

Quantum Algorithms (3)

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The implementation of rotation

P-Gate

$$P(\phi) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix}$$

S-Gate

$$S = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{2}} \end{bmatrix}, \quad S^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & e^{-\frac{i\pi}{2}} \end{bmatrix}$$

T-Gate

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{bmatrix}, \quad T^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & e^{-\frac{i\pi}{4}} \end{bmatrix}$$

U-Gate

$$U(\theta, \phi, \lambda) = \begin{bmatrix} \cos(\frac{\theta}{2}) & -e^{i\lambda} \sin(\frac{\theta}{2}) \\ e^{i\phi} \sin(\frac{\theta}{2}) & e^{i(\phi+\lambda)} \cos(\frac{\theta}{2}) \end{bmatrix}$$

$$U(\frac{\pi}{2}, 0, \pi) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = H$$

$$U(0, 0, \lambda) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\lambda} \end{bmatrix} = P$$

Quantum Fourier Transform (QFT)

DFT and QFT

- The discrete Fourier transform acts on a vector (x_0, \dots, x_{N-1}) and maps it to the vector (y_0, \dots, y_{N-1}) according to the formula

$$y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j \omega_N^{jk}, \text{ where } \omega_N^{jk} = e^{2\pi i \frac{jk}{N}}$$

- the quantum Fourier transform acts on a quantum state $|X\rangle = \sum_{j=0}^{N-1} x_j |j\rangle$ and maps it to the quantum state

$$|Y\rangle = \sum_{k=0}^{N-1} y_k |k\rangle \text{ according to the formula}$$

$$y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j \omega_N^{jk}$$

- Only the amplitudes of the state were affected by this transformation.

$$|j\rangle \mapsto \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega_N^{jk} |k\rangle$$

$$U_{QFT} = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \omega_N^{jk} |k\rangle \langle j|$$

QFT

- Transforms between two bases, the computational (Z) basis, and the Fourier basis.

$$| \text{State in Computational Basis} \rangle \xrightarrow{\text{QFT}} | \text{State in Fourier Basis} \rangle$$

$$\text{QFT} |x\rangle = |\tilde{x}\rangle$$

1-qubit QFT

- $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, $x_0 = \alpha$, $x_1 = \beta$, and $N = 2$.

- Then,

$$y_0 = \frac{1}{\sqrt{2}} \left(\alpha \exp \left(2\pi i \frac{0 \times 0}{2} \right) + \beta \exp \left(2\pi i \frac{1 \times 0}{2} \right) \right) = \frac{1}{\sqrt{2}} (\alpha + \beta)$$

and

$$y_1 = \frac{1}{\sqrt{2}} \left(\alpha \exp \left(2\pi i \frac{0 \times 1}{2} \right) + \beta \exp \left(2\pi i \frac{1 \times 1}{2} \right) \right) = \frac{1}{\sqrt{2}} (\alpha - \beta)$$

$$\begin{aligned} \bullet \quad U_{QFT}|\psi\rangle &= \frac{1}{\sqrt{2}}(\alpha + \beta)|0\rangle + \frac{1}{\sqrt{2}}(\alpha - \beta)|1\rangle \\ &= H|\psi\rangle = H|\alpha|0\rangle + \beta|1\rangle = \frac{1}{\sqrt{2}}(\alpha + \beta)|0\rangle + \frac{1}{\sqrt{2}}(\alpha - \beta)|1\rangle \equiv \tilde{\alpha}|0\rangle + \tilde{\beta}|1\rangle \end{aligned}$$

Generalized QFT

• Given $|x\rangle = |x_1 \dots x_n\rangle$

$$\begin{aligned}
 QFT_N |x\rangle &= \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} \omega_N^{xy} |y\rangle \\
 &= \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{2\pi i xy/2^n} |y\rangle \text{ since } \omega_N^{xy} = e^{2\pi i \frac{xy}{N}} \text{ and } N = 2^n \\
 &= \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{2\pi i \left(\sum_{k=1}^n y_k/2^k\right)x} |y_1 \dots y_n\rangle \text{ rewriting in fractional binary notation } y = y_1 \dots y_n, y/2^n = \sum_{k=1}^n y_k/2^k \\
 &= \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} \prod_{k=1}^n e^{2\pi i x y_k/2^k} |y_1 \dots y_n\rangle \text{ after expanding the exponential of a sum to a product of exponentials} \\
 &= \frac{1}{\sqrt{N}} \bigotimes_{k=1}^n \left(|0\rangle + e^{2\pi i x/2^k} |1\rangle \right) \text{ after rearranging the sum and products, and expanding } \sum_{y=0}^{N-1} = \sum_{y_1=0}^1 \sum_{y_2=0}^1 \dots \sum_{y_n=0}^1 \\
 &= \frac{1}{\sqrt{N}} \left(|0\rangle + e^{\frac{2\pi i}{2}x} |1\rangle \right) \otimes \left(|0\rangle + e^{\frac{2\pi i}{2^2}x} |1\rangle \right) \otimes \dots \otimes \left(|0\rangle + e^{\frac{2\pi i}{2^{n-1}}x} |1\rangle \right) \otimes \left(|0\rangle + e^{\frac{2\pi i}{2^n}x} |1\rangle \right)
 \end{aligned}$$

