k), t_{k-1} \leq \tau_k \leq t_k$. Write

$$F(t) = \phi(t) + i\psi(t),$$

Integrations along Regular Arcs

where $\phi(t)$ and $\psi(t)$ are real-valued functions of parameter t. Then we have

$$\begin{aligned}\sum &= \sum_{k=1}^{n} \phi(\tau_k)(x_k - x_{k-1}) + i\sum_{k=1}^{n} \psi(\tau_k)(x_k - x_{k-1}) + i\sum_{k=1}^{n} \phi(\tau_k)(y_k - y_{k-1}) - \sum_{k=1}^{n} \psi(\tau_k)(y_k - y_{k-1}) \\ &= \Sigma_1 + i\Sigma_2 + i\Sigma_3 - \Sigma_4, \text{ say.}\end{aligned}$$

By the mean value theorem of differential calculus, the first sum is

$$\Sigma_1 = \sum_{k=1}^{n} \phi(\tau_k)\dot{x}(\tau_k')(t_k - t_{k-1}),$$

where $t_{k-1} \le \tau_k' \le t_k$. Let $\Sigma_1' = \sum_{k=1}^{n} \phi(t_k)\dot{x}(t_k)(t_k - t_{k-1})$, by making the norm of P, i.e., $|P|$ sufficiently small, we show that

$$|\Sigma_1 - \Sigma_1'| < M(\epsilon)\epsilon.$$

Now, by the hypothesis, $\phi(t)$ and $\dot{x}(t)$ are continuous and, since every continuous function is bounded, there exists a positive number M such that the inequalities:

$$|\phi(t)| \le M, \quad |\dot{x}(t)| \le M$$

hold for $a \le t \le b$.

Integrations along Regular Arcs

Again, since a continuous function is necessarily uniformly continuous, we can preassign an arbitrary positive number ϵ and then we can choose a positive number $\delta = \delta(\epsilon)$ such that

$$\left|\phi(t) - \phi\left(t'\right)\right| < \epsilon, \quad \left|\dot{x}(t) - \dot{x}\left(t'\right)\right| < \epsilon,$$

where $|t - t'| < \delta$. Hence, if $|P| < \delta$, then we have

$$\begin{aligned} &| \,\phi\left(\tau_k\right)\dot{x}\left(\tau_k'\right) - \phi\left(t_k\right)\dot{x}\left(t_k\right) = \left|\phi\left(\tau_k\right)\left\{\dot{x}\left(\tau_k'\right) - \dot{x}\left(t_k\right)\right\} + \dot{x}\left(t_k\right)\left\{\phi\left(\tau_k\right) - \phi\left(t_k\right)\right\}\right| \\ &\leq \left|\phi\left(\tau_k\right)\right|\left|\dot{x}\left(\tau_k'\right) - \dot{x}\left(t_k\right)\right| + \left|\dot{x}\left(t_k\right)\right|\left|\phi\left(\tau_k\right) - \phi\left(t_k\right)\right| < 2M\epsilon \end{aligned}$$

and so it follows that $\left|\Sigma_1 - \Sigma_1'\right| < 2M\epsilon(b-a)$.
By the definition of the integral of functions of a real variable, Σ_1' tends to the limit

$$\int_a^b \phi(t)\dot{x}(t)dt = \lim_{n\to\infty, |P|\to 0} \Sigma_1'.$$

The remaining $\Sigma's$ tend to corresponding limits in the same manner. Then Σ tends to the limit

$$\int_a^b \{\phi(t)\dot{x}(t) - \psi(t)\dot{y}(t)\}dt + i\int_a^b \{\psi(t)\dot{x}(t) + \phi(t)\dot{y}(t)\}dt = \int_a^b F(t)\{\dot{x}(t) + i\dot{y}(t)\}dt.$$

Examples

Example

1. Evaluate $\int_C \frac{dz}{z}$, where C is the circle with center at the origin and radius r.
2. Evaluate $\int_C \frac{dz}{z-\alpha}$, where C represents a circle $|z-\alpha| = r$.
3. Evaluate $\int_C f(z)dz$, if $f(z) \equiv 1$, and C is any smooth curve.

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3. Evaluate $\int_C f(z)dz$, if $f(z) \equiv 1$, and C is any smooth curve.

Solutions:

1. $$\int_C \frac{dz}{z} = \int_0^{2\pi} \frac{1}{re^{it}} rie^{it} dt = i \int_0^{2\pi} dt = 2\pi i.$$
2. $$\int_C \frac{dz}{z-\alpha} = \int_0^{2\pi} \frac{1}{re^{it}} rie^{it} dt = i \int_0^{2\pi} dt = 2\pi i.$$
3. $$\int_C f(z)dz = \int_a^b \dot{z}(t)dt = z(b) - z(a).$$

An important integral

Theorem

Let C be a circular path with center z_0 and radius $r > 0$ traversed in the anticlockwise direction. Then

$$\int_C (z - z_0)^n \, dz = \begin{cases} 2\pi i & \text{if } n = -1, \\ 0 & \text{if } n \neq -1. \end{cases}$$

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Proof.

We have $C(t) = z_0 + r\exp(it), t \in [0, 2\pi]$, and so $C'(t) = ir\exp(it), t \in [0, 2\pi]$.
1 : When $n = -1$, we have

$$\int_C (z - z_0)^n \, dz = \int_C (z - z_0)^{-1} \, dz = \int_0^{2\pi} \frac{1}{r\exp(it)} \cdot ir\exp(it) dt = \int_0^{2\pi} i dt = 2\pi i.$$

2 : When $n \neq -1$, we have

$$\begin{aligned}\int_C (z - z_0)^n \, dz &= \int_0^{2\pi} r^n \exp(nit) \cdot ir\exp(it) dt = \int_0^{2\pi} ir^{n+1} \exp(i(n+1)t) dt \\ &= -r^{n+1} \int_0^{2\pi} \sin((n+1)t) dt + ir^{n+1} \int_0^{2\pi} \cos((n+1)t) dt = 0 + 0 = 0.\end{aligned}$$

□

Integrations along Regular Arcs

Theorem

Let C be the given curve. Then

$$\int_{-C} f(z)dz = -\int_C f(z)dz.$$

Proof.

