Quantum Error Correction

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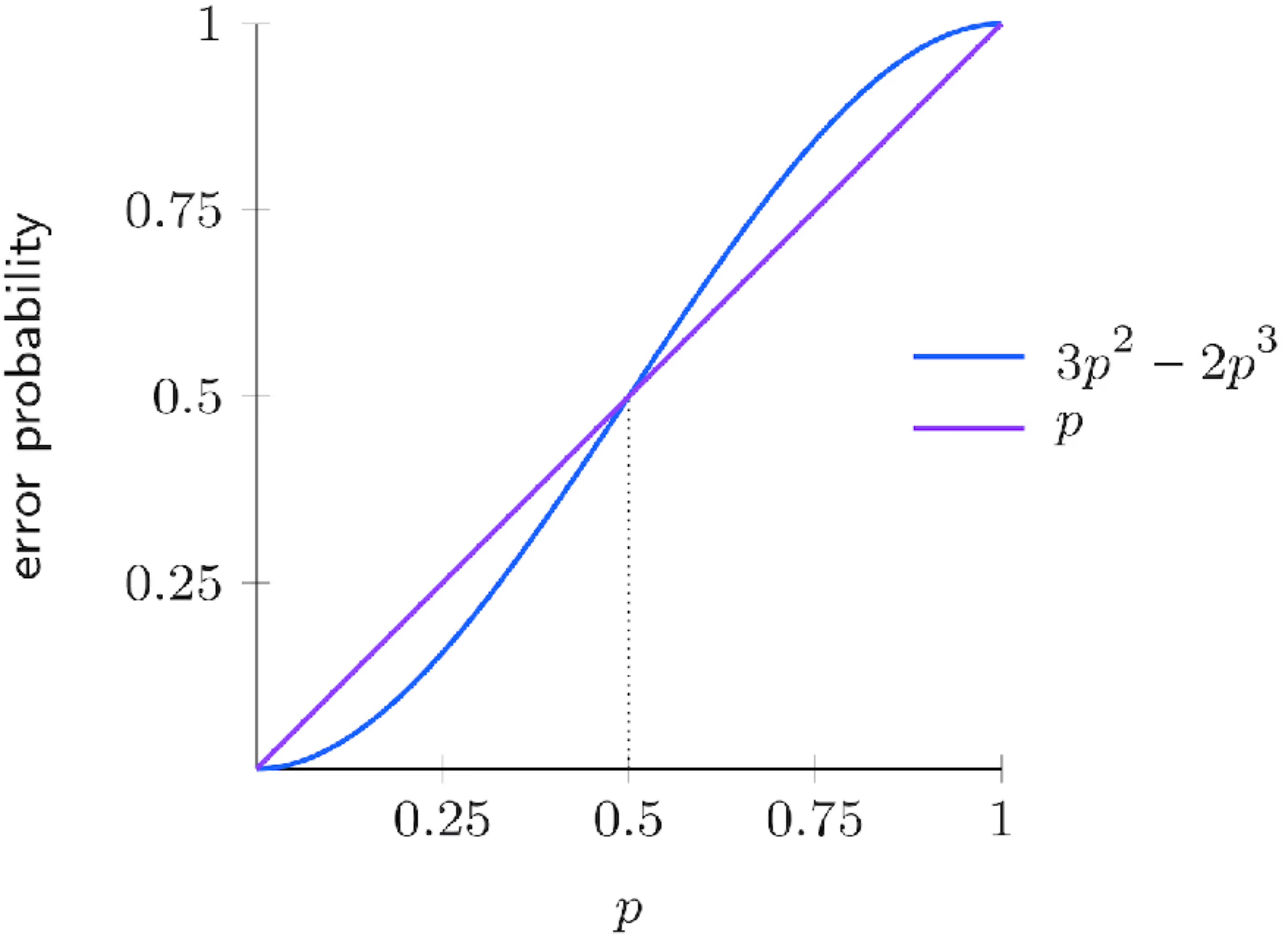
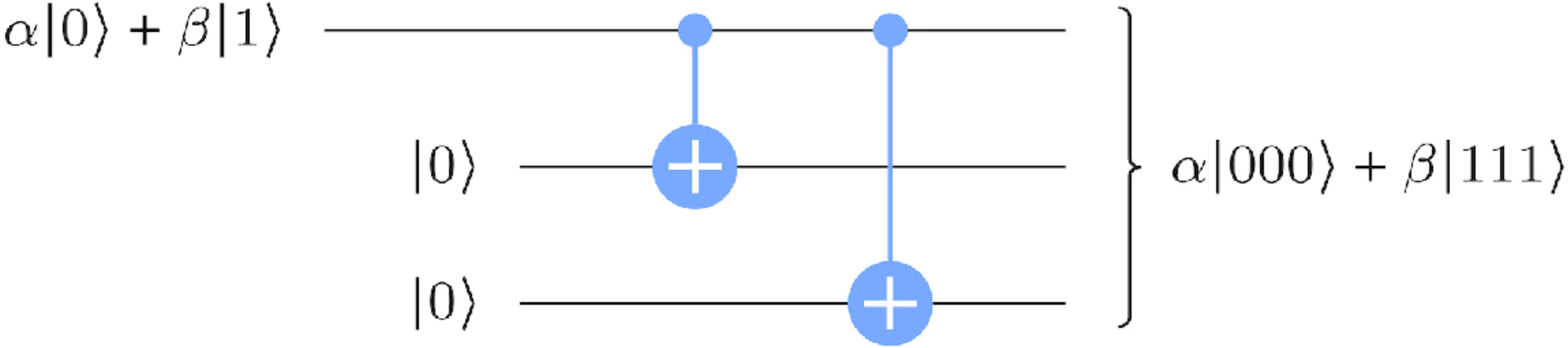
Classical repetition codes

- If the error rate is 1% — with one qubit, the error rate is 1%
- What if we use 3 repetitive qubits to encode a 0 as 000 and 1 as 111
  - 0 — If 2 and more qubits are 0s
  - 1 — Otherwise
  - $3p^2(1-p)+p^3 = 3p^2 - 2p^3 = 0.000298 = 0.0298\,\%$

