MP 2 Case Study 4 - Predator-Prey Models and Chaos

November 24, 2017

1 Background

Environmental engineers deal with a variety of problems involving systems of nonlinear ordinary differential equations. In this section, we will focus on two of these applications. The first relates to the so-called predator-prey models, that are used to study the cycling of nutrient and toxic pollutants in aquatic food chains and biological treatment systems. The second are equations derived from fluid dynamics that are used to simulate the atmosphere. Aside from their obvious application to weather prediction, such equations have also been used to study air pollution and global climate change.

1.1 Tasks

1. search the net for an interesting model or research paper or case study that makes use of Lotka-Volterra equations.
2. briefly discuss the model you found on Lotka-Volterra, cite your references and show pictures
3. search the net for an interesting model or research paper or case study that makes use of Lorenz equations.
4. briefly discuss the model you found on Lorenz, cite your references and show pictures

2 Lotka-Volterra Models

Predator-prey models were developed independently in the early part of the twentieth century by the Italian mathematician Vito Volterra and the American biologist Alfred J. Lotka. These equations are commonly called ***Lotka-Volterra equations***. The simplest example is the following pair of ODEs:

$$\frac{dx}{dt} = ax - bxy \tag{1}$$

$$\frac{dy}{dt} = -cy + dxy \tag{2}$$

where x and y = the number of prey and predators, respectively, a = the prey growth rate, c = the predator death rate, and b and d = the rate characterizing the effect of the predator- prey interaction on prey death and predator growth, respectively. The multiplicative terms (that is, those involving xy) are what make such equations nonlinear.

Use numerical methods to obtain solutions for these equations. Plot the results to visualize how the dependent variables change temporally. In addition, plot the dependent variables versus each other to see whether any interesting patterns emerge.

Use the following parameter values for the predator-prey simulation: a = 1.2, b = 0.6, c = 0.8, and d = 0.3. Employ initial conditions of x = 2 and y = 1 and integrate from t = 0 to 30. Use the fourth-order RK method with double precision floating point data type to obtain solutions. Use step size of 0.1 or smaller.

