Example

Calculate the second fundamental forms of cylindrical surface and plane

Answer:

Assume plane S_1 is a plane in Oxy plane and its parametric function is

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = (u, v, 0)$$

Then its unit normal vector is

$$\boldsymbol{n} = (0,0,1)$$

So

$$I = d\boldsymbol{r}.d\boldsymbol{r} = (du)^2 + (dv)^2$$

$$\Pi = -d\boldsymbol{r}.d\boldsymbol{n} = 0$$

Example

Assume the parametric function of cylindrical surface is

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = \left(\mathrm{acos}\left(\frac{u}{a}\right), \mathrm{asin}\left(\frac{u}{a}\right), v\right)$$

Then

$$\boldsymbol{r}_u = \left(-sin\left(\frac{u}{a}\right), \cos\left(\frac{u}{a}\right), 0\right),$$

$$\boldsymbol{r}_v = (0,0,1)$$

$$\Longrightarrow \boldsymbol{r}_u \times \boldsymbol{r}_v = \left(\cos\left(\frac{u}{a}\right), \sin\left(\frac{u}{a}\right), 0\right) = \boldsymbol{n}$$

$$\boldsymbol{r}_{uu} = \left(-\frac{1}{a}\cos\left(\frac{u}{a}\right), -\frac{1}{a}\sin\left(\frac{u}{a}\right), 0\right)$$

$$\boldsymbol{r}_{vv} = \boldsymbol{r}_{uv} = \boldsymbol{r}_{vu} = 0$$

$$\Longrightarrow E = r_u . r_u = 1, F = r_u . r_v = 0, r_v . r_v = 1$$

$$L = r_{uu} . \boldsymbol{n} = -\frac{1}{a}, M = r_{uv} . \boldsymbol{n} = 0, N = r_{vv} . \boldsymbol{n} = 0$$

So $I = (du)^2 + (dv)^2, \;\; \Pi = -\frac{1}{a}(du)^2$

Example

In conclusion

The first fundamental forms of plane and cylindrical are the same which means we can construct an isometric correspondence between them.

However, their second fundamental forms are different which means they have different appearances and shapes.

Theorem

A regular surface is a plane (or part of plane) if and only if its second fundamental form $\Pi = 0$.

Proof:

The necessary condition has been given in the above example. So we only prove the sufficient condition.

Assume the parametric function of the regular surface S is

$$\boldsymbol{r} = \boldsymbol{r}(u, v)$$

Its second fundamental form $\Pi = 0$, that is

$$L = -\boldsymbol{r}_u . \boldsymbol{n}_u = 0$$

$$M = -\boldsymbol{r}_u . \boldsymbol{n}_v = -\boldsymbol{r}_v . \boldsymbol{n}_u = 0$$

$$N = -\boldsymbol{r}_v . \boldsymbol{n}_v = 0$$

Next, we need to prove the unit normal of S is a constant.

Theorem

$$L = -\boldsymbol{r}_u . \boldsymbol{n}_u = 0$$
$$M = -\boldsymbol{r}_u . \boldsymbol{n}_v = -\boldsymbol{r}_v . \boldsymbol{n}_u = 0$$
$$N = -\boldsymbol{r}_v . \boldsymbol{n}_v = 0$$

Since $\boldsymbol{n}$ is a unit vector field, we have

$$\boldsymbol{n}_u . \boldsymbol{n} = \boldsymbol{n}_v . \boldsymbol{n} = 0$$

We note that $\{\boldsymbol{r}; \boldsymbol{r}_u, \boldsymbol{r}_v, \boldsymbol{n}\}$ is a frame in E^3.
From the above equations we get that
the projections of $\boldsymbol{n}_u, \boldsymbol{n}_v$ on $\boldsymbol{r}_u, \boldsymbol{r}_v, \boldsymbol{n}$ are zeros.
Then

$$\boldsymbol{n}_u = \boldsymbol{n}_v = 0$$

That is

$$d\boldsymbol{n} = \boldsymbol{n}_u du + \boldsymbol{n}_v dv = 0$$
$$\Longrightarrow \boldsymbol{n} = \boldsymbol{n}_0 (\text{constant})$$

Since

$$d\boldsymbol{r} . \boldsymbol{n} = 0$$

So

$$d(\boldsymbol{r} . \boldsymbol{n}) = d\boldsymbol{r} . \boldsymbol{n} + \boldsymbol{r} . d\boldsymbol{n} = 0$$

Theorem

$$d(\boldsymbol{r}.\boldsymbol{n}) = d\boldsymbol{r}.\boldsymbol{n} + \boldsymbol{r}.d\boldsymbol{n} = 0$$

$$\Rightarrow \boldsymbol{r}.\boldsymbol{n} = \boldsymbol{c}\ \text{(constant)}$$

So

$$\boldsymbol{r}(u,v).\boldsymbol{n} = \boldsymbol{r}(u_0,v_0).\boldsymbol{n}$$

$$\Rightarrow \big(\boldsymbol{r}(u,v) - \boldsymbol{r}(u_0,v_0)\big).\boldsymbol{n} = \boldsymbol{0}$$

It proves that S lies on a plane that passes through the point $\boldsymbol{r}(u_0,v_0)$ and has normal $\boldsymbol{n}$.

Theorem

A regular surface is part of a sphere if and only if its second fundamental form and first fundamental form satisfy the following equation for each point

$$\Pi = c(u, v) I$$

where $c(u, v) \neq 0, \forall (u, v)$

Proof:

(Necessary condition)

Assume surface $S: \boldsymbol{r} = \boldsymbol{r}(u, v)$ lies on a sphere whose center is r_0 and radius is R. Then the parametric function of sphere satisfy

$$(\boldsymbol{r}(u, v) - \boldsymbol{r}_0)^2 = R^2$$

Then we have

$$\boxed{d\boldsymbol{r}} . (\boldsymbol{r}(u, v) - \boldsymbol{r}_0) = 0$$

Tangent vector

Theorem

So we know that $\boldsymbol{r}(u,v)-\boldsymbol{r}_0$ is the normal of S, that is

$$\boldsymbol{n}=\frac{1}{R}(\boldsymbol{r}(u,v)-\boldsymbol{r}_0)$$

So

$$\begin{aligned}\Pi &= -d\boldsymbol{r}.\,d\boldsymbol{n}\\ &= -\frac{1}{R}d\boldsymbol{r}.\,d\boldsymbol{r}\\ &= -\frac{1}{R}I\end{aligned}$$

