Methods of Mathematical Physics

— Lecture 5 Singularities, Residue Theory —

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Contents

Zeros

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Let $f(z)$ be analytic in a domain D and let a be a point of D. Then $f(z)$ can be expanded as a Taylor series about $z = a$ in the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n, \text{ where } a_n = \frac{f^{(n)}(a)}{n!}.$$

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$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n, \text{ where } a_n = \frac{f^{(n)}(a)}{n!}.$$

If $f(a) = 0$, i.e., a is a zero of $f(z)$, we have $a_0 = 0$.

It may also happen that more of the coefficients a_n vanish. If $a_n = 0$ for $n < m$, but $a_m \neq 0$, then we have

$$\begin{aligned} f(z) &= \sum_{n=m}^{\infty} a_n (z-a)^n = \sum_{n=0}^{\infty} a_{n+m} (z-a)^{m+n} = (z-a)^m \sum_{n=0}^{\infty} a_{n+m} (z-a)^n \\ &= (z-a)^m \phi(z), \end{aligned}$$

where $\phi(z) = \sum\limits_{n=0}^{\infty} a_{n+m}(z-a)^n$ is analytic within the region of convergence of Taylor's expansion of $f(z)$ and $\phi(a) \neq 0$.

Zeros

In such case, we say that $f(z)$ has a zero of order m at $z = a$. A zero of order one is said to be a simple zero. If a is a zero of $f(z)$ of order m, then we have

$$f(a) = 0, \quad f'(a) = \cdots = f^{(m-1)}(a) = 0,$$

but $f^{(m)}(a) \neq 0$. This is obvious from Taylor's expansion formula.

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Theorem

Zeros are isolated points.

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Theorem

Zeros are isolated points.

Proof.

Let $f(z)$ be analytic in a domain D. Then we show that, unless $f(z)$ is identically zero, there exists a neighborhood of each point in D throughout which the function has no zero, except possibly at the point itself. Suppose that $f(z)$ has a zero of order m at a. Then, as above,

$$f(z) = (z-a)^m \sum_{n=0}^{\infty} a_{m+n}(z-a)^n = (z-a)^m \phi(z). \tag{1}$$

Now, we have

$$\phi(z) = \sum_{n=0}^{\infty} a_{m+n}(z-a)^n \text{ and } \phi(a) = a_m \neq 0.$$

Zeros

Since the series in (1) is uniformly convergent and each term of the series is continuous at a, it follows that $\phi(z)$, being a sum function, is also continuous at a. Hence, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|z - a| < \delta \Longrightarrow |\phi(z) - \phi(a)| < \epsilon. \tag{2}$$

Take $\epsilon = \frac{|a_m|}{2}$ and let δ_0 be the corresponding value of δ. Then (2) gives

$$|z - a| < \delta_0 \Longrightarrow |\phi(z) - a_m| = |\phi(z) - \phi(a)| < \frac{1}{2} |a_m| . \tag{3}$$

It follows that $\phi(a) \neq 0$ at any point in the neighborhood $|z - a| < \delta_0$. For, if $\phi(z) = 0$, then the equality (3) is contradicted. The argument remains valid when $m = 0$. In this case, the two functions ϕ and f are equal and $f(a) \neq 0$. □

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Definition (Singular Points)

A singular point of a function $f(z)$ is the point at which the function creases to be analytic.

For example, the function $f(z) = \frac{1}{z-1}$ has a singularity at $z = 1$.

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A singular point of a function $f(z)$ is the point at which the function creases to be analytic.

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Definition (Isolated Singularities)

A point a is said to be an isolated singularity of function $f(z)$ if $f(z)$ is analytic at each point in some neighborhood $|z - a| < \delta$ of a, except at the point a itself. Otherwise, it is called non-isolated.

Examples:

1. The function $f(z) = \frac{z+1}{z(z^2+2)}$ possesses three isolated singular points $z = 0, z = \sqrt{2}i$ and $z = -\sqrt{2}i$.
2. The function $\mathrm{Ln}\, z$ has a singularity at the origin, but it is not isolated since every neighborhood of zero contains points on the negative real axis where $\mathrm{Ln}\, z$ ceases to be analytic.

Definitions

Suppose f has an isolated singularity at $z = a$.

Definition (Removable Singularities)

If there a function g, analytic at a and such that $f(z) = g(z)$ for all x in some deleted neighborhood of a, we say that f has a removable singularity at a i.e., if the value of f is connected at the point $z = a$, it becomes analytic there.

Definition (Poles)

If, for $z = a$, $f(z)$ can be written as $f(z) = \frac{\phi(z)}{\psi(z)}$ where ϕ and ψ are analytic at $a, \phi(a) \neq 0$, and $\psi(a) = 0$, we say that f has a pole at a. In other words, if ψ has a zero of order m at a, we say that f has a pole of order m.