We have

$$\int_{-C} f(z)dz = -\int_a^b f(z(b+a-t))\dot{z}(b+a-t)dt$$

Now expanding the integral into real and imaginary parts and applying the change of variable theorem to each real integral, we obtain

$$\int_{-C} f(z)dz = \int_b^a f(z(t))\dot{z}(t)dt = -\int_C f(z)dz.$$

□

Complex Integrals as Sum of Two Real Line Integrals

Example

1. Prove that the value of the integral of $\frac{1}{z}$ along a semi-circular arc $|z| = a$ from $-a$ to $+a$ is $-\pi i$ or πi if the arc lies above or below the real axis.
2. Find the value of the integral $\int_0^{1+i} (x - y + ix^2)\, dz$
   - along the straight line from $z = 0$ as $z = 1 + i$; $(\frac{-1+i}{3})$
   - along the real axis from $z = 0$ to $z = 1$ and then along a line parallel to the imaginary axis from $z = 1$ to $z = 1 + i$. $(\frac{-1}{2} + \frac{5i}{6})$
3. Evaluate the integral $\int_0^{1+i} z^2 dz$.
4. Evaluate the integral $\int_{-2+i}^{5+3i} z^3 dz$.

The Absolute Value of Complex Integrals

Theorem

Let $f(z)$ be continuous on a contour L of length l and let $|f(z)| \leq M$ for every point z on L. Then we have

$$\left|\int_L f(z)dz\right| \leq Ml.$$

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Proof.

Without loss of generality, we may assume that L is a regular arc. Now, we have

$$\Sigma = \sum_{k=1}^{n} f(z_k)\left(z_k - z_{k-1}\right).$$

Since the modulus of the sum is less than or equal to the sum of the moduli, we have

$$|\Sigma| = \left|\sum_{k=1}^{n} f(z_k)\left(z_k - z_{k-1}\right)\right| \leq \sum_{k=1}^{n}\left|f(z_k)\left(z_k - z_{k-1}\right)\right| = \sum_{k=1}^{n}|f(z_k)|\left|\left(z_k - z_{k-1}\right)\right| \leq M\sum_{k=1}^{n}\left|\left(z_k - z_{k-1}\right)\right|.$$

Therefore, we have

$$\lim_{n\to\infty}\left|\sum_{k=1}^{n} f(z_k)\left(z_k - z_{k-1}\right)\right| \leq M\lim_{n\to\infty}\sum_{k=1}^{n}\left|\left(z_k - z_{k-1}\right)\right| \Rightarrow \left|\int_L f(z)dz\right| \leq M\int_L |dz| = Ml.$$

Line Integrals as Functions of Arcs

Observe that a line integral $\int_L f(z)dz$ over an arc L can be put in the form, i.e.,

$$\int_L (u+iv)(dx+idy), \quad \text{or} \int_L pdx+qdy.$$

General line integrals of the form $\int_L pdx+qdy$ are often studied as functions (or functionals) of the arc L under the assumption that p, q are defined and continuous in a domain D such that L is free to vary in D. An important class of integrals is characterized by the property that the integral over an arc depends only on its end points. This means that, if the two arcs L_1 and L_2 have the same initial point and the same end point, then we have

$$\int_{L_1} pdx+qdy = \int_{L_2} pdx+qdy.$$

Notice that an integral depends only on the end points is equivalent to saying that the integral over any closed curve is zero. Indeed, if L is a closed curve, then L and $-L$ have the same end points and, if the integral depends only on the end points, then we obtain

$$\int_L = \int_{-L} = -\int_L$$

and, consequently, $\int_L = 0$. Conversely, if L_1 and L_2 have the same end points, then $L_1 - L_2$ is a closed curve and, if the integral over any closed curve vanishes, then we see that

$$\int_{L_1} = \int_{L_2}.$$

Line Integrals as Functions of Arcs

The following theorem gives a necessary and sufficient condition under which a line integral depends only on the end points.

Theorem

The line integral $\int_L pdx + qdy$, defined in a domain D, depends only on the end points of L if and only if there exists a function $U(x, y)$ in D with the partial derivatives $\frac{\partial U}{\partial x} = p$ and $\frac{\partial U}{\partial y} = q$.

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Proof.

Sufficiency: For, if the condition is fulfilled and a, b are the end points of L, then we can write, with the usual notations,

$$\begin{aligned}\int_L pdx + qdy &= \int_L \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy \\ &= \int_a^b \left(\frac{\partial U}{\partial x} x'(t) + \frac{\partial U}{\partial y} y'(t) \right) dt \\ &= \int_a^b \left(\frac{d}{dt} U(x(t), y(t)) \right) dt \\ &= U(x(b), y(b)) - U(x(a), y(a))\end{aligned}$$

and the value of the difference depends only on the end points.

Line Integrals as Functions of Arcs

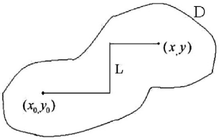
Necessity: we choose a fixed point $(x_0, y_0) \in D$, join it to (x, y) by a polygon L, contained in D, whose sides are parallel to the coordinate axes (see Fig. 2). Now, we define a function U by

$$U(x, y) = \int_L p dx + q dy.$$

By the hypothesis, the integral depends only on the end points and so it is well defined. Further, if we choose the last segment of L horizontal, we can keep y constant and let x vary without changing the other segments. Choosing x as a parameter on the last segment, we obtain

$$U(x, y) = \int^x p(x, y) dx + \text{ constant}, \tag{1}$$

the lower limit of the integral being irrelevant. From (1). it follows at once that $\frac{\partial U}{\partial x} = p$. In the same way, by choosing the last segment vertical, we can show that $\frac{\partial U}{\partial y} = q$. □



Line Integrals as Functions of Arcs

It is customary to write $dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy$, and an expression $pdx + qdy$ which can be written in this form is an exact differential. Using this terminology, the above theorem can be stated as:

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Now, we determine the conditions under which

$$f(z)dz = f(z)dx + if(z)dy$$

is an exact differential.

