Methods of Mathematical Physics

—Lecture 1 Introduction—

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1 Introduction

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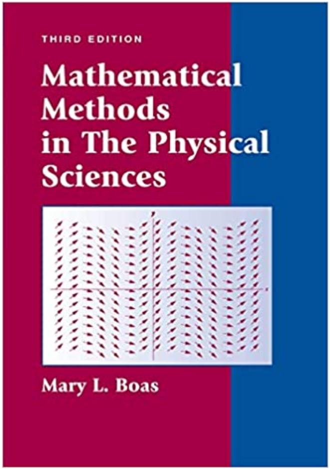
General Information

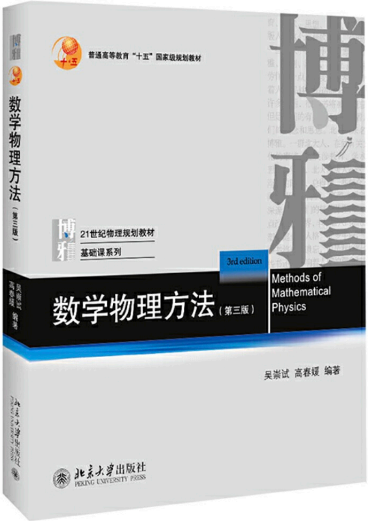
- Class schedule: 10:05 AM-11:40; Tue & Thur. (64 credits)
- Instructor: Lei Du [http://faculty.dlut.edu.cn/dulei]
- TA:
- QQ Group ID: 578890920
- Evaluation: 60% Final exam, 30% homework, 10% Class participation and attendance

Textbooks


THIRD EDITION
Mathematical Methods in The Physical Sciences
Mary L. Boas

Textbooks

Main Textbook



Mathematical Physics

1. What is mathematical physics?
2. Why should we learn from mathematical physics?

What is Complex Analysis?

1. In real analysis, one studies calculus in the setting of real numbers. Thus one studies concepts such as the convergence of real sequences, continuity of real-valued functions, differentiation and integration.

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What is Complex Analysis?

1. In real analysis, one studies calculus in the setting of real numbers. Thus one studies concepts such as the convergence of real sequences, continuity of real-valued functions, differentiation and integration.
2. In complex analysis, does one study similar concepts in the setting of complex numbers? Partly true!!!.

   The subject of complex analysis departs radically from real analysis when one studies differentiation. Complex analysis is the study of "complex differentiable" functions.

What is Complex Analysis?

Example

Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \begin{cases} x^2 \text{ if } x \geq 0 \\ -x^2 \text{ if } x < 0 \end{cases}$.

Figure: Graphs of the functions f and its derivative f'.

Obvious, f is differentiable everywhere, but f is not differentiable at 0.

What is Complex Analysis?

In contrast, we will later learn that if $F : \mathbb{C} \to \mathbb{C}$ is a complex differentiable function in $\mathbb{C}$, then it is infinitely many times complex differentiable!!!

The reason this miracle takes place in complex analysis is that complex differentiability imposes some "rigidity" on the function which enables this phenomenon to occur.

Why study complex analysis?

Complex analysis is fundamental in all of mathematics, and plays an important role in the applied sciences as well. A main purpose of extending from the real axis to the complex plane is to greatly simplify a vast number of tasks in applied mathematics, engineering, etc. The renowned mathematician Jacques Hadamard expressed this succinctly:

"The shortest path between two truths in the real domain passes through the complex domain."

Here is a list of a few reasons to study complex analysis:

- **PDEs**. If $f : \mathbb{C} \to \mathbb{C}$ is a complex differentiable function in $\mathbb{C}$, then we have two associated real-valued functions $u, v : \mathbb{R}^2 \to \mathbb{R}$, namely the real and imaginary parts of f: for $(x, y) \in \mathbb{R}^2$, $u(x, y) := \mathrm{Re}(f(x, y))$ and $v(x, y) := \mathrm{Im}(f(x, y))$. It turns out that u, v satisfy an important basic PDE, called the Laplace equation:

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Why study complex analysis?

