Projection

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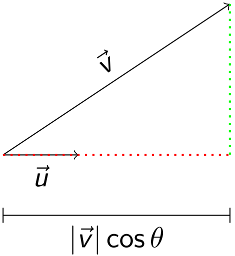
Outline

Suppose $|\vec{u}| = 1$, then

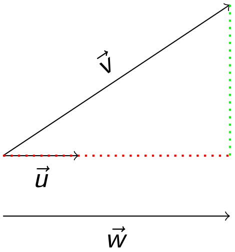
$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta = |\vec{v}| \cos\theta$$



How do we get a vector with the same direction as $\vec{u}$ but with the magnitude of $|\vec{v}| \cos\theta$?

Suppose $|\vec{u}| = 1$, and let

$$\vec{w} = (\vec{u} \cdot \vec{v})\vec{u} = (|\vec{v}| \cos\theta)\vec{u}$$



This is what we call “projection”.

What if $|\vec{u}| \neq 1$? Consider the normalized vector $\vec{u_N}$.

$$\vec{u_N} = \|\vec{u}\| = \frac{\vec{u}}{|\vec{u}|}$$

So we can get the same resultant vector by computing $(\vec{u_N} \cdot \vec{v})\vec{u_N}$. Expanding this:

$$\begin{aligned}(\vec{u_N} \cdot \vec{v})\vec{u_N} &= \left(\frac{\vec{u}}{|\vec{u}|} \cdot \vec{v}\right)\frac{\vec{u}}{|\vec{u}|} \\ &= \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|^2}\vec{u} \\ &= \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\vec{u}\end{aligned}$$

The Projection Function

Definition

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ with $\vec{u} \neq 0$, the projection of vector $\vec{v}$ into $\vec{u}$ defined as:

$$\text{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \vec{u}$$

Remark

If $|\vec{u}| = 1$ then the function simplifies to

$$\text{proj}_{\vec{u}}(\vec{v}) = (\vec{u} \cdot \vec{v}) \vec{u}$$

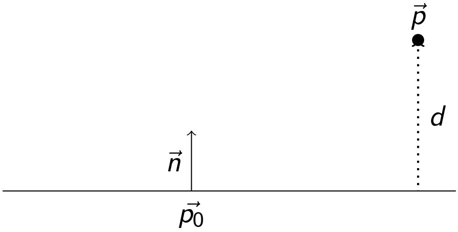
Suppose we have a vector $u \in \mathbb{R}^m$ and a matrix $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$ where $v_1, v_2 \ldots v_n \in \mathbb{R}^m$ are linearly independent vectors. How do we come up with a matrix function $\text{proj}_u(A)$ such that:

$$\text{proj}_u(A) = \begin{bmatrix} \text{proj}_u(v_1) & \text{proj}_u(v_2) & \cdots & \text{proj}_u(v_n) \end{bmatrix}?$$

$$\text{proj}_u(A) = u\frac{u^T A}{u^T u}$$

Computing Distances from a Point to a Line/Hyperplane

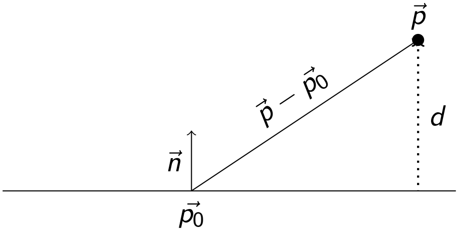
Suppose we have a point $\vec{p}$ and a line/hyperplane passing through point $\vec{p_0}$ with normal $\vec{n}$.



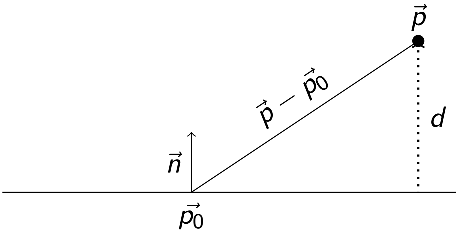
To find the distance between point $\vec{p}$ and line L. Simply compute

Point-Hyperplane Distance Formula

$$d = \frac{\vec{n} \cdot (\vec{p} - \vec{p_0})}{|\vec{n}|} = \pm \left| \, \mathrm{proj}_{\vec{n}} (\vec{p} - \vec{p_0}) \, \right|$$



$$d = \frac{\vec{n} \cdot (\vec{p} - \vec{p_0})}{|\vec{n}|}$$



Exercise

- What does the sign of the result mean?
- How would you do this for the implicit form?

Exercise

Suppose we have a line $L : \vec{l} = \vec{p_0} + t\vec{u}$ and a point $\vec{p}$. What is the distance from $\vec{p}$ to line L?

Solution

Simply compute

$$\frac{(\vec{p} - \vec{p_0}) \cdot \vec{u}^{\perp}}{|\vec{u}|}$$

Or alternatively, if we're in $\mathbb{R}^2$:

$$\frac{\vec{u} \times (\vec{p} - \vec{p_0})}{|\vec{u}|}$$

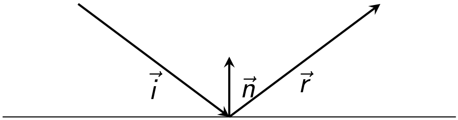
To find the interesection between a line L and a hyperplane P, simply find the solution for:

$$\frac{(\vec{p} - \vec{p_L}) \cdot \vec{u}}{|\vec{u}|} = 0$$

Where $\vec{u}$ is the normal of P, $\vec{p}$ is a point in P and $\vec{p_L}$ is the point in L to be solved.

