常用的连续型随机变量

3. 正态分布

3. 正态分布　　如果连续型随机变量 X 的密度函数为

$$f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\qquad(-\infty<x<+\infty)$$

(其中 $-\infty<\mu<+\infty$，$\sigma>0$ 为参数)，

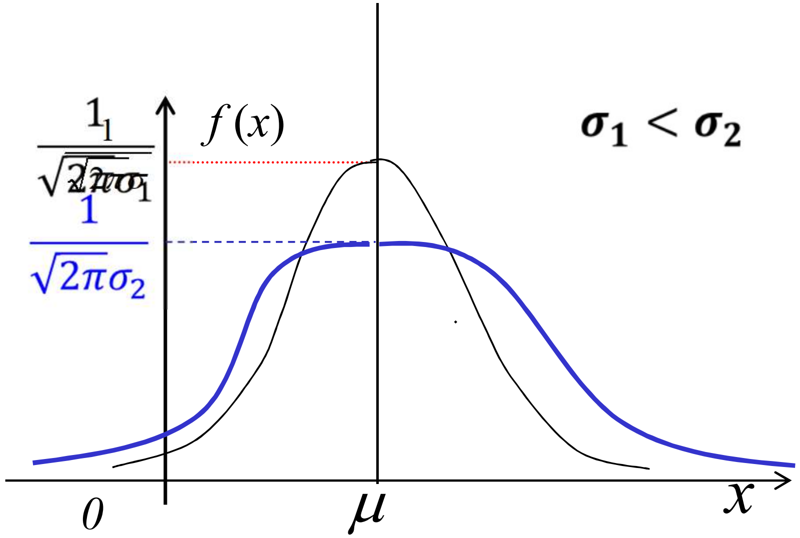
则称随机变量 X 服从参数为 μ、σ^2 的正态分布，记为：$X\sim N(\mu,\ \sigma^2)$

- **μ**: 位置参数
- **σ**: 形状参数

分布函数

$$F(x)=\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx$$

分布函数不是初等函数，计算概率存在困难！



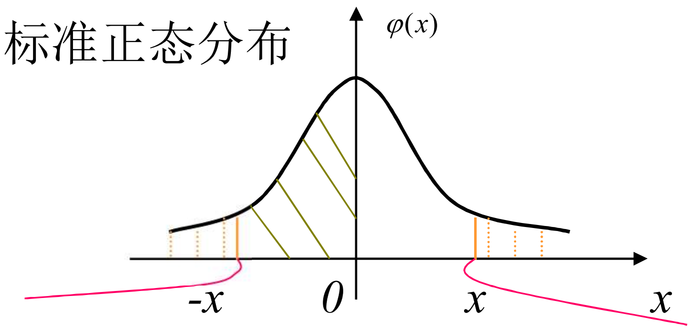
标准正态分布

$$\boxed{f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad (-\infty < x < +\infty)}$$

若$\mu=0$，$\sigma=1$，我们称$N(0, 1)$为标准正态分布

标准正态分布的密度函 数为

$$\varphi(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}},\ x\in R.$$



其分布函数为

$$\Phi(x)=\int_{-\infty}^{x}\varphi(t)dt=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{t^2}{2}}dt \qquad (-\infty < x < +\infty)$$