Similarly $\Delta v = 0$ in $\mathbb{R}^2$ as well. The Laplace equation itself is important because many problems in applications, for example, in physics, give rise to this equation. It occurs for instance in electrostatics, steady-state heat conduction, incompressible fluid flow, Brownian motion, etc.

- **Real analysis**. Using complex analysis, we can calculate some integrals in real analysis, for example
$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx \quad \text{or} \quad \int_0^{\infty} \cos\left(x^2\right) dx.$$
- **Applications**. Many tools used for solving problems in applications, such as the Fourier/Laplace/z-transform, rely on complex function theory. These tools in turn are useful to solve differential equations. Complex analysis plays an important in applied subjects such as mathematical physics and engineering, for example in control theory, signal processing and so on.

Why study complex analysis?

- **Analytic number theory**. Many questions about the natural numbers can be answered using complex analytic tools. For example,

Theorem (Prime Number Theorem)

$$\lim_{n\to\infty} \frac{\pi(n)}{n/(\log n)} = 1,$$

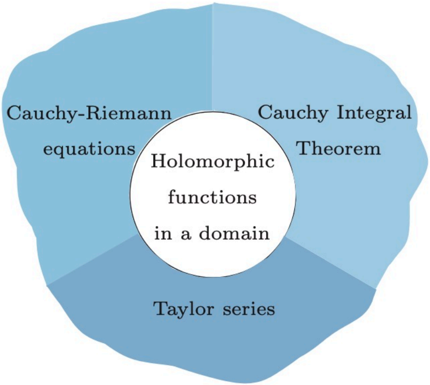
where $\pi(n)$ denotes the number of primes less than n for large n.

The Prime Number Theorem can be proved by using complex analytic computations with a certain complex differentiable function called the Riemann zeta function[1] $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$.

[1] Associated with the Riemann zeta function is also a famous unsolved problem in analytic number theory, namely the Riemann Hypothesis, saying that all the "nontrivial" zeros of the Riemann zeta function lie on the line $\mathrm{Re}(s) = \frac{1}{2}$ in the complex plane.

What will we learn in Complex Analysis

The central object of study in complex analysis will be holomorphic functions in a domain, that is, complex differentiable functions $f\colon D \to \mathbb{C}$, where D is a "domain".



1 Introduction

2 Complex Numbers

Different types of numbers: Integers, rational numbers, and real numbers

- For a very long time, the only numbers used were positive integers, in modern notation 1, 2, 3, ...
- The earliest number system extension was to allow for rational numbers, such as 3/7 or 11/8. These are clearly useful in measuring things like distances and weights.
- Zero and negative numbers originally made little sense. However, much algebra—such as subtraction—became much easier if these were allowed.
- The Greeks, over 2000 years ago, noted that even the rational numbers were insufficient to measure all distances, such as the length of the diagonal of a square with side length one. The discovery of the irrational numbers which, together with rational numbers, form the set of real numbers was kept secret for quite some time.

Historical development of complex numbers

- The phrase "complex number" first occurs in the literature in 1831 with the work of C.F. Gauss.
- Contrary to popular belief, historically, it wasn't the need for solving quadratic equations, but rather cubic equations, that led mathematicians to take complex numbers seriously. Girolamo Cardano (1501-1576), an Italian mathematician, described a formula for solving a general cubic equation.
- L. Euler (1707-1783) was the first mathematician to introduce the symbol i with the property that $i^2 = 1$, i.e., $i = \sqrt{-1}$, and called i the imaginary unit. Subsequently, C.F. Gauss (1777-1855) studied algebraic equations with real coefficients having complex roots of the form $a + ib$, where a and b are real numbers.

Historical development of complex numbers

Around the 16th century, one viewed solving equations like

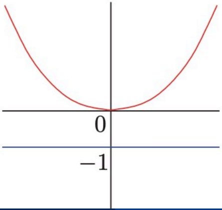
$$ax^2 + bx + c = 0$$

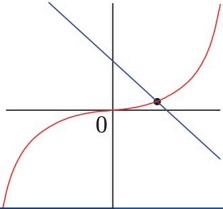
as the geometric problem of finding the intersection point of the parabola $y = x^2$ with the line $y = -bx - c$. It was easy to dismiss the lack of solvability in reals of a quadratic such as $x^2 + 1 = 0$.