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By the definition of an exact differential, there must exist a function $F(z)$ in D with the partial derivatives

$$\frac{\partial F(z)}{\partial x} = f(z), \quad \frac{\partial F(z)}{\partial y} = if(z).$$

It follows that

$$\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y},$$

which is a Cauchy-Riemann equation.

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which is a Cauchy-Riemann equation.
Also, $f(z)$ is, by the assumption, continuous (otherwise, $\int_L f(z)dz$ would not be defined). Hence $F(z)$ is analytic with the derivative $f(z)$.

Line Integrals as Functions of Arcs

From the above discussion, we conclude:

Theorem

The integral $\int_L f(z)dz$, with continuous f, depends only on the end points of L if and only if f is the derivative of an analytic function in D.

Green's Theorem for Two Dimensions

Theorem (Green's Theorem)

Let $P(x,y)$ and $Q(x,y)$ be single-valued and continuous functions of x and y, possessing continuous partial derivatives with respect to both the variables x and y in a simply connected region containing a closed contour C. Then the double integral taken over the simply connected region D enclosed by C, i.e.,

$$\iint_D \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) dxdy,$$

equals the curvilinear integrals taken round C, i.e.,

$$\int_C (Pdx + Qdy).$$

Elementary Form of Cauchy's Theorem

Theorem (Cauchy's Theorem)

Let $f(z)$ be a regular function and let $f'(z)$ be continuous at each point within and on a closed contour C. Then

$$\int_C f(z)dz = 0.$$

Elementary Form of Cauchy's Theorem

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Let $f(z)$ be a regular function and let $f'(z)$ be continuous at each point within and on a closed contour C. Then

$$\int_C f(z)dz = 0.$$

Proof.

Let D denote the closed domain consisting of all points within and on C. Assume that $f(z) = u(x,y) + iv(x,y)$. Then we see that

$$\int_C f(z)dz = \int_C (udx - vdy) + i\int_C (vdx + udy). \tag{2}$$

By the Cauchy-Riemann equations, we have $f'(z) = u_x + iv_x = v_y - iu_y$. Since $f'(z)$ is continuous in D, the four partial derivatives u_x, u_y, v_x and v_y exist are all continuous in D. Thus the conditions of Green's theorem are satisfied. Hence, from (2) and the Cauchy-Riemann equations, it follows that

$$\begin{aligned}\int_C f(z)dz &= -\iint_D \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dxdy + i\iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy \\ &= \iint_D \left(-\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\right) dxdy + i\iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dxdy\end{aligned}$$

Index of Closed Curves with Respect to a Point

It is well known that, if $\gamma(t) = a + e^{2\pi int}$, then

$$\frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-a} = n.$$

Let γ be a closed rectifiable curve and $\{\gamma\}$ be its graph. Then we have the following result.

Theorem

If $\gamma : [0,1] \to \mathbb{C}$ is a closed rectifiable curve, n is an integer and $a \notin \{\gamma\}$, then

$$\frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-a} = n.$$

Index of Closed Curves with Respect to a Point

It is well known that, if $\gamma(t) = a + e^{2\pi i n t}$, then

$$\frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a} = n.$$

Let γ be a closed rectifiable curve and $\{\gamma\}$ be its graph. Then we have the following result.

Theorem

If $\gamma : [0,1] \to \mathbb{C}$ is a closed rectifiable curve, n is an integer and $a \notin \{\gamma\}$, then

$$\frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a} = n.$$

Proof.

We prove this result by assuming that γ is smooth. Define

$$g(t) = \int_0^t \frac{\gamma'(s)}{\gamma(s) - a} ds.$$

continue to prove.

By the definition of g, we have

$$g(0) = 0, \quad g(1) = \int_\gamma \frac{dz}{z-a}.$$

Also, we have $g'(t) = \frac{\gamma'(t)}{\gamma(t)-a} \quad (0 \le t \le 1)$.
But this gives

$$\frac{d}{dt}e^{-g}(\gamma - a) = e^{-g}\gamma' - g'e^{-g}(\gamma - a) = e^{-g}\left[\gamma' - \gamma'(\gamma - a)^{-1}(\gamma - a)\right] = 0$$

and so $e^{-g}(\gamma - a)$ is the constant function $e^{-g(0)}(\gamma(0) - a) = \gamma(0) - a = e^{-g(1)}(\gamma(1) - a)$. Since $\gamma(0) = \gamma(1)$, it follows that $e^{-g(1)} = 1$ or $g(1) = 2\pi$ in for some integer n. Thus we have

$$\int_\gamma \frac{dz}{z-a} = 2\pi in.$$

Therefore, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} = n.$$

Index of Closed Curves with Respect to a Point

Now, we introduce the concept of the index of a closed curve with respect to a point.
Let γ be a closed rectifiable curve and a be a point not on the graph of γ. Then the index or winding number $n(\gamma; a)$ of γ with respect to a point a is defined by the integral

$$n(\gamma; a) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z-a}$$

Geometrically speaking, the winding number counts the number of rounds of a path around the point. It may easily be observed that the winding number is a positive integer with a positively oriented contour and a negative integer for contours with negative orientation. If the point a is enclosed by a simple closed contour, then $n(\gamma; a) = 1$ and, if a is outside of the path, then $n(\gamma; a) = 0$.
As a direct consequence of the definition, we have the following assertions: If $\{\gamma\} = \{\gamma_1\} \cup \{\gamma_2\}$ is the sum of paths of two curves γ_1 and γ_2 in a complex plane, then, for all $a \notin \{\gamma\}$,

1. $n(\gamma; a) = n(\gamma_1; a) + n(\gamma_2; a)$
2. $n(-\gamma; a) = -n(\gamma; a)$.

