#### Hypothesis Testing

February 28, 2019
Data Science CSCI 1951A
Brown University

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HTAs: Wennie Zhang, Maulik Dang, Gurnaaz Kaur

#### Announcements

- Nothing...
- Do you have any announcements?

#### Follow up from last time

- That expected value question...
  - Was going for:  $0 = 0.1(100 \cos t) 0.9(\cos t)$
- Question: Continuous pdfs! Does it make sense to talk about these (as opposed to just the cdf)?
  - Yes!

#### Today

- Hypothesis Testing and Pvalues
- Law of Large Numbers/Central Limit Theorem
- Z-tests and T-tests
- Chi-Squared Tests

I will not talk about biased coins. I will not talk about biased coins.

"I swear literally like 80% of the answers are just (b)"

c d b d b a d c b c b d

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c d b d b a d c b c b d

Probability of this?

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 d
 b
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Probability of this?

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AND/OR

 Set parameters of the model based on data, try to make predictions

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- Make assumptions about the underlying model
- Now -
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AND/OR

 Set parameters of the model based on data, try to make predictions

LUCE

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Probability of this?

$$\langle \Omega, F, P \rangle$$

$$\langle \Omega, F, P \rangle$$
 {b, not b}

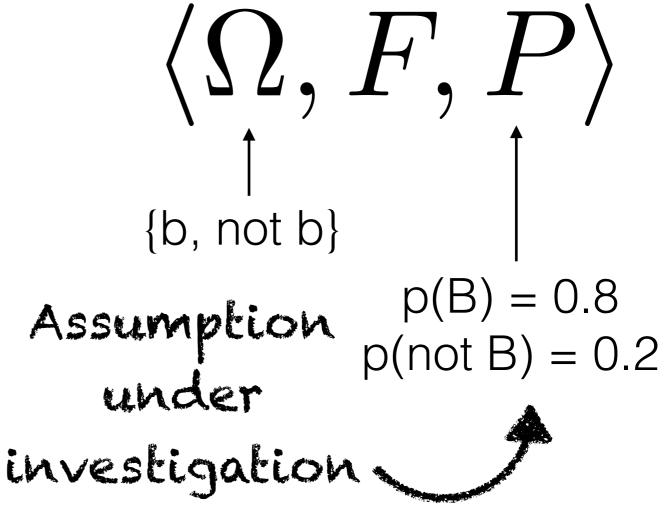
$$\langle \Omega, F, P \rangle$$
 {b, not b} 
$$p(B) = 0.8$$
 
$$p(not B) = 0.2$$

"I swear literally like 80% of the answers are just (b)"

 c
 d
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 b
 c
 b
 d



"I swear literally like 80% of the answers are just (b)"

$$\langle \Omega, F, P \rangle$$
 $\uparrow$ 
{b, not b}  $p(B) = 0.8$ 

(almost like a...biased coin...? no wait what who said that ew stop.)

$$\langle \Omega, F, P \rangle$$
 $\uparrow$ 
{b, not b}  $p(B) = 0.8$ 

$$0.2 \times 0.2 \times 0.8 \times 0.2 \times 0.8 \times 0.2 \times 0.8 \times 0.2 \times 0.2 \times 0.8 \times 0.2 \times 0.8 \times 0.2 = 0.00000105$$

"I swear literally like 80% of the answers are just (b)"

cdbd

There is literally only like a 0.000105% chance you are right!

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 $\uparrow$ 
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"I swear literally like 80% of the answers are just (b)"

#### What is the probability of this event?

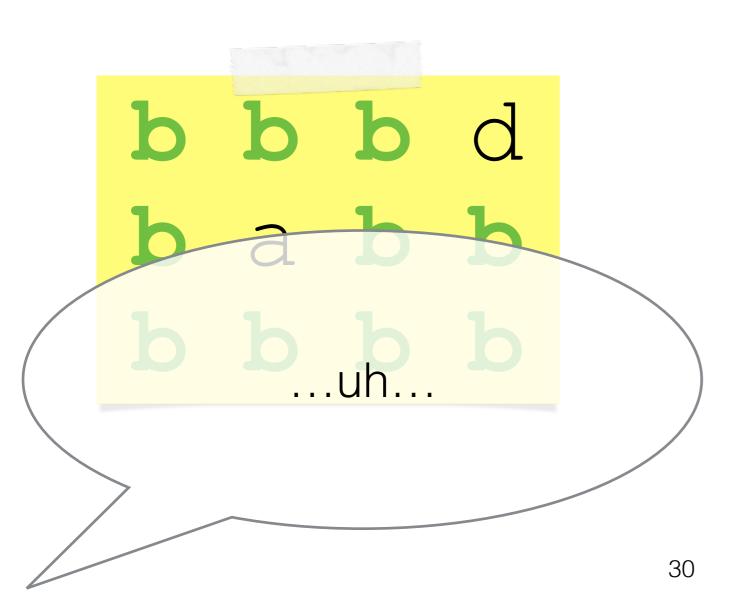
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#### What is the probability of this event?

$$0.8 \times 0.8 \times 0.8 \times 0.2 \times 0.8 \times 0.2 \times 0.8 = 0.004$$



$$\langle \Omega, F, P \rangle$$
 $\uparrow$ 
 $\{ b, \text{ not b} \} p(B) = 0.8$ 
 $0.8 \times 0.8 \times 0.8 \times 0.2 \times 0.8 \times 0.2 \times 0.8 \times 0.8$ 

"I swear literally like 80% of the answers are just (b)"

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Probability of this?

"I swear literally like 80% of the answers are just (b)"

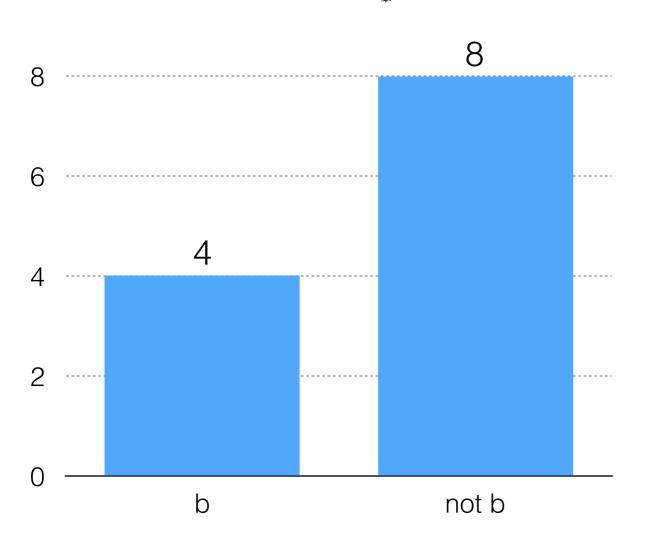
 c
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 d

 b
 a
 d
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 d

Proparity of this?

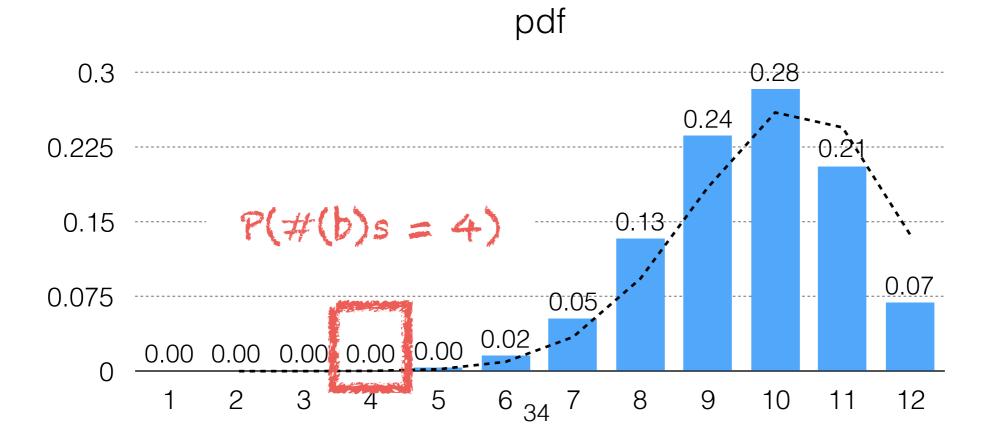
Probability of anything as surprising as this



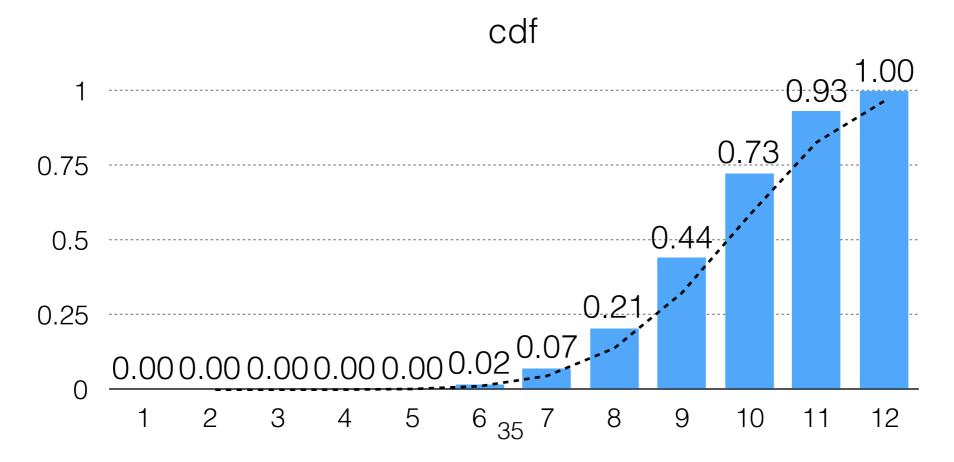
$$\langle \Omega, F, P \rangle$$
 {b, not b} p(B) = 0.8

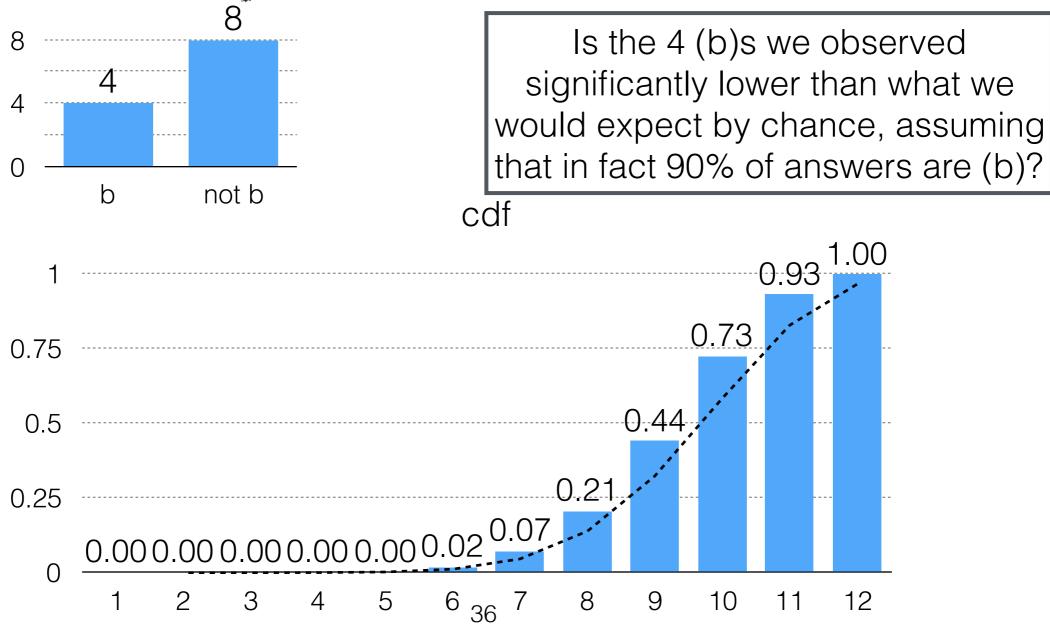
$$X = number of (b)s$$

$$f(k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad \qquad \begin{cases} \langle \Omega, F, P \rangle \\ & \uparrow \\ \text{\{b, not b\} p(B) = 0.8} \end{cases}$$

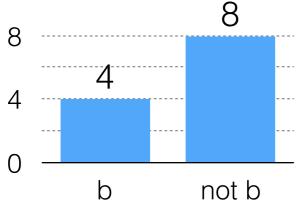


$$\langle \Omega, F, P \rangle$$
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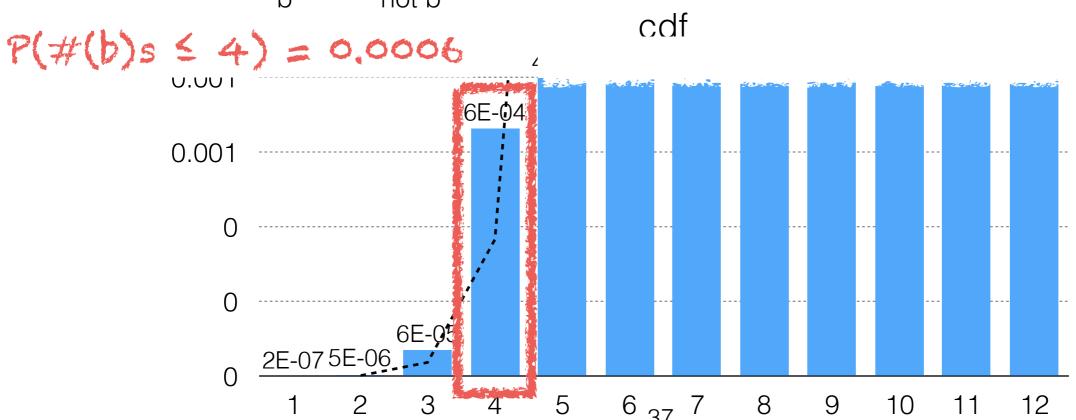


