Theory Questions

Question1

The order matters. For example, the point A is $(1, 0, 0)$.

If we rotate 90 degrees around the x-axis, because A is on the x-axis, the position will not change, and then rotate 90 degrees around the y-axis, the position will be $(0, 0, -1)$.

But if we rotate 90 degrees around the y-axis, then rotate 90 degrees around the x-axis, the position will be $(0, 1, 0)$.

Question2

$$A = \begin{bmatrix} 2 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

We compute the singular values σ_i by finding the eigenvalues of AA^T.

$$AA^T = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}$$

The characteristic polynomial is $\det(AA^T - \lambda I) = (8 - \lambda)(2 - \lambda)$, so the singular values are $\sigma_1 = \sqrt{8} = 2\sqrt{2}$ and $\sigma_2 = \sqrt{2}$.

And the eigenvalues of $A^T A$ are 8, 2, and 0, and since $A^T A$ is symmetric, the eigenvectors will be orthogonal.

For $\lambda = 8$, we have $A^T A - 8I = \begin{bmatrix} -3 & 3 & 0 \\ 3 & -3 & 0 \\ 0 & 0 & -8 \end{bmatrix}$, which row-reduces to $\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. A unit-length vector in the kernel of that matrix is $v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$.

For $\lambda = 2$, we have $A^TA - 8I = \begin{bmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, which row-reduces to $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. A unit-length vector in the kernel of that matrix is $v_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$.

For the last eigenvector, it is easy to get a unit-length vector $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

And we can compute U by the formula $u_i = \frac{1}{\sigma} A v_i$, this gives $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

So the SVD is:

$$A = U \Sigma V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Question3

a. $\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$

b. $\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c. $\begin{bmatrix} s & 0 & t_x \\ 0 & s & t_y \\ 0 & 0 & 1 \end{bmatrix}$

d. $\begin{bmatrix} s & 0 & 0 & t_x \\ 0 & s & 0 & t_y \\ 0 & 0 & s & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Question4

The solution $\bar{x}$ for $Ax = b$ is

$$\bar{x} = (A^T A)^{-1} A^T b = \begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix}$$

The error vector is

$$b - A\bar{x} = \begin{bmatrix} 1/3 \\ 2/3 \\ -1/3 \end{bmatrix}$$

Because $2 * 1/3 + (-1) * 2/3 = 0$, and $1 * 2/3 + 2 * (-1/3) = 0$, the error vector is orthogonal to the columns of A.

Question5

a. The $I_K(x, y)$ can be expressed as:

$$I_K(x,y) = \sum_{p=1}^{k} \sum_{q=1}^{k} K(p,q) I(x + p - (k-1)/2, y + q - (k-1)/2)$$

b. We know that $K(p, q) = g_p h_q$, so the $I_K(x, y)$ is:

$$\begin{aligned}
I_K(x,y) &= \sum_{p=1}^{k} \sum_{q=1}^{k} K(p,q) I(x + p - (k-1)/2, y + q - (k-1)/2) \\
&= \sum_{p=1}^{k} \sum_{q=1}^{k} g_p h_q I(x + p - (k-1)/2, y + q - (k-1)/2) \\
&= \sum_{q=1}^{k} \left(h_q \sum_{p=1}^{k} (g_p I(x + p - (k-1)/2, y + q - (k-1)/2)) \right) \\
&= \sum_{q=1}^{k} h_q I_g(x,y) \\
&= I_{gh}(x,y)
\end{aligned}$$