Meanwhile, Cardano gave a formula for solving the cubic $x^3 = 3px + 2q$,

$$x = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}}.$$

The cubic $y = x^3$ always intersects the line $y = 3px + 2q$.





Historical development of complex numbers

But for an equation like $x^3 = 15x + 4$, that is when $p = 5$ and $q = 2$, we have $q^2 - p^3 = -121 < 0$, and so Cardano's formula fails with real arithmetic, but we do have a real root, namely $x = 4$:

$$4^3 = 64 = 60 + 4 = 15 \cdot 4 + 4.$$

Three decades after the appearance of Cardano's work, Rafael Bombelli (1526-1572) suggested that maybe with the use of complex arithmetic, Cardano's formula would give the desired real root.

$$x = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i} \stackrel{?}{=} 4$$

One can check that $(2 + i)^3 = 2 + 11i$ and $(2 - i)^3 = 2 - 11i$.

Thus Bombelli's work established that even for real problems, complex arithmetic might be relevant. From then on, complex numbers entered mainstream mathematics.

Definitions of complex numbers

- A number of the form $z = x + iy$, where x, y are real numbers[2], is called a complex number.
- If $x = 0$, i.e., $z = iy$, then z is called a pure imaginary number and by iy we mean y units of an imaginary number[3].
- By definition, a complex number is an ordered pair (x, y) of real numbers x and y. For example, $(1,0),(0,1),(0,0),(1,-2)$ are all complex numbers.
- Denote the set of all real numbers by the symbol $\mathbb{R}$. The set $\mathbb{R}\times\mathbb{R}$ of all complex numbers is denoted by $\mathbb{C}$. Thus $\mathbb{C} = \{z = x + iy : x \in \mathbb{R}, y \in \mathbb{R}\}$.

[2] $x = \mathrm{Re}\, z$ and $y = \mathrm{Im}\, z$ are called real and imaginary parts of z respectively.

[3] It is customary to write the complex number $x + i \cdot 0$ as just x, i.e., to treat it as a real number. Although this does not make any difference in analysis, a conceptual distinction must be drawn between the real number x and the complex number $x + i \cdot 0$.

Two operations on $\mathbb{C}$

Let $x_1 + iy_1$ and $x_2 + iy_2$ be two complex numbers. We define the operations of addition "$+$" and multiplication "$\cdot$" on $\mathbb{C}$ by:[4]

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2),$$
$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1),$$

for complex numbers $(x_1, y_1), (x_2, y_2)$.

[4] Assuming that the ordinary rules of arithmetic apply to complex numbers, we find, indeed, the above two identities hold.

Observation

- It is fundamental that real and complex numbers obey the same basic laws of arithmetic.
- Zero is the only number which is at once real and purely imaginary.
- Two complex numbers are equal if and only if they have the same real part and the same imaginary part.
- The phrases "greater than" or "less than" have no meaning in the set of complex numbers.

Fundamental Laws of Addition and Multiplication

If z_1, z_2, z_3 denote complex numbers, then we following:

- Commutative law of addition: $z_1 + z_2 = z_2 + z_1$.
- Associative law of addition: $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$.
- Commutative law of multiplication: $z_1 z_2 = z_2 z_1$.
- Associative law of multiplication: $z_1(z_2 z_3) = (z_1 z_2) z_3$.
- Distributive law of multiplication: $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$.
- Additive identity: $(x, y) + (0, 0) = (x + 0, y + 0) = (x, y)$. Therefore, the complex number $(0, 0)$ is the additive identity and is called the null or zero of the system of complex numbers.
- Additive inverse: $(x, y) + (-x, -y) = (x - x, y - y) = (0, 0)$. The number $(-x, -y)$ is the additive inverse of (x, y) and is called the negative of the complex number (x, y) and we write $-(x, y) = (-x, -y)$.