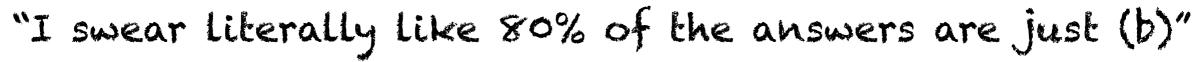


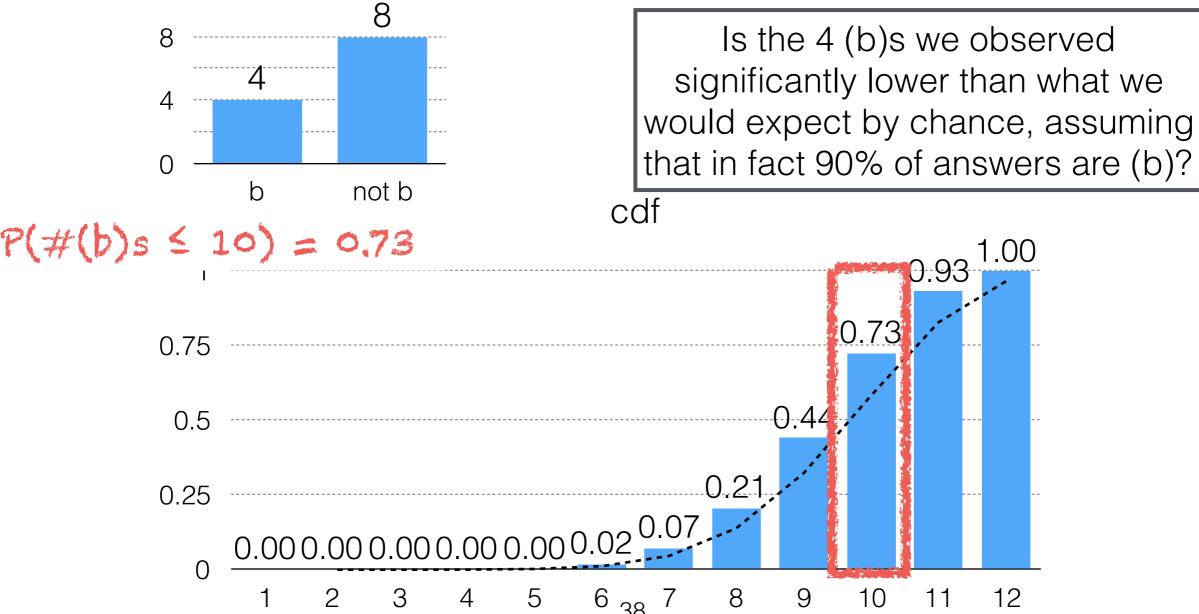
"I swear literally like 80% of the answers are just (b)"



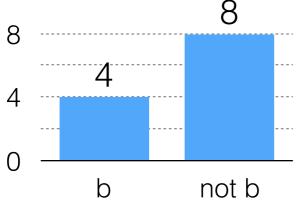
Is the 4 (b)s we observed significantly lower than what we would expect by chance, assuming that in fact 90% of answers are (b)?



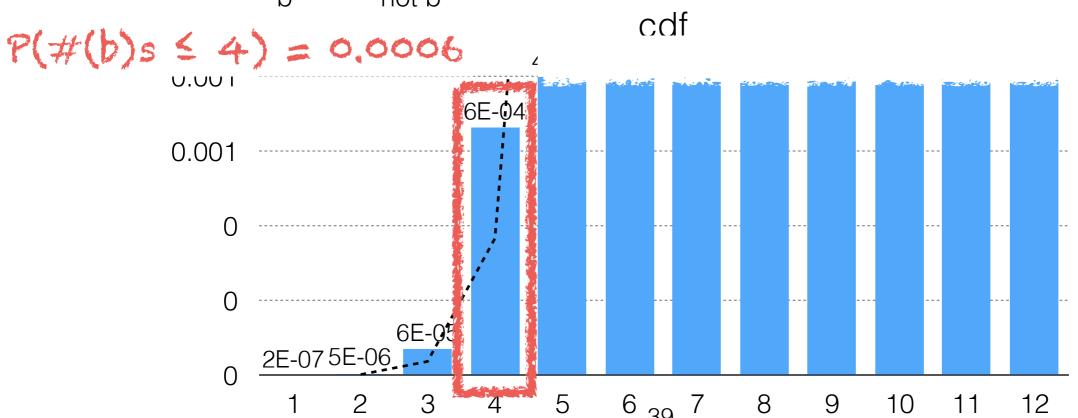




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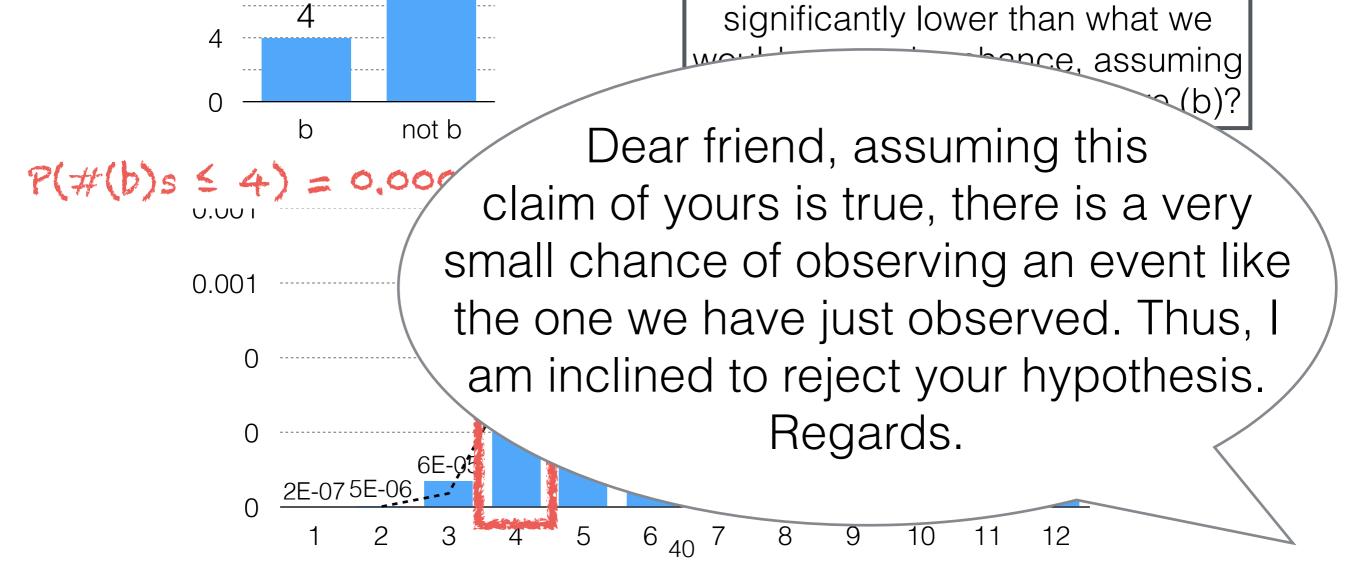


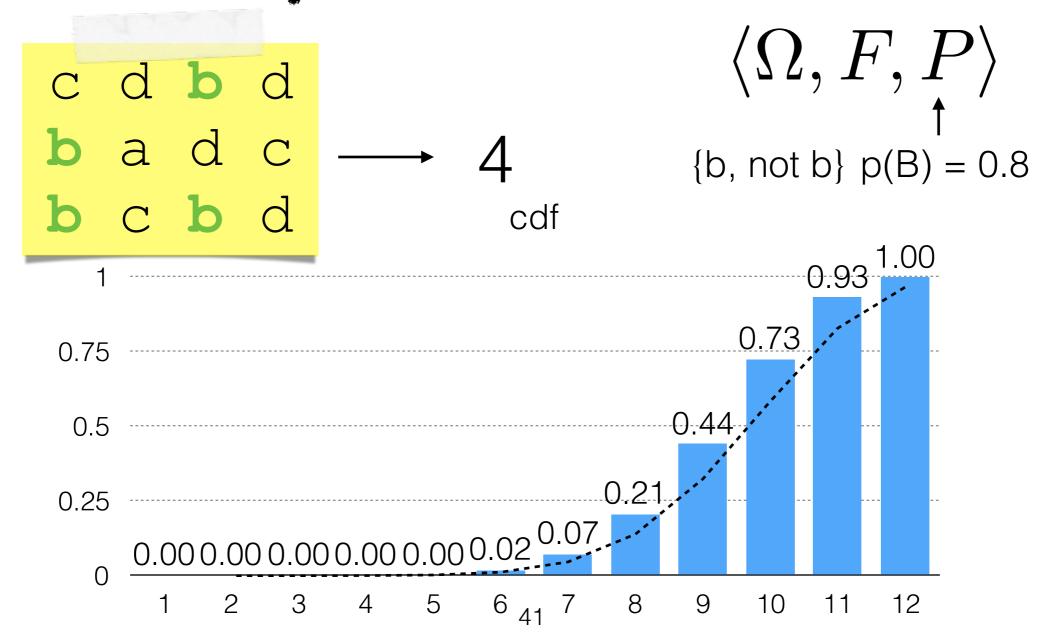
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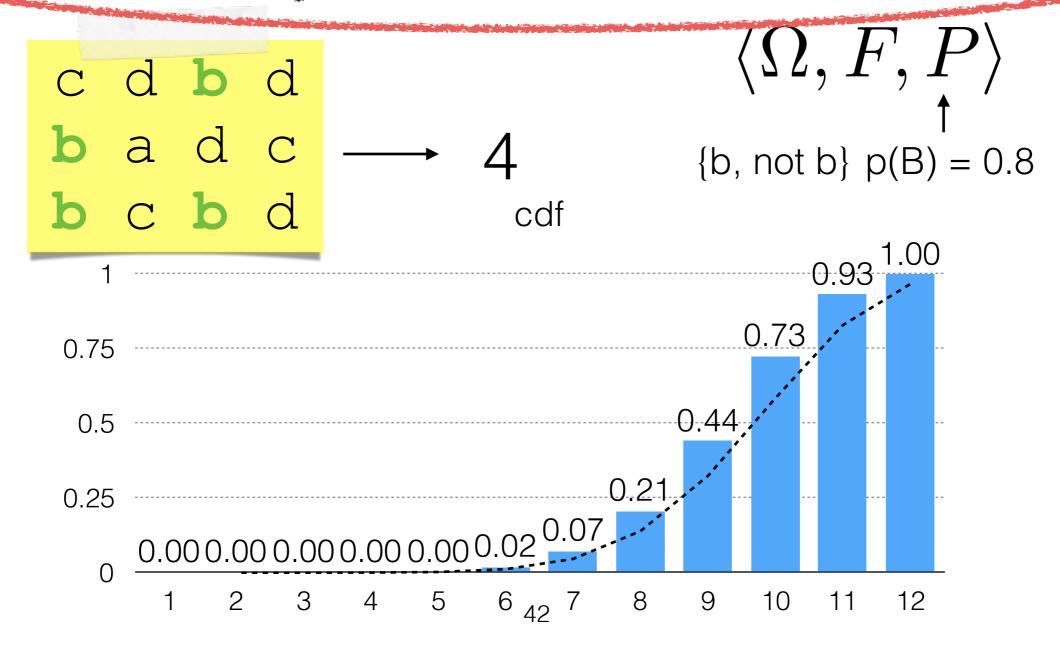


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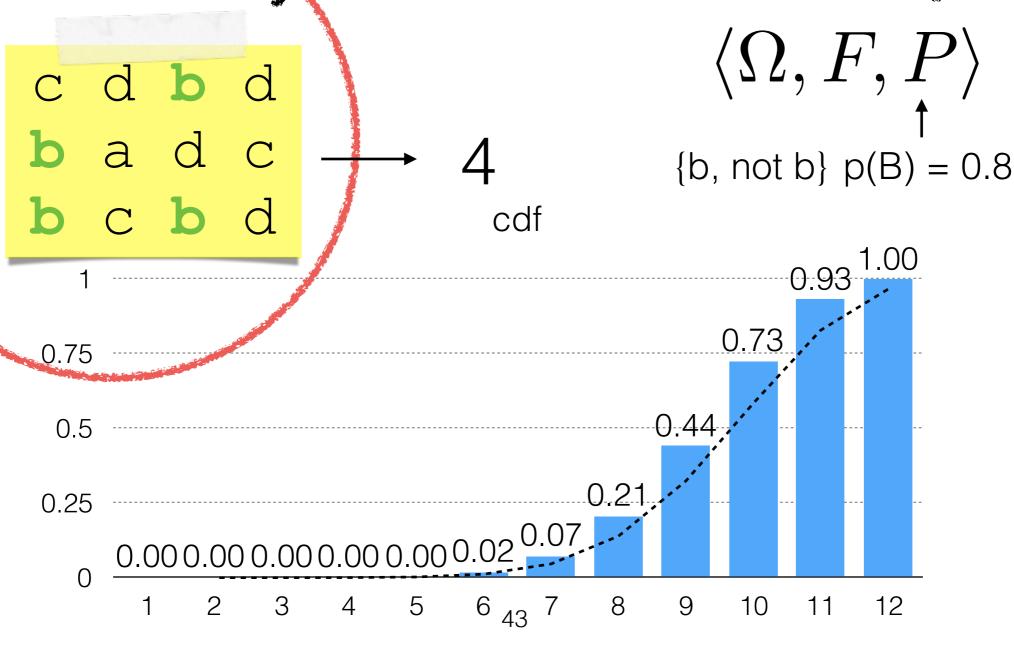
Is the 4 (b)s we observed