标准正态性质

1. $\Phi(0)=1/2$ $\boxed{\varphi(0)}$

2. $\Phi(-x)=1-\Phi(x)$

3. 对于$x\geq 0$我们可直接查表求出 $\Phi(x)=P\{X\leq x\}$

例4　设随机变量$X \sim N(0, 1)$，试求：

$(1) P\{1 \le X < 2\}$；$(2) P\{-1 < X < 2\}$；$(3) P(X > -1.96)$

解：$(1).\ P(1 \le X < 2) = \Phi(2) - \Phi(1)$

$$= 0.97725 - 0.84134 = 0.13591$$

$(2).\ P(-1 \le X < 2) = \Phi(2) - \Phi(-1)$

$$= \Phi(2) - [1 - \Phi(1)]$$

$$= 0.97725 - 1 + 0.84134 = 0.81859$$

$(3).\ P(X > -1.96) = 1 - \Phi(-1.96)$

$$= \Phi(1.96) = 0.975$$

一般正态分布的计算

定理　设$X \sim N(\mu, \sigma^2)$，则$Y = \dfrac{X-\mu}{\sigma} \sim N(0,1)$

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

证明：$\therefore F_Y(y) = P\{Y \le y\} = P(\dfrac{X-\mu}{\sigma} \le y)$

$$= P\{X \le \sigma y + \mu\}$$

$$= F_X(\sigma y + \mu)$$

$$= \int_{-\infty}^{\sigma y+\mu} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

1) 令　$t = \dfrac{x-\mu}{\sigma}$　，则 $dx = \sigma dt$

$$F_Y(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{t^2}{2}} \sigma\, dt = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

2）$f_Y(y) = F_Y'(y) = \dfrac{dF_Y(y)}{dy} = \dfrac{dF_X(\sigma y+\mu)}{dy}$

$$= \frac{dF_X(\sigma y+\mu)}{d(\sigma y+\mu)} \cdot \frac{d(\sigma y+\mu)}{dy} = f_X(\sigma y+\mu) \cdot \sigma$$

$$= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\sigma y+\mu-\mu)^2}{2\sigma^2}} \cdot \sigma = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}},\quad y \in R.$$

一般正态分布的计算（续）

$$F_X(x) = P\{X \le x\} = P\{\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\} = \Phi(\frac{x-\mu}{\sigma})$$

$$P(X > x) = 1 - F(x) = 1 - \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$$\mathrm{P}\{\mathrm{a} < \mathrm{X} < \mathrm{b}\} = F(b) - F(a) = \Phi(\frac{\mathrm{b}-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})$$

例5 设随机变量 $X \sim N(2,\ 9)$，试求：

(1). $P\{X > 0\}$; (2). $P\{1 \le X < 5\}$; (3). $P\{|X-2| > 6\}$.

解： (1) $P\{X > 0\} = 1 - \Phi(\frac{0-2}{3}) = 1 - \Phi\left(-\frac{2}{3}\right)$

$$= \Phi\left(\frac{2}{3}\right) = 0.7486$$

例5续

$$X \sim N(2,\ 9)$$

$$(2)\ P\{1 \le X < 5\} = \Phi(\frac{5-2}{3}) - \Phi(\frac{1-2}{3})$$

$$= \Phi(1) - \Phi\left(-\frac{1}{3}\right) = \Phi(1) + \Phi\left(\frac{1}{3}\right) - 1$$