Recall that, if $\gamma : [0, 1] \to C$ is a curve, then $-\gamma$ is the curve defined by

$$(-\gamma)(t) = \gamma(1-t).$$

If γ_1 and γ_2 are curves defined on $[0, 1]$ with $\gamma_1(1) = \gamma_2(0)$, then $\gamma_1 + \gamma_2$ is the curve defined by

$$(\gamma_1 + \gamma_2)(t) = \begin{cases} \gamma(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \gamma(1-2t), & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

General Form of Cauchy's Theorem

Theorem (Cauchy-Goursat's Theorem)

If a function $f(z)$ is analytic and single-valued inside and on a simple closed contour C, then

$$\int_C f(z)dz = 0.$$

General Form of Cauchy's Theorem

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If a function $f(z)$ is analytic and single-valued inside and on a simple closed contour C, then

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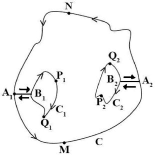
Theorem (The Extension of Cauchy's Theorem to Multiply Connected Regions)

Let $f(z)$ be analytic in the multiply connected region D bounded by the closed contour C and the two interior contours C_1, C_2, then

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz,$$

where C, C_1 and C_2 are all the three traversed in the positive sense (i.e., anticlockwise direction.)



Indefinite Integrals of Primitives

Definition

Let $f(z)$ be a single-valued analytic function on a domain D. Then a function $F(z)$ is said to be an indefinite integral or a primitive or antiderivative of $f(z)$ if single-valued and analytic in D and, for all $z \in D$,

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Let $f(z)$ be defined analytic in a region D. Let us write

$$F(z) = \int_{z_0}^{z} f(\zeta)d\zeta,$$

where z_0 is any fixed point and z is any variable point in D and the path of integration is any contour from z_0 to z lying entirely inside D. We infer at once from Cauchy-Goursat's Theorem that the value of $F(z)$ is independent of the particular contour along which integration is taking place and depends only on the variation of z for its own variation. We call $F(z)$ the indefinite integral of $f(z)$.

Indefinite Integrals of Primitives

Theorem

Let $f(z)$ be analytic in a simply connected region D of the complex plane. Then there exists a function $F(z)$ analytic in D such that

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Indefinite Integrals of Primitives

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for all $z \in D$.

Proof.

Let z_0 be any fixed point and z be any variable point in D. Then by Cauchy-Goursat's Theorem the integral of $f(z)$ along every curve in D joining z_0 to z is the same. Hence we may write

$$F(z) = \int_{z_0}^{z} f(\zeta) d\zeta.$$

Then we have

$$\begin{aligned} \frac{F(z+h) - F(z)}{h} &= \frac{1}{h}\left[\int_{z_0}^{z+h} f(\zeta)d\zeta - \int_{z_0}^{z} f(\zeta)d\zeta\right] = \frac{1}{h}\left[\int_{z_0}^{z+h} f(\zeta)d\zeta + \int_{z}^{z_0} f(\zeta)d\zeta\right] \\ &= \frac{1}{h}\int_{z}^{z+h} f(\zeta)d\zeta, \end{aligned}$$

Indefinite Integrals of Primitives

Proof.

where, by Cauchy's theorem, we may assume without loss of generality that the integral is taken along the straight line joining z and $z+h$. Thus we have

$$\begin{aligned}\frac{F(z+h)-F(z)}{h}-f(z)&=\frac{1}{h}\int_z^{z+h}f(\zeta)d\zeta-\frac{f(z)}{h}h\\&=\frac{1}{h}\left[\int_z^{z+h}f(\zeta)d\zeta-f(z)\int_z^{z+h}f(\zeta)\right]\\&=\frac{1}{h}\int_z^{z+h}[f(\zeta)-f(z)]d\zeta.\end{aligned}$$

On account of the continuity of $f(z)$, for any given positive number ϵ, there exists a positive number δ such that

$$|\zeta-z|<\delta\Rightarrow|f(\zeta)-f(z)|<\epsilon.$$

Therefore, if $0<|h|<\delta$, then we have

$$\begin{aligned}\left|\frac{F(z+h)-F(z)}{h}-f(z)\right|=\frac{1}{|h|}\left|\int_z^{z+h}[f(\zeta)-f(z)]d\zeta\right|&\leq\frac{1}{|h|}\int_z^{z+h}|f(\zeta)-f(z)||d\zeta|\\&<\frac{1}{|h|}\int_z^{z+h}\epsilon|d\zeta|=\frac{1}{|h|}\epsilon|h|=\epsilon.\end{aligned}$$

Indefinite Integrals of Primitives

Hence we have

$$\lim_{h\to 0}\frac{F(z+h)-F(z)}{h}=f(z),$$

i.e., $F'(z)$ exists and $F'(z)=f(z)$ for all $z\in D$. □

Indefinite Integrals of Primitives

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Theorem

A necessary and sufficient condition for a function $f(z)$ to possess an indefinite integral in a simply connected domain D is that the function $f(z)$ is analytic in D. Further, any two indefinite integrals differ by a constant.

Proof.

Suppose that $f(z)$ possesses an indefinite integral $F(z)$. Then we have

$$F'(z) = f(z). \tag{3}$$

This shows that $F(z)$ possesses a derivative $f(z)$ for all $z \in D$ and so $F(z)$ is analytic in D. But the derivative of an analytic function is analytic. If follows from (3) that $f(z)$ is analytic in D.

Indefinite Integrals of Primitives

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Conversely, refer to the proof of Theorem above. Further, let $F(z)$ and $G(z)$ be two indefinite integrals of $f(z)$. Then we have, for all $z \in D$,

Indefinite Integrals of Primitives

$$F'(z) = G'(z) \quad \text{or} \quad G'(z) - F'(z) = 0.$$

Now, writing $G(z) - F(z) = u + iv$, we have, as in the proof of the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0.$$

Hence u and v are constants and so $F(z)$ and $G(z)$ differ from each other by a constant. Thus, let $G(z) = F(z) + c$. Hence the general indefinite integral of an analytic function $f(z)$ is given by $F(z) + c$, where

$$F(z) = \int_{z_0}^{z} f(\zeta)d\zeta.$$

$\square$

Indefinite Integrals of Primitives

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$$F(z) = \int_{z_0}^{z} f(\zeta) d\zeta.$$

□

Theorem (Fundamental Theorem of Integral Calculus for Complex Functions)

Let $f(z)$ be a single-valued analytic function in a simply connected domain D. If $a, b \in D$, then

$$\int_a^b f(z) dz = F(b) - F(a),$$

where $F(z)$ is any indefinite integral of $f(z)$.

