Methods of Mathematical Physics

—Lecture 2 Functions of a Complex Variable—

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Contents

Introduction

The concepts in ordinary calculus in the setting of $\mathbb{R}$, like convergence of sequences, or continuity and differentiability of functions, all rely on the notion of closeness of points in $\mathbb{R}$.

In order to do calculus with complex numbers, we need a notion of distance $d(z_1, z_2)$ between for pairs of complex numbers (z_1, z_2), and the first order of business is to explain what this notion is.

Metric on $\mathbb{C}$

A metric space is a pair (X, d), where X is a set and $d : X \times X \to \mathbb{R}$ is a function called a distance function or metric that satisfies the following conditions: for $x, y, z \in X$,

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ (symmetry);
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

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Example

Let $X = \mathbb{C}$, $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2) \in X$ and define
$d(z_1, z_2) = |z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$,
or $d(z_1, z_2) = |x_1 - x_2| + |y_1 - y_2|$,
or $d(z_1, z_2) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$.
Then $(\mathbb{C}, d)$ is a metric space.

Open discs, open sets, closed sets, compact sets, connected sets

- An open ball/disc $D(z_0, r)$ with center z_0 and radius $r > 0$ is defined by $D(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}$.
- A subset U of $\mathbb{C}$ is called open if for every $z \in U$, there exists an $r_z > 0$ such that $D(z, r_z) \subset U$. (z is an interior point)
- A set S is said to be closed when every limit point of S belongs to S. (A set $F \subset X$ is said to be closed if its complement, $X - F$, is open.)
- A subset S of $\mathbb{C}$ is called bounded if there exists a $M > 0$ such that for all $z \in S, |z| \leq M$. Thus S is contained in a big enough disc in the complex plane.
- A subset $K \subset \mathbb{C}$ is called compact if it is both closed and bounded.
- An open set is said to be connected if it cannot be represented as the union of two nonempty disjoint open sets. A nonempty open set in the complex plane is connected if and only if any two of its points can be joined by a polygonal arc[1] lying entirely in the set.

[1] By a polygonal arc we mean a continuous chain of a finite number of line segments.

Open and Closed Domain (or Region), Curves

- A nonempty **open connected** subset of the complex plane is called an open domain or an open region or, simply, a region.
- A curve or a continuous arc Γ in the complex plane is the set of points z in the complex plane determined by the equation

$$z = z(t) = x(t) + iy(t)$$

where $x(t)$ and $y(t)$ are real continuous functions of a real variable t defined on a real interval $\alpha \le t \le \beta$ where $\alpha \le \beta$. We call $z(\alpha)$ and $z(\beta)$ the end points of Γ, $z(\alpha)$ being the initial point and $z(\beta)$ the terminal point of Γ. If $z(\alpha) = z(\beta), \Gamma$ is called a closed curve.
If the equation $z_0 = x(t) + iy(t)$ is satisfied by more than one value of t in the given range $I : \alpha \le t \le \beta$, then z_0 is said to be a multiple point. In particular, the multiple point is called a double point when the above equation is satisfied by two values of t in the given range I.

Jordan Arc and Simple Closed Jordan Curve

- A curve Γ is called a Jordan arc or a simple curve if it has no multiple points, i.e., if there exists some parametric representation

$$z = z(t) = x(t) + iy(t), \quad \alpha \leq t \leq \beta,$$

  such that, if $t_1 \neq t_2$, then $z(t_1) \neq z(t_2)$, i.e., $z(t)$ is one-to-one. The simplest example of a Jordan arc is a straight line segment.
- If, in a Jordan arc, the initial and terminal points coincide, that is, if there is a double point corresponding to the end points (α and β) of the interval $I : \alpha \leq t \leq \beta$ and there is no other multiple point on it, then it is called a simple closed Jordan curve or simply a closed Jordan curve.

Convergence and continuity

A sequence $(z_n)_{n\in\mathbb{N}}$ is said to be convergent with limit L if for every $\epsilon > 0$, there exists an index $N \subset \mathbb{N}$ such that for every $n > N$, there holds that $|z_n - L| < \epsilon$. It follows from the triangle inequality that for a convergent sequence the limit is unique, and we write

$$\lim_{n\to\infty} z_n = L.$$

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Let S be a subset of $\mathbb{C}, z_0 \in S$ and $f: S \to \mathbb{C}$. Then f is said to be continuous at z_0 if for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $z \in S$ satisfies $|z - z_0| < \delta$, there holds that $|f(z) - f(z_0)| < \epsilon$.

f is said to be continuous if for every $z \in S$, f is continuous at z.

Definitions

Let D be an arbitrary non-empty point set of the complex plane. If z is allowed to denote any point of D, z is called a complex variable and D is called the domain of definition of z or simply the domain.