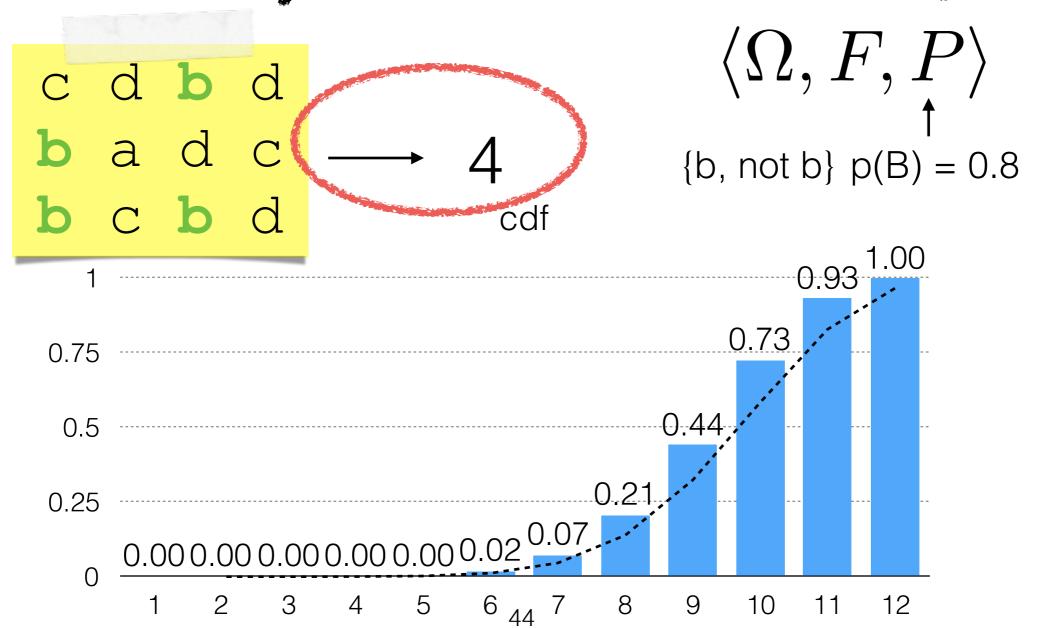




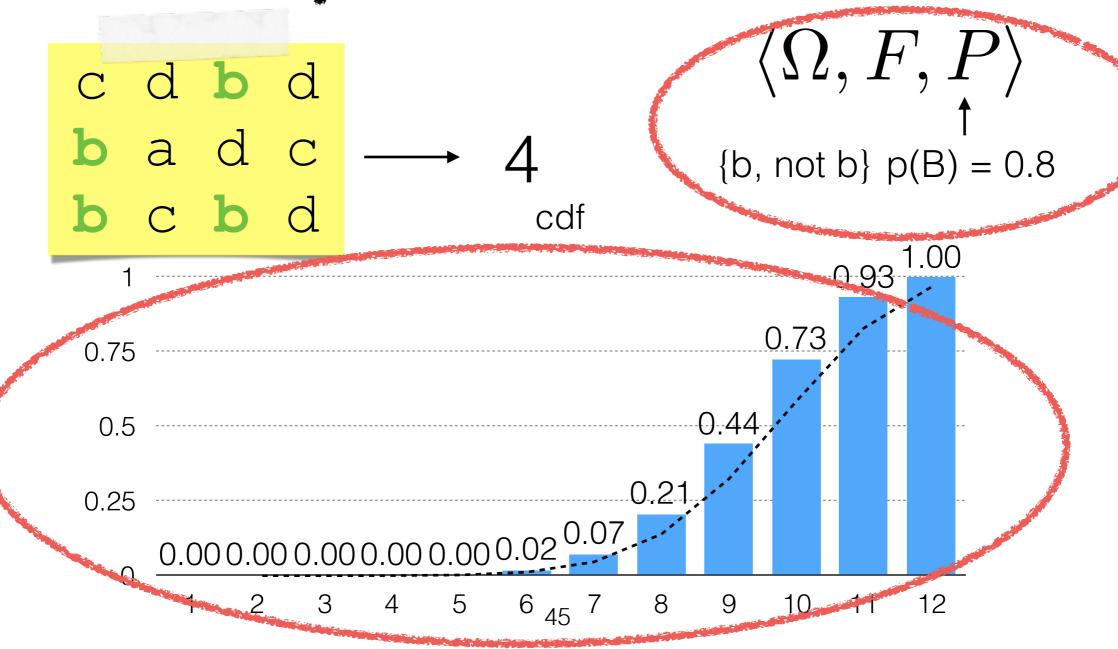
### Observation/Sample

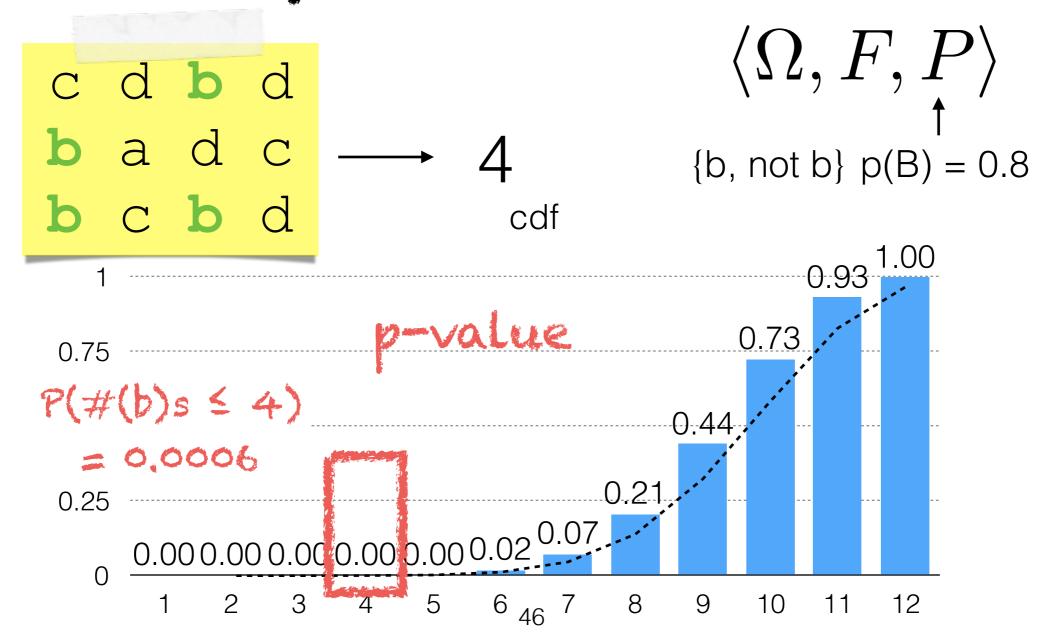


#### Test Statistic



#### Theoretical Distribution





### Given all of this, is your friend wrong?

a) Yes! b) No...

c d b d 
$$\langle \Omega, F, P \rangle$$
 b a d c  $\langle \Omega, F, P \rangle$  b c b d  $\langle \Omega, F, P \rangle$  b, not b} p(B) = 0.8 cdf  $\langle \Omega, F, P \rangle$  b, not b} p(B) = 0.8 cdf  $\langle \Omega, F, P \rangle$  b, not b} p(B) = 0.8 cdf  $\langle \Omega, F, P \rangle$  cdf  $\langle \Omega$ 

### **Piscussion Question!**

### Given all of this, is your friend wrong?

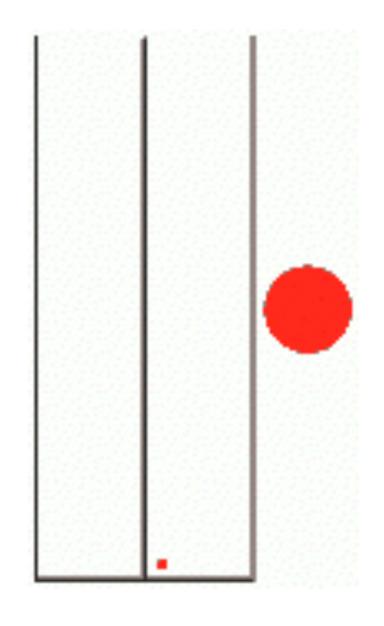
a) Yes! b) No...

c d b d 
$$\langle \Omega, F, P \rangle$$
 b a d c  $\rightarrow$  4 {b, not b} p(B) = 0.8 cdf  $\rightarrow$  4 {b, not b}  $\rightarrow$  0.93  $\rightarrow$  0.00  $\rightarrow$  0.25  $\rightarrow$  0.00  $\rightarrow$ 

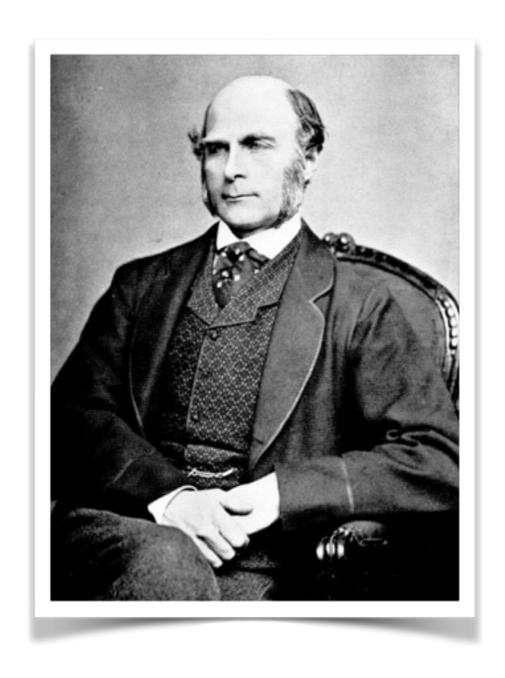
## Law of Large Numbers

- If you perform the same experiment a large number times, the average will converge to the expected value
- Assumes that errors are "random" and uncorrelated, so will balance out over time

$$\bar{X}_n = \frac{1}{n}(X_1 + \dots + X_n)$$
$$\bar{X}_n \to \mu \text{ as } n \to \infty$$

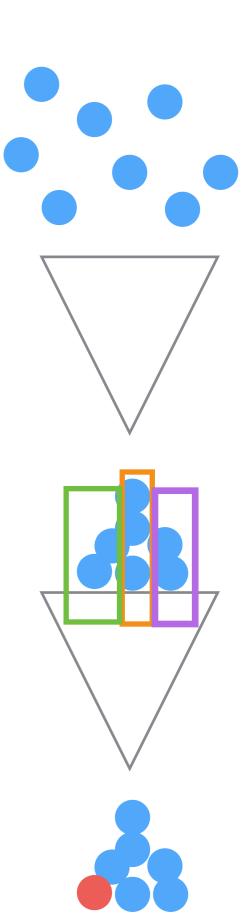






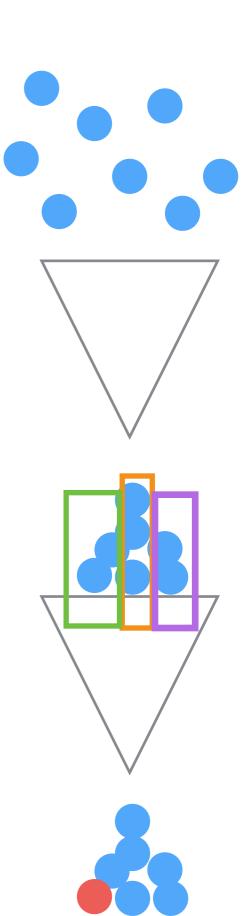
## Where is the red ball most likely to have come from?

- a) Green Region
- b) Orange Region
- c) Purple Region



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a) Green Region b) Orange Region c) Purple Region

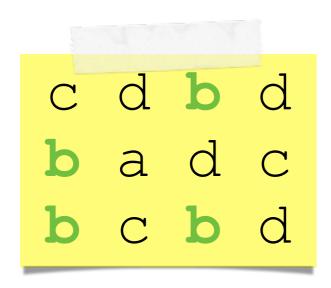


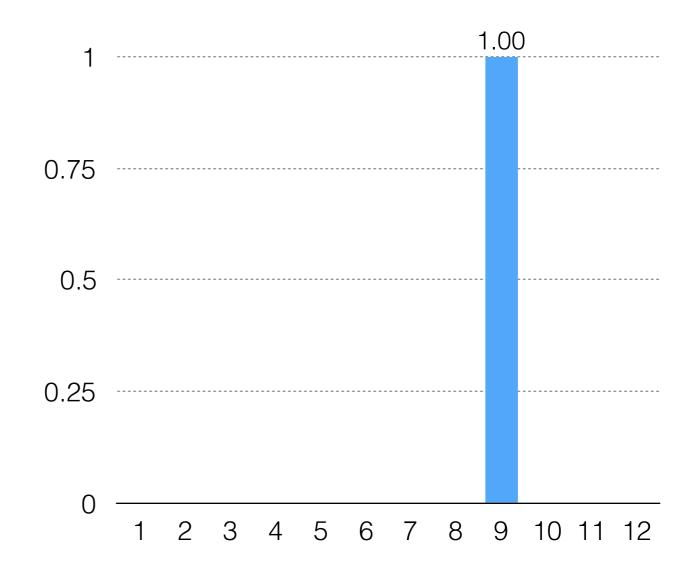
 On repeated measurement, extreme values are likely to become closer to the mean

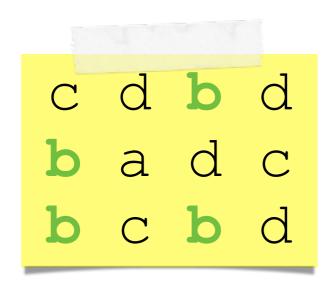
- On repeated measurement, extreme values are likely to become closer to the mean
- Usually applied to repeated measurements of random variables

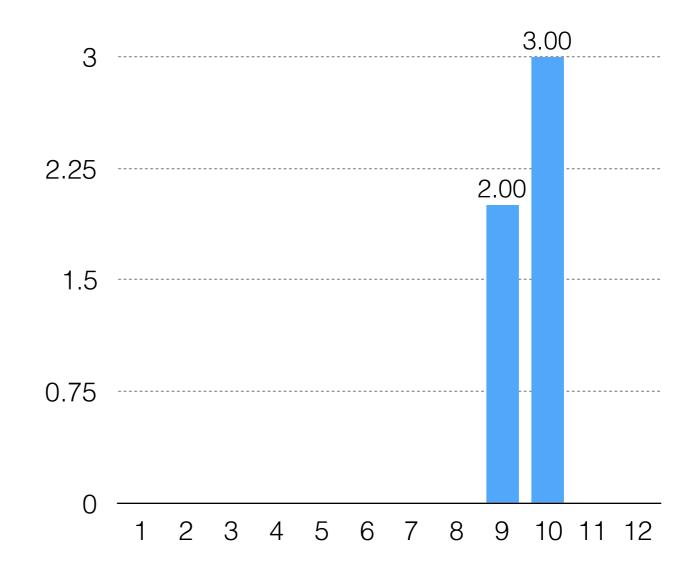
- On repeated measurement, extreme values are likely to become closer to the mean
- Usually applied to repeated measurements of random variables
- Historical observations related to population genetics