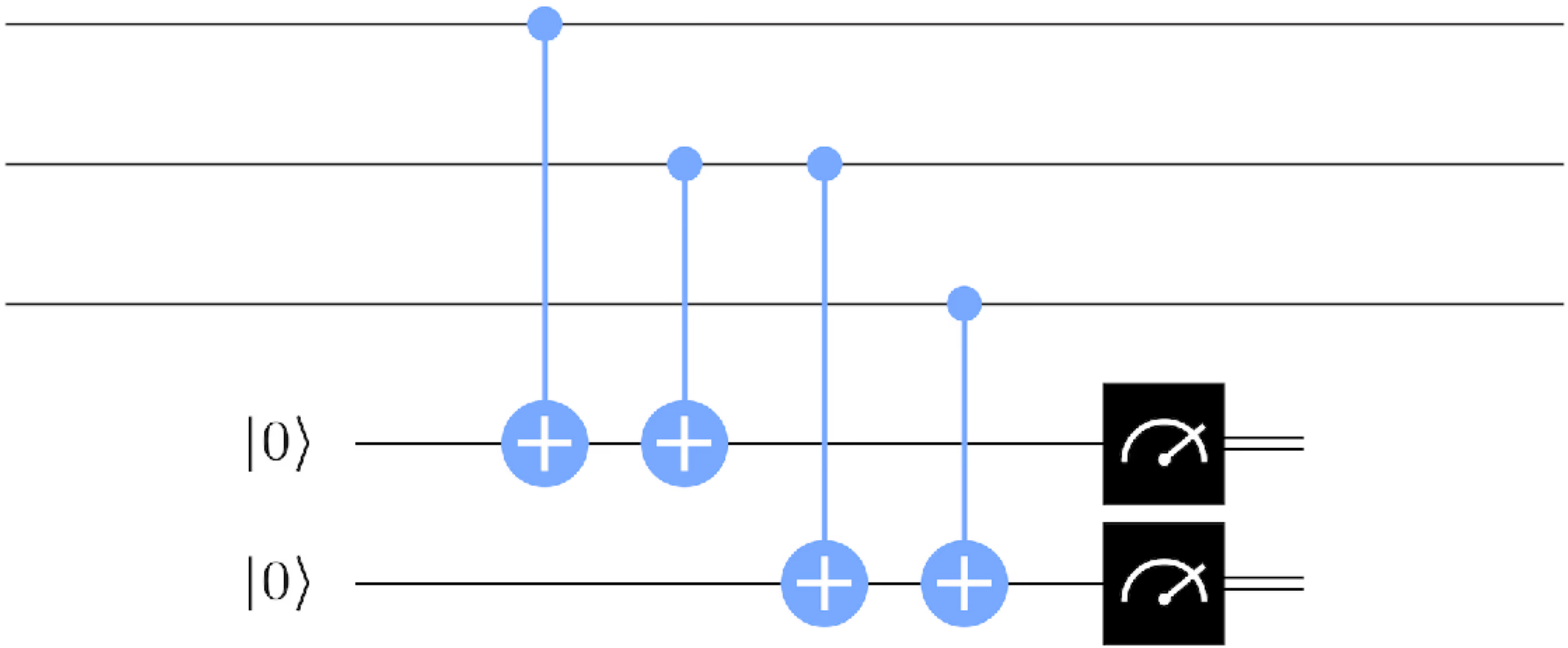


1
0.75
0.5
0.25
error probability
0.25
0.5
0.75
1
p
3p^2 − 2p^3
p

The encoding circuit



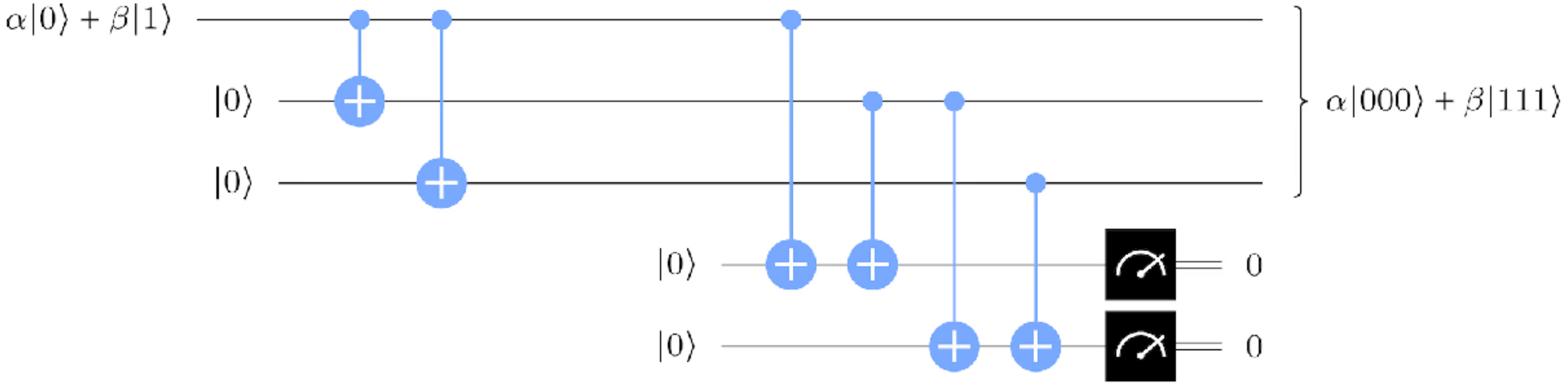
Error detection circuit



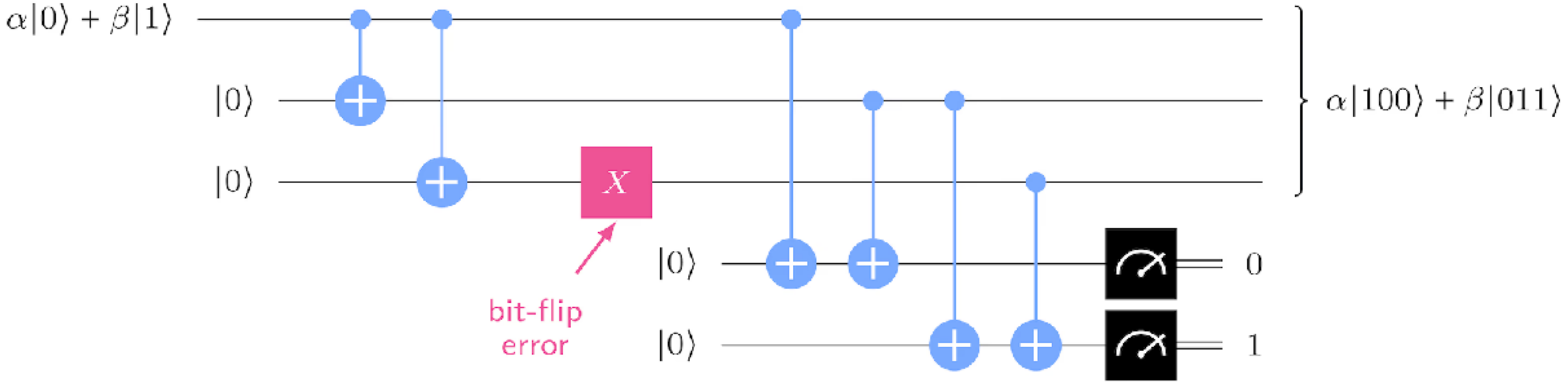
How to correct errors

State	Syndrome	Correction
$\alpha\|000\rangle+\beta\|111\rangle$	00	$I\otimes I\otimes I$
$\alpha\|100\rangle+\beta\|011\rangle$	10	$X\otimes I\otimes I$
$\alpha\|010\rangle+\beta\|101\rangle$	11	$I\otimes X\otimes I$
$\alpha\|001\rangle+\beta\|110\rangle$	01	$I\otimes I\otimes X$

