CS 131 Finals Questions

RVLC - S1 2016-17

December 6, 2016

Abstract

Here are your instructions:

The finals is to be done individually, in pairs or in groups of three. It will scored over 100 points and will go towards your MP 2 grade.

1 Solving a first-order ODE using Euler's IMPLICIT method

A chemical compound decays over time when exposed to air, at a rate proportional to its concentration to the power of 3/2 . At the same time, the compound is produced by another process. The differential equation for its instantaneous concentration is:

$$\frac{dn(t)}{dt} = -0.8n^{3/2} + 10n_1(1 - e^{-3t})$$

where $n(t)$ is the instantaneous concentration and $n_1 = 2000$ is the initial concentration at $t = 0$.

1.1 Modified Euler's Method

Solve the differential equation to find the concentration as a function of time from $t = 0$ until $t = 0.5$ seconds , using the Improved Euler's method (you will call on your classmate's java class) for solving for the roots of a nonlinear equation. Use a step size of $h = 0.002$ seconds, and plot n versus time.

1.2 Implicit Euler's Method

Solve the differential equation to find the concentration as a function of time from $t = 0$ until $t = 0.5$ seconds , using Euler's Implicit method (you will program this) and Newton's method (you will call on your classmate's java class) for solving for the roots of a nonlinear equation. Use a step size of $h = 0.002$ seconds, and plot n versus time. Use 0.001 as the tolerance for the Newton's method.

1.3 (NOT Graded) Tips on the usage of Implicit Euler

The form of the implicit Euler method is:

$$x_{i+1} = x_i + h \tag{1}$$

$$y_{i+1} = y_i + f\left(x_{i+1}, y_{i+1}\right) h \tag{2}$$

In order to solve this using Newton's method, which is a root or zero-finding method, we convert eq. 2 into a form that equates to 0.

$$y_{i+1} - \left[y_i + f\left(x_{i+1}, y_{i+1}\right) h\right] = 0 \tag{3}$$

We can call this form $g(y)$ and use it as input to Newton's Root finding method. g will only be a function of y_{i+1} since x_{i+1} will be a constant equal to $x_i + h$ and y_i is a constant equal to the current value of y for the duration of the Newton method. This means that you will execute the Newton method at every point.

$$g\left(y_{i+1}\right) = y_{i+1} - \left[y_i + f\left(x_{i+1}, y_{i+1}\right) h\right] = 0 \tag{4}$$

The formulation of Newton's method becomes :

$$y_{i+1} = y_i - \frac{g\left(y_{i+1}\right)}{g'\left(y_{i+1}\right)} \tag{5}$$

1.4 (NOT Graded) Tips on applying Implicit Euler to our problem

The following tips are free and come without guarantees. You may thank me if they work, but be sure to check... :)

1. t is the independent variable
2. n is the dependent variable
3. the function $f(t,n)$ is given by
$$f(t,n) = -0.8n^{3/2} + 10n_1(1 - e^{-3t}) \tag{6}$$
4. if you substitute $f(t,n)$ to eq. 2 the implicit Euler setup might look like
$$\begin{aligned} t_{i+1} &= t_i + h \\ n_{i+1} &= n_i + \left[-0.8n_{i+1}^{3/2} + 10n_1(1 - e^{-3t_{i+1}})\right] h \end{aligned} \tag{7}$$
5. at every step of the solution, eq. 7 can be solved numerically via Newton's method. first setup $g(y) = 0$ where $y = n_{i+1}$.
$$g(y) = y + 0.8y^{3/2}h - 10n_1\left(1 - e^{-3t_{i+1}}\right)h - n_i = 0 \tag{8}$$
6. next, setup $g'(y)$
$$g'(y) = 1 + 0.8 \cdot \frac{3}{2} \cdot y^{1/2} \cdot h \tag{9}$$
7. Newton's iteration is done by substituting eq. 8 and eq. 9 into eq. 5 repeatedly until the solution converges with the required tolerance.
$$y_{i+1} = y_i - \frac{y + 0.8y^{3/2}h - 10n_1\left(1 - e^{-3t_{i+1}}\right)h - n_i}{1 + 0.8 \cdot \frac{3}{2} \cdot y^{1/2} \cdot h} \tag{10}$$

1.5 A few questions for discussion

1. Is 20 iterations enough as a maximum for the Newtons Method in section 1.2 ?
2. Does the resulting plots look like the figure 1? What are the differences if any?