Fundamental Laws of Addition and Multiplication

- Multiplicative identity: $(x, y) \cdot (1, 0) = (x \cdot 1 - y \cdot 0, x \cdot 0 + y \cdot 1) = (x, y)$. The complex number $(1, 0)$ is the multiplicative identity and is called unity or one of the system of complex numbers.
- Multiplicative inverse: The complex number $\left(x', y'\right)$ is called the inverse of the complex number (x, y) if

$$(x, y) \cdot \left(x', y'\right) = (1, 0),$$

i.e., if $\left(xx' - yy', xy' + x'y\right) = (1, 0)$, then

$$xx' - yy' = 1, \quad xy' + x'y = 0.$$

These give

$$x' = \frac{x}{x^2 + y^2}, \quad y' = \frac{-y}{x^2 + y^2},$$

provided $(x, y) \neq (0, 0)$.

Difference and Division of Two Complex Numbers

The difference of two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ is defined by the following equality

$$\begin{aligned} z_1 - z_2 &= z_1 + (-z_2) \\ &= (x_1, y_1) + (-x_2, -y_2) \\ &= (x_1 + (-x_2), y_1 + (-y_2)) \\ &= (x_1 - x_2, y_1 - y_2). \end{aligned}$$

The division of a complex number z_1 with z_2 is defined by the following equality

$$\begin{aligned} \frac{z_1}{z_2} &= z_1 z_2^{-1} = (x_1, y_1)(x_2, y_2)^{-1} \\ &= (x_1, y_1)\left(\frac{x_2}{x_2^2 + y_2^2}, -\frac{y_2}{x_2^2 + y_2^2}\right) \\ &= \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}, \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}\right) \end{aligned}$$

provided $x_2^2 + y_2^2 \neq 0$.

Geometric representations of complex numbers

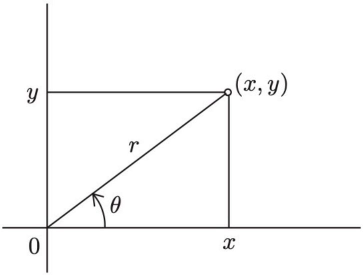


Figure: Cartesian and polar representations of $z = x + iy = r(\cos\theta + i\sin\theta)$ in plane.

Geometric meaning of complex addition

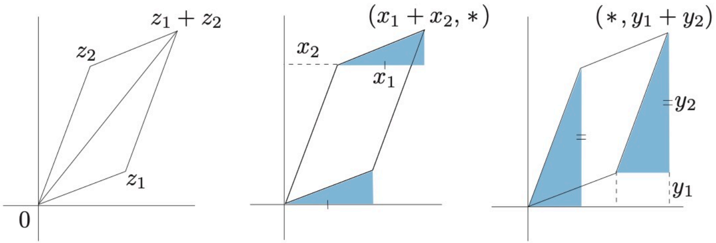


Figure: Addition of two complex numbers.

Geometric meaning of complex subtraction

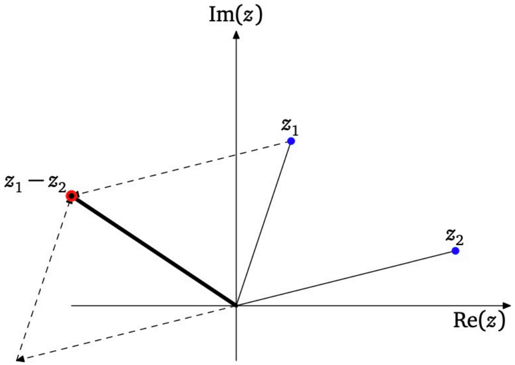


Figure: Subtraction of two complex numbers.