Indefinite Integrals of Primitives

Proof.

Let

$$\varphi(z) = \int_a^z f(\zeta)d\zeta. \tag{4}$$

By Theorem above, the indefinite integral $F(z)$ of $f(z)$ is given by

$$F(z) = \varphi(z) + c.$$

Therefore, we have

$$F(b) = \varphi(b) + c, \quad F(a) = \varphi(a) + c.$$

Hence we have

$$F(b) - F(a) = \varphi(b) - \varphi(a). \tag{5}$$

From (4), we have

$$\varphi(b) = \int_a^b f(\zeta)d\zeta, \quad \varphi(a) = \int_a^a f(\zeta)d\zeta = 0.$$

Then (5) leads to the required result, i.e.,

$$F(b) - F(a) = \int_a^b f(\zeta)d\zeta = \int_a^b f(z)dz.$$

Cauchy's Integral Formula

The most important consequence of Cauchy's Integral Theorem is Cauchy's Integral Formula. This formula is, indeed, useful for evaluating integrals. Equally important is its key role in proving the fact that analytic functions have derivatives of all orders.

Theorem (Cauchy's Integral Formula)

Let $f(z)$ be analytic within and on a closed contour C, and let a be any point within C. Then

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz.$$

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$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz.$$

Proof.

We describe a circle γ defined by the equation $|z-a| = \rho$, where $\rho < d$ and d is the distance from a to the nearest point of C. Consider the function

$$\phi(z) = \frac{f(z)}{z-a}.$$

Thus,

$$\int_C \phi(z) dz = \int_\gamma \phi(z) dz \quad \text{or} \quad \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} dz, \tag{6}$$

where C and γ are both traversed in the positive sense as shown in Fig. 4 below.

Cauchy's Integral Formula

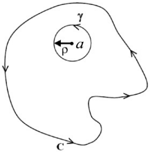

Figure: 4. A doubly connected region bounded by a closed C and a circle γ.

Since $f(z)$ is continuous at a, for any $\epsilon > 0$, there exists $\delta > 0$, whenever $|z-a| < \delta$, such that

$$|f(z) - f(a)| < \epsilon, \tag{7}$$

Now, we have

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz = \frac{1}{2\pi i}\int_\gamma \frac{f(z)-f(a)}{z-a}dz + \frac{1}{2\pi i}\int_\gamma \frac{f(a)}{z-a}dz. \tag{8}$$

Writing $z - a = \rho r^{i\theta}$ and $dz = \rho i e^{i\theta} d\theta$, we have

$$\frac{1}{2\pi i}\int_C \frac{f(a)}{z-a}dz = \frac{f(a)}{2\pi i}\int_0^{2\pi} \frac{\rho i e^{i\theta}}{\rho e^{i\theta}} d\theta = \frac{f(a)}{2\pi}\int_0^{2\pi} d\theta = f(a).$$

Cauchy's Integral Formula

Hence (8) becomes

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz = \frac{1}{2\pi i}\int_\gamma \frac{f(z)-f(a)}{z-a}dz + f(a).$$

or

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz - f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)-f(a)}{z-a}dz. \tag{9}$$

Since we may choose ρ as small as we please, we take $\rho < \delta$. Thus the inequality (7) is satisfied for all points on γ. Hence, by (7), we have

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_\gamma \frac{f(z)-f(a)}{z-a}dz\right| &= \left|\frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z)-f(a)}{\rho e^{i\theta}}\cdot \rho i e^{i\theta} d\theta\right| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z)-f(a)| d\theta \\ &< \frac{1}{2\pi}\int_0^{2\pi} \epsilon d\theta = \frac{1}{2\pi}\cdot 2\pi\epsilon = \epsilon.\end{aligned}$$

Thus we have

$$\left|\frac{1}{2\pi i}\int_\gamma \frac{f(z)-f(a)}{z-a}dz\right| < \epsilon. \tag{10}$$

Then it follows from (9) and (10) that $\left|\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz - f(a)\right| < \epsilon.$ □

Cauchy's Integral Formula

Now, we observe that ϵ is arbitrary and the left-hand side is independent of it. This implies that

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f(z)}{z-a}dz - f(a) = 0 \quad \text{or} \quad \frac{1}{2\pi i}\int_{\gamma}\frac{f(z)}{z-a}dz = f(a). \tag{11}$$

Finally, from (6) and (11), it follows that

$$f(a) = \frac{1}{2\pi i}\int_{C}\frac{f(z)}{z-a}dz.$$

□

Cauchy's Integral Formula

Corollary (Gauss's Mean Value Theorem)

If $f(z)$ is an analytic function on a domain D and the circular region $|z-a| \leq \rho$ is contained in D, then

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f\left(a+\rho e^{i\theta}\right)\,d\theta.$$

In other words, the value of $f(z)$ at the point a equals the average of its values on the boundary of the circle $|z-a| = \rho$.

Cauchy's Integral Formula

Corollary (Gauss's Mean Value Theorem)

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In other words, the value of $f(z)$ at the point a equals the average of its values on the boundary of the circle $|z-a| = \rho$.

Proof.

Let γ denote the circle $|z-a| = \rho$. Then we can write

$$z = a + \rho e^{i\theta} \quad (0 \leq \theta \leq 2\pi)$$

and so

$$dz = \rho i e^{i\theta} d\theta.$$

Hence, by Cauchy's Integral Formula, we have

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f\left(a + \rho e^{i\theta}\right) i\rho e^{i\theta}}{\rho e^{i\theta}} d\theta = \frac{1}{2\pi} \int_0^{2\pi} f\left(a + \rho e^{i\theta}\right) d\theta.$$

Cauchy's Integral Formula

Corollary

The extension of Cauchy's Integral Formula to multiply connected regions.

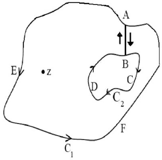

Figure: 5. A doubly connected region bounded by two closed curves C_1 and C_2.

