Geometry and Linear Algebra

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Outline

Notations/Conventions

- $[a_1, a_2, \ldots, a_n]^T$ is equal to $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$.
- $\langle a_1, a_2, \ldots, a_n \rangle$ is equivalent to $[a_1, a_2, \ldots, a_n]^T$.
- a, b, c, d, s, t will usually mean scalars (i.e. elements of $\mathbb{R}$).
- $\vec{u}, \vec{v}$ will usually mean vectors (the arrows are there to emphasize).
- $|\vec{u}|$ means the magnitude of vector $\vec{u}$.
- A, B will usually mean matrices.
- n, m will usually mean natural numbers (i.e. elements of $\mathbb{N}$).
- A "euclidean space" basically means $\mathbb{R}^n$ given a geometric interpretation.

Geometric Interpretation of Linear Combinations

Consider $\mathbb{R}^2$. Let $\vec{i} = \langle 1, 0 \rangle$ and $\vec{j} = \langle 0, 1 \rangle$. Then,

$$\langle a, b \rangle = a\vec{i} + b\vec{j}.$$

What if we retain a and b and try another pair of vectors? Say $\vec{u} = \langle 2, 0 \rangle$ and $\vec{v} = \langle 0, 2 \rangle$.

$$a\vec{u} + b\vec{v} = \begin{bmatrix} \vec{u} & \vec{v} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \langle 2a, 2b \rangle$$

What do you think is the effect visually?

Remark

By using different bases, we can "transform" the points.

A basis is also known as a coordinate frame.
One common question is that, suppose we have a basis $\{u_1, u_2\}$ and a vector $p = au_1 + bu_2$. What's the equivalent point to another set of basis $\{v_1, v_2\}$? That is, find s, t such that

$$au_1 + bu_2 = sv_1 + tv_2.$$

This basically boils down to the following matrix equation:

$$\begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$$

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1} \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix}$$

For certain cases, the inverse is easy to compute.

Dot Product in $\mathbb{R}^2$

Definition

Let $\vec{u} = \langle x_0, y_0 \rangle, \vec{v} = \langle x_1, y_1 \rangle$. The dot product of $\vec{u}$ and $\vec{v}$, denoted by $\vec{u} \cdot \vec{v}$ is defined as

$$\vec{u} \cdot \vec{v} = x_0 x_1 + y_0 y_1.$$

Note that the dot product maps $\mathbb{R}^2$ to $\mathbb{R}$.

Theorem

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$, then

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \theta$$

where θ is the angle between $\vec{u}$ and $\vec{v}$.

Example

Suppose we have $\vec{u} = \langle 1, 1 \rangle, \vec{v} = \langle \frac{\sqrt{3}-1}{2}, \frac{\sqrt{3}+1}{2} \rangle$. Find the angle between $\vec{u}$ and $\vec{v}$.

Example

Let A be an $n \times n$ matrix, what will the elements of $A^T A$ contain?

Example

What is $\vec{u} \cdot \vec{u}$?

Dot Products in $\mathbb{R}^n$

Definition

Let $\vec{u} = \langle a_1, a_2, \ldots, a_n \rangle$ and $\vec{v} = \langle b_1, b_2, \ldots, b_n \rangle$. Then

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} a_i b_i.$$

Equivalently, if $u, v \in \mathbb{R}^n$ then the dot product (also called the standard inner product) is written as

$$u^T v.$$

Definition

$u, v \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ are orthogonal (i.e. perpendicular) if and only if

$$u^T v = 0$$

Example

Is $\langle \frac{\sqrt{3}+1}{2}, -\frac{\sqrt{3}-1}{2} \rangle$ and $\langle \frac{\sqrt{3}-1}{2}, \frac{\sqrt{3}+1}{2} \rangle$ perpendicular to each other?

Properties of the Dot Product

Let $u, v, w \in \mathbb{R}^n$, $a \in \mathbb{R}$:

- $u^T v = v^T u$ (Commutativity)
- $(au)^T v = a(u^T v)$ (Homogeneity)
- $u^T \mathbf{0} = 0$
- $|u|^2 = u^T u \geq 0$
- $u^T u = 0 \Leftrightarrow u = \mathbf{0}$
- $w^T(u + v) = w^T u + w^T v$ (Distributivity over addition)
- $u^T v = |u||v| \cos\theta \leq |u||v|$