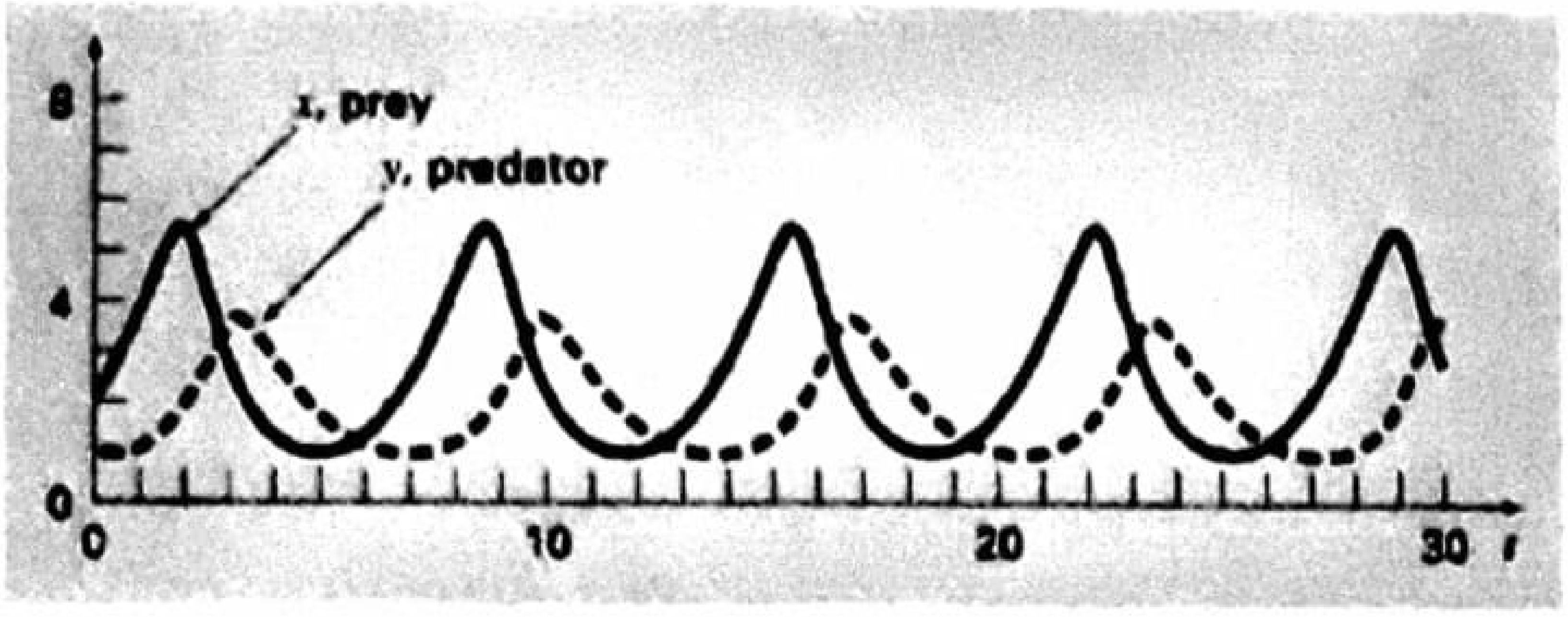

Figure 1: Time-domain representation of numbers of prey and predators for the Lotka-Volterra model.

2.1 Tasks

1. Check the RK4 class in our Git repo if it meets the criteria required by the problem. Modify and create a pull request for the changes you make, if the current class needs editing.
2. Run the predator-prey simulation
3. Graph the results
4. Briefly explain what the graphs suggest about the relationship of predator and prey populations.

3 Time Series Solution

The results using a step size of 0.1 are shown in Fig. 1. Note that a cyclical pattern emerges. Thus, because predator population is initially small, the prey grows exponentially. At a certain point, the prey become so numerous, that the predator population begins to grow. Eventually, the increased predators cause the prey to decline. This decrease, in turn, leads to a decrease of the predators. Eventually, the process repeats. Notice that, as expected, the predator peak lags the prey. Also, observe that the process has a fixed period, that is, it repeats in a set time.

Now, if the parameters used to simulate Fig. 1 were changed, although the general pattern would remain the same, the magnitudes of the peaks, lags, and period would change. Thus, there are an infinite number of cycles that could occur.

3.1 Tasks

1. Redo the original simulation with the population of predators double the original and graph the results
2. Redo the original simulation with the population of prey double the original and graph the results
3. Redo the original simulation with the population of predators half the original and graph the results
4. Redo the original simulation with the population of prey half the original and graph the results
5. Redo the original simulation with the population of both predator and prey double the original and graph the results
6. Redo the original simulation with the population of both predator and prey half the original values and graph the results
7. You may perform more variations of the simulation