Cauchy's Integral Formula

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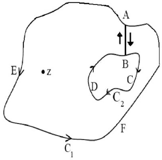

Figure: 5. A doubly connected region bounded by two closed curves C_1 and C_2.

Proof.

Let us consider the case of a doubly connected region D bounded by two closed curves C_1 and C_2. Let a be any point of D. Then we prove that

$$f(a) = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)dz}{z-a} - \frac{1}{2\pi i}\int_{C_2} \frac{f(z)dz}{z-a},$$

where C_1 is the outer curve.

Cauchy's Integral Formula

Make a cross-cut AB connecting the curve C_1 and C_2 as shown in Fig. 5 above. Then $f(z)$ is analytic in the region bounded by $ABCDBAEFA$. Now, by Cauchy's Integral Formula, we have

$$
\begin{aligned}
f(a) =& \frac{1}{2\pi i}\int_{ABCDBAEFA} \frac{f(z)}{z-a}dz \\
=& \frac{1}{2\pi i}\int_{AEFA} \frac{f(z)}{z-a}dz + \frac{1}{2\pi i}\int_{AB} \frac{f(z)}{z-a}dz \\
& + \frac{1}{2\pi i}\int_{BCDB} \frac{f(z)}{z-a}dz + \frac{1}{2\pi i}\int_{BA} \frac{f(z)}{z-a}dz \\
=& \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-a}dz - \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z-a}dz
\end{aligned}
$$

since the integrals along AB and BA cancel. □

Derivatives of Analytic Functions

By Cauchy's Integral Formula, we show that complex analytic functions have derivatives of all orders, which differs from the situation in real analysis. Indeed, if a real function is once differentiable, then nothing follows about the existence of second and higher-order derivatives.

Theorem

Let $f(z)$ be analytic within and on the boundary C of a simply connected region D and let a be any point within C. Then the value of the derivative at a is given by the formula

$$f'(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^2} dz.$$

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$$f'(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^2} dz.$$

Proof.

Let $a+h$ be a point in the neighborhood of the point a so that h is at our choice. Then, by Cauchy's Integral Formula, we have

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz \quad \text{and} \quad f(a+h) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a-h} dz.$$

Therefore, we have

$$f(a+h) - f(a) = \frac{1}{2\pi i} \int_C f(z) \left[\frac{1}{z-a-h} - \frac{1}{z-a} \right] dz$$

Derivatives of Analytic Functions

or

$$\frac{f(a+h)-f(a)}{h}=\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a-h)(z-a)}dz.$$

Evidently, as $|h|\to 0$, the integrals $\frac{f(z)}{(z-a)(z-a-h)}$ tend to $\frac{f(z)}{(z-a)^2}$. Thus we need only to show that we can proceed to the limit under the sign of integration. So we consider

$$\begin{aligned}\frac{f(a+h)-f(a)}{h}-\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}dz &= \frac{1}{2\pi i}\int_C \left[\frac{1}{(z-a-h)(z-a)}-\frac{1}{(z-a)^2}\right]dz \\ &= \frac{1}{2\pi i}\int_C \frac{hf(z)}{(z-a)^2(z-a-h)}dz.\end{aligned} \tag{12}$$

Now, we describe a circle γ with center a and radius ρ such that γ lies entirely within C. Then, by Corollary of the Cauchy-Goursat theorem (The Extension of Cauchy's Theorem to Multiply Connected Regions), we have

$$\frac{1}{2\pi i}\int_C \frac{hf(z)}{(z-a)^2(z-a-h)}dz=\frac{1}{2\pi i}\int_\gamma \frac{hf(z)}{(z-a)^2(z-a-h)}dz. \tag{13}$$

Let us choose h so small that the point $a+h$ lies within γ and that $|h|<\frac{1}{2}\rho$. The equation of γ is $|z-a|=\rho$. Hence, for any point z on γ, we have

Derivatives of Analytic Functions

$$|z-a-h| \geq |z-a| - |h| \geq \rho - \frac{1}{2}\rho = \frac{1}{2}\rho.$$

Again, since $f(z)$ is analytic in D, it is bounded in D. Let us suppose that $|f(z)| \leq M$ in D, where M is an absolute positive constant. Using these facts, it follows from (12) and (13) that

$$\begin{aligned}
&\left|\frac{f(a+h)-f(a)}{h} - \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}dz\right| = \left|\frac{1}{2\pi i}\int_\gamma \frac{hf(z)}{(z-a)^2(z-a-h)}dz\right| \\
&\leq \frac{|h|}{2\pi}\int_\gamma \frac{|f(z)|}{|z-a|^2|z-a-h|}|dz| \leq \frac{|h|}{2\pi}\int_\gamma \frac{M}{\rho^2\left(\frac{1}{2}\rho\right)}|dz| \\
&= \frac{|h|M}{\pi\rho^2}\cdot 2\pi\rho = \frac{2|h|M}{\rho},
\end{aligned}$$

since the length of γ is $2\pi\rho$. It follows that the right-hand side of (13) tends to zeros as $|h| \to 0$ and so

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}dz.$$

Hence $f(z)$ is differentiable at a and

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}dz. \tag{14}$$

Derivatives of Analytic Functions

It is evident from (14) that the derivative $f'(a)$ can be written formally by differentiating the integral in Cauchy's Integral Formula.

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$$

with respect to a under the integral sign. Thus we have

$$f'(a) = \frac{1}{2\pi i} \int_C \frac{d}{da} \left(\frac{f(z)}{z-a} \right) dz = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^2} dz,$$

which is Cauchy's Integral Formula for $f'(a)$ of points within C. This result has far-reaching consequences. The significance of this result lies in the fact that $f'(a)$ is itself analytic in C.

Hence it is also established that the derivative of an analytic function $f(z)$ is an analytic function of z.

Higher-Order Derivatives

Theorem

Let $f(z)$ be analytic within and on the boundary C of a simply connected region D and let a be any point within C. Then derivatives of all orders are analytic and are given by

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)dz}{(z-a)^{n+1}}.$$

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Proof.