Definition (Essential Singularities)

If f has neither a removable singularity nor a pole at a, we say that f has an essential singularity at a.

Removable Singularities

Let $z = a$ be an isolated singularity of a function $f(z)$. Since the singularity is isolated, there exists a deleted neighborhood N_a defined by

$$0 < |z - a| < \delta$$

in which $f(z)$ is analytic. Then, by Laurent's theorem, we can expand $f(z)$ in a series of non-negative and negative powers of $(z - a)$ in N_a. Thus, with suitable definitions of a_n and b_n in the region N_a, we have

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} b_n(z-a)^{-n}.$$

The part $b_n(z-a)^{-n}$ of Laurent's series is called the principal part of $f(z)$ at $z = a$.

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Now, there arise three distinct possibilities:

1. **Removable Singularity**. If the principal part of $f(z)$ at $z = a$ consists of no terms, then a is said to be a removable singularity of $f(z)$.
   Alternative Definition. A singularity $z = a$ is called a removable singularity of $f(z)$ if $\lim_{z \to a} f(z)$ exists finitely.

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   Alternative Definition. A singularity $z = a$ is called a removable singularity of $f(z)$ if $\lim_{z \to a} f(z)$ exists finitely.
   For example, the function $f(z) = \frac{\sin z}{z}$ has a removable singularity at $z = 0$ since

$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

   has no term containing negative powers of z. However, this singularity can be removed and the function be made analytic by defining $\frac{\sin z}{z} = 1$ at $z = 0$.

Poles

2. **Pole**. If the principal part of a function $f(z)$ at $z = a$ consists of a finite number of terms, say m, we say that a is a pole of order m of $f(z)$. For example, if b_m is the last coefficient that does not vanish, then we have

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_m}{(z-a)^m}.$$

Poles of order 1 and 2 are called, respectively, single and double poles.
Alternate Definition. If there exists a positive integer m such that

$$\lim_{z \to a}(z-a)^m f(z) = b \neq 0, \text{ but } \lim_{z \to a}(z-a)^{m+1} f(z) = 0,$$

then $z = a$ is called a pole of order m.

Examples:

- Let $f(z) = \frac{1}{(z-1)^2(z-3)^5}$. Then $z = 1$ is a pole of order 2 and $z = 3$ is a pole of order 5.
- $\csc^2 z$ has one double pole and an infinite number of simple poles.

Isolated Essential Singularities

3. **Isolated Essential Singularity**. If the principal part of $f(z)$ at $z = a$ contains an infinite number of terms, then a is called an isolated essential singularity of $f(z)$. In such a case

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n},$$

the last series being convergent for all values of z in $|z-a| < \delta$ except at $z = a$.
Alternate Definition. If there exists no finite value of m such that

$$\lim_{z \to a} (z-a)^m f(z) = b = \text{ a finite non-zero constant,}$$

then $z = a$ is called an isolated essential singularity.

Examples:

- $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$ has an isolated essential singularity at $z = 0$.
- The function $f(z) = (z-3)\sin\frac{1}{z+2}$ has Laurent's expansion $f(z) = 1 - \frac{5}{z+2} - \frac{1}{6(z+2)^2} + \frac{5}{6(z+2)^3} + \frac{1}{120(z+2)^4} - \cdots$. Thus $z = -2$ is an essential singularity of $f(z)$.

Isolated Essential Singularities

We must take utmost care while classifying a given point a as an isolated essential singularity of a function $f(z)$ on the basis of Laurent's expansion of $f(z)$ in which the series of negative powers of $z - a$ does not terminate. It is important to bear in mind that the series should be convergent for all values of z in $|z - a| < \delta$, except at $z = a$, for some $\delta > 0$.

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For example, the following series contains an infinite number of terms in the principal part

$$\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}+\sum_{n=1}^{\infty}z^{-n}. \tag{4}$$

But on this ground alone, we should not declare that $z=0$ is an isolated essential singularity of the sum-function of the series (4). We must also test whether the series (4) converges in some deleted neighborhood of the origin, say, $0<|z|<\delta$.

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Since

$$\frac{1}{2-z} = \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}, \quad \frac{1}{z-1} = \sum_{n=1}^{\infty} z^{-n},$$

and the first series converges for $|z|<2$, while the second series converges for $|z|>1$. Thus, the domain of convergence of the series (4) is the annular region $1<|z|<2$, but it is not a neighborhood of the origin. Indeed, the sum-function of (4) in $1<|z|<2$ is given by

$$f(z) = \frac{1}{z-1} + \frac{1}{2-z} = \frac{1}{3z-2-z^2}.$$

$f(z)$ is a function of which the only singularities are the simple poles at $z=1$ and $z=2$.

Let us consider another example of the series

$$f(z) = \sum_{n=1}^{\infty} (z-1)^{-n}, \tag{5}$$

which gives an impression, at first sight, that the point $z = 1$ is an isolated essential singularity of the sum-function of the series (5). The crux of the matter is that the series converges for $|z-1| > 1$ and this does not define a neighborhood of 1 .