If there is no error

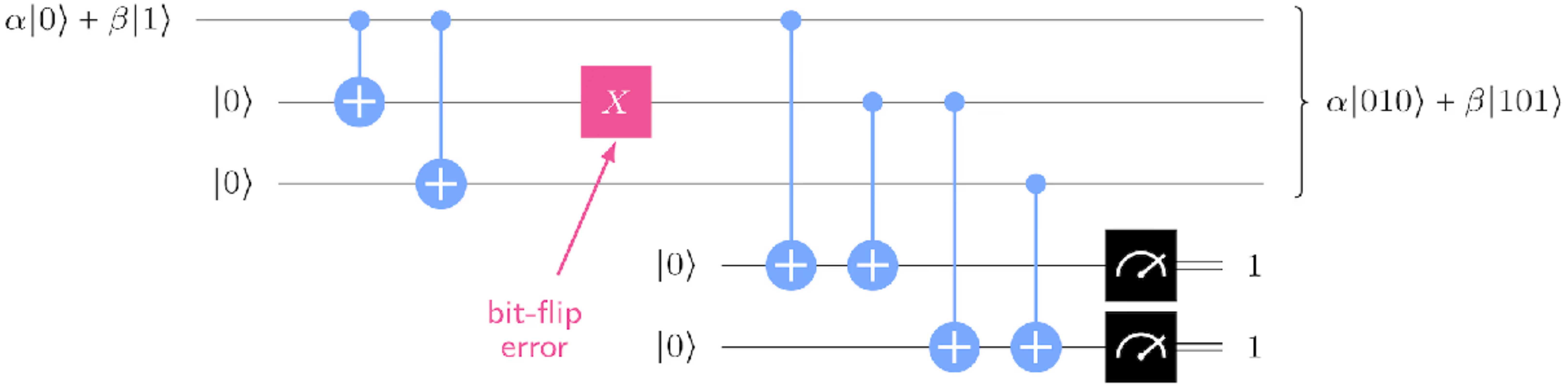


If q_2 gets wrong



$\alpha|100\rangle+\beta|011\rangle$	10	$X\otimes I\otimes I$
	10	

Or if q_1 gets wrong



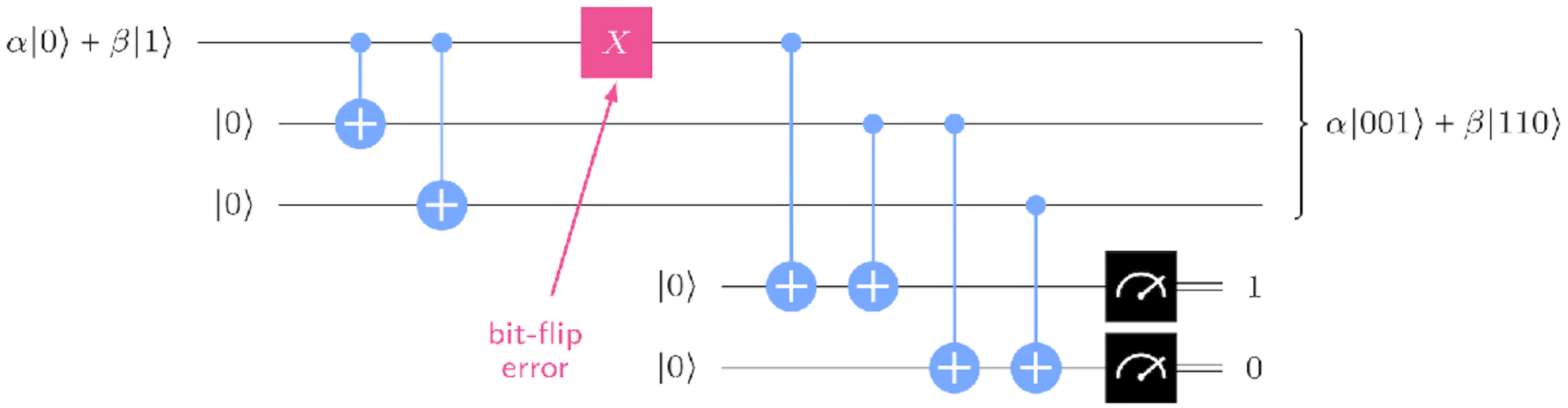
$\alpha|010\rangle+\beta|101\rangle$

11

$I \otimes X \otimes I$

11

If q_0 gets wrong

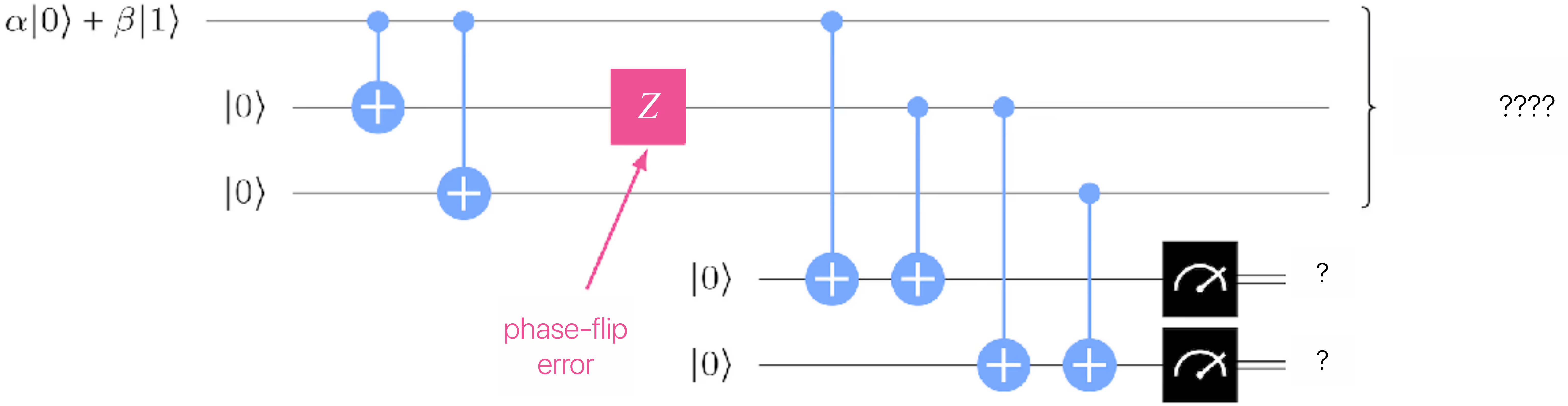


$\alpha|001\rangle+\beta|110\rangle$

01

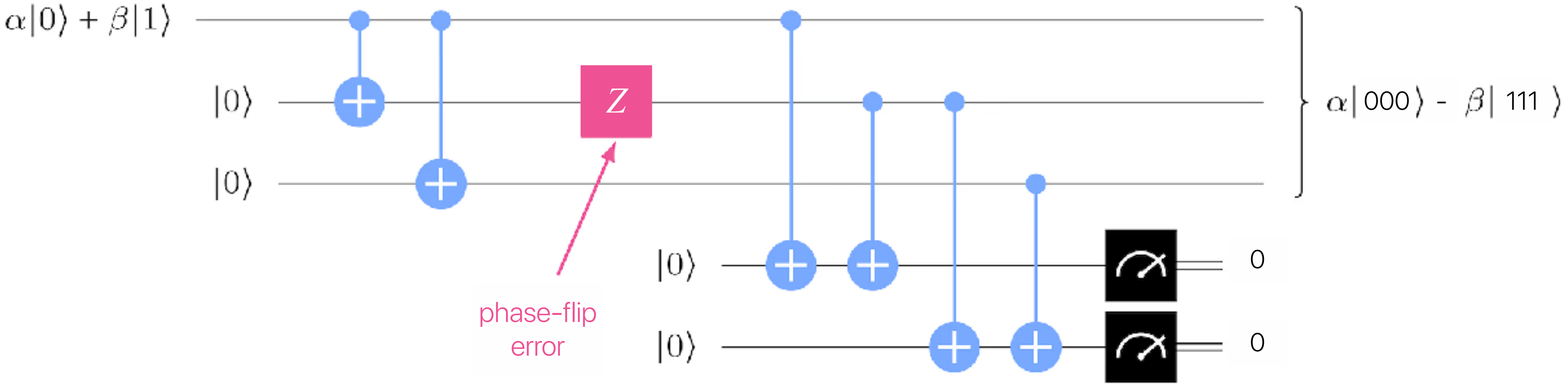
