Geometry and Linear Algebra 2

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Outline

1. Line Representations
   - Explicit Form
   - Two-point Form
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   - Implicit Form

2. Plane Representations
   - Three-point Form
   - Parametric Form
   - Implicit Form

Explicit Form

First, we have the classic representation of a line in $\mathbb{R}^2$ from your High School math class:

$$y = mx + b,$$

or the calculus version:

$$f(x) = mx + b.$$

Where m is the slope and b is the y-intercept. This only requires 2 numbers for storage. However, this has one big problem: it cannot represent vertical lines. Another problem is that it cannot represent a line going from right to left.

One way to fix this problem is to avoid division altogether, so we have this familiar point-slope form:

$$(y - y_0)b = (x - x_0)a$$

However, we now lose the explicitness and now require 4 numbers for storage.

The $\mathbb{R}^3$ version of the explicit form is also not very "explicit":

$$(x - x_0)a = (y - y_0)b = (z - z_0)c$$

where x_0, y_0, z_0 is the x, y, and z-intercepts respectively and a, b, c is the slope of the line in the x, y and z direction.

Two-point Form

From your geometry lessons, two points always determines a line (even if a line contains an infinite number of points). So we can just store two points, $p_0, p_1 \in \mathbb{R}^n$.

Pros:

- Can represent any line.
- Can represent line segments.
- Lines can have a direction. Reverse p_0 and p_1 and you have a line going in the opposite direction.

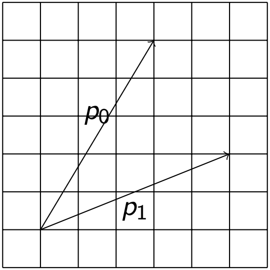
However, this form is usually not useful for computations because it is not a function/equation. It needs to be converted to other forms first.

Parametric Form

Do a minor modification to the Two-point Form to turn it to a function. Let $p_0, p_1 \in \mathbb{R}^n$ be points as in the two-point form, then

$$\vec{l}(t) = (1-t)p_0 + tp_1.$$

This is basically the linear interpolation function between any two points.



Rearranging the terms, we get

$$\vec{l}(t) = p_0 + t(p_1 - p_0).$$

Functions like $\vec{l}(t)$ are called "vector functions" since they are functions that return a vector as opposed to a single value as what you've learned from your previous math classes.

Definition

Let $p_0, \vec{u} \in \mathbb{R}^n$ where p_0 is any point on the line and $\vec{u}$ is the direction of the line. Then, the parametric representation of the line is

$$\vec{l}(t) = p_0 + t\vec{u}.$$

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- Similar to the explicit form, this form makes it easy to generate points on the line.
- Note that this form generalizes easily to lines in any dimension.
- Requires $2n$ numbers for storage where n is the dimension (e.g. 4 numbers for $\mathbb{R}^2$).
- The direction of the line can be flipped by negating $\vec{u}$.
- In $\mathbb{R}^2$, the perpendicular of the line can be obtained by computing $\vec{u}^\perp$.
- This form can represent a line segment by limiting the range of t to $[0, 1]$.

Implicit Form

The implicit form of a line in $\mathbb{R}^2$ should look familiar from high school:

$$ax + by + c = 0$$

Where a point x, y is on the line when the above equation is true. What your teachers, may not have told you, is where this equation REALLY comes from (no, it's not just a rearrangement of the explicit form).
Note that in $\mathbb{R}^3$, this actually becomes an equation for a plane and in higher dimensions, an equation of this form in $\mathbb{R}^n$ is the equation of an $n - 1$ dimensional *hyperplane*. In $\mathbb{R}^2$, this just happens to be a line.

Let $\vec{n} \in \mathbb{R}^2$ and consider all $\vec{p} \in \mathbb{R}^2$ that are orthogonal to $\vec{n}$:

$$\vec{p} \cdot \vec{n} = 0.$$

What does this look like?
One issue with this form is that the line is always centered at the origin. This can be fixed introducing an offset, $\vec{p_0}$:

$$(\vec{p} - \vec{p_0}) \cdot \vec{n} = 0.$$

The $(\vec{p} - \vec{p_0})$ term basically shifts the entire loci of points to the origin.