Indeed, the sum-function of (5), in the domain of its convergence, is $\frac{1}{z-2}$, which is analytic at $z = 1$ and of which the only singularity is the simple pole at $z = 2$.

Classification of singularities via limits

Suppose z_0 is an isolated singularity of $f(z)$. Then

z_0 *is removable*	$\Leftrightarrow$	$\lim\limits_{z\to z_0}(z-z_0)f(z)=0.$
z_0 *is a pole*	$\Leftrightarrow$	$(a)\ \neg\left(\lim\limits_{z\to z_0}(z-z_0)f(z)=0\right)$ *and* $(b)\ \exists n\in\mathbb{N}$ *such that* $\lim\limits_{z\to z_0}(z-z_0)^{n+1}f(z)=0.$ (*The smallest such* n *is called* the order *of the pole* z_0 *of* f.)
z_0 *is essential*	$\Leftrightarrow$	$\forall n\in\mathbb{N}\ \neg\left(\lim\limits_{z\to z_0}(z-z_0)^n f(z)=0\right).$

(Here $\neg$ is the symbol for negation, to be read as "it is not the case that".)

Classification via Laurent coefficients

Let

1. z_0 be an isolated singularity of $f(z)$, and
2. $f(z) = \sum\limits_{n\in\mathbb{Z}} c_n (z - z_0)^n$ for $0 < |z - z_0| < R$, for some $R > 0$.

Then

z_0 *is removable*	$\Leftrightarrow$	*For all* $n < 0$, $c_n = 0$
z_0 *is a pole*	$\Leftrightarrow$	*There exists an* $m \in \mathbb{N}$ *such that* (a) $c_{-m} \neq 0$ *and* (b) *for all* $n < -m$, $c_n = 0$ *Then the order of the pole* z_0 *is* m.
z_0 *is essential*	$\Leftrightarrow$	*There are infinitely many negative indices* n *such that* $c_n \neq 0$.

Introduction

The inspiration behind this section is the desire to obtain possible values for the integrals $\int_C f(z)dz$, where f is analytic inside the closed curve C and on C, except for a inside C.

- If f has a removable singularity at a, then it is clear that the integral will be zero.
- If $z = a$ is a pole or an essential singularity, then the answer is not always zero, but can be found with little difficulty.

In this section, we show the very surprising fact that Cauchy's residue theorem yields a very elegant and simple method for evaluation of such integrals.

The Residues at Singularities

We know that, in the neighborhood of an isolated essential singularity $z = a$, a single-valued analytic function $f(z)$ can be expanded in Laurent's series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}.$$

Thus the coefficient b_1, which is called the residue of $f(z)$ at $z = a$, is given by the formula

$$b_1 = \frac{1}{2\pi i} \int_{\gamma} f(z) dz,$$

where γ is any circle with center at a, which includes singularities of $f(z)$. We denote the residue of $f(z)$ at $z = a$ by

$$\mathrm{Res}_{z=a} f(z) \quad \text{or} \quad \mathrm{Res}(f, z_0).$$

If $z = a$ is a single pole, then we also have

$$b_1 = \lim_{z \to \alpha} (z-a) f(z).$$

The Residues at Singularities

A more general definition of the "residue" of a function $f(z)$ at a point is the following.

If $z = a$ is the only singularity of an analytic function $f(z)$ inside a closed contour C and

$$\frac{1}{2\pi i}\int_C f(z)dz$$

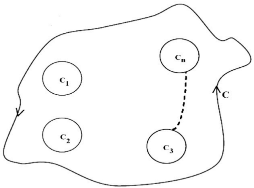
has a definite value, then the value is called the residue of $f(z)$ at $z = a$.

If C includes a finite number of singularities of $f(z)$ which is an analytic function elsewhere inside and on C, then the sum of the residues at singularities is given by

$$\frac{1}{2\pi i}\int_C f(z)dz.$$

The Residues at Singularities

If $f(z)$ is analytic in a multiply connected region bounded by and including the contours C and $C_1, C_2, \cdots, C_n$ contained within C as shown in the following figure.



then the sum of the residues of $f(z)$ at the included essential singularities is easily seen to be given by

$$\frac{1}{2\pi i}\left[\int_C f(z)dz - \sum_{r=1}^{n}\int_{C_r} f(z)dz\right].$$