Exercise

How would you find the intersection of two lines in parametric form?



Definition

The reflection of an incident vector $\vec{i}$, hitting a plane with normal $\vec{n}$ is:

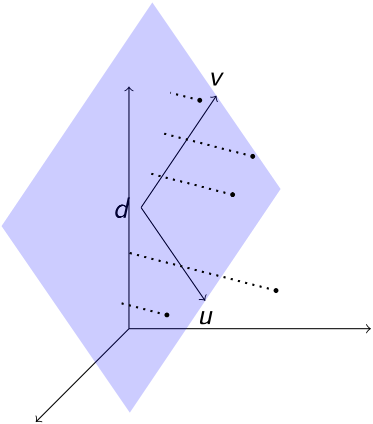
$$\vec{r}(\vec{i}, \vec{n}) = \vec{i} - 2\,\mathrm{proj}_{\vec{n}}\left(\vec{i}\right)$$

Projection of Points into a Plane

Suppose we have a set of points $S = \{p_1, p_2, \ldots, p_n\} \subset \mathbb{R}^n$ and a plane $P(s, t) = su + tv + d$ (where $u, v, d \in \mathbb{R}^n$). How do we project S orthogonally to P? That is, find the set $S' = \{p'_1, p'_2, \ldots, p'_n\}$ such that:

1. All points in S' lie in P.
2. $p_i - p'_i$ is orthogonal to both u and v for all i (i.e. $|p_i - p'_i|$ is minimized).

Point Projection Visualization



This is how many algorithms compute shadows.

To make the derivation easier, suppose that the plane passes through the origin (so $d = 0$). Let $p_i \in S$, we need to find an $a, b \in \mathbb{R}$ such that:

$$p_i = au + bv = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

However, since p_i is not in the span(u, v), we have an inconsistent solution. However, we know that $p_i - p_i'$ is orthogonal to both u and v so:

$$\left.\begin{aligned} (p_i - (au + bv)) \cdot u = 0 \\ (p_i - (au + bv)) \cdot v = 0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} p_i \cdot u = (au + bv) \cdot u \\ p_i \cdot v = (au + bv) \cdot v \end{aligned}\right\}$$

The following system

$$\left.\begin{aligned} p_i \cdot u &= (au + bv) \cdot u \\ p_i \cdot v &= (au + bv) \cdot v \end{aligned}\right\} \Rightarrow \left.\begin{aligned} u^T p_i &= u^T (au + bv) \\ v^T p_i &= v^T (au + bv) \end{aligned}\right\}$$

can be expressed in terms of a matrix equation:

$$\begin{bmatrix} u & v \end{bmatrix}^T p_i = \begin{bmatrix} u & v \end{bmatrix}^T \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

Note how this differs from the initial system that we formed that was inconsistent. Let $B = \begin{bmatrix} u & v \end{bmatrix}$, the system looks like:

$$B^T p_i = B^T B \begin{bmatrix} a \\ b \end{bmatrix}$$

Since u and v are linearly independent, B^TB is non-singular (verify). So, B^TB has an inverse, therefore:

$$\begin{bmatrix} a \\ b \end{bmatrix} = (B^TB)^{-1}B^Tp_i$$

Here, a and b are the coordinates of p_i' on the basis $\{u, v\}$. To get the actual value of p_i', simply do a linear combination:

$$p_i' = au + bv = B\begin{bmatrix} a \\ b \end{bmatrix} = B(B^TB)^{-1}B^Tp_i$$

So the projection of point p_i to a plane defined by u, v passing through the origin is:

$$\operatorname{proj}_B(p_i) = B(B^T B)^{-1} B^T p_i$$

where $B = \begin{bmatrix} u & v \end{bmatrix}$.

Remark

This formula is applicable for any matrix B formed by linearly independent vectors. (i.e. if $B = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$ such that $u_i \in \mathbb{R}^m$ for all i and $p_i \in \mathbb{R}^m$).

If the plane passes through a point p_0, then the function can be modified to:

$$\operatorname{proj}_B(p_i - p_0) + p_0 = B(B^T B)^{-1} B^T (p_i - p_0) + p_0.$$

We've managed to define

$$\text{proj}_u\,(v)$$

when u and v are both vectors. We also defined

$$\text{proj}_B\,(v) \quad \text{and} \quad \text{proj}_u\,(A)$$

when B and A is a matrix.
Can we combine both to form

$$\text{proj}_B\,(A)?$$

General Orthogonal Projection Function

Let $S = \{u_1, u_2, \ldots, u_n\} \subset \mathbb{R}^m$ be a linearly independent set and let $T = \{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^m$. Construct $B = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$ and $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$. The projection of T into span(S) is then:

$$\mathrm{proj}_B\,(A) = B(B^T B)^{-1} B^T A$$

Remarks

Note that the projection function simple degrades to the vector-vector and vector-matrix and matrix-vector projections when A or B is a $1 \times m$ matrix (i.e. a vector).