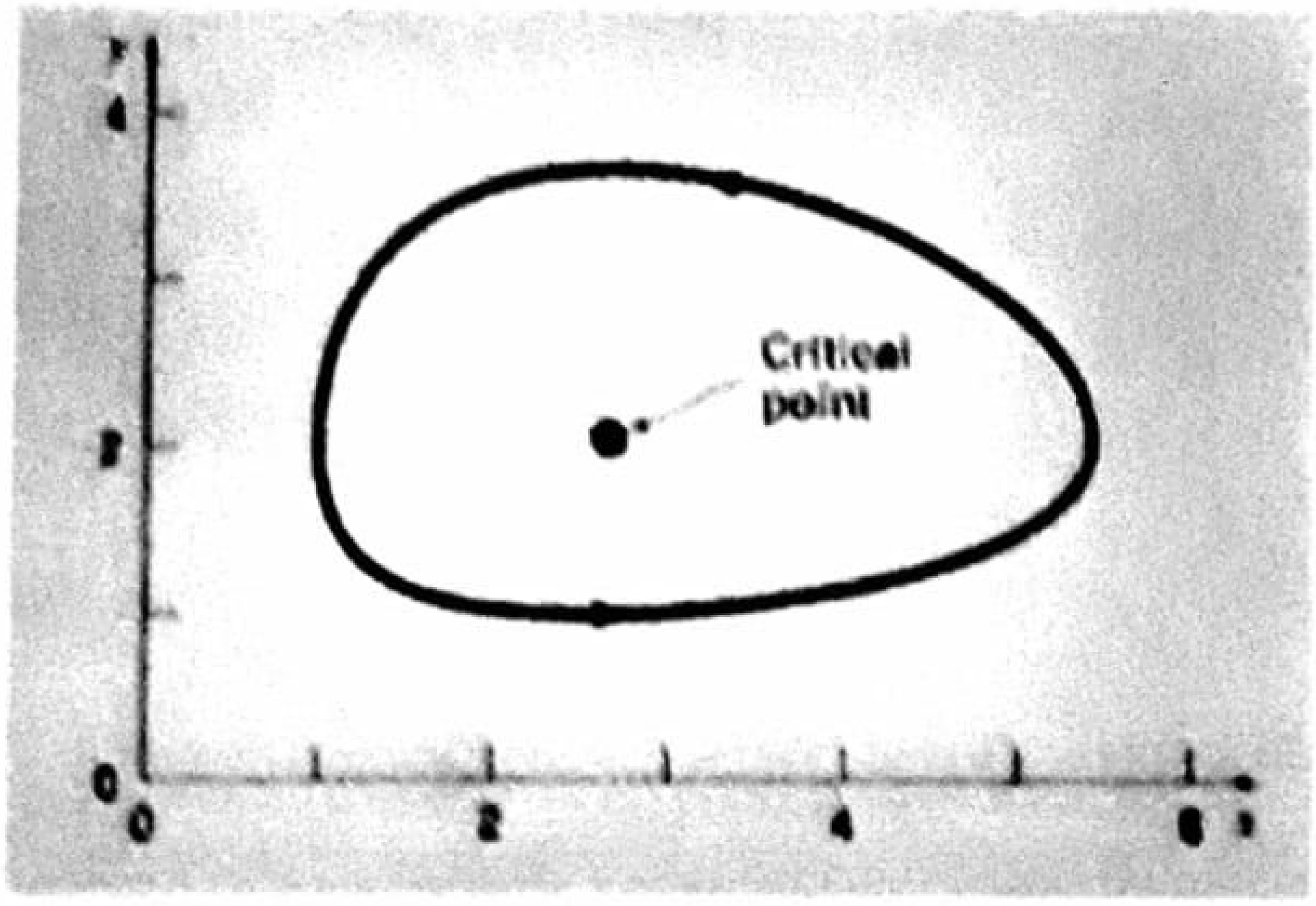

Figure 2: Phase-plane representation for the Lotka-Volterra model.

8. Compare all your graphs and explain your findings on how sensitive the system is to the initial population of predators

9. Compare all your graphs and explain your findings on how sensitive the system is to the initial population of prey

3.2 Phase-plane Solution

A phase-plane representation is useful in discerning the underlying structure of the model. Rather than plotting x and y versus t, we can plot x versus y. This plot illustrates the way that the state variables (x and y) interact, and is referred to as a *phase-plane representation*.

Figure 2 shows the phase-plane representation for the case we are studying. Thus, the interaction between the predator and the prey defines a closed counterclockwise orbit. Notice that there is a critical or rest point at the center of the orbit. The exact location of this point can be determined by setting Eqs. (1) and (2) to steady state ($dy/dt = dx/dt = 0$) and solving for $(x, y) = (0, 0)$ and $(c/d, a/b)$. The former is the trivial result that if we start with neither predators nor prey, nothing will happen. The latter is the more interesting outcome that if the initial conditions are set at $x = c/d$ and $y = a/b$, the derivative will be zero and the populations will remain constant.

3.3 Tasks

1. Plot the phase plane of the system using the values computed from RK4.

2. Determine the critical point and use both $(x, y) = (0, 0)$ and $(c/d, a/b)$

3. Find out and explain how it would be possible to change the size of the orbit to make it bigger or smaller

4. Show what will happen if the orbit touches either the x or the y axis

5. Explain what touching the axis means in terms of the populations involved

6. create a guide for 2D plotting numerical method results for the benefit of those using our Git repo