$I \otimes I \otimes X$

But, what if?



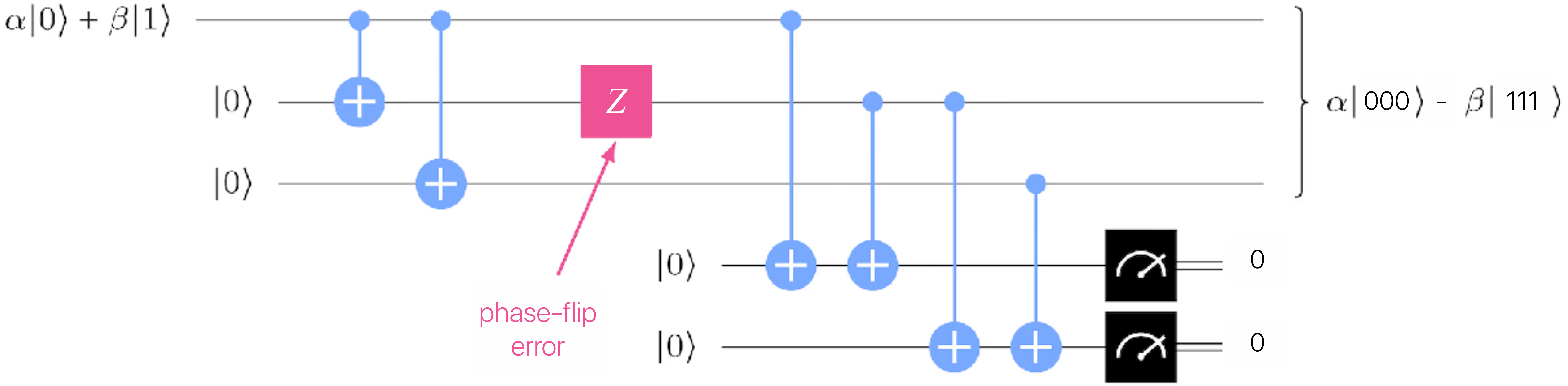
$\alpha|0\rangle + \beta|1\rangle$
$|0\rangle$
$|0\rangle$
Z
phase-flip error
$|0\rangle$
$|0\rangle$
?
?
????

But, what if?

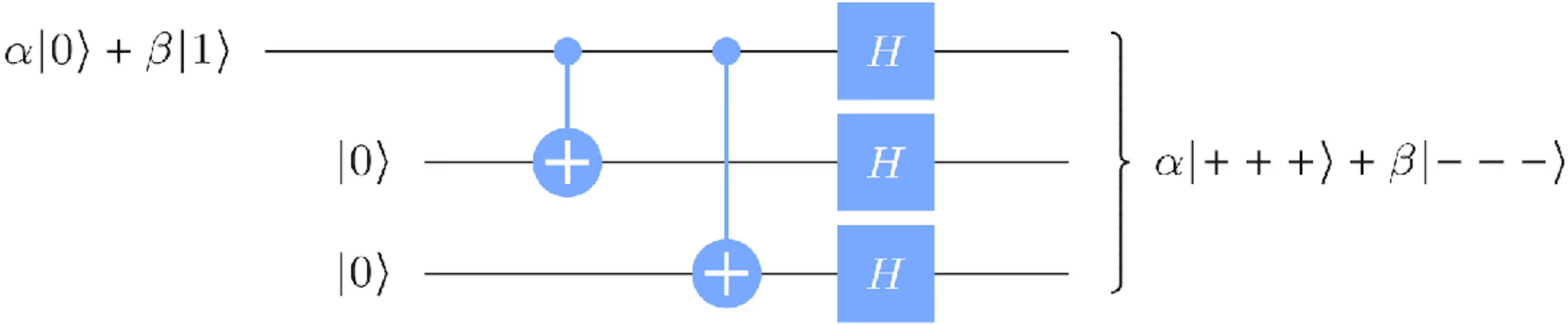


$\alpha|0\rangle + \beta|1\rangle$
$|0\rangle$
$|0\rangle$
Z
phase-flip
error
$|0\rangle$
$|0\rangle$
0
0
$\alpha|000\rangle$ - $\beta|111\rangle$