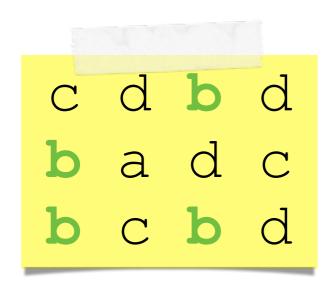
- Given  $X_1 \dots X_n$
- Not only does a  $\bar{X}_n o \mu \text{ as } n o \infty$
- But the distribution approaches a normal distribution

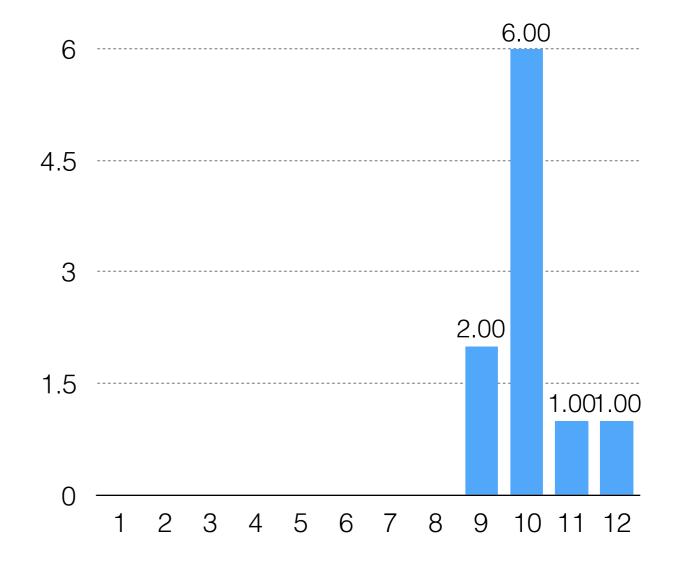


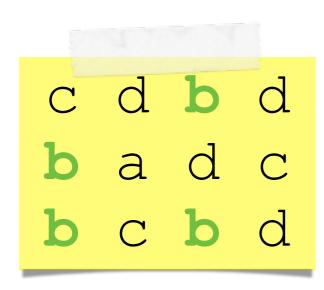


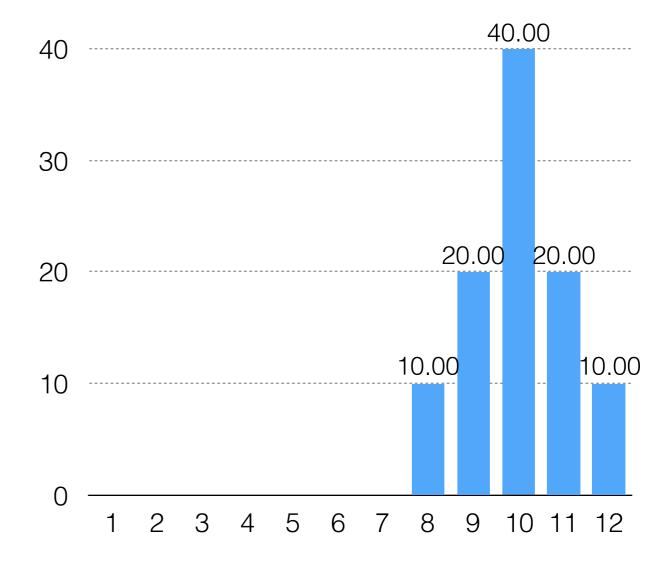


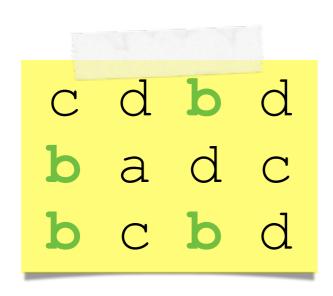




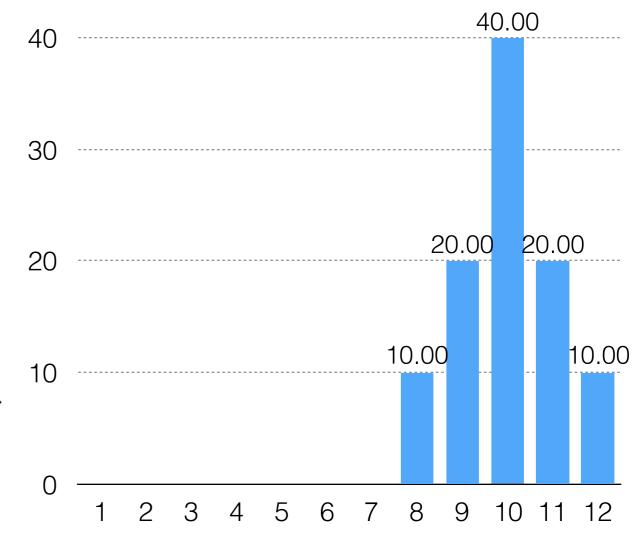




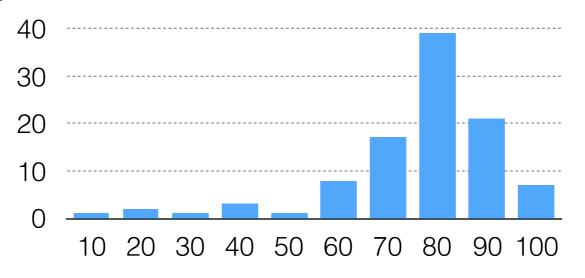




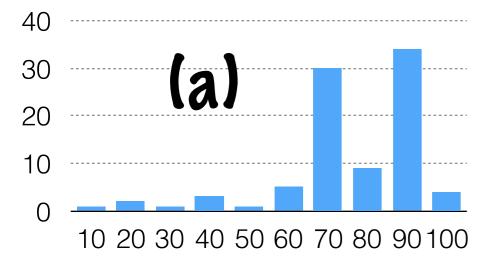
Can apply statistical methods designed for normal distributions even when underlying distribution is not normal

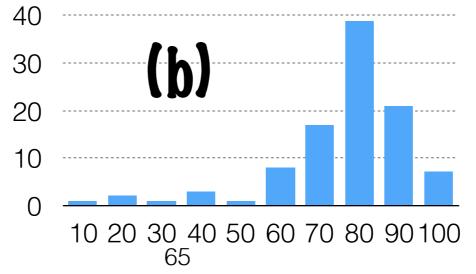


Every year, I compute the mean grade in my class. I never change the material or my methods for evaluating because, lazy. Over the 439 years that I have been teaching this class, this has resulted in the below distribution.



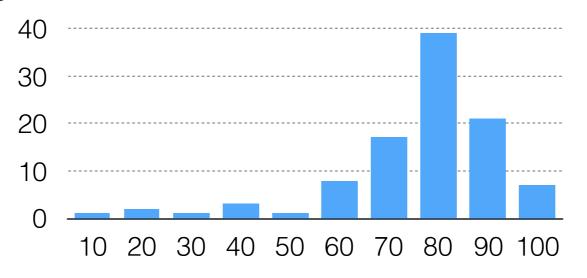
Which of these is mostly like the typical distribution on any given year?



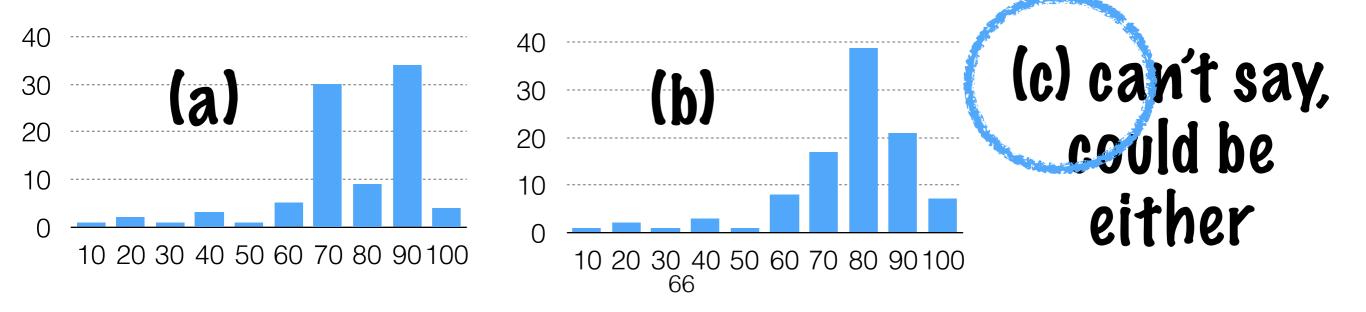


(c) can't say, could be either

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Which of these is mostly like the typical distribution on any given year?



mater have Learning the man grade in my class I reversebance the mater have Central Limit Theorem: repeated on.

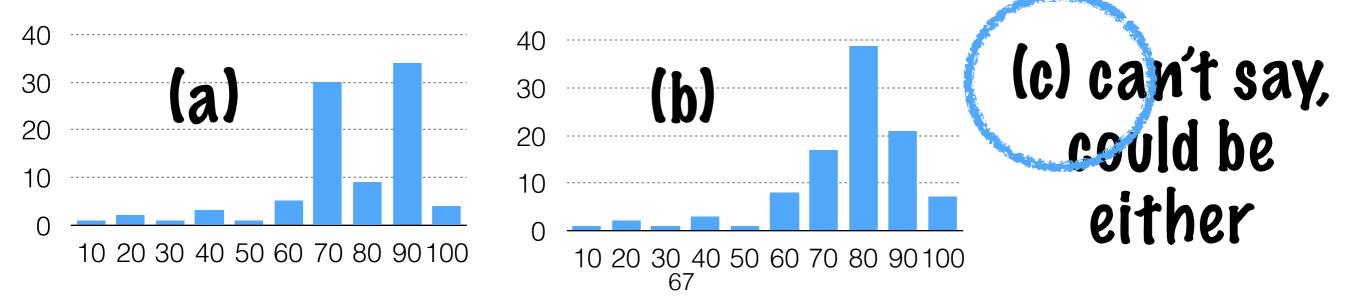
measures of mean will be normally

distributed, doesn't assume the

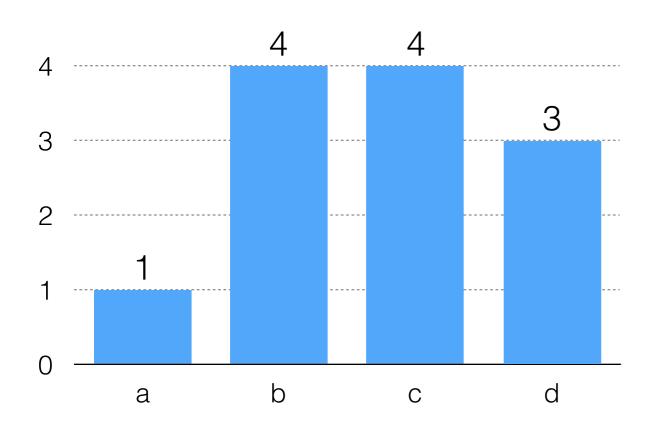
population over which you are taking

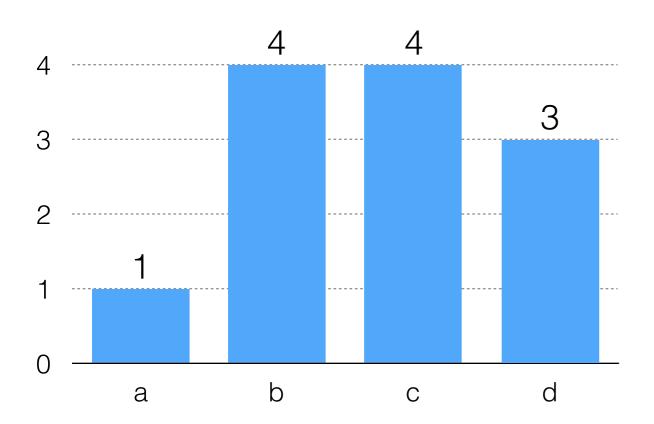
the mean is normally distributed.