Calculation of Residues in Some Special Cases

1. **Residues at Simple Poles**:
   When $f(z) = \frac{\phi(z)}{\psi(z)}$*, where* $\phi(a) \neq 0$ *and* $\psi(z)$ *has a simple zero at* $z = a$*.*
   Since $\psi(z)$ has a simple zero at $z = a, \psi(a) = 0$, but $\psi'(a) \neq 0$. Then it is evident that $f(z)$ has a simple pole at $z = a$. Therefore, we have

$$
\begin{aligned}
\mathrm{Res}_{z=a} f(z) = \mathrm{Res}_{z=a} \frac{\phi(z)}{\psi(z)} &= \lim_{z \to a} (z-a) \frac{\phi(z)}{\psi(z)} \\
&= \lim_{z \to a} \frac{\phi(z)}{\frac{\psi(z)-\psi(a)}{z-a}} = \frac{\phi(a)}{\psi'(a)}.
\end{aligned}
$$

Calculation of Residues in Some Special Cases

1. **Residues at Simple Poles**:
When $f(z) = \frac{\phi(z)}{\psi(z)}$, where $\phi(a) \neq 0$ and $\psi(z)$ has a simple zero at $z = a$.
Since $\psi(z)$ has a simple zero at $z = a, \psi(a) = 0$, but $\psi'(a) \neq 0$. Then it is evident that $f(z)$ has a simple pole at $z = a$. Therefore, we have

$$\begin{aligned}\mathrm{Res}_{z=a} f(z) = \mathrm{Res}_{z=a} \frac{\phi(z)}{\psi(z)} &= \lim_{z \to a} (z-a) \frac{\phi(z)}{\psi(z)} \\ &= \lim_{z \to a} \frac{\phi(z)}{\frac{\psi(z)-\psi(a)}{z-a}} = \frac{\phi(a)}{\psi'(a)}.\end{aligned}$$

2. **Residues of Poles of Order Greater than Unity**:
When $f(z)$ has a pole of order $m (m > 1)$ at $z = a$.
Laurent's expansion of $f(z)$ in the neighborhood of the point $z = a$ is given by

$$f(z) = \frac{b_m}{(z-a)^m} + \ldots + \frac{b_1}{z-a} + a_0 + a_1(z-a) + \cdots .$$

Hence we have

$$\begin{aligned}(z-a)^m f(z) =& b_m + b_{m-1}(z-a) + \cdots + b_1(z-a)^{m-1} \\ &+ a_0(z-a)^m + \cdots\end{aligned}$$

Differentiating both sides with respect to $z, (m-1)$ times, we have

$$D^{m-1}(z-a)^m f(z) = (m-1)! b_1 + m(m-1) \cdots 2 a_0 (z-a) + \cdots .$$

Taking the limit as $z \to a$, we have

$$\begin{aligned}(m-1)!b_1 &= \lim_{z\to a}\left[D^{m-1}\left\{(z-a)^m f(z)\right\}\right] \\ &= \lim_{z\to a}\left[\phi^{(m-1)}(z)\right] \\ &= \phi^{(m-1)}(a),\end{aligned}$$

where $f(z) = \frac{\phi(z)}{(z-a)^m}$. Hence we have

$$\mathrm{Res}_{z=a} f(z) = b_1 = \frac{\phi^{(m-1)}(a)}{(m-1)!}$$

In particular, if $\frac{\phi(z)}{(z-a)^2}$, then we have

$$\mathrm{Res}_{z=a} f(z) = \phi'(a).$$

If $\frac{\phi(z)}{(z-a)^3}$, then we have

$$\mathrm{Res}_{z=a} f(z) = \frac{\phi''(a)}{2!}$$

and so on.

Calculation of Residues in Some Special Cases

3. **Another Method**:
Since residue at $z=a$ is the coefficient of $\frac{1}{z-a}$ in Laurent's expansion of $f(z)$, it follows that the residue is the coefficient of $1/t$ in the expansion of $f(a+t)$ as a power series where t is considered sufficiently small.
When $f(z)=\frac{\phi(z)}{z\psi(z)}$*, where the numerator and the denominator have no common factor while* $\psi(0)\neq 0$*.*
In this case, $f(z)$ has a simple pole at the origin, due to the factor $\frac{1}{z}$, and $f(z)$ also has a number of simple poles arising from the zeros of $\psi(z)$. Hence we have

$$\mathrm{Res}_{z=0} f(z)=\frac{\phi(0)}{\psi(0)}.$$

Suppose that $z=a_m$ is a simple pole of $\frac{1}{\psi(z)}$. Then we have

$$\begin{aligned}\mathrm{Res}_{z=a_m} f(z) &= \lim_{z\to a_m}\left[\left(z-a_m\right)\frac{\phi(z)}{z\psi(z)}\right] \\ &= \frac{\phi\left(a_m\right)}{a_m\psi'\left(a_m\right)}\end{aligned}$$

provided $a_m\neq 0$.

Definition: Residues at Infinity

The definition of residue can be extended to include the point at infinity. If $f(z)$ is analytic or has an isolated essential singularity at infinity and C is a circle enclosing within it all other singularities of $f(z)$ in the finite regions of the z-plane, then the residue at infinity is defined by

$$\frac{1}{2\pi i}\int_C f(z)dz$$

where the integral is taken round C in the negative sense (clockwise direction), provided that this integral has a definite value.