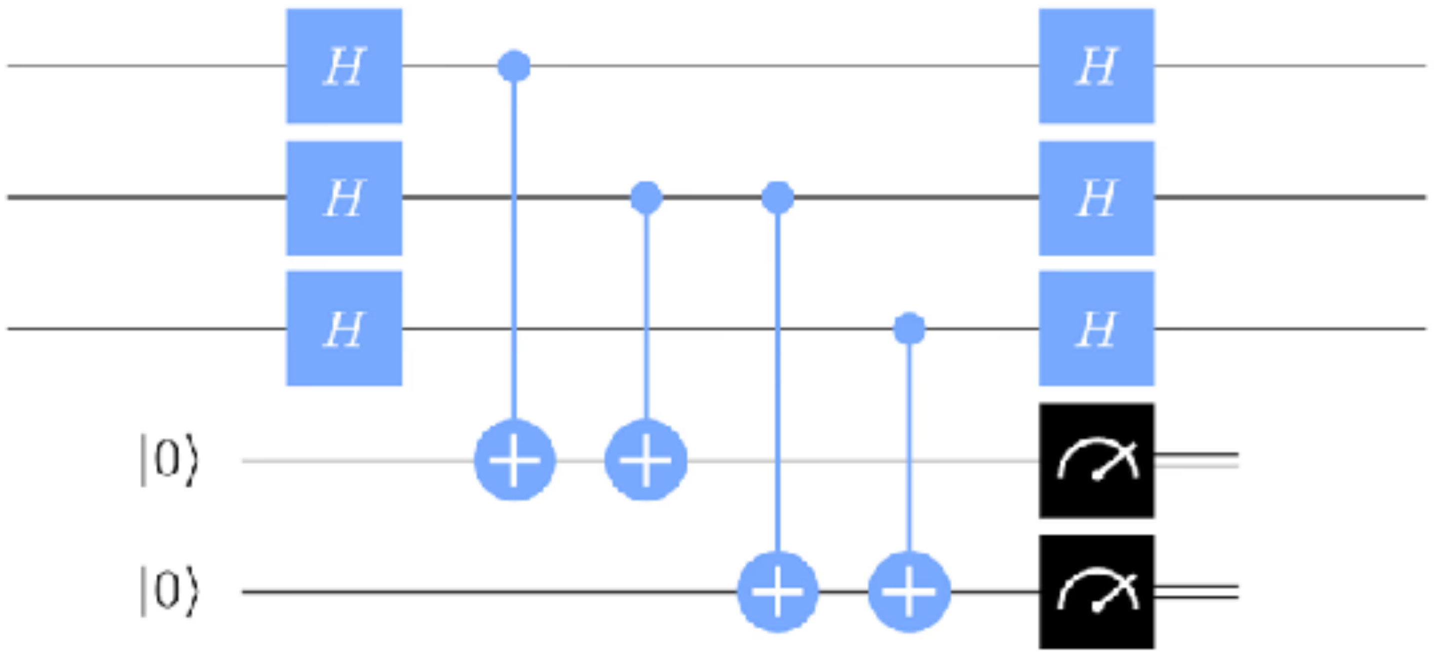
How can we express the “phase”?

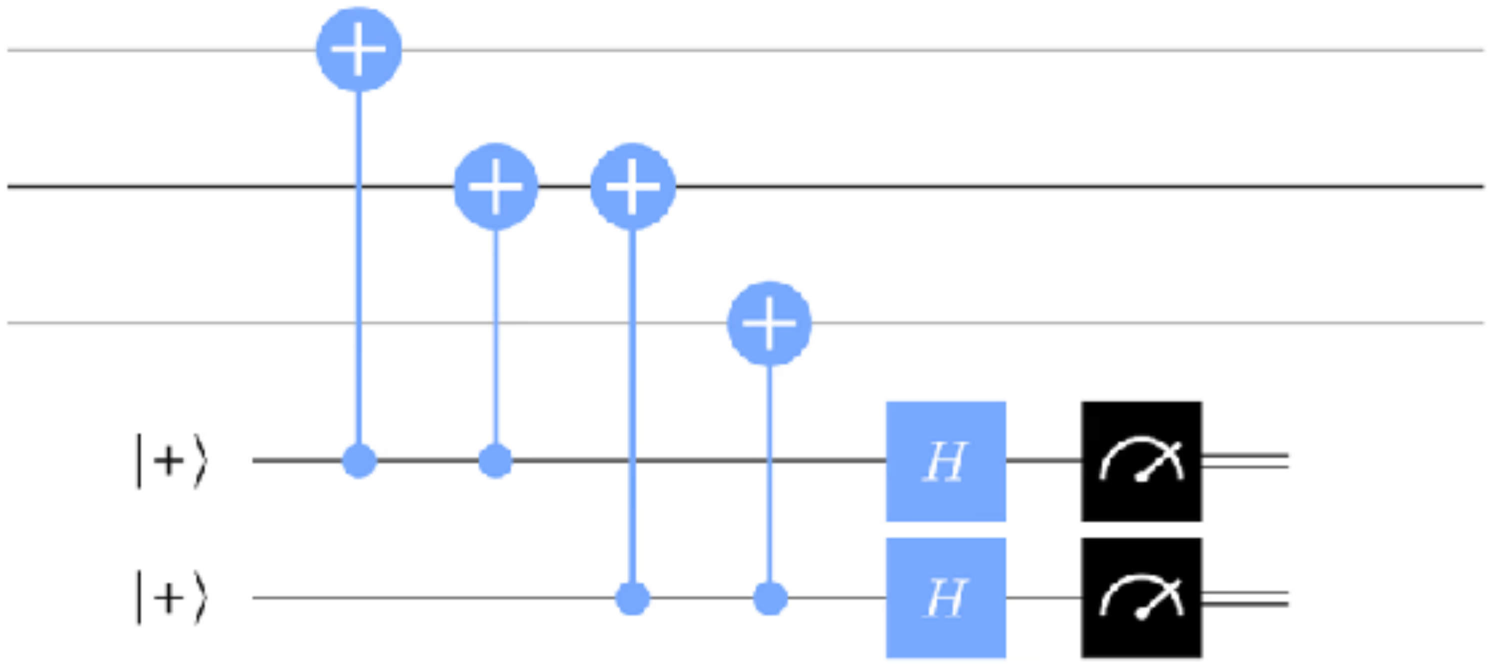


Use superpositions!