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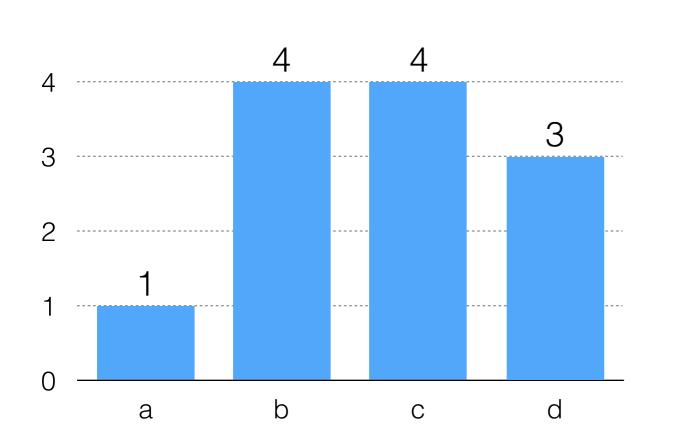
c d b d b a d c b c b d





Xi = count of answer i  

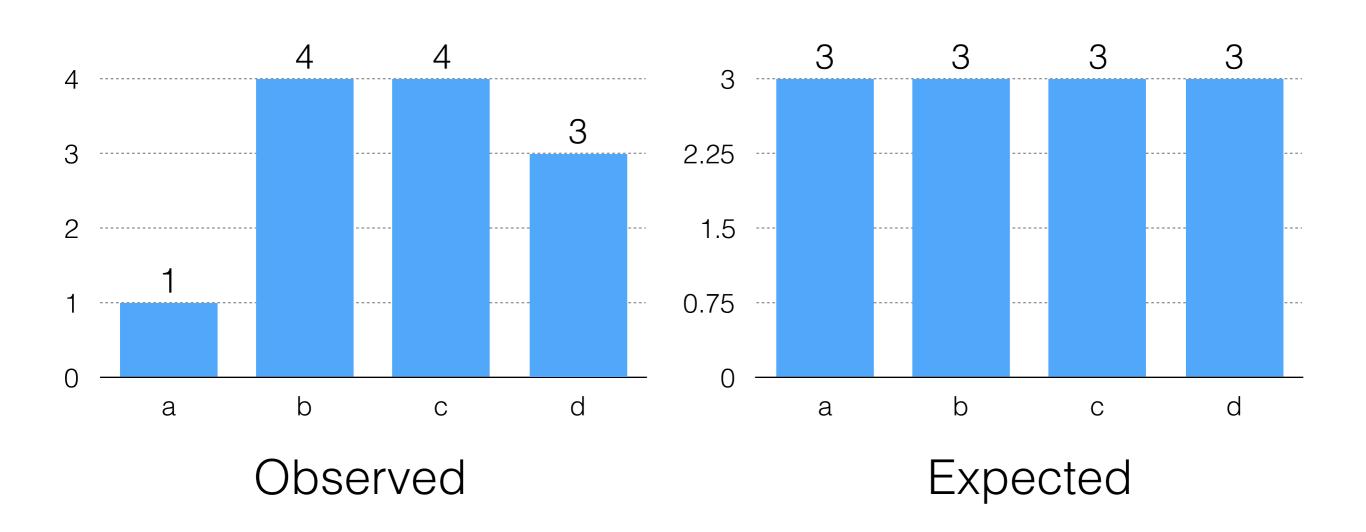
$$p(a) = p(b) = p(c) = p(d) = 0.25$$

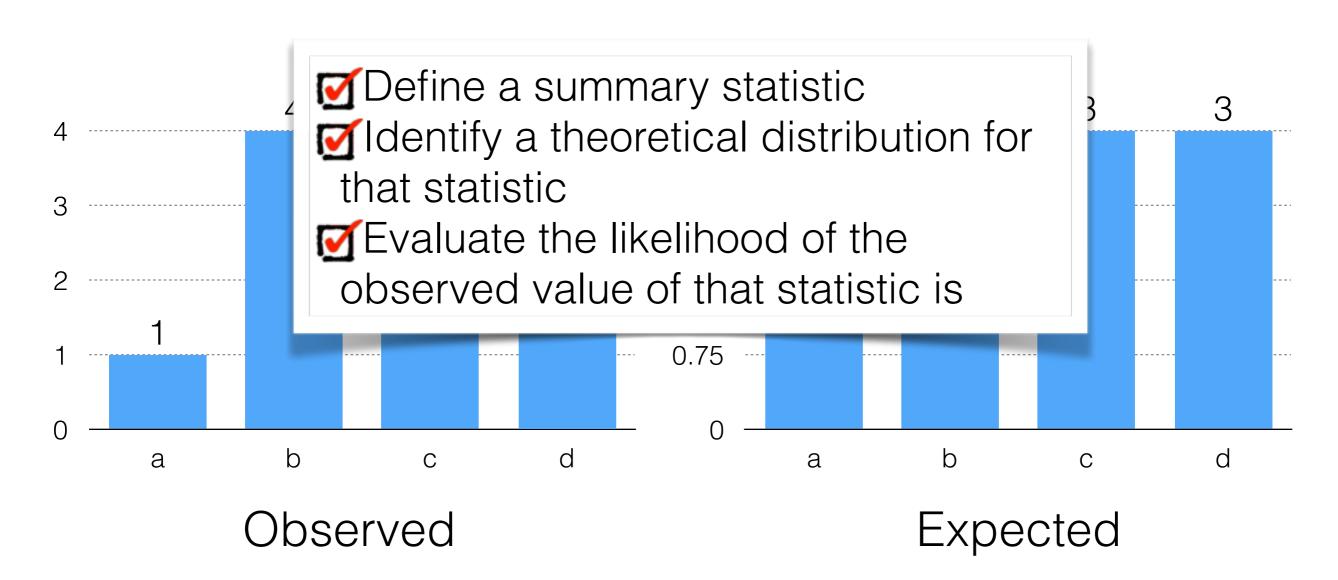


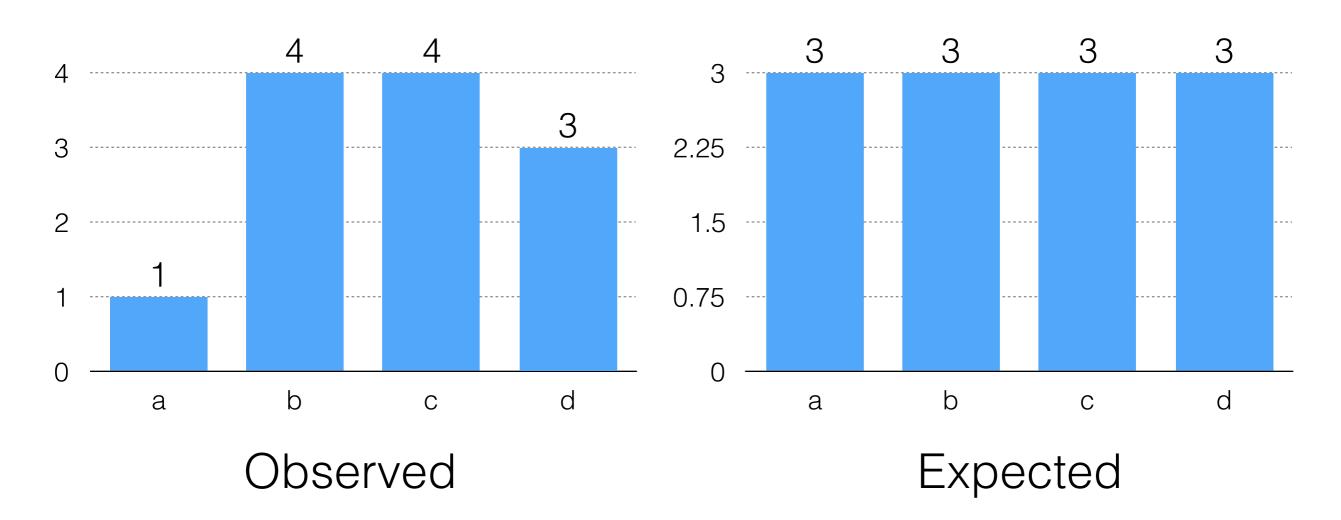
Is this distribution significantly different than what we would expect by chance, assuming that in fact all answers are equally likely?

Xi = count of answer i  

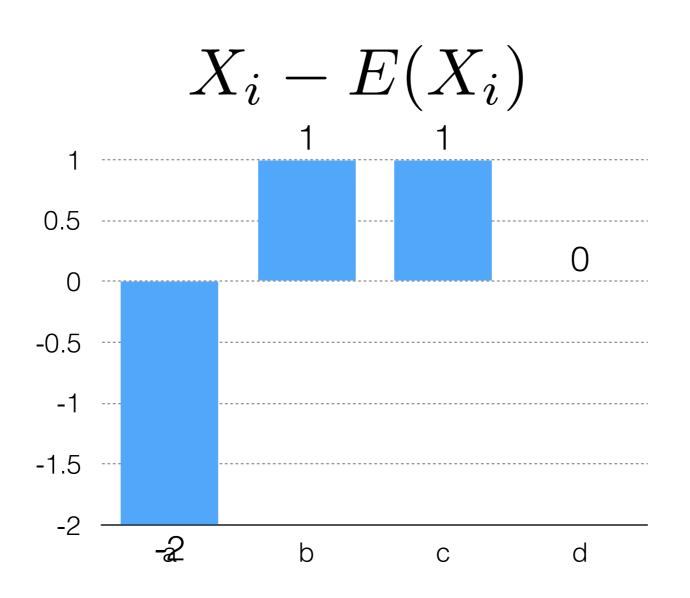
$$p(a) = p(b) = p(c) = p(d) = 0.25$$







Want to model the difference between these



Should I use the total difference between observed and expected as my summary statistic? I.e.

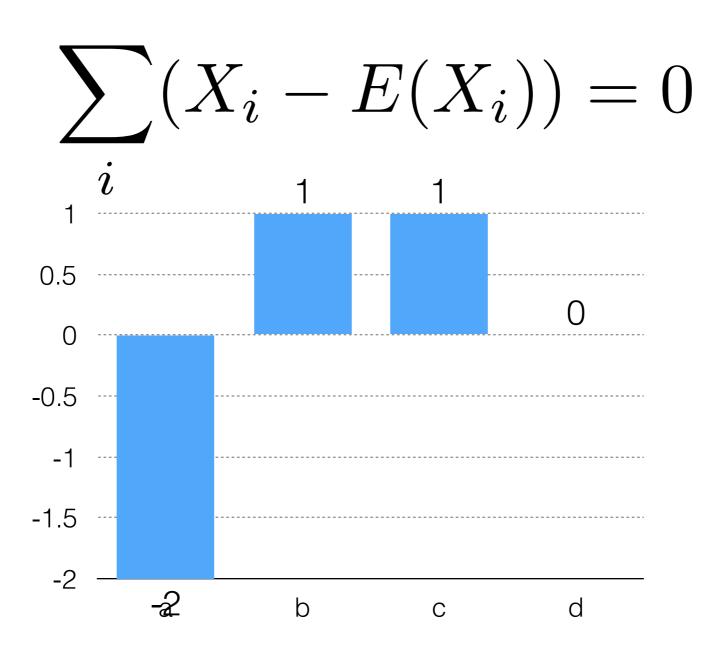
$$\sum_{i} (X_i - E(X_i))$$

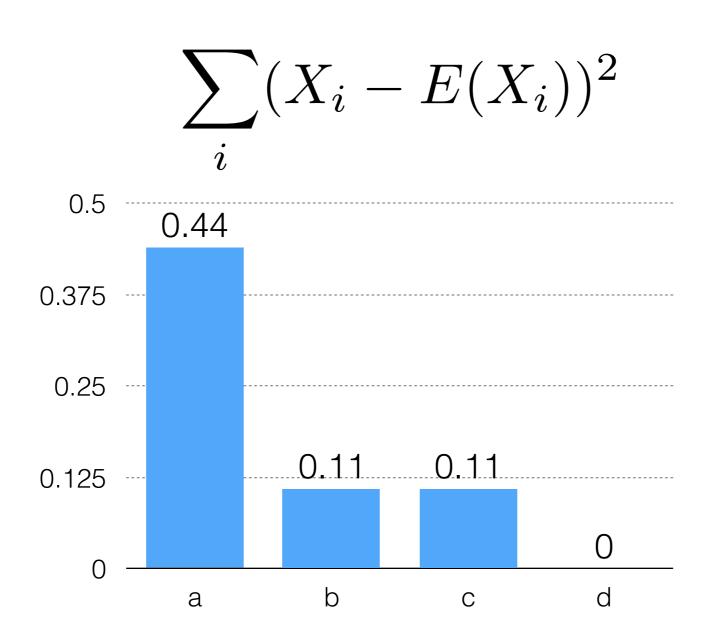
a) Yes! That sounds good. b) No! I have qualms...

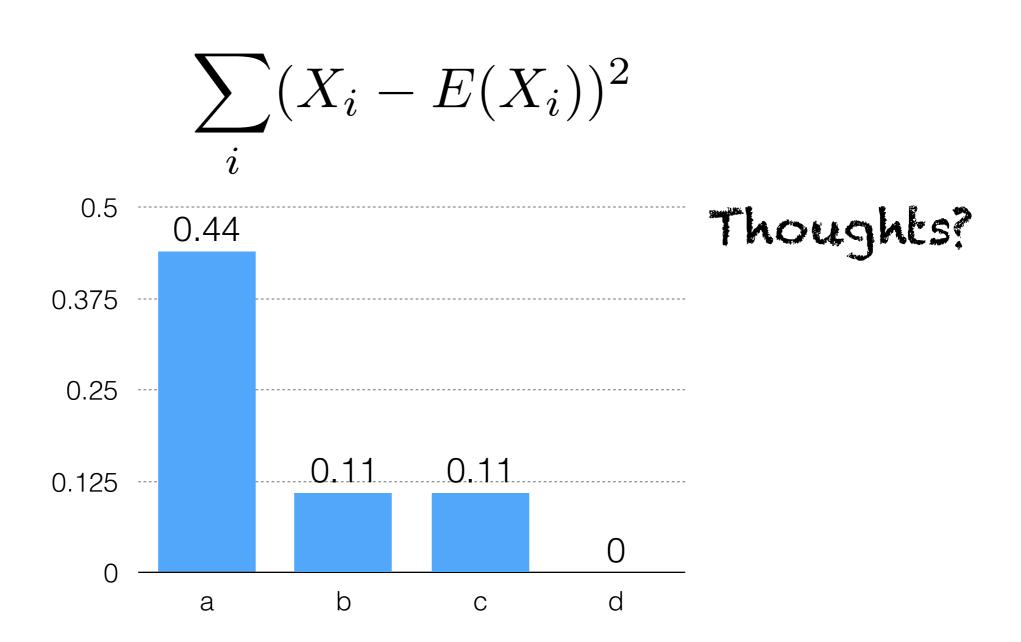
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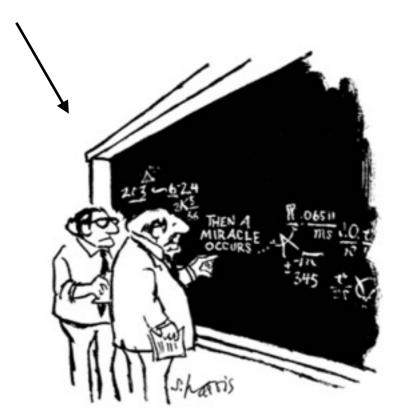