We first show that $f'(z)$ is analytic inside C. To prove this, it is enough to show that $f'(z)$ has a differential coefficient at every point a inside C. By Cauchy's Formula for $f'(a)$ and $f'(a+h)$, we have

$$\begin{aligned}\frac{f'(a+h)-f'(a)}{h} &= \frac{1}{2\pi i}\int_C \left[\frac{1}{(z-a-h)^2} - \frac{1}{(z-a)^2}\right]\frac{f(z)}{h}dz \\ &= \frac{1}{2\pi i}\int_C \frac{2h(z-a)-h^2}{(z-a-h)^2(z-a)^2}\frac{f(z)}{h}dz \\ &= \frac{2!}{2\pi i}\int_C \frac{(z-a)-(h/2)}{(z-a-h)^2\cdot(z-a)^2}f(z)dz.\end{aligned}$$

□

Higher-Order Derivatives

It follows that

$$
\begin{aligned}
&\frac{f'(a+h)-f'(a)}{h} - \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}\,dz \\
&= \frac{2!}{2\pi i}\int_C f(z)\left[\frac{(z-a)-(h/2)}{(z-a-h)^2(z-a)^2} - \frac{1}{(z-a)^3}\right] dz \\
&= \frac{2!}{2\pi i}\int_C f(z)\left[\frac{\frac{3}{2}h(z-a)-h^2}{(z-a)^3(z-a-h)^2}\right] dz = \frac{2!}{2\pi i}\int_\gamma \frac{h\left[\frac{3}{2}(z-a)-h\right]}{(z-a)^3(z-a-h)^2} f(z)dz,
\end{aligned}
$$

where γ is the circle $|z-a|=\rho$ lying entirely within C. Hence, by means of arguments parallel to those used in the proof of Cauchy's Formula for $f'(a)$, we have

$$
\begin{aligned}
&\left|\frac{f'(a+h)-f'(a)}{h} - \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}\,dz\right| \\
&\le \frac{2!|h|}{2\pi}\int_\gamma \frac{\frac{3}{2}|z-a|+|-h|}{|z-a|^3|z-a-h|^2}|f(z)||dz| \\
&\le \frac{2!|h|}{2\pi}\cdot\frac{\frac{3}{2}\rho+|h|}{\rho^3\left(\frac{1}{2}\rho\right)^2}\cdot M\cdot 2\pi\rho = \frac{2!|h|\left(\frac{3}{2}\rho+|h|\right)M}{\frac{1}{4}\rho^4},
\end{aligned}
\tag{15}
$$

Higher-Order Derivatives

where M is the upper bound of $f(z)$ in D. Hence, when $|h| \to 0$, the right-hand side of (15) also tends to zero. Then we have

$$\lim_{h \to 0} \frac{f'(a+h) - f'(a)}{h} = \frac{2!}{2\pi i} \int_C \frac{f(z)}{(z-a)^3} dz.$$

Thus $f'(z)$ has a derivative at a by the formula

$$f''(a) = \frac{2!}{2\pi i} \int_C \frac{f(z)}{(z-a)^3} dz.$$

Therefore, the formula is true for $n = 2$. Now, suppose that the formula is true for $n = m$, i.e.,

$$f^{(m)}(a) = \frac{m!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{m+1}} dz$$

Therefore, we have

$$\frac{f^{(m)}(a+h) - f^{(m)}(a)}{h} = \frac{1}{h} \cdot \frac{m!}{2\pi i} \left[\int_C \frac{f(z)}{(z-a-h)^{m+1}} dz - \int_C \frac{f(z)}{(z-a)^{m+1}} dz \right]$$

Higher-Order Derivatives

$$= \frac{1}{h} \cdot \frac{m!}{2\pi i} \int_C \left[\frac{1}{(z-a)^{m+1}} \left\{ \left[1 - \frac{h}{z-a} \right]^{-(m+1)} - 1 \right\} \right] f(z) dz$$

$$= \frac{1}{h} \cdot \frac{m!}{2\pi i} \int_C \left[\frac{1}{(z-a)^{m+1}} \left\{ (m+1) \cdot \frac{h}{z-a} + \frac{(m+1)(m+2)}{2!} \cdot \frac{h^2}{(z-a)^2} \right. \right.$$

$$\left. \left. + \text{ terms with higher powers of h } \right\} \right] f(z) dz.$$

Taking the limit as $h \to 0$, we have

$$\lim_{h \to 0} \frac{f^{(m)}(a+h) - f^{(m)}(a)}{h} = \frac{(m+1)!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{m+2}} dz$$

i.e.,

$$f^{(m+1)}(a) = \frac{(m+1)!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{m+2}} dz.$$

This shows that the formula is also true for $n = m+1$. Hence the formula holds for all values of n, i.e.,

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz.$$

Thus $f(z)$ has derivatives of all orders and these are all analytic at a. □

Morera's Theorem

Theorem (Morera's Theorem)

Let $f(z)$ be continuous in a simply connected domain D and let, for every closed contour C in the domain D, $\int_C f(z)dz = 0$. Then $f(z)$ is analytic in D.

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Proof.

Let z_0 be a fixed point and z be a variable point inside the domain D. We have already seen that the value of the function

$$F(z) = \int_{z_0}^{z} f(\zeta)d\zeta$$

is independent of the path of integration joining z_0 and z and depends on z only.
Let $z + h$ be a point in the neighborhood of z, we have

$$F(z+h) = \int_{z_0}^{z+h} f(\zeta)d\zeta,$$

and

$$F(z+h) - F(z) = \int_{z}^{z+h} f(\zeta)d\zeta - \int_{z_0}^{z} f(\zeta)d\zeta = \int_{z}^{z+h} f(\zeta)d\zeta. \tag{16}$$

Since the integral in (16) is path-independent, it may be taken along the straight line segment joining z to $z + h$. Now, we have

Morera's Theorem

$$\begin{aligned}\frac{F(z+h)-F(z)}{h}-f(z)&=\frac{1}{h}\int_z^{z+h}f(\zeta)d\zeta-\frac{f(z)}{h}h=\frac{1}{h}\left[\int_z^{z+h}f(\zeta)d\zeta-f(z)\int_z^{z+h}d\zeta\right]\\&=\frac{1}{h}\int_z^{z+h}\left(f(\zeta)-f(z)\right)d\zeta.\end{aligned}\tag{17}$$