Example

$$w^T(au + bv) = w^T(au) + w^T(bv) = au^T w + bv^T w$$

Example

$$|-u|^2 = (-u)^T(-u) = -(u^T(-u)) = -((-u)^T u) = u^T u$$

Example

$$\begin{aligned}
|u-v|^2 &= (u-v)^T(u-v) \\
&= (u-v)^T u - (u-v)^T v \\
&= u^T u - v^T u - u^T v + v^T v \\
&= u^T u - u^T v - u^T v + v^T v \\
&= |u|^2 + |v|^2 - 2u^T v \\
&= |u|^2 + |v|^2 - 2|u||v|\cos\theta \qquad \text{(Law of Cosines)}
\end{aligned}$$

Theorem

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ then

$$\vec{u} \cdot \vec{v} \begin{cases} > 0 & \textit{If the angle between } \vec{u} \textit{ and } \vec{v} \textit{ is acute.} \\ = 0 & \textit{If the angle between } \vec{u} \textit{ and } \vec{v} \textit{ is right.} \\ < 0 & \textit{If the angle between } \vec{u} \textit{ and } \vec{v} \textit{ is obtuse.} \end{cases}$$

Cross Product in $\mathbb{R}^2$

Definition

Let $\vec{u} = \langle x_0, y_0 \rangle, \vec{v} = \langle x_1, y_1 \rangle$. The cross product of u and v, denoted by $u \times v$ is defined as

$$\vec{u} \times \vec{v} = x_0 y_1 - x_1 y_0.$$

Note that the cross product maps $\mathbb{R}^2$ to $\mathbb{R}$.

Theorem

Let $\vec{u}, \vec{v} \in \mathbb{R}^2$, then

$$\vec{u} \times \vec{v} = |\vec{u}||\vec{v}| \sin \theta$$

where θ is the angle between $\vec{u}$ and $\vec{v}$.

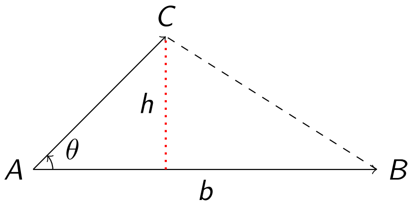
Properties of the Cross Product

Let $u, v, w \in \mathbb{R}^2$, $a \in \mathbb{R}$:

- $u \times v = -(v \times u)$ (Anticommutativity)
- $u \times (v + w) = u \times v + u \times w$
- $(v + w) \times u = v \times u + w \times u$
- $au \times v = u \times av = a(u \times v)$
- Let $u = [x_0, y_0]^T, v = [x_1, y_1]^T$ then $u \times v = \det \begin{pmatrix} x_0 & y_0 \\ x_1 & y_1 \end{pmatrix}$

Area of a Triangle

Suppose we have triangle ABC. Find the area of the triangle.

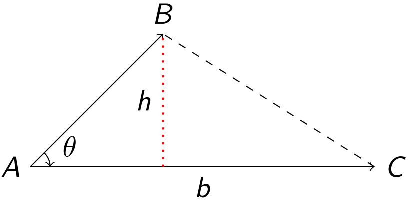


Let $\vec{u} = B - A$, $\vec{v} = C - A$.

$$\text{Area} = \frac{bh}{2} = \frac{|\vec{u}|(|\vec{v}|\sin\theta)}{2} = \frac{\vec{u}\times\vec{v}}{2}$$

Triangle Order

$\frac{\vec{u}\times\vec{v}}{2}$ is signed, when will the computation of the area be negative? Suppose we have the same triangle ABC but labeled differently:



Again, let $\vec{u} = B - A$, $\vec{v} = C - A$:

$$\frac{\vec{u}\times\vec{v}}{2} = \frac{|\vec{u}||\vec{v}|\sin(-\theta)}{2} = -\frac{bh}{2} = -\text{Area}$$

Theorem

Given triangle ABC, let $\vec{u} = B - A$ *and* $\vec{v} = C - A$.

$$u \times v = \begin{cases} > 0 & \text{if } ABC \text{ is counter-clockwise} \\ < 0 & \text{if } ABC \text{ is clockwise} \\ = 0 & \text{if } ABC \text{ is degenerate} \end{cases}$$

Cross Product in $\mathbb{R}^3$

Definition

Let $\vec{u} = \langle x_0, y_0, z_0 \rangle, \vec{v} = \langle x_1, y_1, z_1 \rangle$. The cross product of u and v, denoted by $u \times v$ is defined as

$$\vec{u} \times \vec{v} = \langle y_0 z_1 - y_1 z_0, -(x_0 z_1 - x_1 z_0), x_0 y_1 - x_1 y_0 \rangle.$$

This maps $\mathbb{R}^3$ to $\mathbb{R}^3$.