Remark

$\vec{n}$ here is what we call the "normal" of the line/plane.

$$(\vec{p} - \vec{p_0}) \cdot \vec{n} = 0.$$

Now let $\vec{n} = \langle a, b \rangle$, $\vec{p} = \langle x, y \rangle$ and $\vec{p_0} = \langle x_0, y_0 \rangle$. The above equation then becomes:

$$\begin{aligned} \langle x - x_0, y - y_0 \rangle \cdot \langle a, b \rangle &= 0 \\ a(x - x_0) + b(y - y_0) &= 0 \\ ax + by - (ax_0 + by_0) &= 0 \end{aligned}$$

Now let $c = -(ax_0 + by_0) = -\vec{p_0} \cdot \vec{n}$ and you have the familiar:

$$ax + by + c = 0$$

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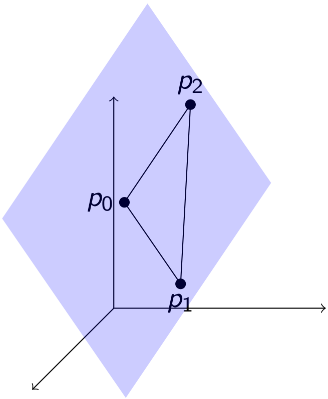
- The form requires $n + 1$ numbers for storage where n is the dimension (e.g. for $\mathbb{R}^2$ we need 3 numbers).
- Can represent any line.
- Manipulating $\langle a, b \rangle$ changes the direction of the line.
- Manipulating c changes the position (but not the direction) of the line.
- $\langle a, b \rangle$ points to the "left" of the line.
- It's easy to convert from parametric to implicit. (How?)
- Easy to query whether a point is on the line.
- Hard to use when you want to generate points on the line.

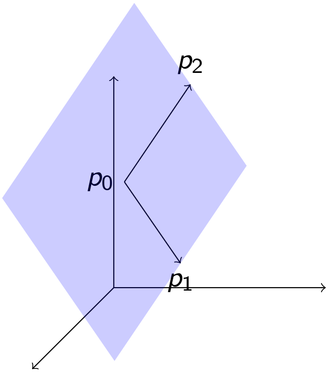
Three-point Form

Three non-collinear points determine a plane in $\mathbb{R}^3$. Like lines in $\mathbb{R}^2$, planes can have one of two facing sides depending on the order of specification (clockwise or counter-clockwise).



Similar to the line in $\mathbb{R}^2$, a parametric representation can be formed from the three-point form. Let $\vec{p_0}, \vec{p_1}, \vec{p_2} \in \mathbb{R}^3$ be points on the plane. Then, the parametric representation $\vec{p}(s, t)$ is:

$$\vec{p}(s, t) = s(\vec{p_1} - \vec{p_0}) + t(\vec{p_2} - \vec{p_0}) + \vec{p_0}$$



Let $\vec{u} = (\vec{p_1} - \vec{p_0})$ and $\vec{v} = (\vec{p_2} - \vec{p_0})$, then

$$\vec{p}(s,t) = s\vec{u} + t\vec{v} + \vec{p_0}$$

- Since a plane is a 2-dimensional vector space, we now have 2 parameters.
- Determining whether a point is in the plane now involves solving a (3 x 2) linear system which is non-trivial.
- The representation can be used to represent a parallelogram by restricting the range of s, t to $[0, 1]$.
- The normal of the plane is obtained by computing $\vec{u} \times \vec{v}$.

Implicit Form

The implicit form of a line in $\mathbb{R}^2$ can be extended to form a plane in $\mathbb{R}^3$.

$$ax + by + cz + d = 0$$

where $\langle a, b, c \rangle$ is the normal of the plane.

Exercise

Given 3 points $\vec{p_0}, \vec{p_1}, \vec{p_2}$, how do we obtain the parameters for the implicit form?