A complex variable w is said to be a function of the complex variable z if, to every value of z in a certain domain D, there corresponds **one or more values** of w. Thus, if w is a function of z, it is written as $w = f(z)$. We also say that f defines a mapping of D into the w-plane. The totality of values $f(z)$ corresponding to all z in D constitutes another set R of complex numbers, known as the range of the function f.

Since $z = x + iy$, $f(z)$ will be of the form $u + iv$, where u and v are functions of two real variables x and y. We may then write

$$w = f(z) = u(x, y) + iv(x, y).$$

Single-valued and multiple-valued Functions

A function $f(z)$ of the complex variable z with domain of definition D and range R is said to be single-valued or one-valued if w takes only one value in R for each value of z in D.

If there correspond two or more values of $f(z)$ in R for some or all values of z in D, then $f(z)$ is called a multiple-valued or many-valued function of z.

Limits of Functions

Let $f(z)$ be a function of z defined in some neighborhood of a point z_0. The function $f(z)$ is said to have the limit ℓ as z tends to z_0 if, to each positive arbitrary number ϵ, there exists a positive number δ depending upon ϵ with the property that

$$|f(z) - \ell| < \epsilon$$

for all z such that $0 < |z - z_0| < \delta$ and $z \neq z_0$. In other words, there exists a deleted neighborhood of the point $z = z_0$ in which $|f(z) - \ell|$ can be made as small as we please. Symbolically, we write $\lim_{z\to z_0} f(z) = \ell$.

Continuity

Let G be an open set in $\mathbb{C}$ and let $f\colon G \to \mathbb{C}$. Then f is said to be continuous at a point z_0 in G if, given any positive number ϵ, we can find a member $\delta > 0$ depending in general on ϵ and z_0 such that

$$|f(z) - f(z_0)| < \epsilon$$

for all $z \in G$ in the neighborhood $|z - z_0| < \delta$ of z_0.

It follows from the above definition and the definition of limit that f is continuous at $z = z_0$ if

$$\lim_{z \to z_0} f(z) = f(z_0)\,.$$

If a function is continuous at every point of G, it is said to be continuous in G.

Continuity in terms of Real & Imaginary Parts of $f(z)$

If $f(z) = u(z, y) + iv(x, y)$, then it can be easily shown that f is a continuous function of z if and only if $u(x, y)$ and $v(x, y)$ are separately continuous functions of x and y.

Let f and g be continuous functions from X into $\mathbb{C}$ and let $a, b \in \mathbb{C}$. Then $af + bg$ and fg are both continuous. Also, f/g is continuous provided $g(x) \neq 0$ for every x in X.

A continuous function of a continuous function is a continuous function; that is, if $f : X \to Y$ and $g : Y \to Z$ are continuous functions, then $g \circ f$ where $(g \circ f)(x) = g(f(x))$ is a continuous function from X into Z.

Complex differentiability

In this section we will learn three main things:

1. The definition of complex differentiability.
2. The Cauchy-Riemann equations.
3. The geometric meaning of the complex derivative $f'(z_0)$.

The central result in this section is the necessity and (under mild conditions) sufficiency of the Cauchy-Riemann equations for the complex differentiability of a function in an open set.

Differentiability

If G is an open set in $\mathbb{C}$ and $f\colon G \to \mathbb{C}$ is a function, then f is said to be differentiable at a point z_0 in G if, for any positive number ϵ, we can find a positive number δ depending on ϵ and possibly on z_0 such that

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \epsilon$$

for all $z \in G$ in the neighborhood of z_0 defined by $|z - z_0| < \delta$.

If f is differentiable at each point of G, then we say that f is differentiable on G.

An example

Example

If $f(z) = \frac{x^3 y(y-ix)}{x^6+y^2}$ $(z \neq 0), f(0) = 0$, prove that $\frac{f(z)-f(0)}{z-0} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in any manner.

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Proof.

Let $z \to 0$ along $y = mx$ (radius vector). Then we have

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{z\to 0} \frac{x^3y(y-ix)}{(x^6+y^2)(x+iy)} = \lim_{x\to 0} \frac{x^3mx(mx-ix)}{(x^6+m^2x^2))(x+imx)}$$

$$= \lim_{x\to 0} \frac{m(m-i)\cdot x^2}{(m^2+x^4)(1+im)} = 0.$$

Now, let $z \to 0$ along the path $y = x^3$. Then, for $x \neq 0$

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{x\to 0} \frac{x^6\left(x^3-ix\right)}{(x^6+x^6)(x+ix^3)} = \lim_{x\to 0} \frac{\left(x^2-i\right)}{2\left(1+ix^2\right)} = -\frac{i}{2}.$$

Differentiability

Theorem

If $f : G \to C$ is differentiable at a point z_0 in G, then f is continuous at z_0.