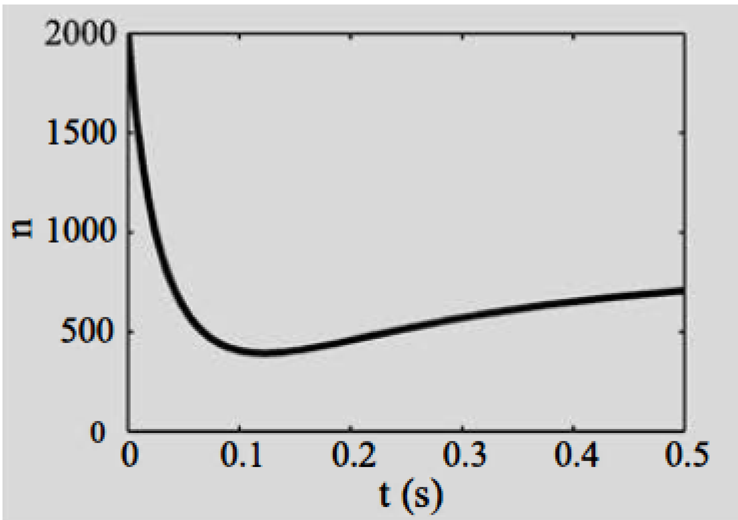

Figure 1:

2 Hypnotic Tools: Damped, nonlinear motion of a pendulum

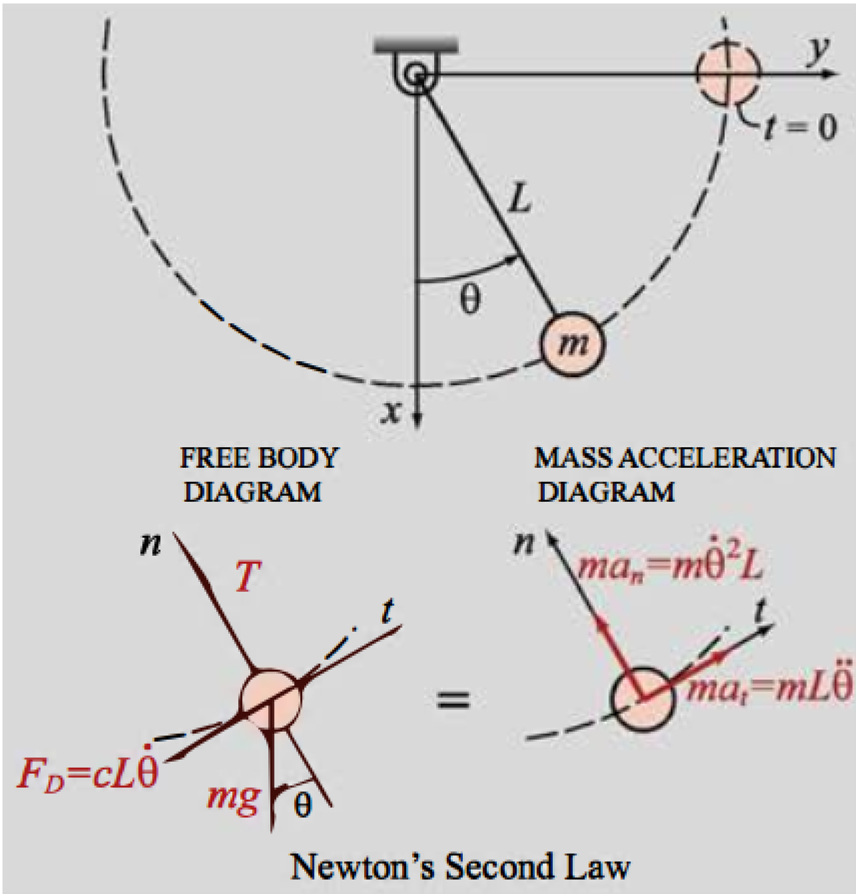

Figure 2:

A pendulum is modeled by a mass that is attached to a weightless rigid rod. According to Newton's second law, as the pendulum swings back and forth, the sum of the forces that are acting on the mass equals the mass times acceleration (see the fee body diagram in the figure). Writing the equilibrium equation in the tangential direction gives:

$$\sum F_t = -cL\frac{d\theta}{dt} - mg\sin(\theta) = \mathrm{mL}\frac{\mathrm{d}^2\theta}{\mathrm{dt}^2} \tag{11}$$

where $\theta$is the angle of the pendulum (with respect to the vertical axis, as shown in figure 2) , $c = 0.16\ (N \cdot m)/s$ is the damping coefficient, $m = 0.5\ kg$ is the mass, $L = 1.2m$ is the length, and $g = 9.81\ m/s^2$ is the acceleration due to gravity.

Equation 11can be rewritten as a second-order differential equation.

$$\frac{d^2\theta}{dt^2} = -\frac{c}{m}\frac{d\theta}{dt} - \frac{g}{L}sin(\theta) \tag{12}$$

2.1 Your mission

The pendulum initially displaced such that $\theta = 90°$, and then at $t = 0$ it is released from rest , $\frac{d\theta}{dt} = 0$ (zero initial velocity). Determine the angle of the pendulum as a function of time, $\theta(t)$, for the first 18 seconds after it is released. Graph your results.

2.2 (NOT Graded) Tips

The following tips are free and come without guarantees. You may thank me if they work, but be sure to check... :)

1. Reduce the 2nd-order equation to a system of 1st-order equations. Maybe we can introduce a new dependent variable w

$$w = \frac{d\theta}{dt} \quad , \quad \frac{dw}{dt} = \frac{d^2\theta}{dt^2} \tag{13}$$

2. Setup the system to solve

$$\frac{d\theta}{dt} = w \quad , \quad \frac{dw}{dt} = -\frac{c}{m}w - \frac{g}{L}sin(\theta) \tag{14}$$

3. Setup the initial conditions

$$\theta(0) = \frac{\pi}{2} \quad , \quad w(0) = 0$$

4. Solve using an accurate numerical method made for a system of at least 2 equations that one of your classmates made.

2.3 Discussion

1. Does your graph resemble figure 3?

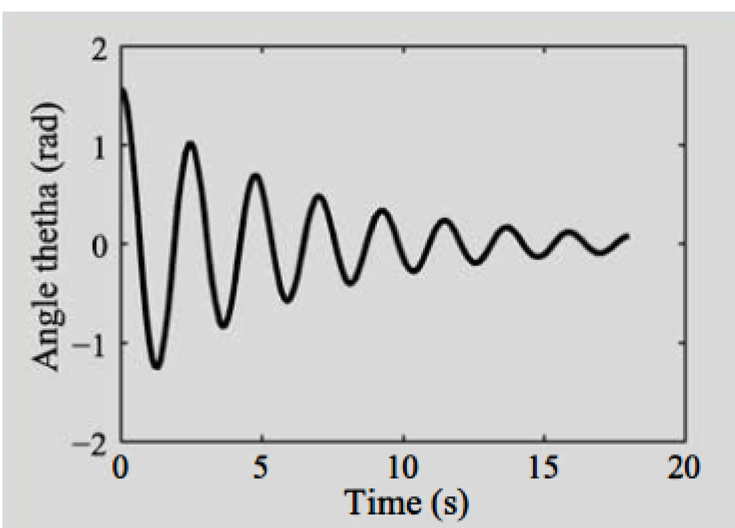

Figure 3:

2. What is the difference, if any?
3. Can you explain what's happening to the pendulum in relation to what your plot is showing?
4. (bonus) can you animate a virtual pendulum that follows the physics of what you just solved for?