Theorem

(Sufficient condition)

Assume there is a function $c(u,v) \neq 0, \forall(u,v)$, so that

$$\Pi = c(u,v)I$$

Then we have

$$\begin{aligned} &L(du)^2 + 2Mdudv + N(dv)^2 \\ &= c(u,v)(E(du)^2 + 2Fdudv + G(dv)^2) \end{aligned}$$

$$\Longrightarrow (L - cE)(du)^2 + 2(M - cF)dudv + (N - cG)(dv)^2 = 0$$

Since the above equation is satisfied for each point (u,v), then

$$(L - cE)(du)^2 + 2(M - cF)dudv + (N - cG)(dv)^2 = 0$$

$$\Leftrightarrow \begin{cases} L(u,v) = c(u,v)E(u,v) \\ M(u,v) = c(u,v)F(u,v) \\ N(u,v) = c(u,v)G(u,v) \end{cases}$$

$$\Leftrightarrow \begin{cases} -\boldsymbol{n}_u.\boldsymbol{r}_u = c\boldsymbol{r}_u.\boldsymbol{r}_u \\ -\boldsymbol{n}_u.\boldsymbol{r}_v = -\boldsymbol{n}_v.\boldsymbol{r}_u = c\boldsymbol{r}_u.\boldsymbol{r}_v \\ -\boldsymbol{n}_v.\boldsymbol{r}_v = c\boldsymbol{r}_v.\boldsymbol{r}_v \end{cases}$$

Theorem

$$\Leftrightarrow \begin{cases} \boxed{\begin{gathered}(\boldsymbol{n}_u + c\boldsymbol{r}_u).\boldsymbol{r}_u = 0 \\ (c\boldsymbol{r}_u + \boldsymbol{n}_u).\boldsymbol{r}_v = (c\boldsymbol{r}_v + \boldsymbol{n}_v).\boldsymbol{r}_u = 0 \\ (\boldsymbol{n}_v + c\boldsymbol{r}_v).\boldsymbol{r}_v = 0\end{gathered}} \end{cases}$$

In another way, since $\boldsymbol{n}$ is the unit normal field, so

$$\boxed{(c\boldsymbol{r}_u + \boldsymbol{n}_u).\boldsymbol{n} = (c\boldsymbol{r}_v + \boldsymbol{n}_v).\boldsymbol{n} = 0}$$

Since $\{\boldsymbol{r}; \boldsymbol{r}_u, \boldsymbol{r}_v, \boldsymbol{n}\}$ is a frame in E^3, with the above equations, we know that the projections of $c\boldsymbol{r}_u + \boldsymbol{n}_u$and $c\boldsymbol{r}_v + \boldsymbol{n}_v$ on $\boldsymbol{r}_u, \boldsymbol{r}_v, \boldsymbol{n}$ are zeros, that is

$$c\boldsymbol{r}_u + \boldsymbol{n}_u = 0, c\boldsymbol{r}_v + \boldsymbol{n}_v = 0$$

We calculate the derivatives of above equation regarding to v and u respectively

$$c_v\boldsymbol{r}_u + c\boldsymbol{r}_{uv} + \boldsymbol{n}_{uv} = 0,$$
$$c_u\boldsymbol{r}_v + c\boldsymbol{r}_{vu} + \boldsymbol{n}_{vu} = 0$$

Comparing them we have

$$c_v\boldsymbol{r}_u = c_u\boldsymbol{r}_v$$

Theorem

Since $\boldsymbol{r}_u, \boldsymbol{r}_v$ are linearly independent, so

$$c_v = c_u = 0$$

That is $c = c_0$ is a constant.

Next we calculate

$$\begin{aligned} & d(\boldsymbol{n} + c\boldsymbol{r}) \\ &= \boldsymbol{n}_u du + \boldsymbol{n}_v dv + c\boldsymbol{r}_u du + c\boldsymbol{r}_v dv \\ &= (\boldsymbol{n}_u + c\boldsymbol{r}_u) du + (\boldsymbol{n}_v + c\boldsymbol{r}_v) dv \\ &= 0 \end{aligned}$$

Let assume

$$\boldsymbol{n} + c\boldsymbol{r} = c\boldsymbol{r_0}$$

$$\Longrightarrow \boldsymbol{r} - \boldsymbol{r_0} = \frac{1}{c}\boldsymbol{n}$$

$$\Longrightarrow (\boldsymbol{r} - \boldsymbol{r_0})^{\mathbf{2}} = \frac{1}{c^2}$$

That to say surface S lies on a sphere whose center is $\boldsymbol{r}_0$ and radius is $\left|\frac{1}{c}\right|$.

Note

For surface $\boldsymbol{r} = \{x, y, z(x, y)\}$, its coefficients of the second fundament form are