$$= 0.84134 + 0.62930 - 1 = 0.47064$$

$$(3)\ P\{|X-2| > 6\} = P(X-2 \ge 6) + P(X-2 \le -6)$$

$$= P(X \ge 8) + P(X \le -4)$$

$$= 1 - \Phi(\frac{8-2}{3}) + \Phi(\frac{-4-2}{3}))$$

$$= 1 - \Phi(2) + \Phi(-2)$$

$$= 2 \times [1 - \Phi(2)] = 2 \times (1 - 0.97725) = 0.0455$$

$\boxed{3\sigma-\text{原则}}$

已知 $X \sim N(\mu,\ \sigma^2)$

$X \sim N(1.72,\ 0.15^2)$

$[1.27,2.17]$

$$P(|X-\mu|<\sigma)=P\left(\left|\frac{X-\mu}{\sigma}\right|<1\right)$$

$$=\Phi(1)-\Phi(-1)=2\Phi(1)-1$$

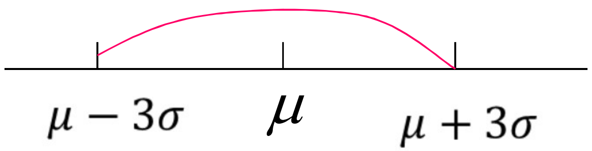
$$=0.6826$$

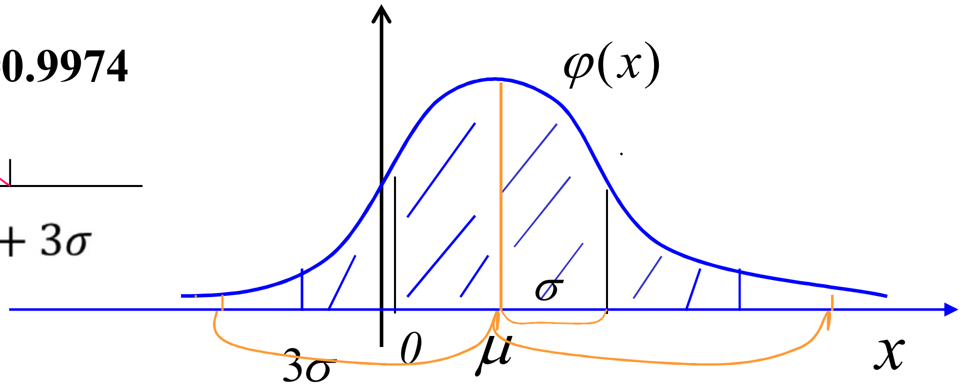
$$P(|X-\mu|<2\sigma)=\Phi(2)-\Phi(-2)=2\Phi(2)-1$$

$$=0.9544$$

$$P(|X-\mu|<3\sigma)=\Phi(3)-\Phi(-3)=2\Phi(3)-1$$

$$=\mathbf{0.9974}$$





例6 已知 $X \sim N(109,9)$ ，试确定 b 满足

1) $P(X > b) = 0.1$ 2) $P(|X-b| \ge b) = 0.9$

解 1) $P(X > b) = 1 - \Phi(\frac{b-109}{3}) = 0.1$

$$\Phi(\frac{b-109}{3}) = 0.9$$

$$\frac{b-109}{3} = 1.28$$

2) $P(|X-b| \ge b) = P(X - b \ge b) + P(X - b \le -b)$

$$= P(X \ge 2b) + P(X \le 0)$$

$$= 1 - \Phi(\frac{2b-109}{3}) + \Phi(\frac{0-109}{3}) = 0.9$$

$$\Phi(\frac{2b-109}{3}) = 0.1 \Rightarrow \frac{2b-109}{3} = -1.28$$

例 7 某单位招工１５５人，现有５２３人应召，用人单位进行统一考核决定考试成绩由高到低录用，假设考试成绩服从正态布，考试结果有１２人超过９０，５３人没到６０分，如果某人考试得了７８分，问他能否被录用。

$\boxed{\dfrac{155}{523}}$

解：令X表示考试成绩，则 $X \sim N\left(\mu, \sigma^2\right)$

$$P(X>90)=\frac{12}{523} \Rightarrow 1-\Phi\left(\frac{90-\mu}{\sigma}\right)=\frac{12}{523}$$

$\boxed{\dfrac{12}{523}}$

$$\Rightarrow \Phi\left(\frac{90-\mu}{\sigma}\right)=0.977 \Rightarrow \boxed{\frac{90-\mu}{\sigma}=2}$$

$\boxed{\dfrac{53}{523}}$

$$P(X<60)=\frac{53}{523} \Rightarrow \Phi\left(\frac{60-\mu}{\sigma}\right)=\frac{53}{523}$$

$$\Rightarrow \Phi\left(-\frac{60-\mu}{\sigma}\right)=1-\Phi\left(\frac{60-\mu}{\sigma}\right)=0.8987 \Rightarrow \boxed{\frac{60-\mu}{\sigma}=-1.27}$$

$$\Rightarrow \mu=71.2; \quad \sigma=9.4$$

$\boxed{X \sim N(71.2,\ 9.4^2)}$ $\boxed{\frac{155}{523}}$

设录取分数为a,则

$$P(X \geq a) = \frac{155}{523} \longrightarrow 1 - \Phi\left(\frac{a-71.2}{9.4}\right) = \frac{155}{523}$$

$$\Phi\left(\frac{a-71.2}{9.4}\right) = 1 - \frac{155}{523} = 0.7036$$

$$\frac{a-71.2}{9.4} \approx 0.53$$

$$a \approx 76.2 < 78$$

录用