3 Sizzling Sisig: Cooling of a hot plate

When a thin hot plate is suddenly taken out of an oven and is exposed to the surrounding air, it cools due to heat loss by convection and radiation. The rate at which the plate's temperature, T is changing with time is given by

$$\frac{dT}{dt} = -\frac{A_s}{\rho V C_v}\left[\sigma_{SB}\varepsilon\left(T^4 - T_\infty^4\right) + h\left(T - T_\infty\right)\right] \tag{15}$$

where A_s is the plate's surface area, $\rho = 300\ kg/m^3$is its mass density, V is its volume, $C_V = 900\ J/kg/K$ is its specific heat at constant volume, and $\varepsilon = 0.8$ is its radiative emissivity. Also, $\sigma_{SB} = 5.67 \times 10^{-8}\ W/m^2/K^4$ is the Stefan-Boltzmann constant, $h = 30\ W/m^2/K$ is the heat transfer coefficient, and $T_\infty$is the ambient air temperature.

3.1 Your mission

Write a program that calculates the temperature of the place as a function of time for the first 180 seconds after the plate is taken out of the oven, and display the results in a figure.

User classical RK 4 and another method of your choice for this and compare your results.

3.2 The specifications

Plot the variation of the temperature of the plate as a function of time for a plate with $V = 0.003m^3$, $A_s = 0.25m^2$, which has an initial temperature of 673 K , when the ambient temperature is 298 K.

3.3 Discussion

1. Describe briefly the following terms:
   (a) mass density
   (b) specific heat
   (c) radiative emissivity
   (d) the Stefan-Boltzmann constant, and where it is commonly used.
   (e) who is Stefan, who is Boltzmann, what did they do?
   (f) why T_∞ is a good symbol for ambient temperature
2. Does your graph resemble figure 4?

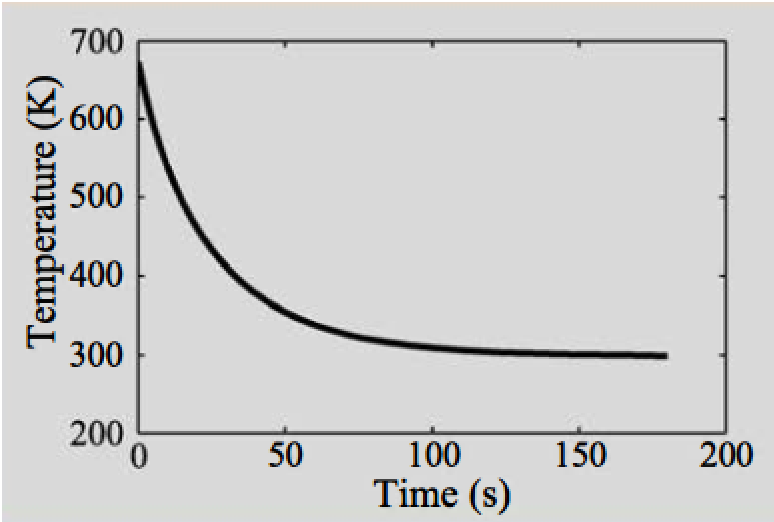

Figure 4:

3. What is the difference, if any?

4 Predator-Prey

The relationship between the population of lions(predators), N_L, and the population of gazelles (prey), N_G, that reside in the same area can be modeled by a system of two ODEs. Suppose a community consists of N_L lions(predators) and N_G gazelles(prey), with b and d representing the birth and death rates of the respective species. The rate of change (growth or decay) of the lion (L) and gazelle (G) populations can be modeled by the equations:

$$\begin{aligned} \frac{dN_L}{dt} &= b_L N_L N_G - d_L N_L \\ \frac{dN_G}{dt} &= b_G N_G - d_G N_G N_L \end{aligned} \qquad (16)$$

4.1 Your mission

Determine the population of the lions and gazelles as a function of time from $0 \leq t \leq 25$ years, if at $t = 0$, $N_G = 3000$, and $N_L = 500$. The coefficients in the are: $b_G = 1.1\ yr^{-1}$, $b_L = 0.00025\ yr^{-1}$, $d_G = 0.0005\ yr^{-1}$, and $d_L = 0.7\ yr^{-1}$.