If we take the integral round C in an anti-clockwise direction, then the residue at infinity is $-\frac{1}{2\pi i}\int_C f(z)dz$.

Calculation: Residues at Infinity

By means of the substitution $z = \frac{1}{Z}$, the integral defining the residue at infinity takes the form

$$\frac{1}{2\pi i}\int\left\{-f\left(\frac{1}{Z}\right)\right\}\frac{dZ}{Z^2}$$

taken in a counterclockwise direction round a sufficiently small circle with center at the origin. It follows that, if

$$\lim_{Z\to 0}\left\{-f\left(\frac{1}{Z}\right)Z^{-1}\right\} \text{ or } \lim_{z\to\infty}\{-zf(z)\}$$

has a definite value, then that value is the residue of $f(z)$ at infinity.

Some Residue Theorems

Theorem (Cauchy's Residue Theorem)

If $f(z)$ is regular, except at a finite number of poles $z_0, z_2, \cdots, z_n$ within a closed contour C where its residues are $R_1, R_2, \cdots, R_n$, respectively, and continuous on the boundary C, then

$$\int_C f(z)dz = 2\pi i(R_1 + R_2 + \cdots + R_n)$$

or

$$\int_C f(z)dz = 2\pi i \text{ (sum of residues at the poles within) } C.$$

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$$\int_C f(z)dz = 2\pi i(R_1 + R_2 + \cdots + R_n)$$

or

$$\int_C f(z)dz = 2\pi i \text{ (sum of residues at the poles within) } C.$$

Theorem

If a single-valued function has only a finite number of singularities, then the sum of residues at these singularities, including the residue at infinity, is zero.

Two Useful Theorems

Before proceeding to the evaluation of definite integrals, we prove two useful theorems.

Theorem

If C is the arc $\theta_1 \leq \theta \leq \theta_2$ of the circle $|z-a| = r$ and $\lim_{z \to a}(z-a)f(z) = A$, then

$$\lim_{r \to 0} \int_C f(z)dz = iA\left(\theta_2 - \theta_1\right).$$

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Proof.

Since $\lim_{z\to a}(z-a)f(z) = A$, it follows that, for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that

$$|(z-a)f(z) - A| < \epsilon$$

whenever $|z-a| < \delta$. But $|z-a| = r$ and so we may take $r < \delta$. Then $|(z-a)f(z) - A| < \epsilon$ on the arc C. Therefore, we have

$$(z-a)f(z) = A + \eta(z),$$

where $|\eta(z)| < \epsilon$, and so

$$f(z) = \frac{A + \eta(z)}{z-a}.$$

Then, putting $z - a = re^{i\theta}$, we have

Two Useful Theorems

$$\begin{aligned}\int_C f(z)dz &= \int_C \frac{A+\eta(z)}{z-a}dz \\ &= \int_{\theta_1}^{\theta_2} \frac{(A+\eta(z))re^{i\theta}\, id\theta}{re^{i\theta}} \\ &= (\theta_2-\theta_1)\, iA + (\theta_2-\theta_1)\, i\eta(z)\end{aligned}$$

so that

$$\left|\int_C f(z)dz - iA\,(\theta_2-\theta_1)\right| = (\theta_2-\theta_1)\,|i\eta(z)| < (\theta_2-\theta_1)\,\epsilon \to 0$$

as $\epsilon \to 0$ and $\epsilon \to 0$ as $z \to a$ and $z \to a$ as $r \to 0$. Hence we have

$$\lim_{r\to 0}\int_C f(z)dz = iA\,(\theta_2-\theta_1)\,.$$

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as $\epsilon \to 0$ and $\epsilon \to 0$ as $z \to a$ and $z \to a$ as $r \to 0$. Hence we have

$$\lim_{r\to 0}\int_C f(z)dz = iA\,(\theta_2-\theta_1)\,.$$

In particular, if $z=a$ is a simple pole of $f(z)$, then A is the residue of $f(z)$ at $z=a$. Thus, if C is a small circle $|z-a|=r$, then we have $\theta_2-\theta_1=2\pi$ and

$$\int_C f(z)dz = 2\pi iA.$$

Particularly, if $(z-a)f(z) \to 0$ as $z \to a$, then we have

Two Useful Theorems

Theorem

If C is the arc $\theta_1 \leq \theta \leq \theta_2$ of the circle $|z| = R$ and $\lim_{R\to\infty} zf(z) = A$, then

$$\lim_{R\to\infty} \int_C f(z)dz = i(\theta_2 - \theta_1)\, A.$$

Two Useful Theorems

Theorem

If C is the arc $\theta_1 \leq \theta \leq \theta_2$ of the circle $|z| = R$ and $\lim\limits_{R\to\infty} zf(z) = A$, then

$$\lim_{R\to\infty} \int_C f(z)dz = i\,(\theta_2 - \theta_1)\,A.$$

Proof.