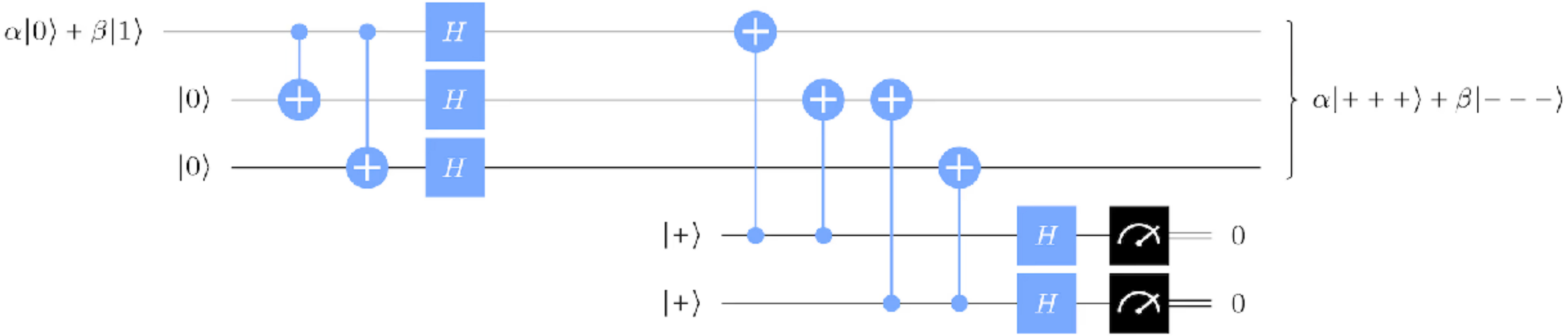


Error detection

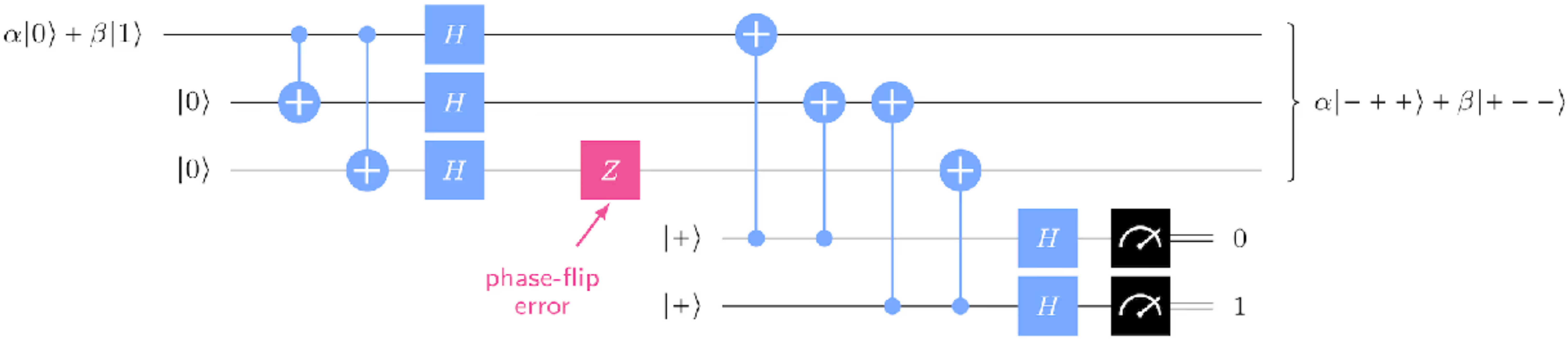




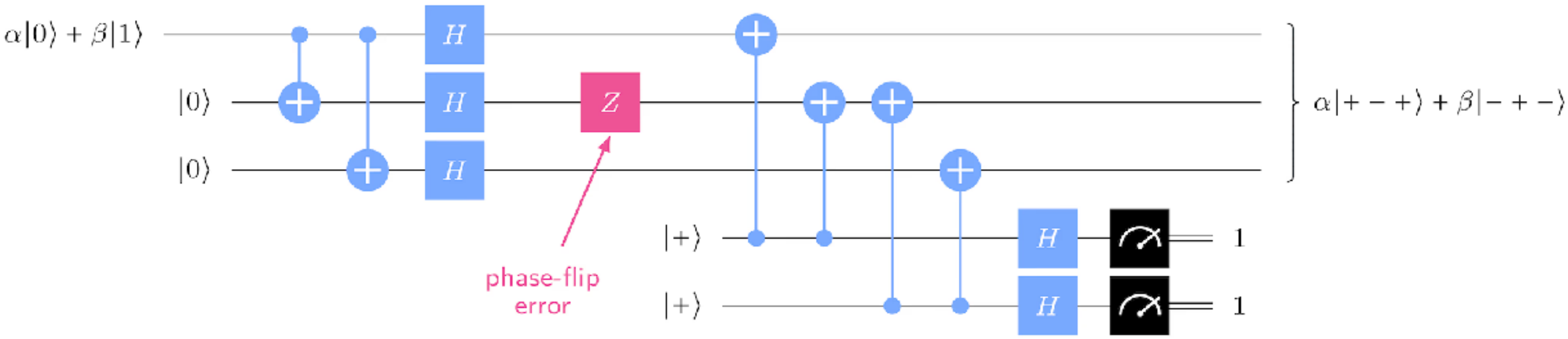
Error detection



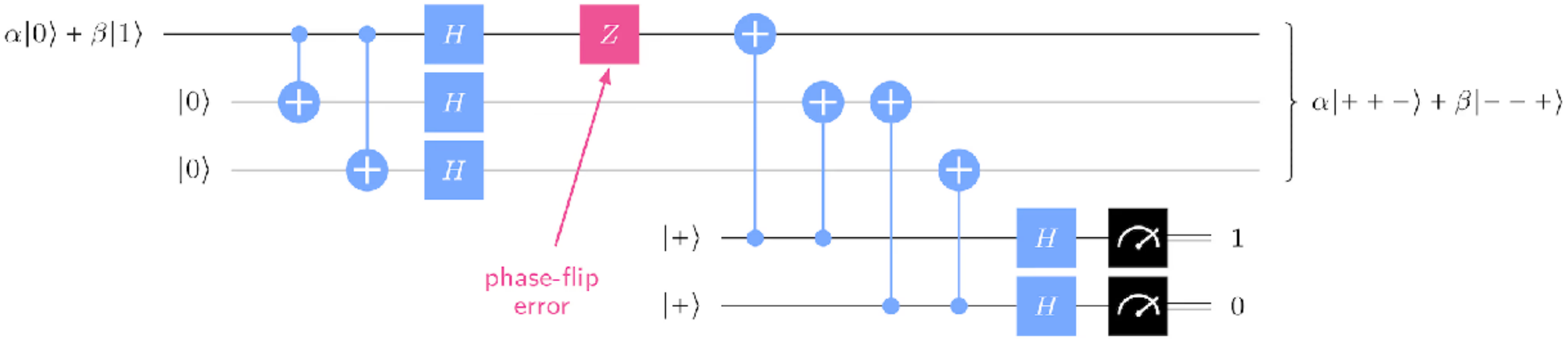
If q_2 gets wrong



If q_1 gets wrong



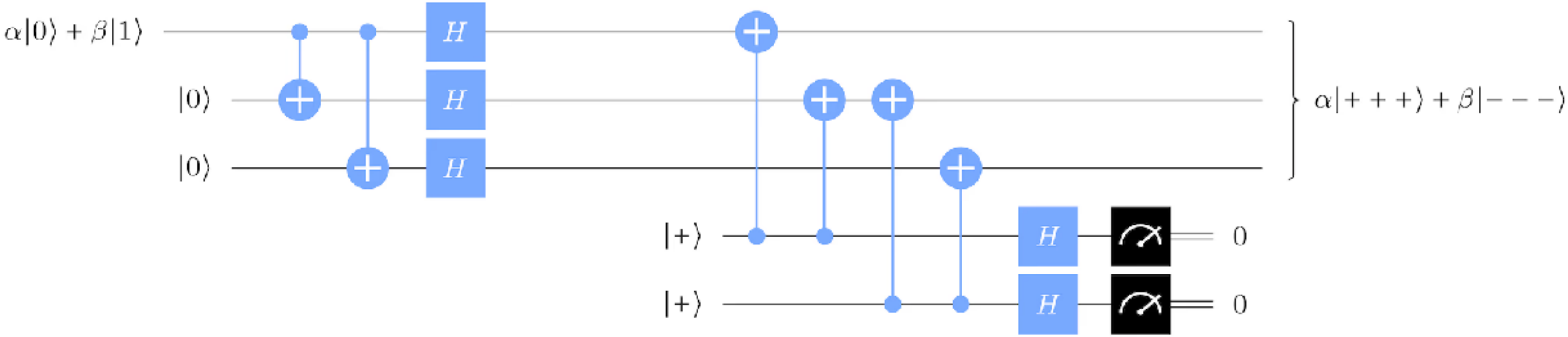
If q_0 gets wrong



How to correct errors

State	Syndrome	Correction
$\alpha\lvert +++\rangle + \beta\lvert ---\rangle$	00	$I \otimes I \otimes I$