Geometric meaning of complex multiplication

For two complex numbers expressed in polar coordinates as

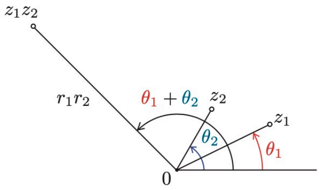
$$z_1 = r_1 (\cos\theta_1 + i \sin\theta_1), z_2 = r_2 (\cos\theta_2 + i \sin\theta_2),$$

we have that

$$\begin{aligned} z_1 \cdot z_2 &= r_1 (\cos\theta_1 + i \sin\theta_1) \cdot r_2 (\cos\theta_2 + i \sin\theta_2) \\ &= r_1 r_2 (\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 + i(\cos\theta_1 \sin\theta_2 + \cos\theta_2 \sin\theta_1)) \\ &= r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)). \end{aligned}$$

Thus,

$$|z_1 z_2| = |z_1||z_2| \text{ and } \arg(z_1 z_2) = \arg(z_1) + \arg(z_2).$$



Geometric meaning of complex multiplication: a special case

Consider multiplication by $\cos\alpha + i\sin\alpha$, which is at a distance of 1 from the origin. If $z \in \mathbb{C}$, then $z \cdot (\cos\alpha + i\sin\alpha)$ is obtained by rotating the line joining 0 to z anticlockwise through an angle of α. In particular, multiplying z by

$$i = 0 + i \cdot 1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

produces a counterclockwise rotation of 90°.

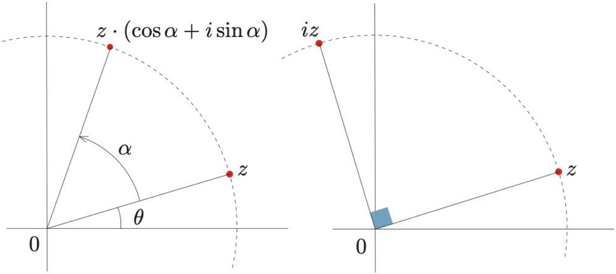


Figure: Multiplication by $\cos\alpha + i\sin\alpha$ produces an anticlockwise rotation through α.

De Moivre's formula and nth roots

We have for all $n \in \mathbb{N}$

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

This is called de Moivre's formula.

De Moivre's formula gives an easy way of finding the nth roots of a complex number z, that is, complex numbers w that satisfy $w^n = z$.
Let $z = r(\cos\theta + i\sin\theta)$ for some $r \geq 0$ and $\theta \in [0, 2\pi)$. If $w^n = z$, where $w = \rho(\cos\alpha + i\sin\alpha)$, then

$$w^n = \rho^n(\cos(n\alpha) + i\sin(n\alpha)) = r(\cos\theta + i\sin\theta) = z.$$

Thus $\rho = \sqrt[n]{r}$. On the other hand, the angle that w^n makes with the positive real axis is $n\alpha$, which is in the set $\{\cdots, \theta - 4\pi, \theta - 2\pi, \theta, \theta + 2\pi, \theta + 4\pi, \theta + 6\pi, \cdots\}$, i.e., $\theta + 2\pi k$ for any integer k.

De Moivre's formula and nth roots

Thus we get that $\alpha \in \left\{\frac{\theta}{n} + \frac{2\pi}{n}k : k \in \mathbb{Z}\right\}$, and this gives distinct w for $\alpha \in \left\{\frac{\theta}{n}, \frac{\theta}{n} + \frac{2\pi}{n}, \frac{\theta}{n} + 2 \cdot \frac{2\pi}{n}, \ldots, \frac{\theta}{n} + (n-1) \cdot \frac{2\pi}{n}\right\}$.

De Moivre's formula and nth roots

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In particular, if $z = 1$, we get the nth roots of unity, which are located at the vertices of an n-sided regular polygon inscribed in a circle.

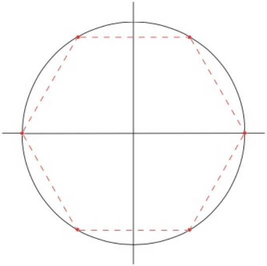

Figure: The six 6th roots of unity.