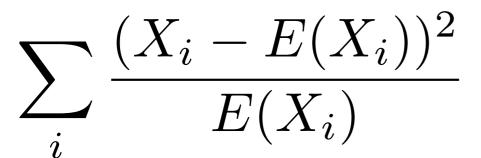
$$\sum_{i} \frac{(X_{i} - E(X_{i}))^{2}}{E(X_{i})}$$
0.16 0.148
0.08
0.04 0.037 0.037
0 a b c d

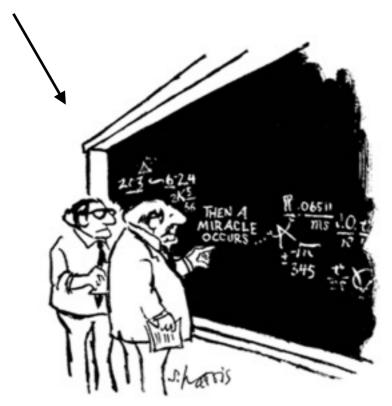
$$\sum_{i} \frac{(X_i - E(X_i))^2}{E(X_i)}$$

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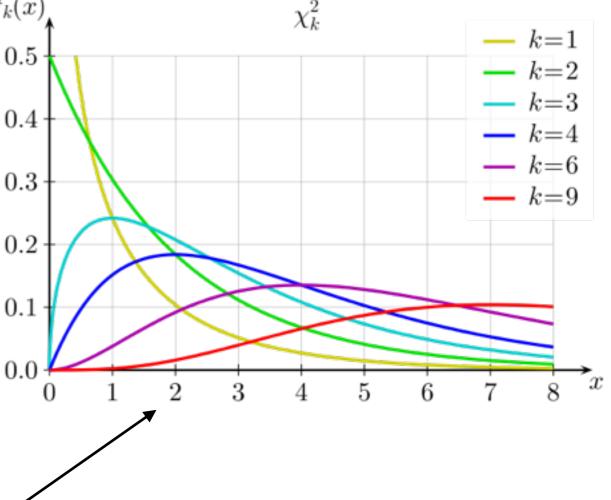


"I think you should be more explicit here in step two."

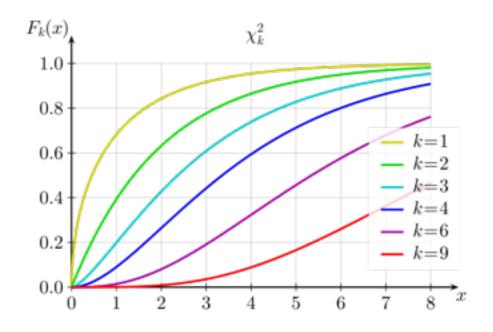




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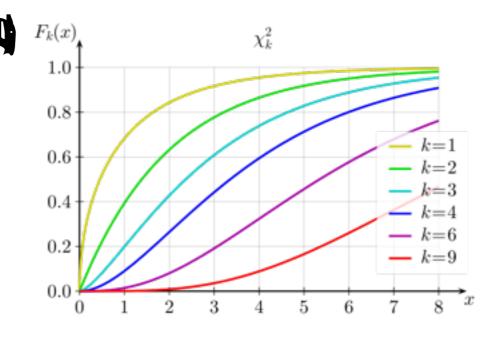
$$\sum_{i} \frac{(X_i - E(X_i))^2}{E(X_i)}$$



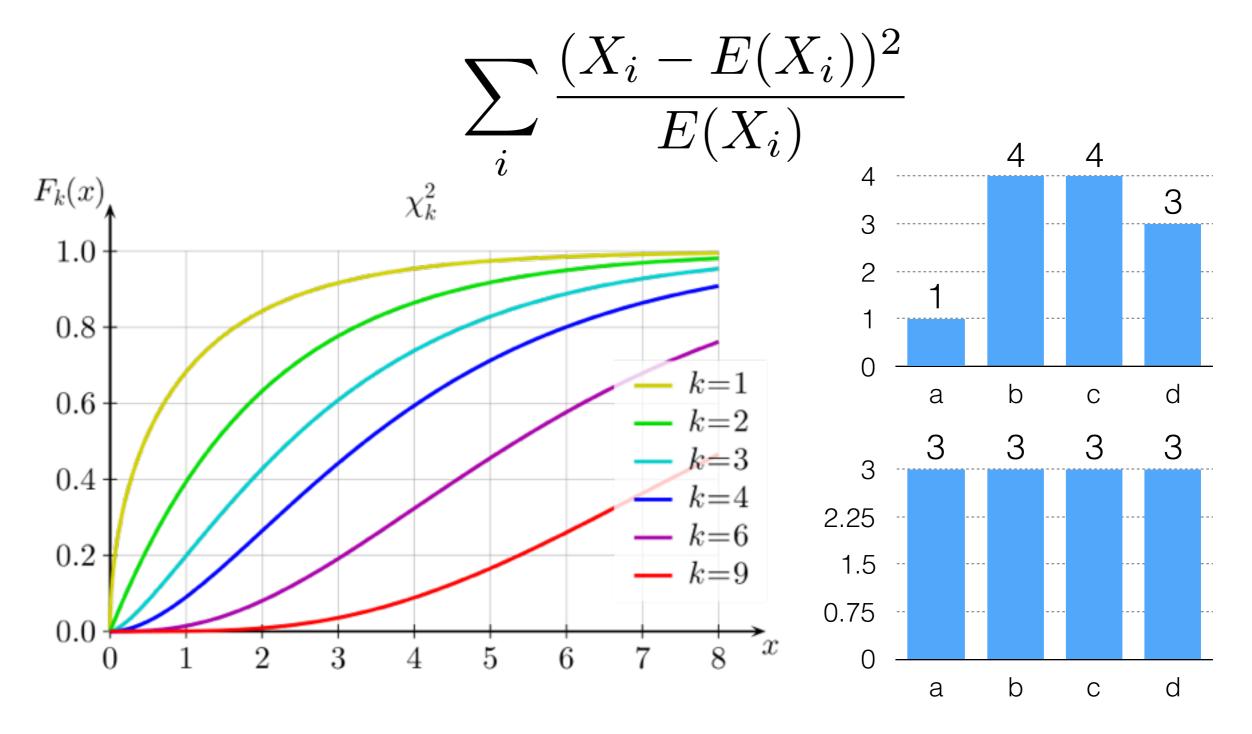
$$\frac{1}{\Gamma(k/2)}\gamma(\frac{k}{2},\frac{x}{2})$$

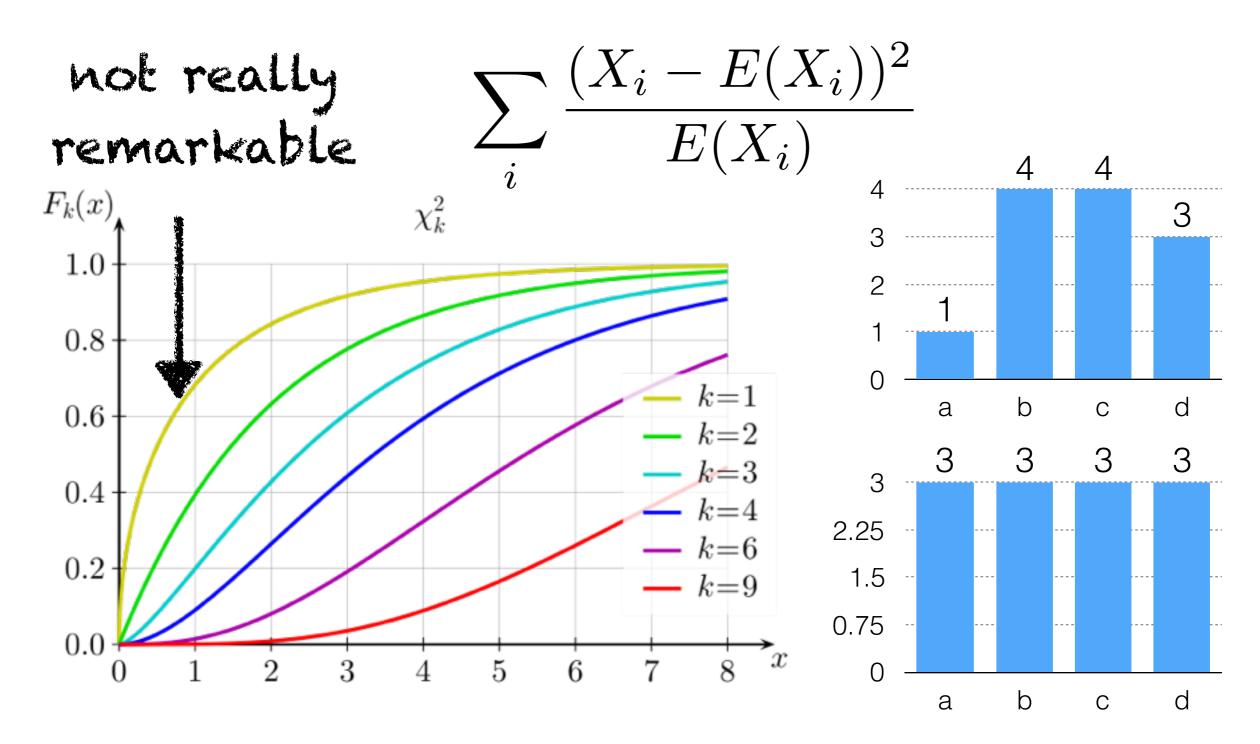
$$\sum_{i} \frac{(X_i - E(X_i))^2}{E(X_i)}$$

that we can compute explicitly



$$\frac{1}{\Gamma(k/2)}\gamma(\frac{k}{2},\frac{x}{2})$$





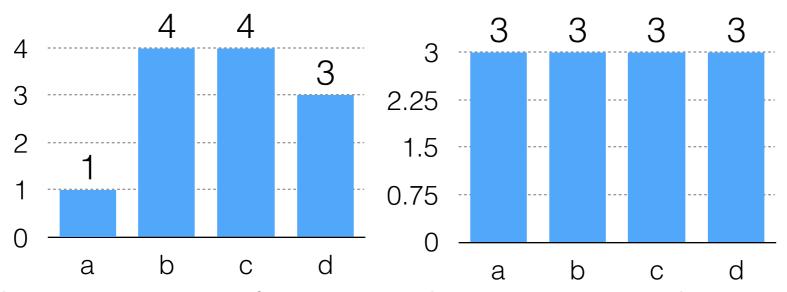
 Null hypothesis (H<sub>0</sub>) — the "nothing to see here" assumption

- Null hypothesis (H<sub>0</sub>) the "nothing to see here" assumption
- Alternative hypothesis (H<sub>a</sub>)—the thing you know will lead to an explosive headline and are really hoping is true but you are a good scientist, so you will look to the data to confirm

Thing you can model

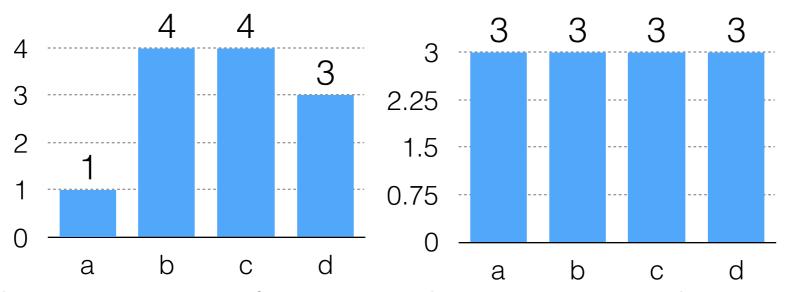
- Null hypothesis (H<sub>0</sub>) the "nothing to see here" assumption
- Alternative hypothesis (H<sub>a</sub>)—the thing you know will lead to an explosive headline and are really hoping is true but you are a good scientist, so you will look to the data to confirm

In the example we just did, what was the null hypothesis?



- The number of (a)s is the same as the number al of (b)s
- The distribution observed is different from b) what is expected
- The distribution that is observed is not c) meaningfully different than expected There are not enough (a)s d)

In the example we just did, what was the null hypothesis?