$$L = \boldsymbol{r}_{xx}.\boldsymbol{n} = \frac{r}{\sqrt{1 + p^2 + q^2}}$$

$$M = \boldsymbol{r}_{xy}.\boldsymbol{n} = \frac{s}{\sqrt{1 + p^2 + q^2}}$$

$$N = \boldsymbol{r}_{yy}.\boldsymbol{n} = \frac{t}{\sqrt{1 + p^2 + q^2}}$$

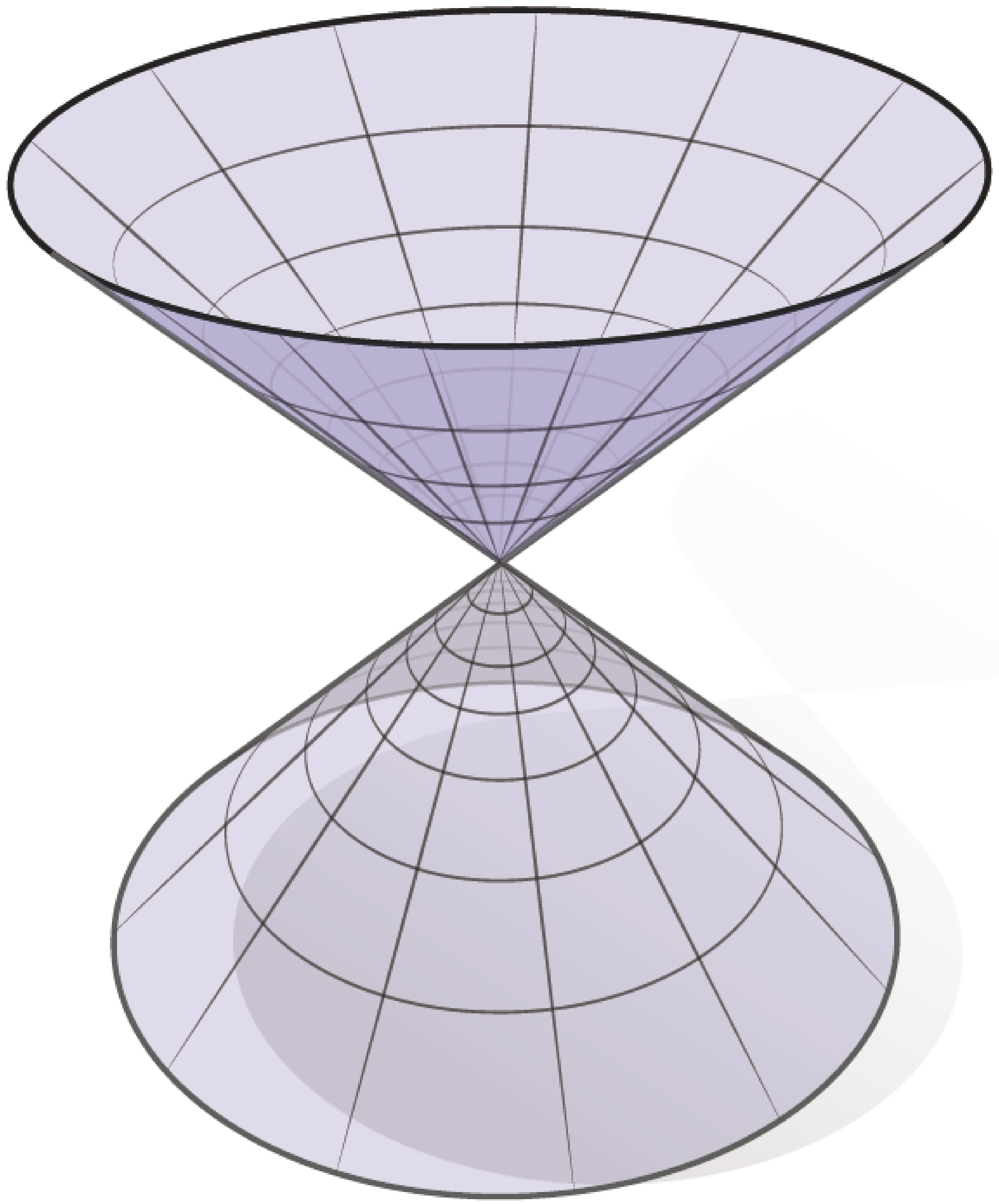
where

$$p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$$

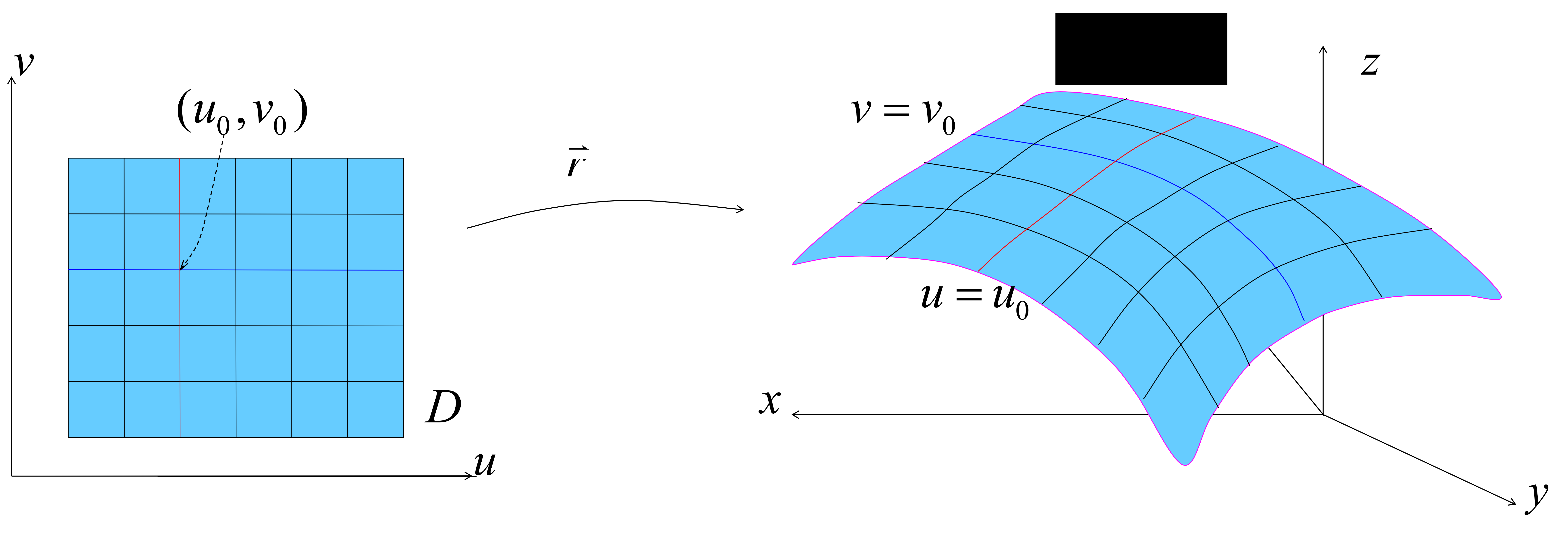
Homework

1. Calculate the first and second fundamental forms of paraboloid surface $z = a(x^2 + y^2)$, a is $constant$.
2. Calculate the first and second fundamental forms of paraboloid surface $2x_3 = 5x_1^2 + 4x_1x_2 + 2x_2^2$ at origin (0,0,0).

Normal curvature



Normal curvature



Normal curvature

Assume the parametric function of surface S is $\boldsymbol{r} = \boldsymbol{r}(u, v)$, then the parametric function of a curve C on S can be represented as

$$u = u(s), v = v(s)$$

Here we assume s is the arc length parameter of C.