4.1.1 time-evolution plot

Plot you the populations of Lions and Gazelles in one graph with the x-axis being time in years.

4.1.2 phase portraits

Create a second graph that with the population of gazelles in the x-axis and the population of lions in the y-axis.

4.2 Discussion

1. Does the resulting graph look like figure 5 ?

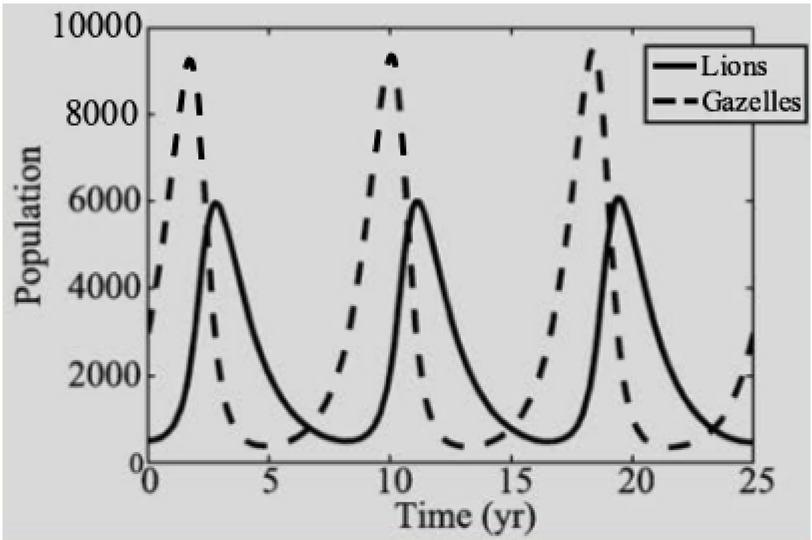

Figure 5:

2. Can you discuss what's happening with the populations of lions and gazelles according to the graphs? Discuss the differences of your output graph with figure 5 , if there are any.
3. Can you interpret the results of the phase portrait produced in section 4.1.2 ? What simple geometrical figure does it resemble?
4. What would happen if either one of the populations become zero at any point in time?
5. Can you change the initial conditions in a way that at least one of the populations will become zero after some time (i.e. they must NOT be zero initially). Plot the resulting time-based graph.

5 Shooting for the temp distribution in a pin fin

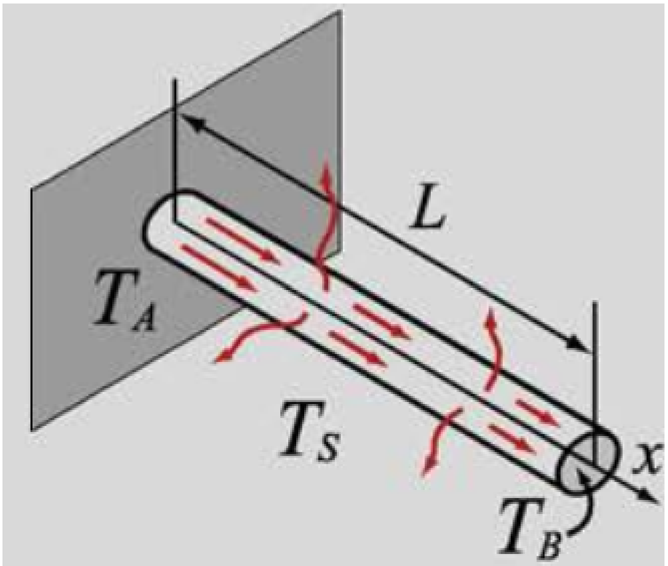

Figure 6:

A pin fin is a slender extension attached to a surface in order to increase the surface area and enable greater heat transfer. When con vection and radiation are included in the analysis, the steady-state temperature distribution, T (x) , along a pin fin can be calculated from the solution of the equation:

$$\frac{d^2T}{dx^2} - \frac{h_c P}{kA_C}(T - T_S) - \frac{\varepsilon \sigma_{SB} P}{kA_C}\left(T^4 - T_S^4\right) = 0 \quad , \quad 0 \leq x \leq L \tag{17}$$

with the boundary conditions: $T(0) = T_A$ and $T(L) = T_B$.