Since $f(\zeta)$ is continuous at z in D, for any positive number ϵ, there exists a positive number δ such that

$$|f(\zeta)-f(z)|<\epsilon\tag{18}$$

for all ζ satisfying $|\zeta-z|<\delta$. Choosing $|h|<\delta$. Then the inequality (18) is satisfied for every point ζ on the line segment joining z to $z+h$. Hence, from (17) and (18), we obtain

$$\left|\frac{F(z+h)-F(z)}{h}-f(z)\right|\le\frac{1}{|h|}\int_z^{z+h}|f(\zeta)-f(z)||d\zeta|<\frac{1}{|h|}\int_z^{z+h}\epsilon|d\zeta|=\epsilon.\tag{19}$$

Thus, $\lim_{h\to 0}\frac{F(z+h)-F(z)}{h}=f(z)$, that is, $F(z)$ is analytic with the derivative $f(z)$, i.e., $F'(z)=f(z)$ for all $z\in D$ and, consequently, $F(z)$ is analytic in D. Since we know that the derivative of an analytic function is analytic. It follows that $f(z)$ is analytic inside D as required. □

Morera's Theorem

Morera's Theorem can also be stated as follows:

Theorem (Morera's theorem)

If $f(z)$ is defined and continuous in a region Ω and if $\int_\gamma f dz = 0$ for all closed curve γ in Ω, then $f(z)$ is analytic in Ω.

[1] Cauchy-Goursat's theorem: If a function $f(z)$ is analytic and single-valued inside and on a simple closed contour C, then $\int_C f(z)dz = 0$.

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Keeping in view the Cauchy-Goursat theorem[1] and Morera's theorem, we may state the following theorem.

Theorem

Theorem. Let $f(z)$ be continuous in a simply connected domain D and let C be any closed contour in the domain D. Then a necessary and sufficient condition for $f(z)$ to be analytic in D is that

$$\int_C f(z)dz = 0.$$

[1] Cauchy-Goursat's theorem: If a function $f(z)$ is analytic and single-valued inside and on a simple closed contour C, then $\int_C f(z)dz = 0$.

Cauchy's Inequality

Theorem (Cauchy's Estimates)

Let $f(z)$ be analytic within and on a circle C defined by $|z - z_0| = r$. If $|f(z)| \le M$ on C, then

$$\left| f^{(n)}(z_0) \right| \le n! \frac{M}{r^n}.$$

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$$\left|f^{(n)}(z_0)\right|\le n!\frac{M}{r^n}.$$

Proof.

By Cauchy's formula for the n^{th} derivative of an analytic function at a point, we have $f^{(n)}(z_0)=\frac{n!}{2\pi i}\int_C\frac{f(z)}{(z-z_0)^{n+1}}dz$. Therefore, we have

$$\left|f^{(n)}(z_0)\right|\le\frac{n!}{2\pi}\int_C\frac{|f(z)|}{|z-z_0|^{n+1}}|dz|\le\frac{M\cdot n!}{2\pi}\int_C\frac{|dz|}{|z-z_0|^{n+1}}. \tag{20}$$

Then, let $z-z_0=re^{i\theta}$ so that $dz=rie^{i\theta}d\theta$. Thus $|dz|=rd\theta$. Hence (20) yields

$$\left|f^{(n)}(z_0)\right|\le\frac{M\cdot n!}{2\pi}\int_0^{2\pi}\frac{rd\theta}{r^{n+1}}=n!\frac{M}{2\pi r^n}2\pi=n!\frac{M}{r^n}.$$

□

Liouville's Theorem

Theorem (Liouville's Theorem)

If $f(z)$ is an integral function and is bounded for all values of z, then it is constant. In other words, a bounded entire function is constant.

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Proof.

Let a be any arbitrary point of the z-plane. Now, $f(z)$ is, by the hypothesis, analytic for all $z \in \mathbb{C}$. Moreover, $|f(z)|$ satisfies the inequality

$$|f(z)| \leq M$$

on any circle $|z-a| = r$, where M denotes the upper bound of $|f(z)|$ for z lying in all finite regions of the z-plane. Hence, by Cauchy's Inequality, we have

$$\left|f'(a)\right| \leq \frac{M}{r}.$$

Letting $r \to \infty$, we obtain

$$f'(a) = 0.$$

Since a is an arbitrary point in the z-plane. Thus the derivative of $f(z)$ vanishes everywhere. It follows that $f(z)$ is a constant. $\square$

Liouville's Theorem

One of the most elegant applications of Liouville's Theorem is a verification of what is known as the Fundamental Theorem of Algebra.

Theorem (The fundamental theorem of algebra)

Let $p(z)$ be a nonconstant polynomial. Then p has a root. That is, there exists an $\alpha \in \mathbb{C}$ such that $p(\alpha) = 0$.

Liouville's Theorem

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Theorem (The fundamental theorem of algebra)

Let $p(z)$ be a nonconstant polynomial. Then p has a root. That is, there exists an $\alpha \in \mathbb{C}$ such that $p(\alpha) = 0$.

Proof.

To understand this result, suppose not. Then $g(z) = 1/p(z)$ is entire. Also, when $|z| \to \infty$, then $|p(z)| \to +\infty$. Thus $1/|p(z)| \to 0$ as $|z| \to \infty$; hence g is bounded. By Liouville's Theorem, g is constant, hence p is constant. Contradiction. $\square$

Harmonic Functions

Definition

A C^2 function u is said to be harmonic if it satisfies the equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u=0.$$

This equation is called Laplace's equation, and is frequently abbreviated as

$$\triangle u=0.$$

Harmonic Functions

Definition

If both functions $u(x, y)$ and $v(x, y)$ are harmonic functions in the simply connected region D, and satisfy the C-R equations: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, then we call the function $v(x, y)$ a harmonic conjugate for $u(x, y)$.

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Theorem

A necessary and sufficient condition for $f(z) = u(x, y) + iv(x, y)$ to be analytic in the simply connected region D is that $v(x, y)$ a harmonic conjugate for $u(x, y)$ in D.