Since C is a curve on S, so its parametric function is

$$\boldsymbol{r} = \boldsymbol{r}\big(u(s), v(s)\big)$$

The unit tangent vector of C is

$$\boldsymbol{T}(s) = \frac{d\boldsymbol{r}}{ds}$$

$$= \boldsymbol{r}_u \frac{du(s)}{ds} + \boldsymbol{r}_v \frac{dv(s)}{ds}$$

Then the curvature vector is

$$\frac{d\boldsymbol{T}(s)}{ds} = k\boldsymbol{N}(s)$$

Normal curvature

$$\boldsymbol{T}(s) = \boldsymbol{r}_u \frac{du(s)}{ds} + \boldsymbol{r}_v \frac{dv(s)}{ds}$$

$$\frac{d\boldsymbol{T}(s)}{ds} = k\boldsymbol{N}(s)$$

$$= \boldsymbol{r}_{uu}\left(\frac{du(s)}{ds}\right)^2 + 2\boldsymbol{r}_{uv}\frac{du(s)}{ds}\frac{dv(s)}{ds} + \boldsymbol{r}_{vv}\left(\frac{dv(s)}{ds}\right)^2$$

$$+\boldsymbol{r}_u \frac{d^2u(s)}{ds^2} + \boldsymbol{r}_v \frac{d^2v(s)}{ds^2}$$

So the orthogonal projection of curvature vector on normal vector $\boldsymbol{n}$ is

$$k_n = \frac{d\boldsymbol{T}(s)}{ds}.\boldsymbol{n}$$

$$= (k\boldsymbol{N}).\boldsymbol{n} = k(\boldsymbol{N}.\boldsymbol{n})$$

$$= L\left(\frac{du(s)}{ds}\right)^2 + 2M\frac{du(s)}{ds}\frac{dv(s)}{ds} + N\left(\frac{dv(s)}{ds}\right)^2$$

Normal curvature

Therefore, k_n is only dependent on the coefficients of the second fundamental form and the unit tangent vector of C $\left(\frac{du(s)}{ds}, \frac{dv(s)}{ds}\right)$.

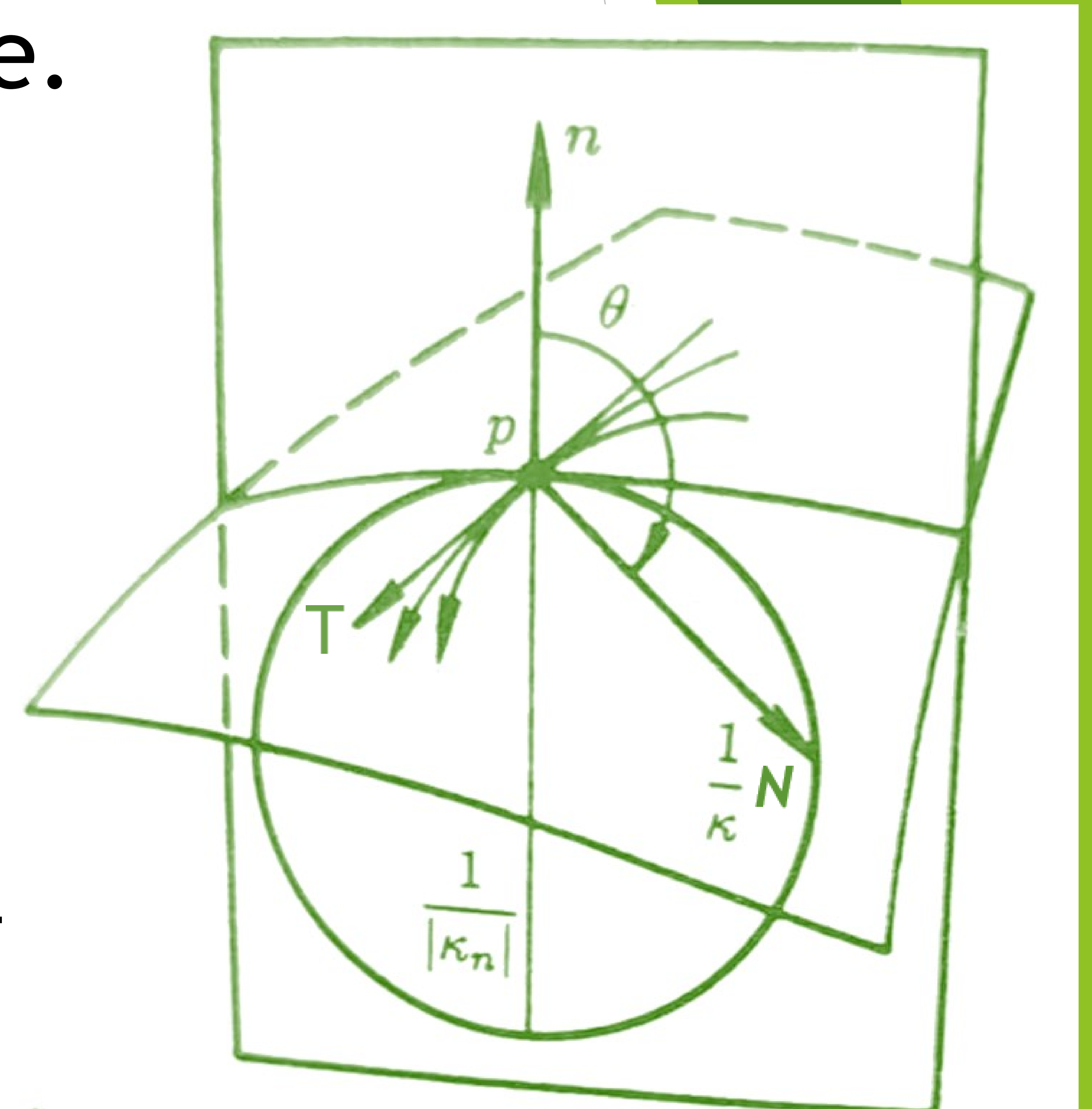
k_n is called as the **normal curvature** (法曲率) of curve C on surface S.

If two curves C_1, C_2 on S have the same tangent vector at point p, they will have the same normal curvature.