Since $\lim\limits_{R\to\infty} zf(z) = A$, it follows that, for any $\epsilon > 0$, we can choose R so large that

$$|zf(z) - A| < \epsilon \text{ on the arc } C, \text{ or } zf(z) - A = \eta, \text{ where } |\eta| < \epsilon, \text{ or } zf(z) = A + \eta.$$

Therefore, putting $z = Re^{i\theta}$, we have

$$\int_C f(z)dz = \int_C \frac{A+\eta}{z}dz = \int_{\theta_1}^{\theta_2} \frac{(A+\eta)Re^{i\theta}\,id\theta}{Re^{i\theta}} = Ai\,(\theta_2 - \theta_1) + \eta i\,(\theta_2 - \theta_1)\,.$$

Letting $\epsilon \to 0$ and, consequently, $R \to \infty$, we have $\lim\limits_{R\to\infty} \int_C f(z)dz = Ai\,(\theta_2 - \theta_1)$. In particular, if $zf(z) \to 0$ as $R \to \infty$, then we have $\int_C f(z)dz \to 0$ as $R \to \infty$. □

Type I: Integrals of the form $\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta$

Let us consider the integrals of the form

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta, \tag{6}$$

where the integrand is a rational function of $\sin\theta$ and $\cos\theta$. The basic idea here is to convert a real trigonometric integral of form (6) into a complex integral, where the contour C is the unit circle $|z| = 1$ centered at the origin. Writing $z = e^{i\theta}$, we have $dz = ie^{i\theta}d\theta$ or $\frac{dz}{iz} = d\theta$ and $\frac{1}{2}(z + z^{-1}) = \cos\theta$, $\frac{1}{2i}(z - z^{-1}) = \sin\theta$ and so

$$\begin{aligned}\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta &= \int_C f\left(\frac{1}{2}(z + z^{-1}), \frac{1}{2i}(z - z^{-1})\right)\frac{dz}{iz} \\ &= \int_C \phi(z)dz,\end{aligned}$$

where C is the unit circle $|z| = 1$. It is evident that $\phi(z)$ is a rational function of z. Hence, by Cauchy's residue theorem, we have

$$\int_C \phi(z)dz = 2\pi i \sum R_C,$$

where $\sum R_C$ is the sum of the residue of the function $\phi(z)$ at its poles inside C.

Type 2: Integrals of the form $\int_{-\infty}^{\infty} f(x)dx$

Suppose $y = f(x)$ is a real function that is defined and continuous on the interval $[0, \infty)$. The improper integral $I_1 = \int_0^{\infty} f(x)dx$ is defined as the limit

$$I_1 = \int_0^{\infty} f(x)dx = \lim_{R\to\infty} \int_0^R f(x)dx. \tag{7}$$

If the limit exists, the integral I_1 is said to be convergent; otherwise, it is divergent. The improper integral $I_2 = \int_{-\infty}^{0} f(x)dx$ is defined similarly:

$$I_2 = \int_{-\infty}^{0} f(x)dx = \lim_{R\to\infty} \int_{-R}^{0} f(x)dx. \tag{8}$$

Finally, if f is continuous on $(-\infty, \infty)$, then $\int_{-\infty}^{\infty} f(x)dx$ is defined to be

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_0^{\infty} f(x)dx = I_1 + I_2 \tag{9}$$

provided both integrals I_1 and I_2 are convergent. If either one, I_1 or I_2, is divergent, then $\int_{-\infty}^{\infty} f(x)dx$ is divergent.

It is important to remember that the right-hand side of (9) is not the same as

$$\lim_{R\to\infty} \left[\int_{-R}^{0} f(x)dx + \int_0^R f(x)dx\right] = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx. \tag{10}$$

Type 2: Integrals of the form $\int_{-\infty}^{\infty} f(x)dx$

For the integral $\int_{-\infty}^{\infty} f(x)dx$ to be convergent, the limits (7) and (8) must exist independently of one another. But, in the event that we know that an improper integral $\int_{-\infty}^{\infty} f(x)dx$ converges, we can then evaluate it by means of the single limiting process given in (10):

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx. \tag{11}$$

The limit in (11), if it exists, is called the Cauchy principal value (P.V.) of the integral and is written

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx. \tag{12}$$

On the other hand, the symmetric limit in (11) may exist even though the improper integral $\int_{-\infty}^{\infty} f(x)dx$ is divergent. For example, the integral $\int_{-\infty}^{\infty} xdx$ is divergent since $\lim_{R\to\infty} \int_0^R xdx = \lim_{R\to\infty} \frac{1}{2}R^2 = \infty$. However, (11) gives

$$\lim_{R\to\infty} \int_{-R}^{R} xdx = \lim_{R\to\infty} \frac{1}{2}\left[R^2 - (-R)^2\right] = 0,$$

which shows that P.V. $\int_{-\infty}^{\infty} xdx = 0$.