In Eq. 17 , h_C is the convective heat transfer coefficient, P is the perimeter bounding the cross-sectional area of the fin, T_S is the temperature of the surrounding air, and $\sigma_{SB} = 5.67 \times 10^{-8}$ $W/(m^2K^4)$ is the Stefan-Boltzmann constant.

5.1 Your mission

Using the Shooting Method, determine the temperature distribution if $L = 0.1m$, $T(0) = 473K$, $T(0.1) = 293K$, and $T_S = 293K$. Use the following values for the parameters in Eq. 17: $h_C = 40W/m^2/K$, $P = 0.016m$, $\varepsilon = 0.4$, $k = 240W/m/K$, and $A_C = 1.6 \times 10^{-5}m^2$.

5.2 Details

1. transform the the 2nd-order problem into a system of 1st-order equations. Introduce an new variable $w = \frac{dT}{dx}$.

$$\begin{array}{ll} \frac{dT}{dx} & = w \\ \frac{dw}{dx} & = \quad \frac{h_C P}{kA_C}(T - T_S) + \frac{\varepsilon \sigma_{SB} P}{kA_C}\left(T^4 - T_S^4\right) \end{array} \tag{18}$$

2. Set the initial condition for $T(0) = 473$.
3. For the shooting method, we shall use 3 guesses, I will provide the first 2 and you shall compute for the 3rd guess by calling the Secant method java class.
   (a) 1st guess: $w(0) = -1000$
   (b) 2nd guess: $w(0) = -3500$
   (c) use the guesses and results from (a) and (b) , as starting values for our Secant method java class.
4. Solve the system using our RK-2 java class for a system of 2 differential equations. Keep iterating the guesses using the secant method until the difference of the numerical solution and the prescribed boundary condition is less than $0.01K$

5.3 Discussion

1. Which one is the convection term and which is the radiation term in the problem?
2. Does the output look like figure 7

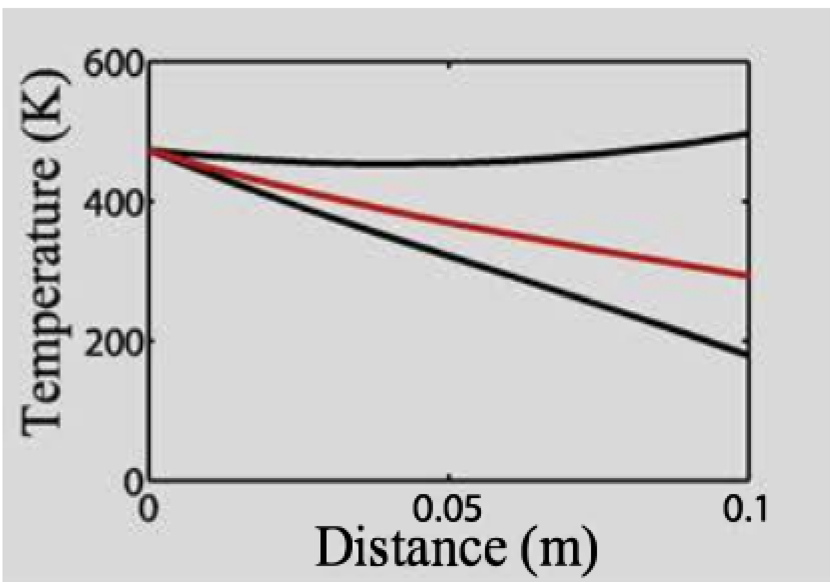

Figure 7:

3. Give probable explanations for the difference, if there are any. Moving forward, what can be done to further improve our results?