Let $\angle(\boldsymbol{N}, \boldsymbol{n}) = \theta$, then

$$k_n = k\cos(\theta)$$

We can observe that for all curves which pass through $p = \boldsymbol{r}(u(s), v(s))$ and have the same tangent vector at p, their curvature centers at p will located on the same circle whose diameter is $\frac{1}{|k_n|}$



Normal curvature

In fact,

$$\frac{1}{k} = \frac{1}{|k_n|}|\cos(\theta)|$$

So the center of curvature is

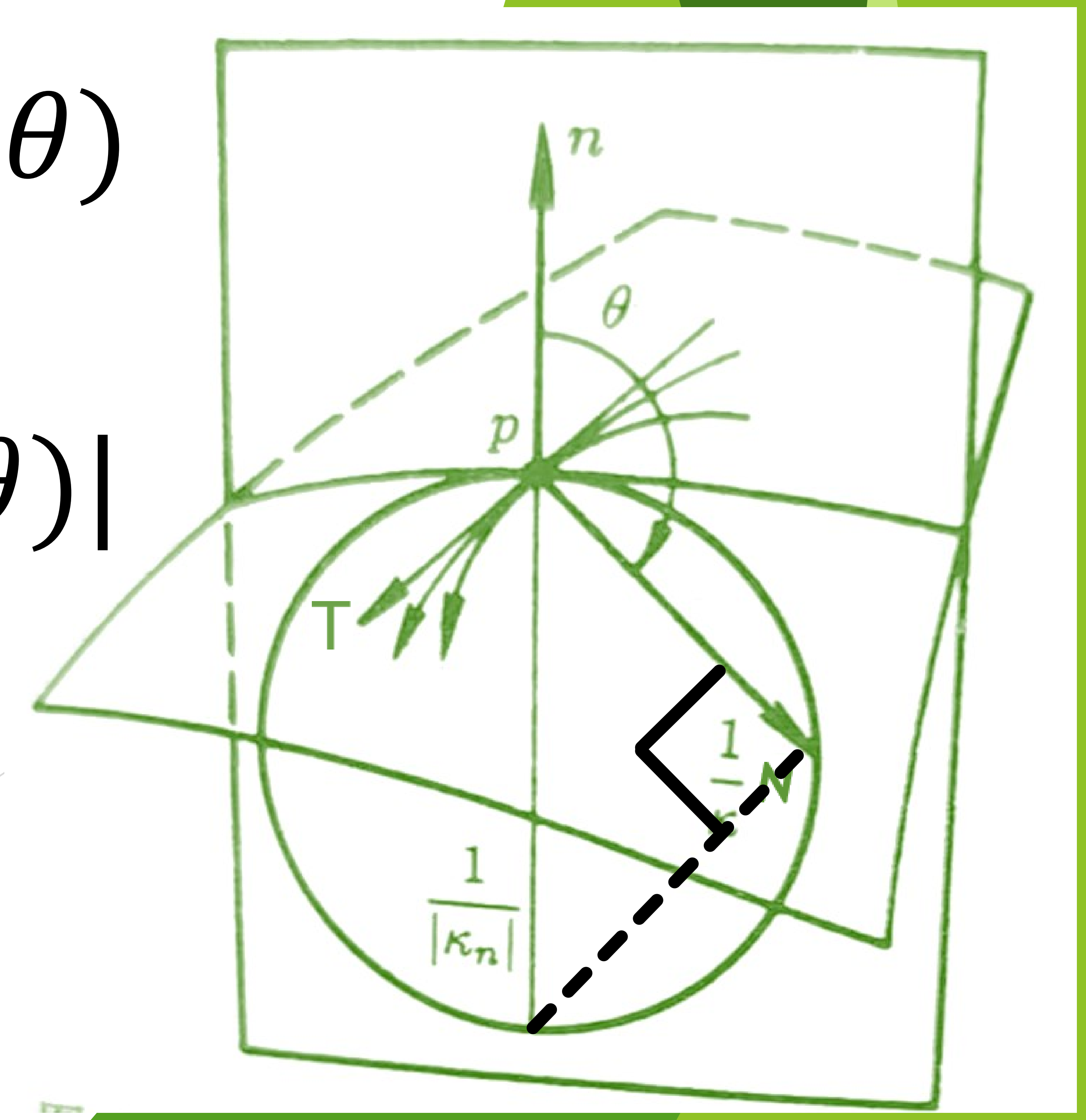
$$c = \boldsymbol{r}\big(u(s), v(s)\big) + \frac{1}{k}N$$

$$= \boldsymbol{r}\big(u(s), v(s)\big) + \frac{1}{|k_n|}|\cos(\theta)|\,N$$

So

$$\big|c - \boldsymbol{r}\big(u(s), v(s)\big)\big|^2 = \frac{1}{k_n^2}\cos^2(\theta)$$

$$\Longrightarrow \frac{\big|c - \boldsymbol{r}\big(u(s), v(s)\big)\big|}{\frac{1}{|k_n|}} = |\cos(\theta)|$$



Normal curvature

Since s is the arc length parameter of curve $r(s) = r\big(u(s), v(s)\big)$, so we have

$$|r'(s)|^2$$

$$= E\left(\frac{du(s)}{ds}\right)^2 + 2F\frac{du(s)}{ds}\frac{dv(s)}{ds} + G\left(\frac{dv(s)}{ds}\right)^2 = 1$$

That is

$$ds^2 = E(du)^2 + 2Fdudv + G(dv)^2 = I$$

As

$$k_n = L\left(\frac{du(s)}{ds}\right)^2 + 2M\frac{du(s)}{ds}\frac{dv(s)}{ds} + N\left(\frac{dv(s)}{ds}\right)^2$$

$$= \frac{L(du)^2 + 2Mdudv + N(dv)^2}{ds^2}$$

$$\frac{L(du)^2 + 2Mdudv + N(dv)^2}{E(du)^2 + 2Fdudv + G(dv)^2} = \frac{\Pi}{I}$$

We can see that k_n is the function of $\frac{du}{dv}$.