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Proof.

Consider the following identity:

$$\begin{aligned}\lim_{z\to z_0} |f(z) - f(z_0)| &= \left[\lim_{z\to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|}\right] \cdot \left[\lim_{z\to z_0} |z - z_0|\right] \\ &= f'(z_0) \cdot 0 \\ &= 0\end{aligned}$$

that is, $\lim_{z\to z_0} f(z) = f(z_0)$. Thus it follows that $f(z)$ is continuous at z_0. This completes the proof. □

Differentiability

The converse of the above theorem is not necessarily true.
For example, take the function $|z|^2$ which is continuous in all finite regions of the z-plane. It has, however, a derivative only at the origin, since, when $z \neq z_0$ and $z_0 \neq 0$, we have, for $f(z) = |z|^2$,

$$\begin{aligned}
\frac{f(z)-f(z_0)}{z-z_0} &= \frac{|z|^2-|z_0|^2}{z-z_0} = \frac{z\bar{z}-z_0\bar{z}_0}{z-z_0} \\
&= \frac{z\bar{z}-z_0\bar{z}+z_0\bar{z}-z_0\bar{z}_0}{z-z_0} = \bar{z}+z_0\frac{\bar{z}-\bar{z}_0}{z-z_0} \\
&= \bar{z}+z_0\frac{\rho(\cos\theta-i\sin\theta)}{\rho(\cos\theta+i\sin\theta)} = \bar{z}+z_0(\cos 2\theta - i\sin 2\theta),
\end{aligned}$$

where $\rho = |z-z_0|$ and $\theta = \arg(z-z_0)$. Clearly, $\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$ does not exist since the limit depends upon $\arg(z-z_0)$. However, when $z_0 = 0$, the expression reduces to $\bar{z}$ which tends to 0 with z tends to 0 .

Definition

Definition

- The function f is analytic at z_0 if $f(z)$ is differentiable in some neighborhood of z_0 (open region including z_0);
- The function f is analytic in a region if it is analytic at all points in that region;
- The function f is holomorphic if it is analytic. The terms are synonyms.
- An analytic function is entire if its region of analyticity includes all points in $\mathbb{C}$, the finite complex plane, excluding infinity.

If we describe a function as analytic, without specifying any point or region, that means there is some region within which it is analytic.

Rules of Differentiation

Theorem

If f and g are analytic on G, where $g(z) \neq 0$, then

1. $(f \pm g)'(z) = f'(z) \pm g'(z)$.
2. $(cf)'(z) = cf'(z)$, *where c is a complex constant.*
3. $(f \cdot g)'(z) = f(z) \cdot g'(z) + g(z) \cdot f'(z)$.
4. $\left(\frac{f}{g}\right)'(z) = \frac{g(z)f'(z) - f(z)g'(z)}{[g(z)]^2}$ *where* $g(z) \neq 0$.

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4. $\left(\frac{f}{g}\right)'(z) = \frac{g(z)f'(z) - f(z)g'(z)}{[g(z)]^2}$ *where $g(z) \neq 0$.*

Theorem (Chain Rule)

If f and g are analytic on G and Ω, respectively, and suppose $f(G) \subset \Omega$, then $g \circ f$ is analytic on G and for all z in G,

$$(g \circ f)'(z) = g'(f(z))f'(z).$$

The chain rule shows that an analytic function of an analytic function is analytic.

Examples

Example

Show that the function $f(z) = z^n$ where n is a positive integer is an analytic function. Furthermore, polynomials and rational Functions are analytic functions.

Cauchy-Riemann Equations

Theorem

A necessary condition for a function $f(z) = u(x, y) + iv(x, y)$ *to be analytic at any point* $z = x + iy$ *of the domain* D *of* f *is that the four partial derivatives* u_x, u_y, u_y *and* v_x *should exist and satisfy the equation*

$$u_x = v_y, \quad u_y = -v_x. \tag{1}$$

The equations given in (1) are known as the Cauchy–Riemann equations.

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Example

Show that the function $f(z) = u + iv$, where

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}\,(z \neq 0), \quad f(0) = 0,$$

is continuous and the Cauchy-Riemann equations are satisfied at the origin, but $f'(0)$ does not exist.

Sufficient conditions

Theorem

The one-valued function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D if the four partial derivatives u_x, v_x, u_y and v_y exist, are continuous and satisfy the Cauchy-Riemann equations at each point D.

$$f'(z) = u_x + iv_x = v_y - iu_y.$$

Conjugate Functions

Definition

If a function $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ is analytic in a domain D, then the functions u and v of two variables x and y are called conjugate functions.