The direction of the resultant vector can be visualized using the right hand rule.

Theorem

Let $\vec{u}, \vec{v} \in \mathbb{R}^3$, then

$$\vec{u} \times \vec{v} = \vec{w}(|\vec{u}||\vec{v}| \sin \theta)$$

where θ is the angle from $\vec{u}$ to $\vec{v}$ and $\vec{w}$ is the normalized vector perpendicular to $\vec{u}$ and $\vec{v}$ using the right hand rule.

remark

The cross product in $\mathbb{R}^3$ is the way to obtain a vector perpendicular to two vectors. This is one of the computations used for backface culling.

Some Additional Properties of the Cross Product in $\mathbb{R}^3$

- $(\vec{u} \times \vec{v}) \cdot \vec{u} = 0$ and $(\vec{u} \times \vec{v}) \cdot \vec{v} = 0$.
- $(\vec{u} \times \vec{v}) \cdot \vec{w} = \vec{u} \cdot (\vec{v} \times \vec{w})$
- Let $\vec{u} = \langle x_0, y_0, z_0 \rangle$ and $\vec{v} = \langle x_1, y_1, z_1 \rangle$ then
$$\vec{u} \times \vec{v} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_0 & y_0 & z_0 \\ x_1 & y_1 & z_1 \end{pmatrix}$$

Let $\vec{i} = \langle 1, 0, 0 \rangle, \vec{j} = \langle 0, 1, 0 \rangle$ and $\vec{k} = \langle 0, 0, 1 \rangle$ then:

- $\vec{i} \times \vec{j} = \vec{k}$.
- $\vec{j} \times \vec{k} = \vec{i}$.
- $\vec{k} \times \vec{i} = \vec{j}$.

Recall that vectors $\vec{u}$ and $\vec{v}$ are orthogonal if $\vec{u} \cdot \vec{v} = 0$.

Pythagorean Theorem

$$|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2.$$

Coordinate Retrieval

If $\vec{w} \in \text{span}(\vec{u}, \vec{v})$ so $\exists a, b : \vec{w} = a\vec{u} + b\vec{v}$ then

$$\begin{aligned} \frac{\vec{w} \cdot \vec{u}}{|\vec{u}|^2} &= \frac{(a\vec{u} + b\vec{v}) \cdot \vec{u}}{|\vec{u}|^2} = \frac{a\vec{u} \cdot \vec{u} + b\vec{v} \cdot \vec{u}}{|\vec{u}|^2} \\ &= \frac{a\vec{u} \cdot \vec{u}}{|\vec{u}|^2} = a \end{aligned}$$

The same goes for $\vec{v}$.

Definition

Let $\mathcal{V}$ be a vector space, the orthogonal complement of $\mathcal{V}$, denoted by $\mathcal{V}^\perp$ is:

$$\mathcal{V}^\perp = \{u : u^T v = 0 \quad \forall v \in \mathcal{V}\}$$

Note: $\mathcal{V}^\perp$ forms a vector space.

Definition

Define the unary operator $\perp$ on $\mathbb{R}^2$ to be:

$$\langle a, b\rangle^\perp = \langle -b, a\rangle$$

Example

Let $\vec{v} \in \mathbb{R}^2$, what would the orthogonal complement of $\text{span}(\vec{v})$ look like?

Example

Let $\vec{v} \in \mathbb{R}^3$, what would the orthogonal complement of $\text{span}(\vec{v})$ look like?

Example

Let $\{\vec{v}, \vec{u}\}$ be a basis in $\mathbb{R}^3$, what would the orthogonal complement of $\text{span}(\vec{v}, \vec{u})$ look like?

Definition

A basis $\{u_1, u_2, \ldots, u_n\}$ is orthogonal if:

$$u_i \cdot u_j = \begin{cases} 0 & i \neq j \\ c_i \neq 0 & i = j \end{cases}$$

for all $i, j \leq n$.
If $c_i = 1 \; \forall i$ (i.e. $|u_i| = 1$) then the basis is also said to be orthonormal.

Let $A = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$ where $u_1, u_2, \ldots, u_n$ form an orthogonal basis.

- What is $A^T A$?
- How do we get the coordinates of a vector v with respect to $\begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$?
- If $u_1, u_2, \ldots, u_n$ form an orthonormal basis, what is $A^T A$?
- How do we extract the coordinates for an orthonormal basis?