$\alpha\lvert -++\rangle + \beta\lvert +--\rangle$	10	$Z \otimes I \otimes I$
$\alpha\lvert +-+\rangle + \beta\lvert -+-\rangle$	11	$I \otimes Z \otimes I$
$\alpha\lvert ++-\rangle + \beta\lvert --+\rangle$	01	$I \otimes I \otimes Z$

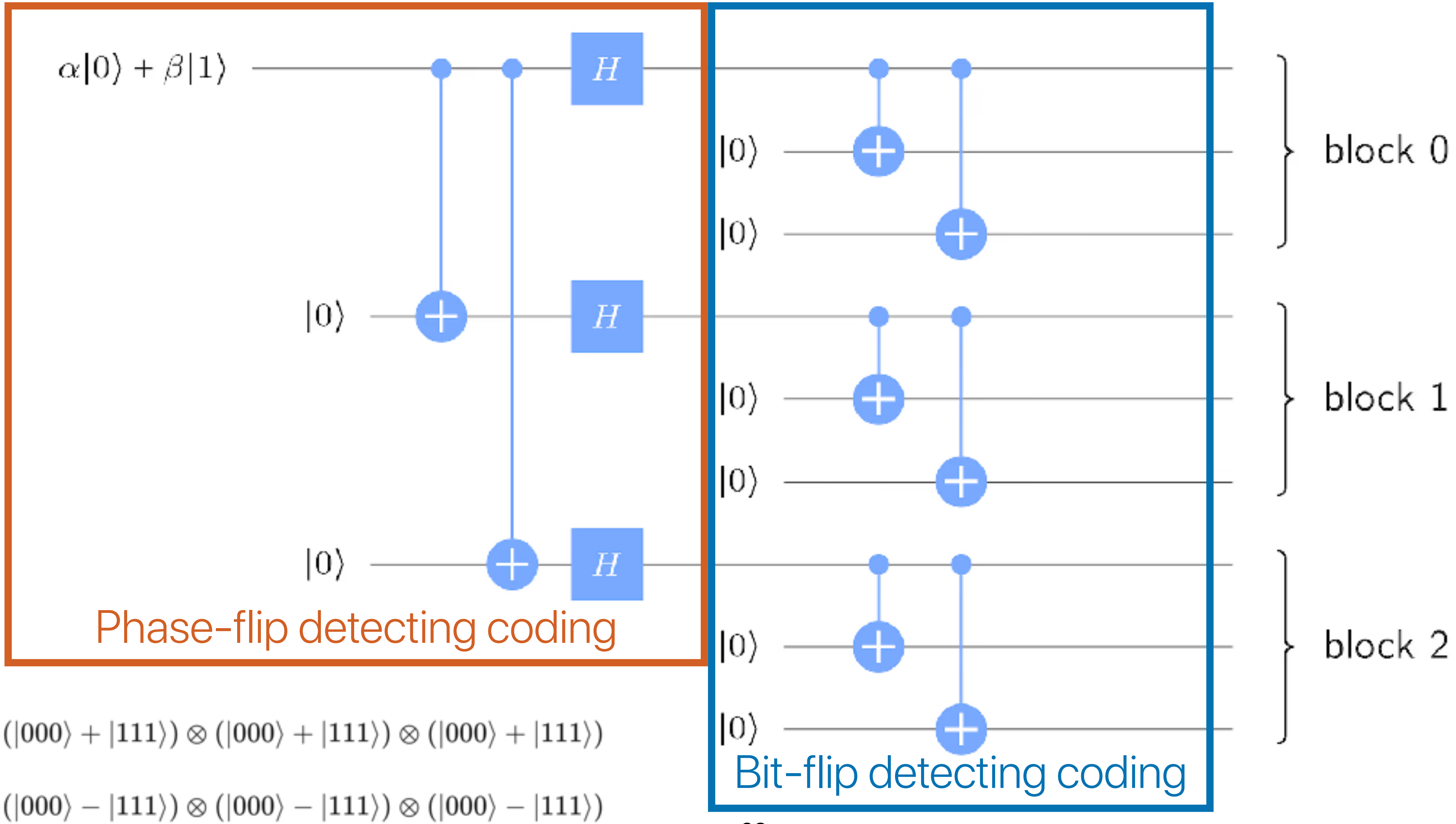
Does this work for bit-flips?



How can we detect/correct both?

- Concatenating both codes!
- First apply one coding, and then encode each encoded qubit using the other coding
- 3×3 qubits for each raw qubit