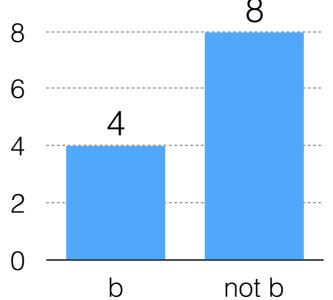


- a) The number of (a)s is the same as the number of (b)s
- b) The distribution observed is different from what is expected

c)

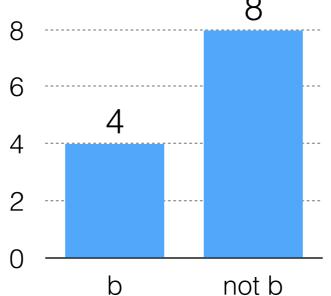
The distribution that is observed is not meaningfully different than expected There are not enough (a)s

In our first example, what was the null hypothesis?



a) 80% of the answers are (b)
b) The answers are evenly distributed
c) The number of (b)s is abnormally low
d) The number of (b)s is expected

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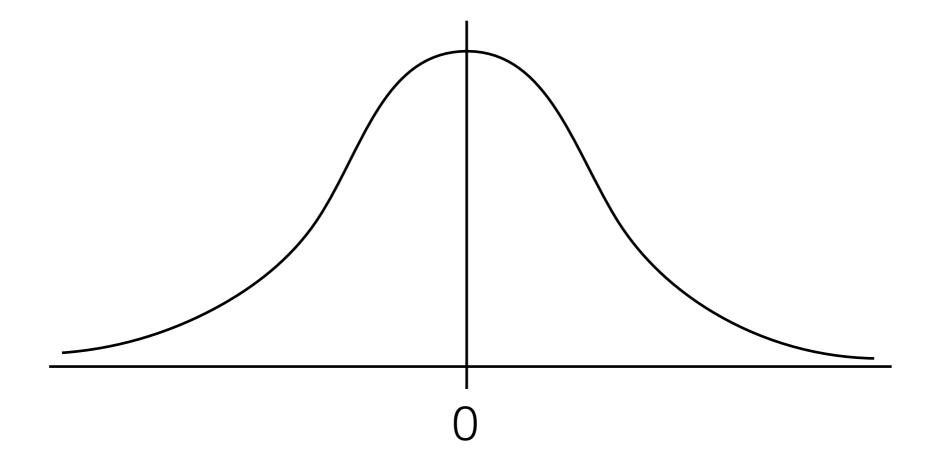
 Assume the null hypothesis is true—i.e., don't deviate from status quo without good reason:)

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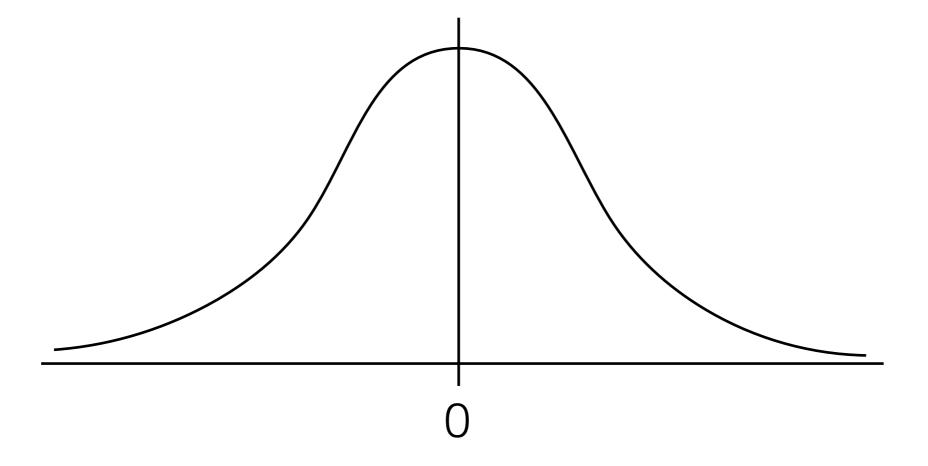
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- If there is not enough evidence, we "fail to reject it"
- We don't "accept" or "prove" H<sub>0</sub> or H<sub>a</sub>

$$arphi(x)=rac{1}{\sqrt{2\pi}}e^{-rac{1}{2}x^2}$$

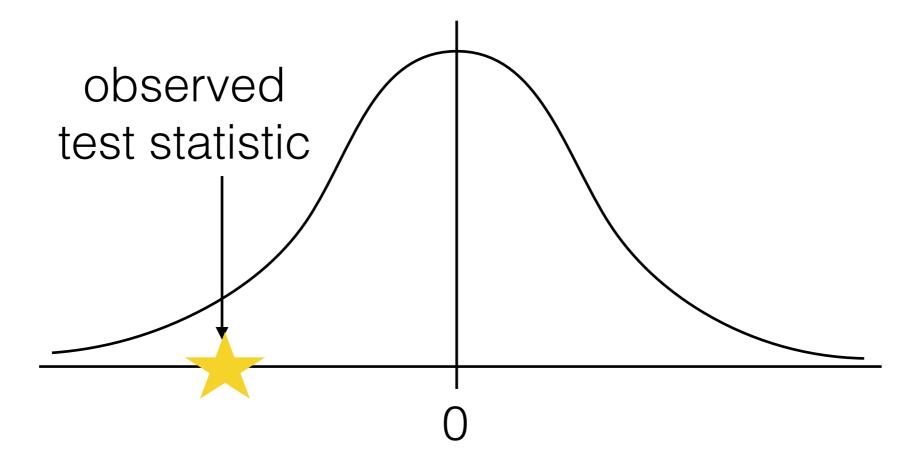


z = distance from mean in std units

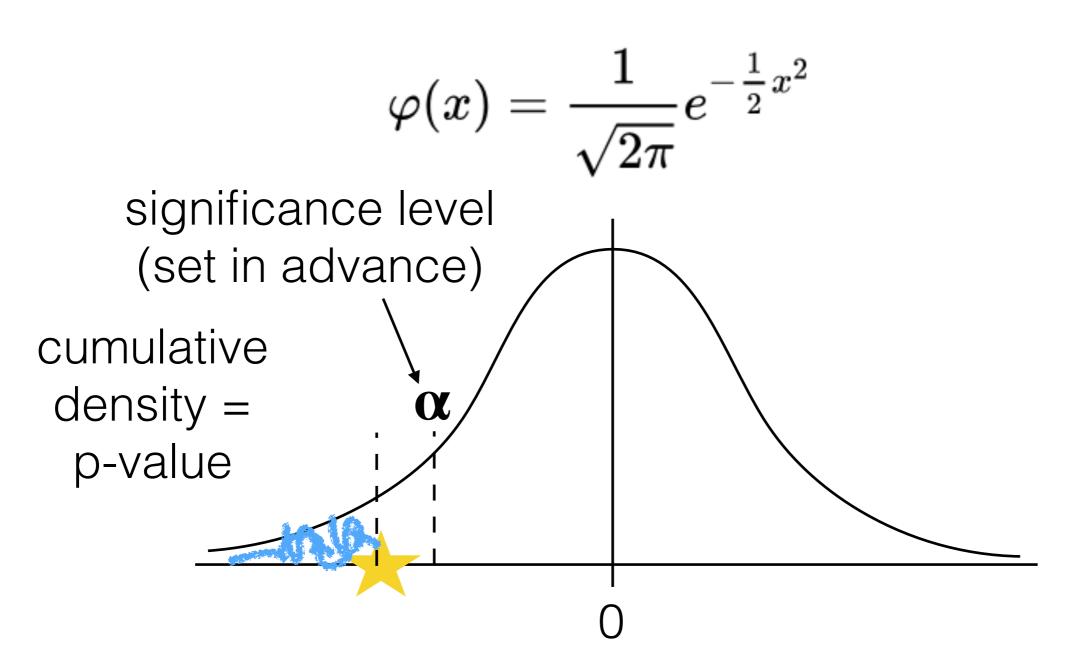
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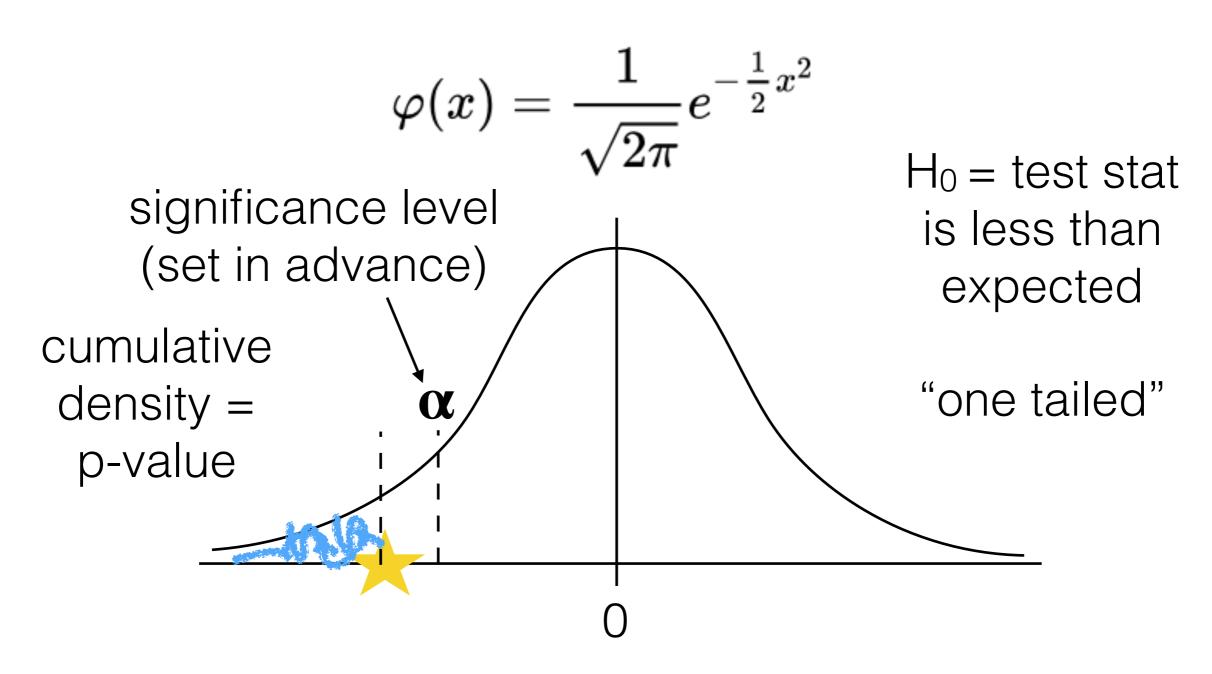


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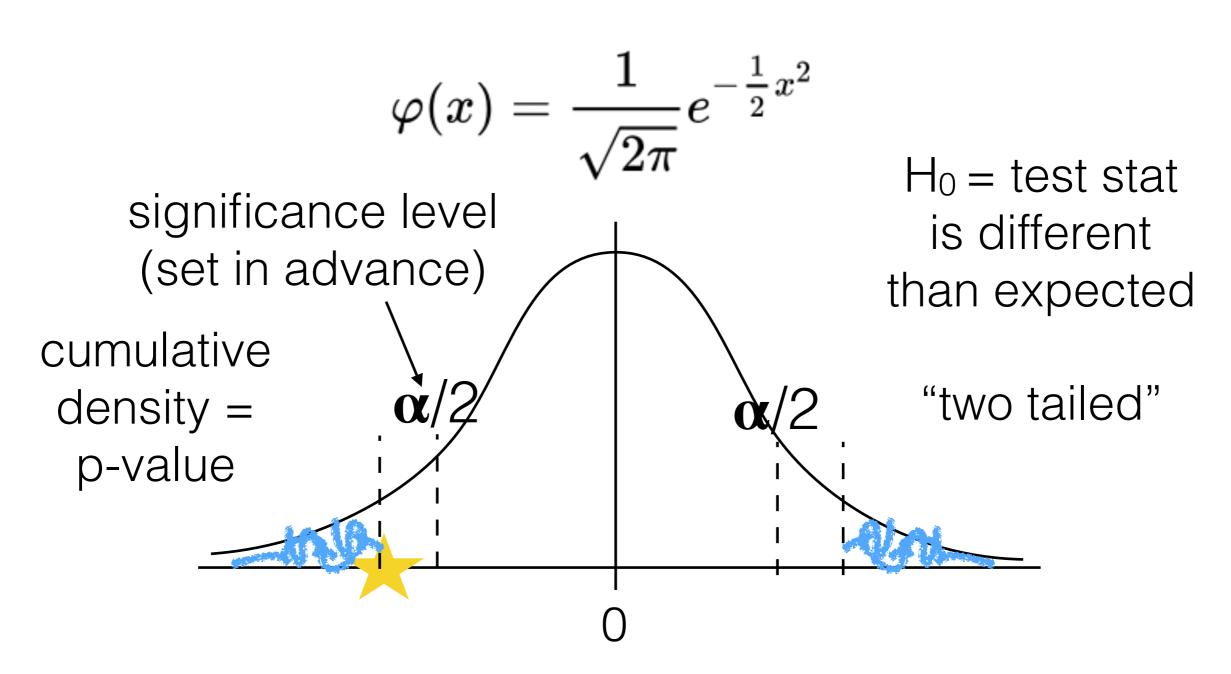


$$\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$$
 cumulative density = p-value





### Standard Normal Distr.



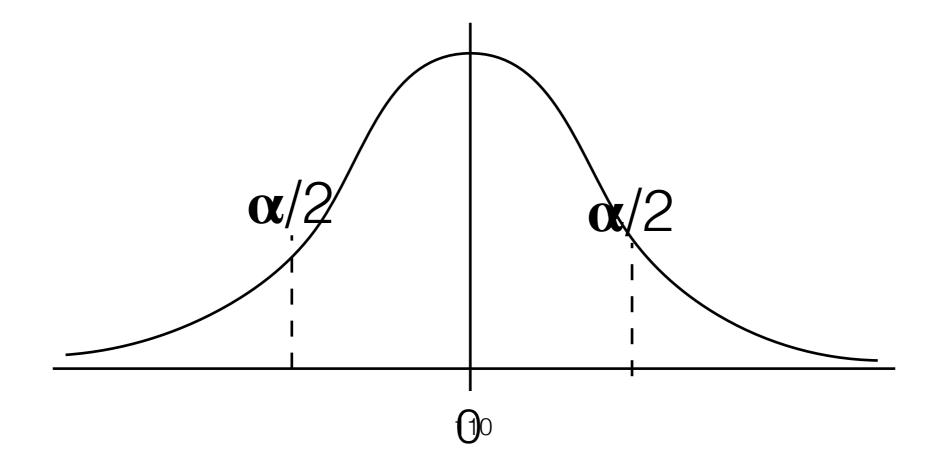
z = (observed - expected) / standard deviation

# Test for population proportion

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1 - p_0)}{n}}}$$

 $H_0$  = proportion of (b) is 80%

 $H_a$  = proportion of (b) is not 80%



### Clicker Question!

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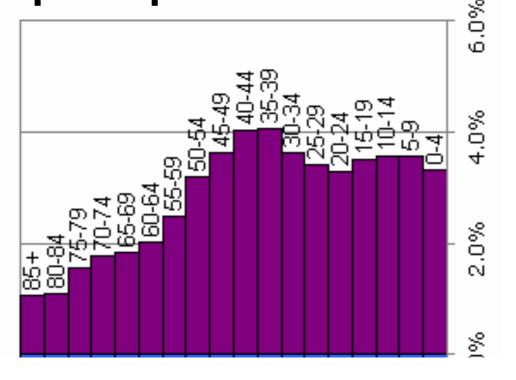
Why can we use a normally-distributed test statistic to evaluate a binomial distribution like this?