Type 2: Integrals of the form $\int_{-\infty}^{\infty} f(x)dx$

Suppose $f(x)$ is continuous on $(-\infty, \infty)$ and is an even function, that is, $f(-x) = f(x)$. Then

$$\int_{-R}^{0} f(x)dx = \int_{0}^{R} f(x)dx \text{ and } \int_{-R}^{R} f(x)dx = \int_{-R}^{0} f(x)dx + \int_{0}^{R} f(x)dx = 2\int_{0}^{R} f(x)dx.$$

Thus, if the Cauchy principal value (12) exists, then both $\int_0^\infty f(x)dx$ and $\int_{-\infty}^{\infty} f(x)dx$ converge. The values of the integrals are

$$\int_0^\infty f(x)dx = \frac{1}{2}\ \text{P.V.} \int_{-\infty}^{\infty} f(x)dx \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)dx = \text{ P.V.} \int_{-\infty}^{\infty} f(x)dx.$$

To evaluate an integral $\int_{-\infty}^{\infty} f(x)dx$, where the rational function $f(x) = p(x)/q(x)$ is continuous on $(-\infty, \infty)$, by residue theory we replace x by the complex variable z and integrate the complex function f over a closed contour C that consists of the interval $[-R, R]$ on the real axis and a semicircle C_R of radius large enough to enclose all the poles of $f(z) = p(z)/q(z)$ in the upper half-plane $\mathrm{Im}(z) > 0$. By Cauchy's Residue Theorem, we have

$$\int_C f(z)dz = \int_{C_R} f(z)dz + \int_{-R}^{R} f(x)dx = 2\pi i \sum_{k=1}^{n} \mathrm{Res}\,(f(z), z_k)$$

where z_k, $k = 1, 2, \ldots, n$ denotes poles in the upper half-plane. If we can show that the integral $\int_{C_R} f(z)dz \to 0$ as $R \to \infty$, then we have

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx = 2\pi i \sum_{k=1}^{n} \mathrm{Res}\,(f(z), z_k)\,.$$

Type 2: Integrals of the form $\int_{-\infty}^{\infty} f(x)dx$

It is often tedious to have to show that the contour integral along C_R approaches zero as $R \to \infty$. Sufficient conditions under which this behavior is always true are summarized in the next theorem.

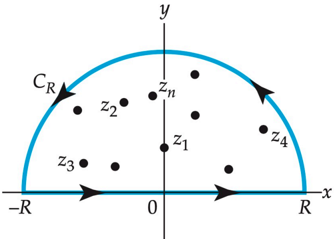
Theorem

Let $f(z) = \frac{p(z)}{q(z)}$, where $p(z)$ and $q(z)$ are polynomials such that

1. *$q(z) = 0$ has no real roots;*
2. *the degree of $p(z)$ is at least two less than that of $q(z)$ so that $\lim\limits_{|z|\to\infty} zf(z) = 0$.*

Then we have

$$P.V. \int_{-\infty}^{\infty} \frac{p(x)}{q(x)} dx = 2\pi i \sum_{k=1}^{n} \mathrm{Res}\,(f(z), z_k)\,.$$



Theorem (Jordan's Inequality)

If $0 \leq \theta \leq \frac{\pi}{2}$, *then*

$$\frac{2\theta}{\pi} \leq \sin\theta \leq \theta.$$

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Proof.

Since $\cos\theta$ decreases steadily as θ increases from 0 to $\pi/2$, the mean ordinate of the graph of $y = \cos\theta$ between $\theta = 0$ to θ is

$$\frac{1}{\theta}\int_0^{\theta} \cos\theta d\theta = \frac{\sin\theta}{\theta}.$$

When $\theta = 0$, the ordinate is $\cos 0 = 1$ and, when $\theta = \frac{\pi}{2}$, the mean ordinate is equal to $\frac{2}{\pi}$. It follows that, when $0 \le \theta \le \frac{\pi}{2}$, the mean ordinate lies between 1 and $\frac{2}{\pi}$, that is,

$$\frac{2}{\pi} \le \frac{\sin\theta}{\theta} \le 1,$$

or

$$\frac{2\theta}{\pi} \le \sin\theta \le \theta.$$

□

Theorem (Jordan's Lemma)

If

1. $f(z) \to 0$ *as* $|z| \to \infty$ *uniformly for* $0 \le \arg z \le \pi$;
2. $f(z)$ *is meromorphic in the upper half-plane, then*

$$\lim_{R\to\infty} \int_{C_R} e^{i\alpha z} f(z)dz = 0 \quad (\alpha > 0),$$

where C_R *denote the semi-circle* $|z| = R$ *and* $\mathrm{Im}(z) > 0$.

Proof.