Normal curvature

Definition

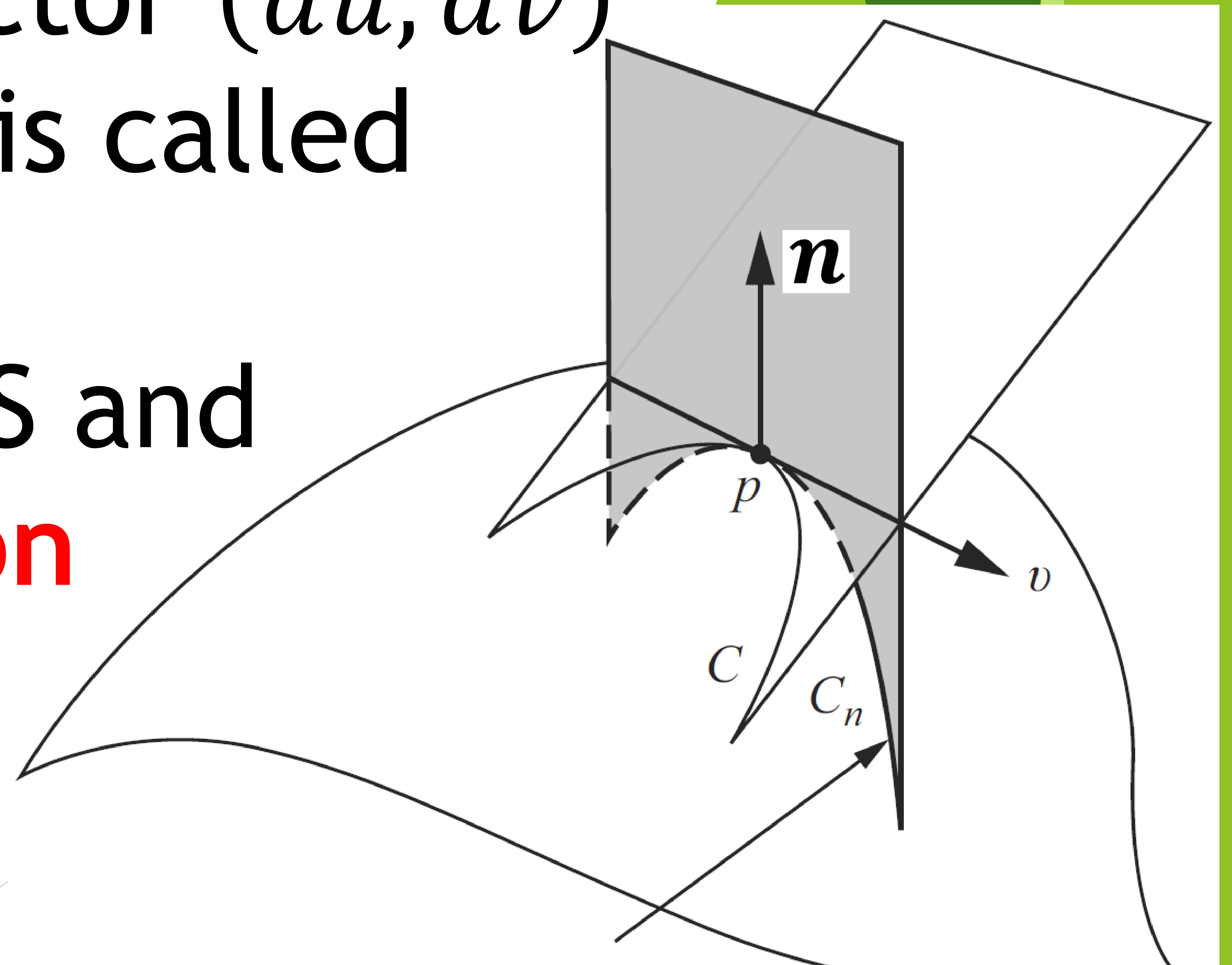
Assume the parametric function of regular surface S is $r = r(u, v)$, I, Π are the first and second fundamental forms of S, then

$$k_n = \frac{\Pi}{I} = \frac{L(du)^2 + 2Mdudv + N(dv)^2}{E(du)^2 + 2Fdudv + G(dv)^2}$$

is called as the normal curvature of S at point (u, v) along tangent direction (du, dv).

The plane determined by the tangent vector (du, dv) and normal vector $\boldsymbol{n}(u, v)$ at point (u, v) is called as **normal section** (法截面).

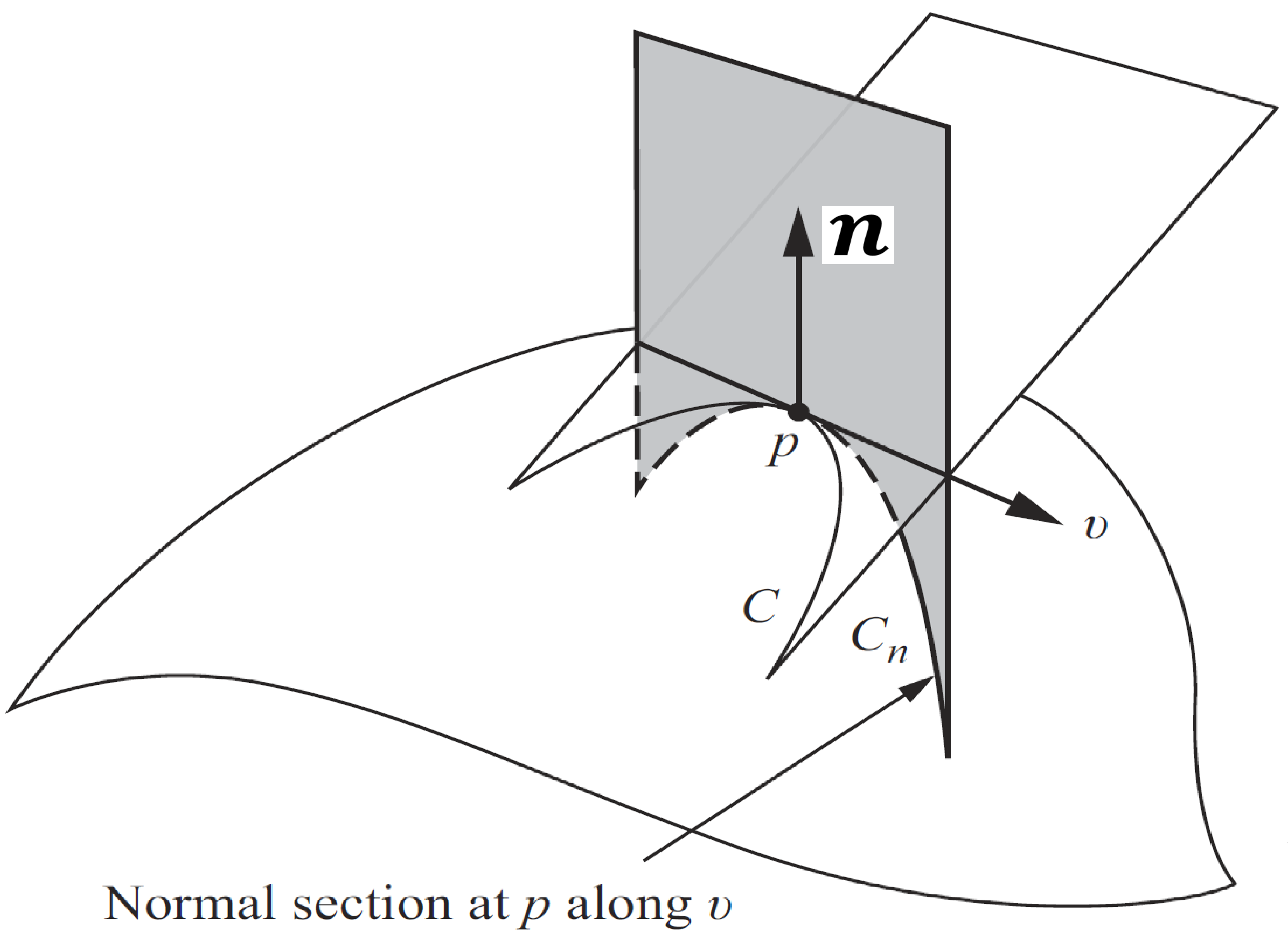
The curve formed by the intersection of S and normal section is called as **normal section line** (法截线).