Division of two complex numbers

For two complex numbers expressed in polar coordinates as

$$z_1 = r_1\left(\cos\theta_1 + i\sin\theta_1\right), z_2 = r_2\left(\cos\theta_2 + i\sin\theta_2\right),$$

we have that

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right).$$

Thus,

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \text{ and } \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2).$$

Modulus and Argument of Complex Numbers

Let $z = x + iy$ be any complex number. If $x = r\cos\theta, y = r\sin\theta$, then $+\sqrt{x^2 + y^2}$ is called the modulus[5] of z, written as $|z|$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ is called the argument or the amplitude of z, written as $\arg z$ or $\operatorname{amp} z$. Geometrically, $|z|$ is the distance of the point z from the origin. Also,

$$\mathrm{Re}(z) = x \leq \sqrt{x^2 + y^2} = |z|.$$

Note that the modulus of any complex number is a single-valued function of its real and imaginary parts, whereas the argument is not.
The value of the argument that lies between $-\pi$ and π, i.e.,

$$-\pi < \theta \leq \pi, \text{ or } -\pi \leq \theta < \pi$$

is called the principal value of the argument. In general, when we write $\arg z$, we mean the principal value of the argument of z.

[5] K. Weierstrass calls the modulus of $x + iy$ as the absolute value of $x + iy$ and writes it as $|x + iy|$.

Complex conjugate

The complex conjugate $\bar{z}$ of $z = x + iy$ where $x, y \in \mathbb{R}$, is defined by

$$\bar{z} = x - iy.$$

It is easy to check the following properties

$$\bar{\bar{z}} = z, \quad z\bar{z} = |z|^2, \quad \mathrm{Re}(z) = \frac{z + \bar{z}}{2}, \quad \mathrm{Im}(z) = \frac{z - \bar{z}}{2i},$$

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} \quad \text{and} \quad \overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}.$$

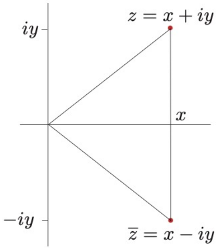

Figure: the complex conjugate is obtained by reflecting z in the real axis.

Stereographic projection

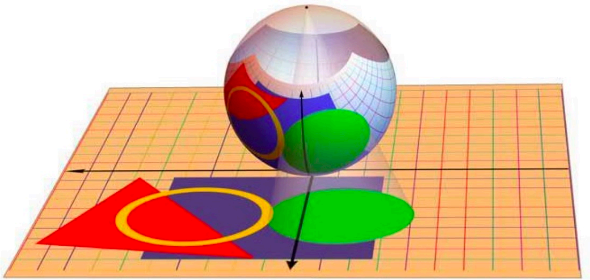

Figure: The stereographic projection of the lines $\text{Re}z = 0$ and $\text{Im}z = 0$ (in bold) and two arbitrary circles (viewed from a location above the complex plane, with $\text{Im}z$ large and positive). The North Pole (denoted by a black dot) maps to infinity, and the South Pole maps to zero.

Stereographic projection

The correspondence between points $z = x + iy$ in the plane and points (X, Y, Z) on the sphere (satisfying $X^2 + Y^2 + (Z-1)^2 = 1$) as follows:

- Plane $\to$ Sphere:

$$X = \frac{4x}{x^2+y^2+4}, \ Y = \frac{4y}{x^2+y^2+4}, \ Z = \frac{2(x^2+y^2)}{x^2+y^2+4}.$$

- Sphere $\to$ Plane:

$$x = \frac{2X}{2-Z}, \ y = \frac{2Y}{2-Z}.$$

Stereographic projection

There is a trade-off in defining complex numbers as points on the sphere instead of on the plane:

- Lose the nice immediate illustrations of the arithmetic operations $+, , , /$.
- Gain a better way to think about divisions by zero.
- Maybe more importantly, the sphere helps in thinking about singularities, such as poles and branch points.

Stereographic projection

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This stereographic projection has the following nice properties:

- Locally, angles between intersecting curves are preserved between the plane and the sphere.
- A circle in the complex plane maps to a circle on the sphere.
- A circle on the sphere which contains the North Pole maps to a straight line in the plane.