6 Convection-only temp distribution in a pin fin

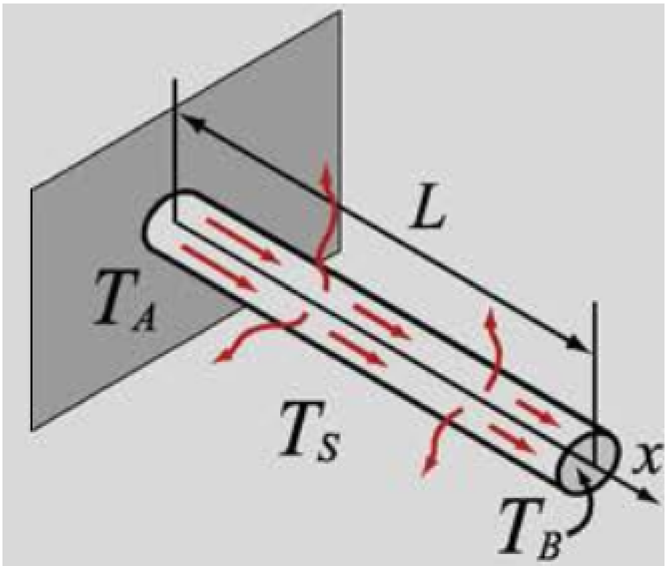

Figure 8:

A pin fin is a slender extension attached to a surface in order to increase the surface area and enable greater heat transfer. When only convection is included in the analysis, the steady state tem perature distribution, T (x) , along a pin fin can be obtained from the solution of the equation:

$$\frac{d^2T}{dx^2} - \frac{h_c P}{kA_C}(T - T_S) = 0 \quad , \quad 0 \le x \le L \tag{19}$$

with the boundary conditions: $T(0) = T_A$ and $T(L) = T_B$.

In Eq. 19 , h_C is the convective heat transfer coefficient, P is the perimeter bounding the cross-sectional area of the fin, T_S is the temperature of the surrounding air.

6.1 Your mission

Using the Finite Difference Method, determine the temperature distribution if $L = 0.1m$, $T(0) = 473K$, $T(0.1) = 293K$, and $T_S = 293K$. Use the following values for the parameters in Eq. 19: $h_C = 40W/m^2/K$, $P = 0.016m$, $\varepsilon = 0.4$, $k = 240W/m/K$, and $A_C = 1.6 \times 10^{-5}m^2$.

6.2 Details

1. Divide the domain of the solution into five equally spaced intervals.

2. Setup the 2nd-order linear ordinary differential equation (Eq 19) into a finite difference using the central difference formula

$$\frac{T_{i-1} - 2T_i + T_{i+1}}{h^2} - \frac{h_C P}{kA_C}(T_i - T_S) = 0 \tag{20}$$

3. If we set $\beta = \frac{h_C P}{k A_C}$, Eq 20 can be written as

$$T_{i-1} - \left(2 + h^2\beta\right) T_i + T_{i+1} = -h^2 \beta T_S \tag{21}$$

4. Next, the domain of the solution is divided into five equally spaced subintervals (defined by six points), as shown in the figure

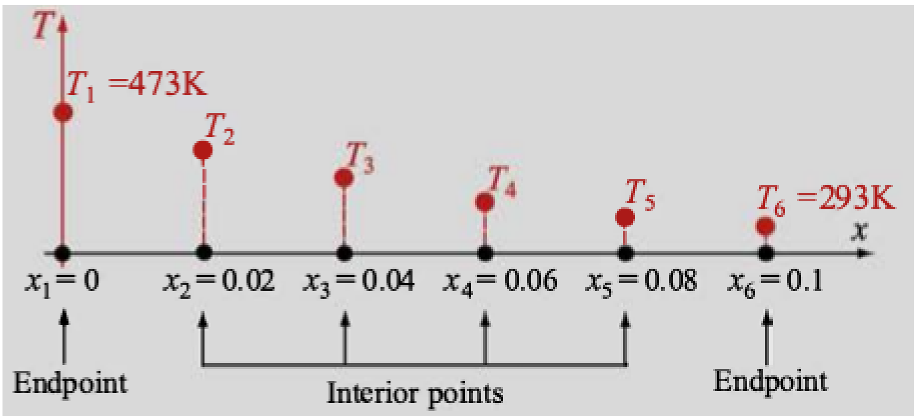

Figure 9:

5. Write Eq 21 for each of the interior points (i.e. i=2,3,4,5). note that T_1 is known.

$$\begin{array}{lllcl} for & i=2 & : & T_1 - \left(2+h^2\beta\right) T_2 + T_3 & = & -h^2\beta T_S \\ for & i=3 & : & T_2 - \left(2+h^2\beta\right) T_3 + T_4 & = & -h^2\beta T_S \\ for & i=4 & : & T_3 - \left(2+h^2\beta\right) T_4 + T_5 & = & -h^2\beta T_S \\ for & i=5 & : & T_4 - \left(2+h^2\beta\right) T_5 + T_6 & = & -h^2\beta T_S \end{array} \tag{22}$$

6. Note that T_6 is known from the boundary condition $T(0.1) = 293K$. This will allow us to rewrite Eq. 22 as a not so big matrix

$$\begin{bmatrix} -\left(2+h^2\beta\right) & 1 & 0 & 0 \\ 1 & -\left(2+h^2\beta\right) & 1 & 0 \\ 0 & 1 & -\left(2+h^2\beta\right) & 1 \\ 0 & 0 & 1 & -\left(2+h^2\beta\right) \end{bmatrix} \begin{bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} -\left(h^2\beta T_S + T_1\right) \\ -h^2\beta T_S \\ -h^2\beta T_S \\ -\left(h^2\beta T_S + T_6\right) \end{bmatrix}$$

7. Use our LU factorization, Crout's algorithm java class to solve the matrix for the values of vector T.

8. Plot the results.

6.3 Discussion

1. Derive / Explain how Eq 20 was formed.

2. Does the output look like figure 10

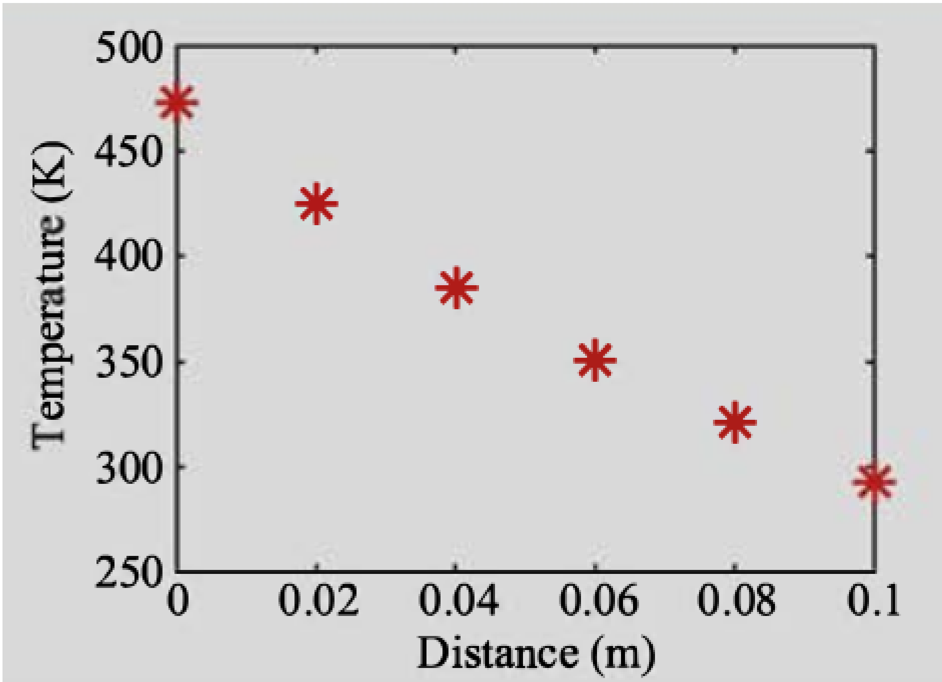

Figure 10:

3. Give probable explanations for the difference, if there are any. Moving forward, what can be done to further improve our results?