Normal section at p along v

Theorem

The normal curvature of surface S at point (u, v) along tangent direction (du, dv) is equal to the relative curvature k_r of the normal section line at (u, v) determined by tangent direction (du, dv).



Example

Calculate the normal curvatures of plane, cylindrical surface, and sphere.

Answer:

Since the second fundamental form of plane $\Pi = 0$, according to

$$k_n = \frac{\Pi}{I}$$

we know $k_n = 0$ for plane.

The parametric function of cylindrical surface is

$$r = \left(a\cos\left(\frac{u}{a}\right), a\sin\left(\frac{u}{a}\right), v\right)$$

Its first and second fundamental forms are

$$I = (du)^2 + (dv)^2, \Pi = -\frac{1}{a}(du)^2$$

Example

$$I = (du)^2 + (dv)^2, \Pi = -\frac{1}{a}(du)^2$$

So the normal curvature is

$$k_n = -\frac{(du)^2}{a\big((du)^2 + (dv)^2\big)}$$

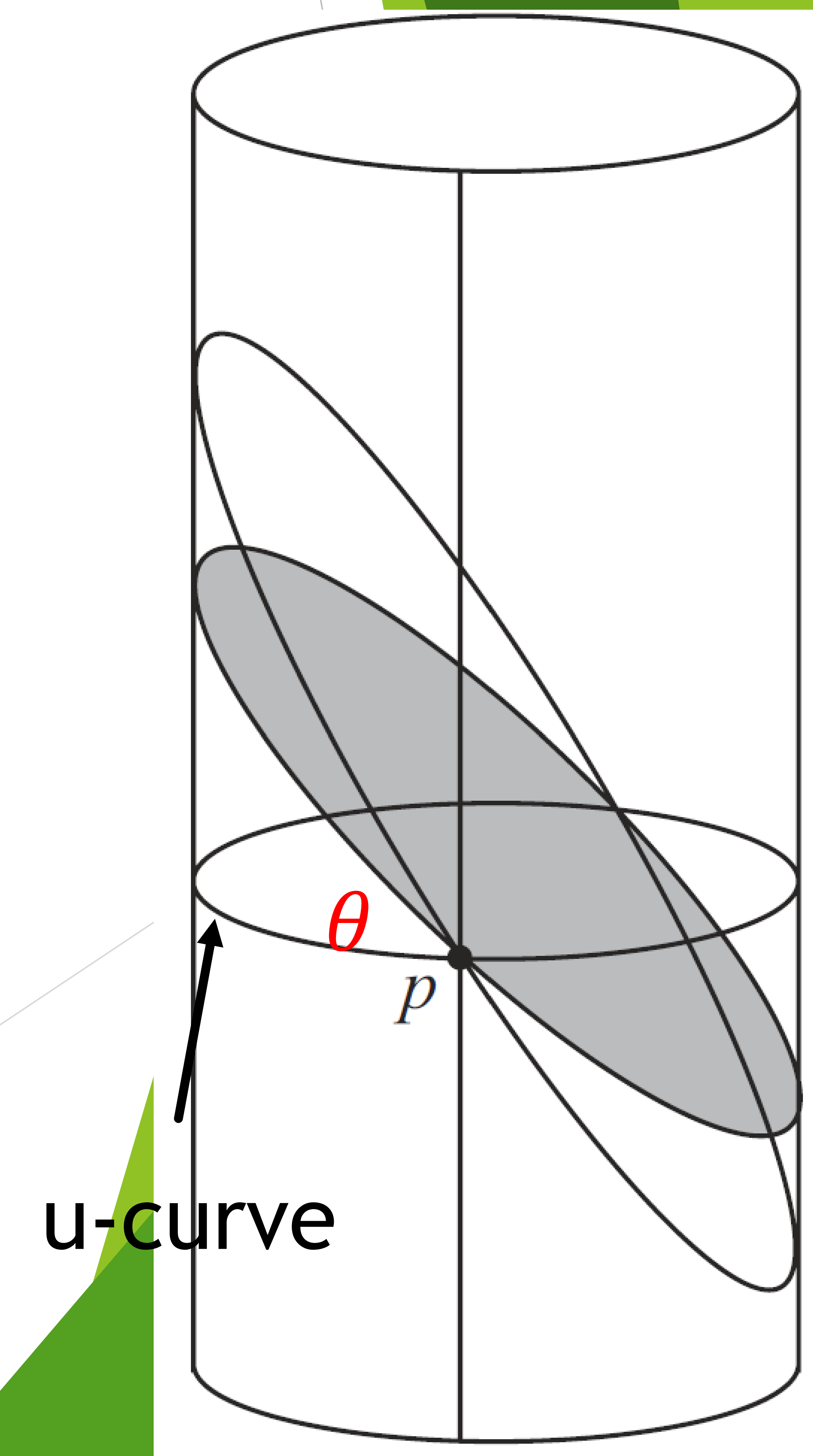
Let θ be the angle between tangent direction (du, dv) and tangent direction of u-curve, then

$$\cos(\theta) = \frac{du}{\sqrt{(du)^2 + (dv)^2}}$$

So

$$k_n = -\frac{\cos^2(\theta)}{a}$$

Here "$-$" means normal section line bends in the opposite direction of normal vector $\boldsymbol{n}$.