Assume that $f(z)$ has no singularities on C_R for a sufficiently large value of R. Since $\lim\limits_{R\to\infty} f(z) = 0$, it follows that, for any $\epsilon > 0$, $|f(z)| < \epsilon$ when $|z| = R \ge R_0$ where $R_0 > 0$.
Let C_R denote any semi-circle with radius $R \ge R_0$. From $|f(z)| < \epsilon$, $z = Re^{i\theta}$ and Jordan's inequality, we have

$$\left|\int_{C_R} e^{i\alpha z} f(z)dz\right| \le \int_{C_R} \left|e^{i\alpha z}\right| |f(z)||dz| < \epsilon \int_{C_R} \left|e^{i\alpha z}\right| |dz| = \epsilon \int_0^{\pi} \left|e^{i\alpha(R\cos\theta + iR\sin\theta)}\right| \left|Rie^{i\theta} d\theta\right|$$

$$= \epsilon \int_0^{\pi} e^{-\alpha R\sin\theta} Rd\theta \le 2\epsilon R \int_0^{\pi/2} e^{-\alpha R\frac{2\theta}{\pi}} d\theta = \frac{\epsilon\pi}{\alpha}\left(1 - e^{-\alpha R}\right) < \frac{\epsilon\pi}{\alpha}.$$

Type 3: Integrals of the Forms $\int_{-\infty}^{\infty} f(x)\cos\alpha x dx$ and $\int_{-\infty}^{\infty} f(x)\sin\alpha x dx$

Because improper integrals of the form $\int_{-\infty}^{\infty} f(x)\sin\alpha x dx$ are encountered in applications of Fourier analysis, they often are referred to as Fourier integrals. Fourier integrals appear as the real and imaginary parts in the improper integral $\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx$. In view of Euler's formula $e^{i\alpha x} = \cos\alpha x + i\sin\alpha x$, where α is a positive real number, we can write

$$\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx = \int_{-\infty}^{\infty} f(x)\cos\alpha x dx + i\int_{-\infty}^{\infty} f(x)\sin\alpha x dx \tag{13}$$

whenever both integrals on the right-hand side converge.

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whenever both integrals on the right-hand side converge.

Suppose $f(x) = p(x)/q(x)$ is a continuous rational function on $(-\infty, \infty)$. Then both Fourier integrals in (13) can be evaluated at the same time by the complex integral $\int_C f(z)e^{i\alpha z}dz$, where $\alpha > 0$, the contour C consists of the interval $[-R, R]$ on the real axis and a semicircular contour C_R with radius large enough to enclose the poles of $f(z)$ in the upper-half plane.

Theorem

Let $f(z) = \frac{p(z)}{q(z)}$, where $p(z), q(z)$ are polynomials and the degree of $q(z)$ exceeds that of $p(z)$ and $q(z) = 0$ has no real roots. Let $\alpha > 0$. Then

$$\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx = 2\pi i\sum_{k=1}^{n} \mathrm{Res}\left(f(z)e^{i\alpha z}, z_k\right),$$

where $\sum_{k=1}^{n} \mathrm{Res}\left(f(z)e^{i\alpha z}, z_k\right)$ denotes the sum of the residues of $e^{i\alpha z}f(z)$ at its poles in the upper half-plane.

Type 4: Poles on the Real Axis

If the integrand has simple poles on the real axis, we have the following theorem:

Theorem

Let $f(z) = \frac{p(z)}{q(z)}$, where $p(z), q(z)$ are polynomials and $q(z)$ has only non-repeated real roots, that is, $f(z)$ has only simple poles on the real axis. Let $m > 0$ and let the degree of $q(z)$ exceed that of $p(z)$. Then

$$P.V. \int_{-\infty}^{\infty} e^{i\alpha x} f(x) dx = 2\pi i \sum_{k=1}^{p} \operatorname{Res}(a_k) + \pi i \sum_{k=1}^{q} \operatorname{Res}(b_k)$$

where $a_1, a_2, \cdots, a_p$ are the zeros of $q(z)$ in the region $Im\, z > 0$ and $b_1, b_2, \cdots, b_q$ its zeros in the real axis, where by Res(c) we mean the residue of $e^{i\alpha z} f(z)$ at c.

The Indenting Method is useful when the integrand has simple poles on the real axis. In such cases, we follow the procedure known as indenting at a point. We exclude the poles on the real axis by enclosing them with a semi-circle of small radii.

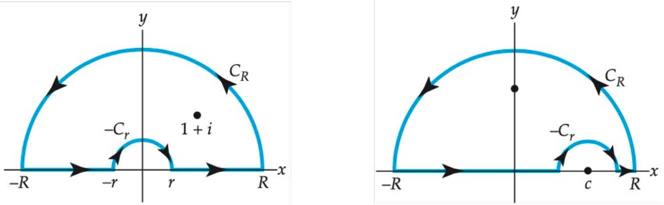

Figure: Two illustrated figures