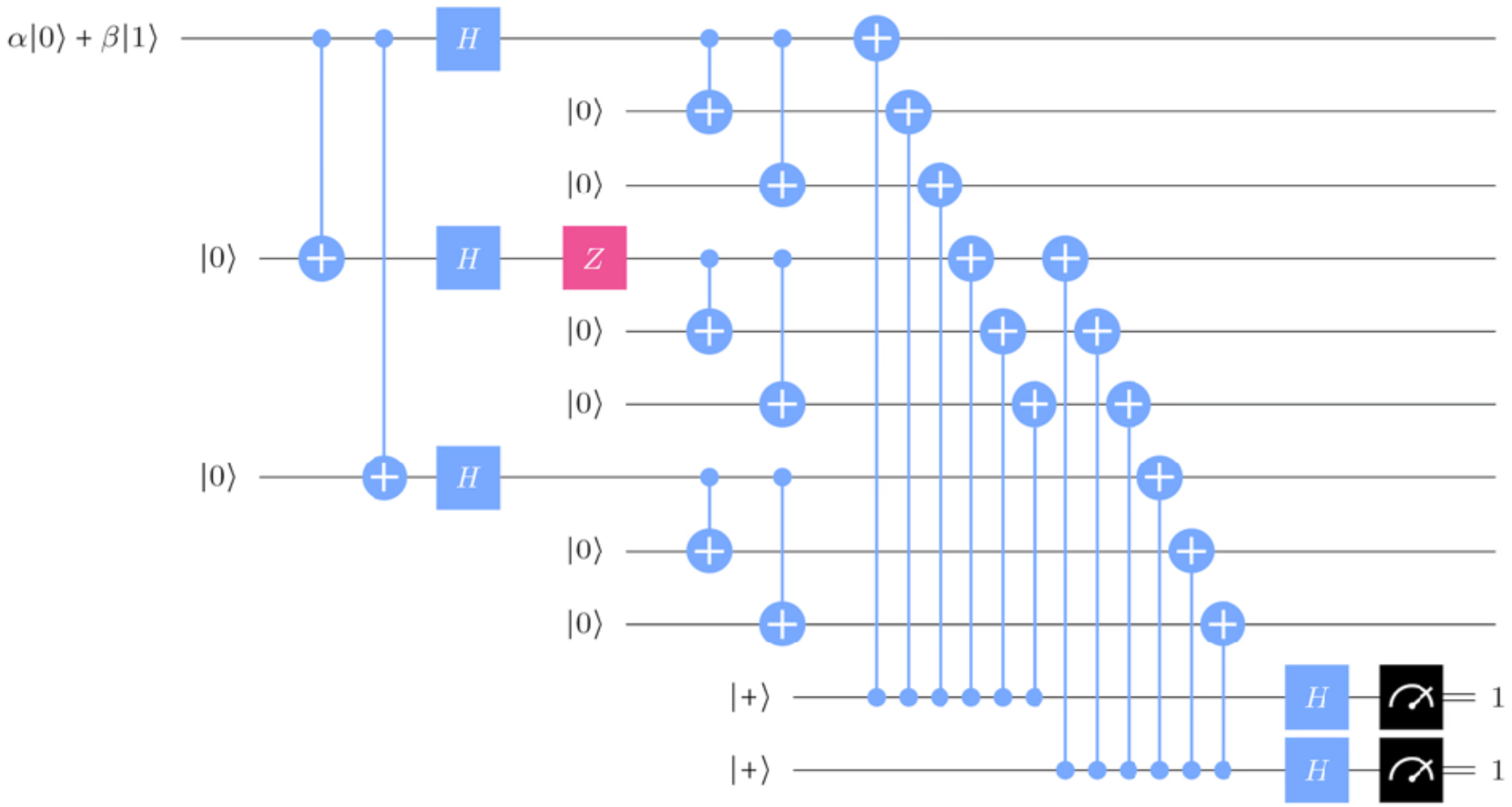
9-qubit Shor code



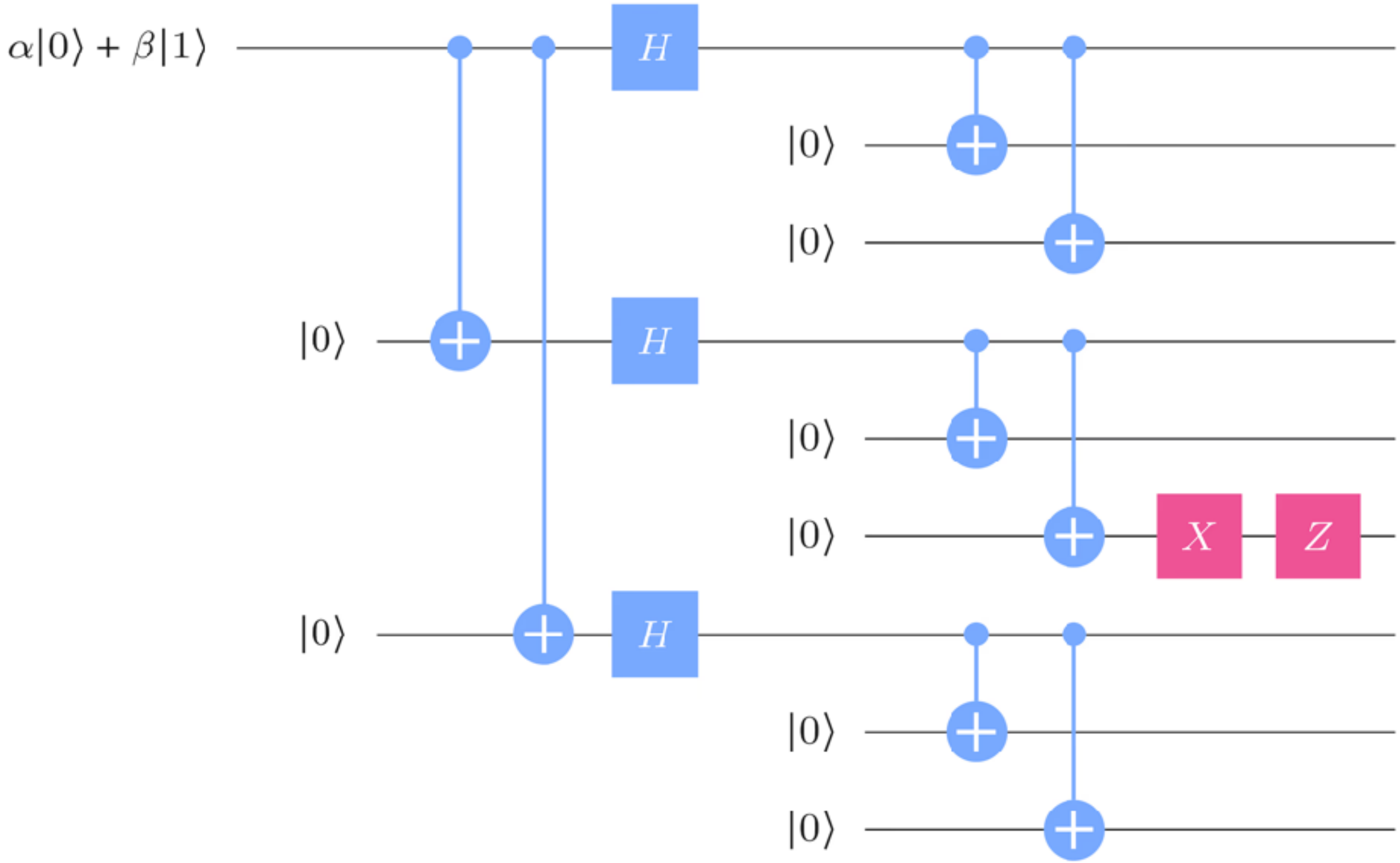
$$|0\rangle \mapsto \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle)$$

$$|1\rangle \mapsto \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle)$$

Shor coding with error corrections



Simultaneous bit- and phase-flip errors



Limitations of Shor code

- Corrects any Pauli error on a single qubit, including a Y, but doesn't properly correct two or more Y errors.
- The probably of at most one qubit error is $(1-p)^9 + 9p(1-p)^8$
- So it still fails for $1 - [(1-p)^9 + 9p(1-p)^8]$ of the time

Only works if the p is small