Example

The normal section line of sphere at any point is a circle whose radius R is the radius of the sphere. Since the relative curvature of a circle is

$$k_r = -\frac{1}{R}$$

so

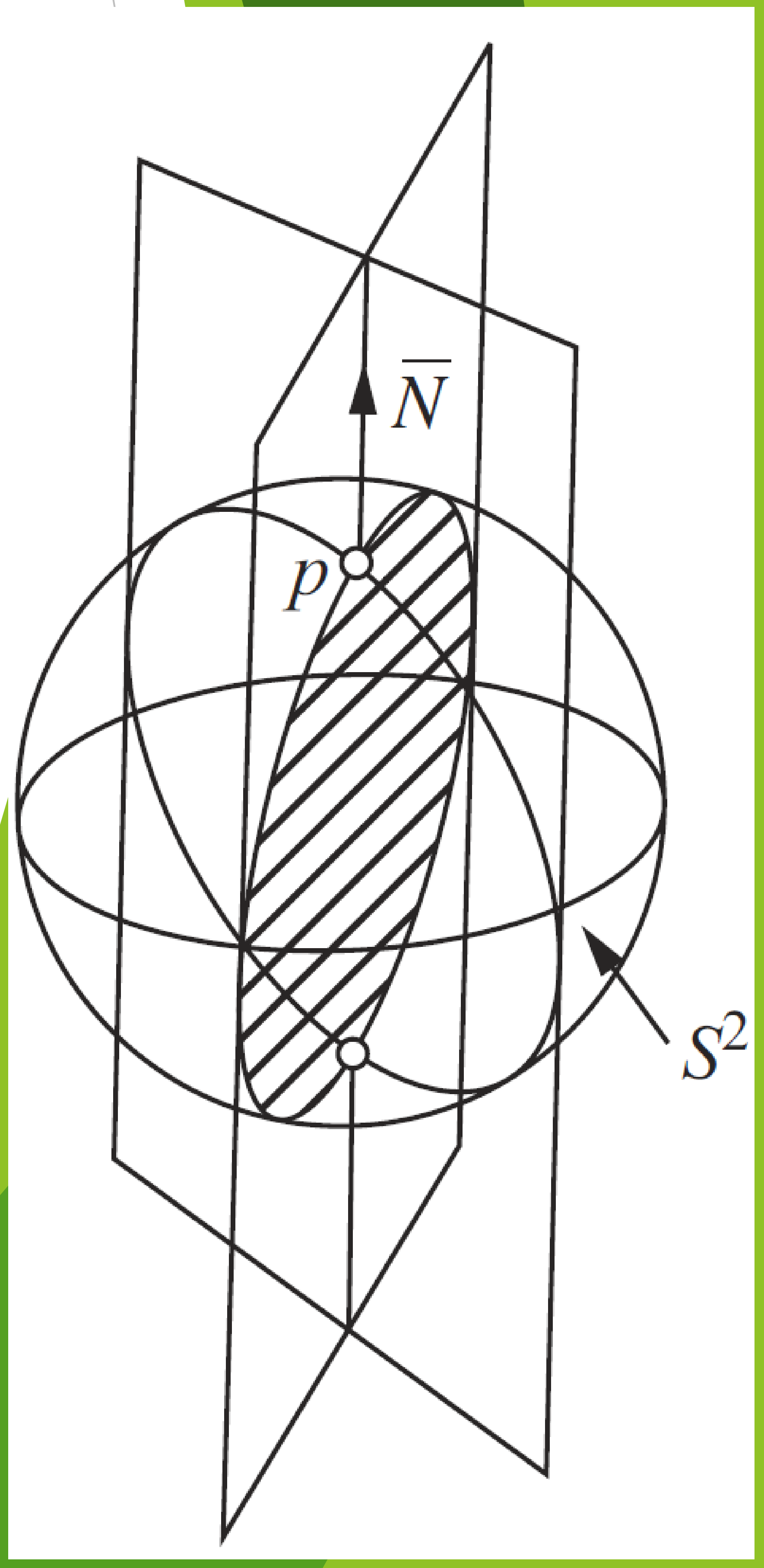
$$k_n = -\frac{1}{R}$$

Here we use "−", since the circle bends in the opposite direction of normal.

Since

$$k_n = k\cos(\theta)$$

so the curvature of the curves on sphere cannot be zeros.



Principal curvatures and directions

Assume surface $S: r = r(u,v)$ is orthogonal at point $r(u_0, v_0)$, so the first and second fundamental forms of S at this point are

$$I = E(du)^2 + G(dv)^2$$

$$\Pi = L(du)^2 + 2Mdudv + N(dv)^2$$

The normal curvature of S at this point is

$$k_n = \frac{\Pi}{I} = \frac{L(du)^2 + 2Mdudv + N(dv)^2}{E(du)^2 + G(dv)^2}$$

$$= \frac{L}{E}\left(\frac{\sqrt{E}du}{\sqrt{E(du)^2 + G(dv)^2}}\right)^2$$

$$+ \frac{2M}{\sqrt{EG}}\frac{\sqrt{E}du}{\sqrt{E(du)^2 + G(dv)^2}}\frac{\sqrt{G}dv}{\sqrt{E(du)^2 + G(dv)^2}}$$

$$+ \frac{N}{G}\left(\frac{\sqrt{G}dv}{\sqrt{E(du)^2 + G(dv)^2}}\right)^2$$