- a) Because its a random variable
- b) Because of regression to the mean
- c) Because of the law of large numbers
- d) Because of the central limit theorem
- e) The limit does not exist!

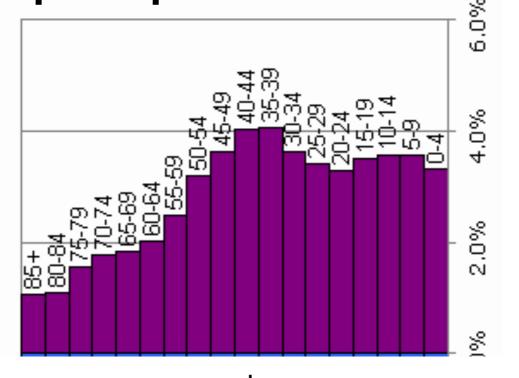
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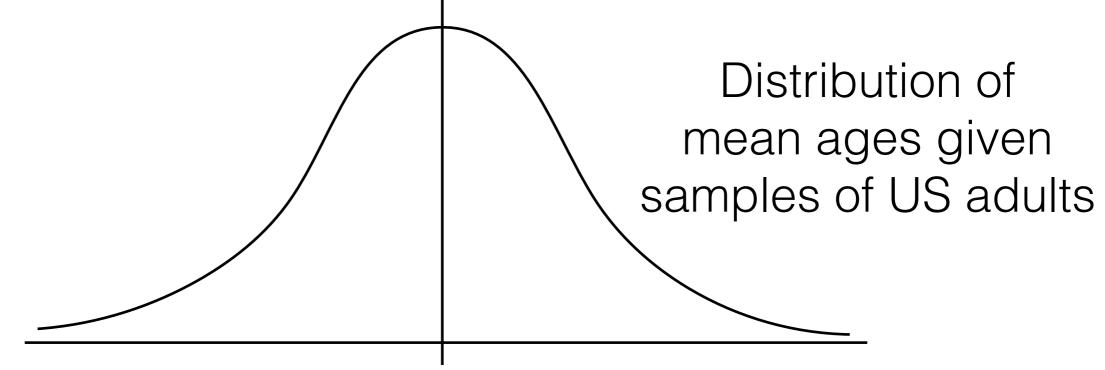
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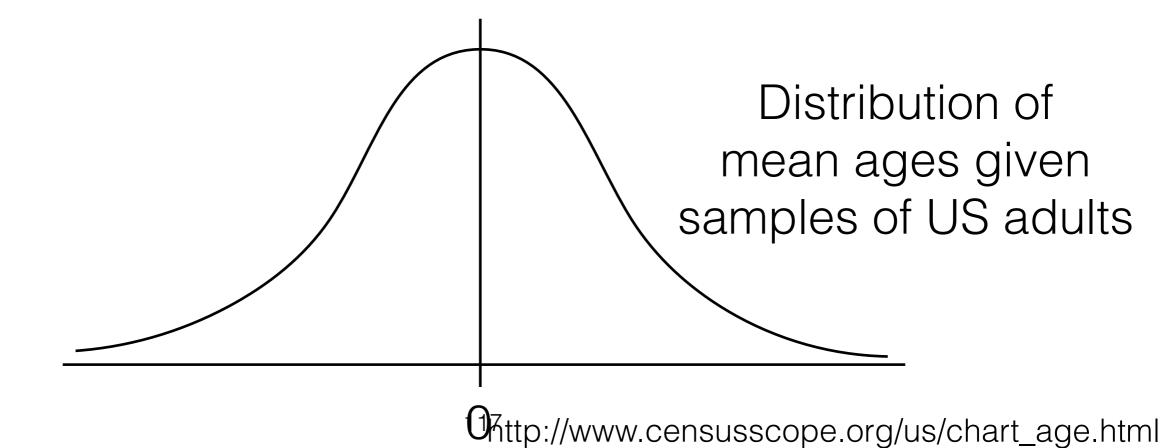
Distribution of ages in the US



Distribution of ages in the US

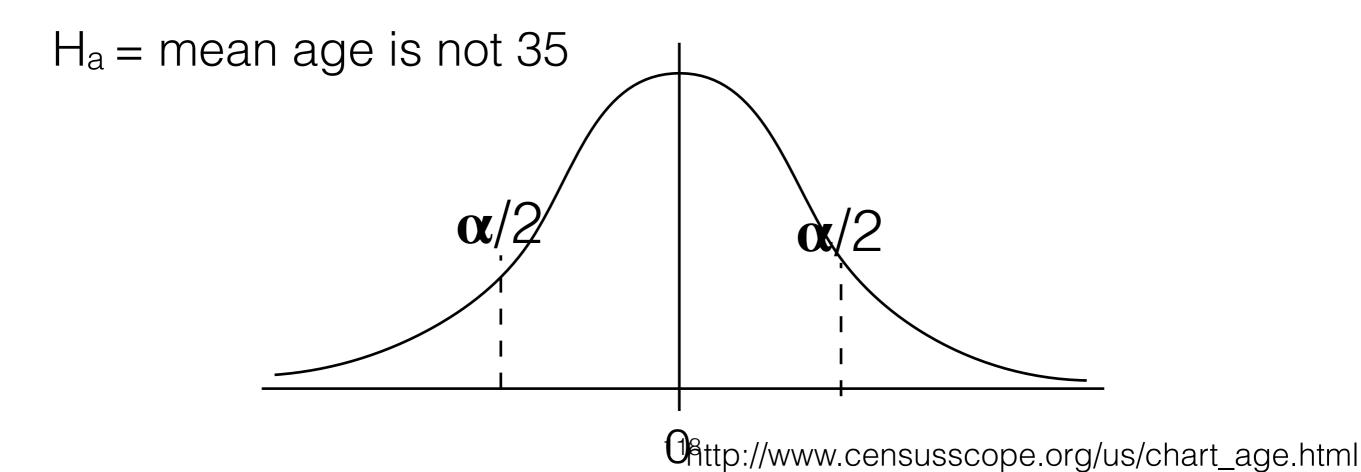


$$z = \frac{\bar{x} - \mu_0}{\sqrt{\frac{\sigma^2}{n}}}$$



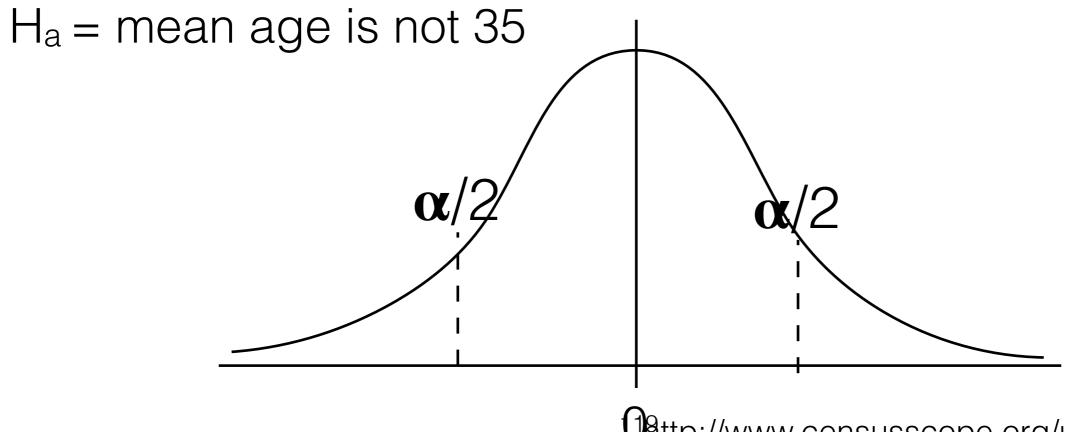
$$z = \frac{\bar{x} - \mu_0}{\sqrt{\frac{\sigma^2}{n}}}$$

 $H_0$  = mean age is 35



$$z = \frac{\bar{x} - \mu_0}{\sqrt{\sigma^2}}$$

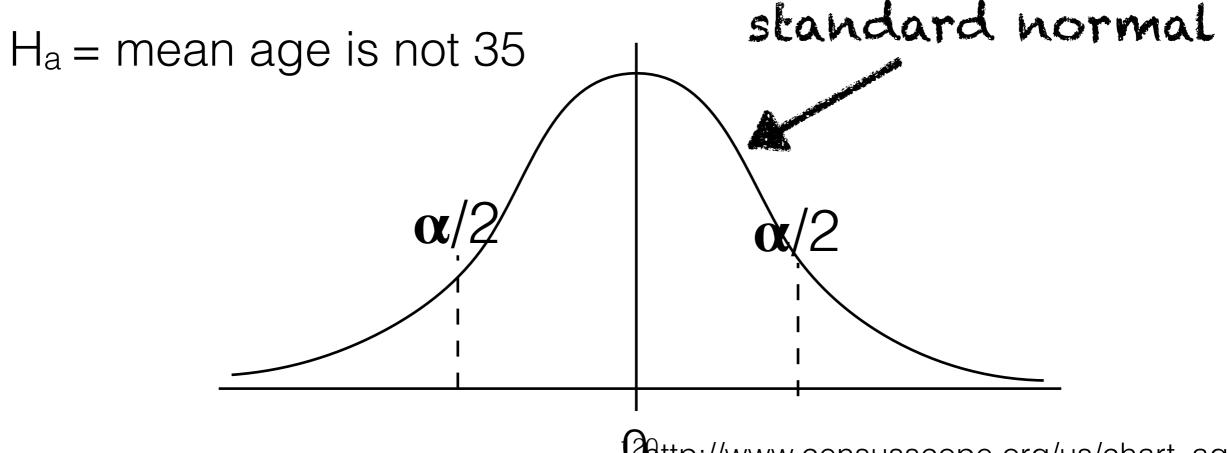
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$$z = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}}$$

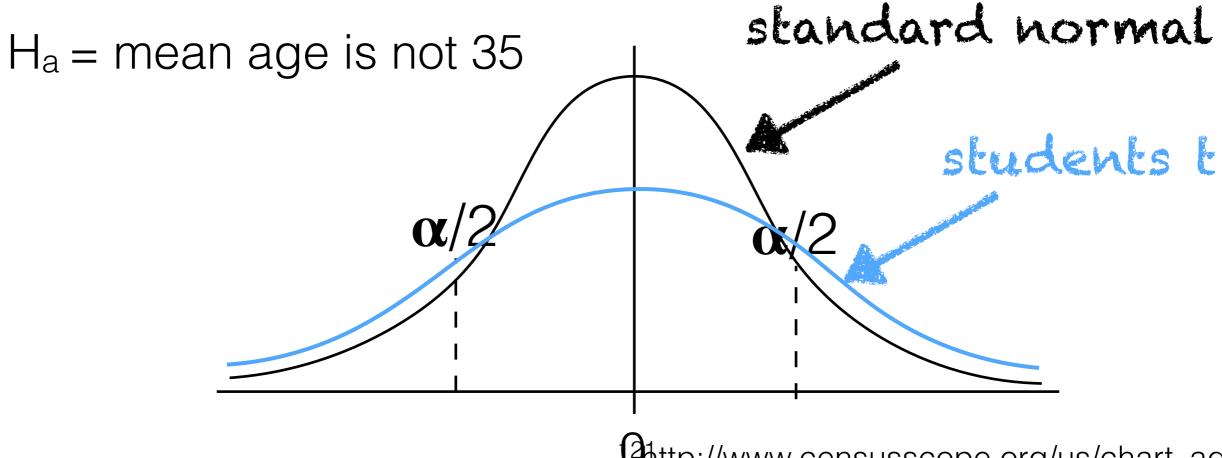
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1999 | Http://www.censusscope.org/us/chart\_age.html

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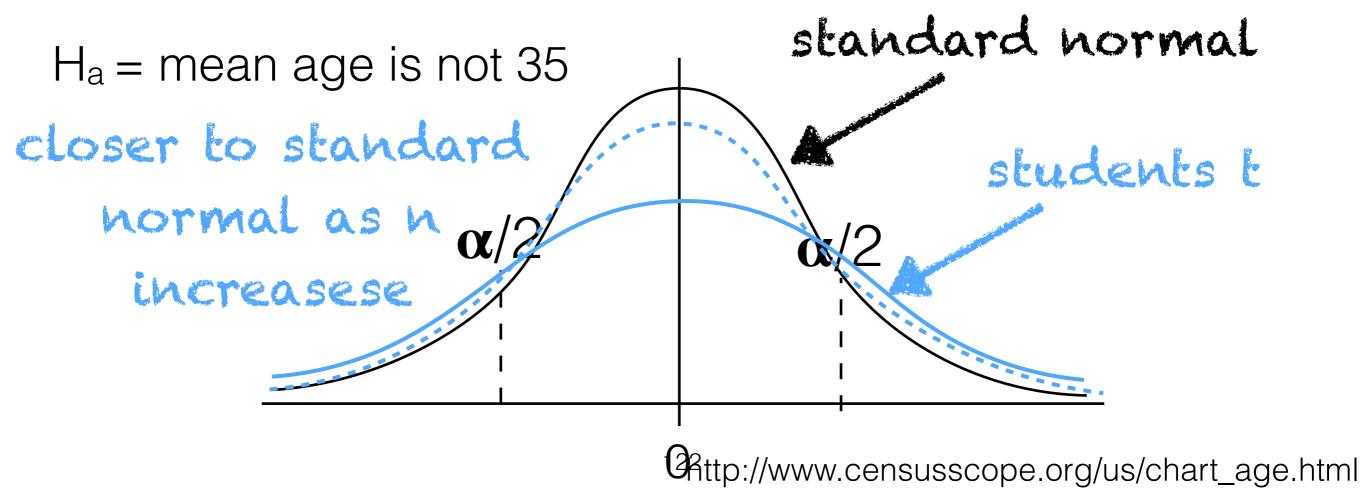
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okie done now