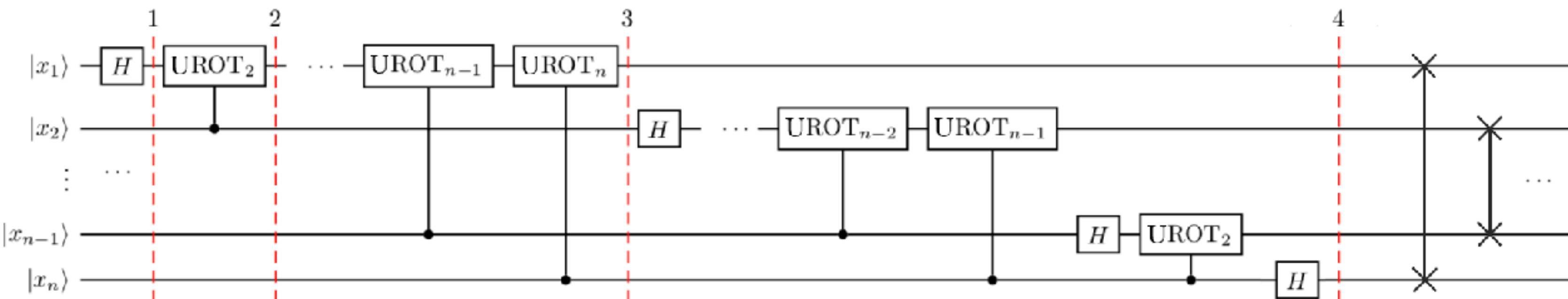
Implementing QFT

$$\bullet H|x_k\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle + \exp\left(\frac{2\pi i}{2}x_k\right) |1\rangle \right)$$

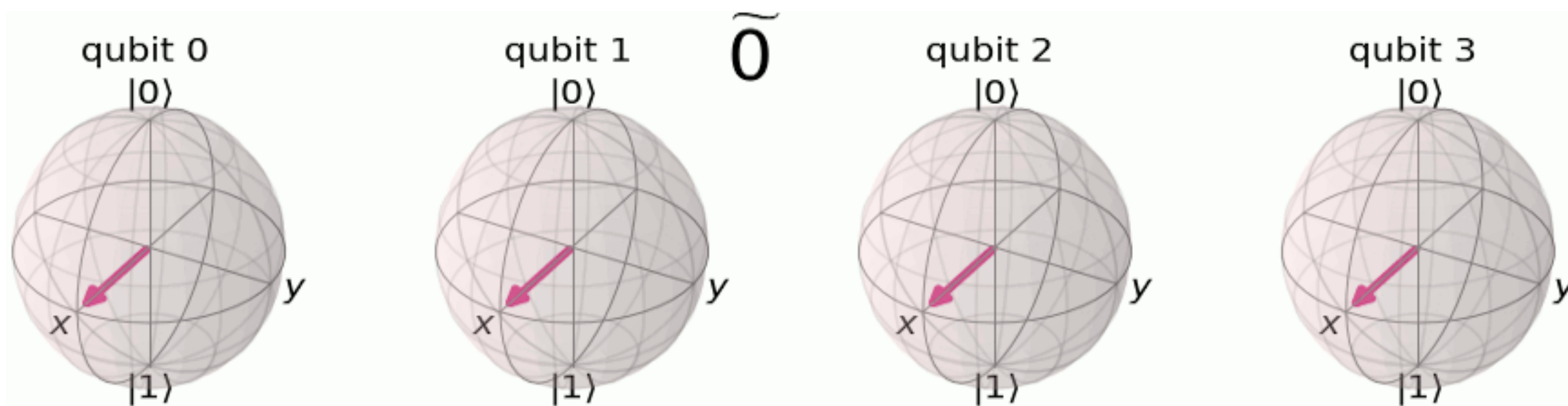
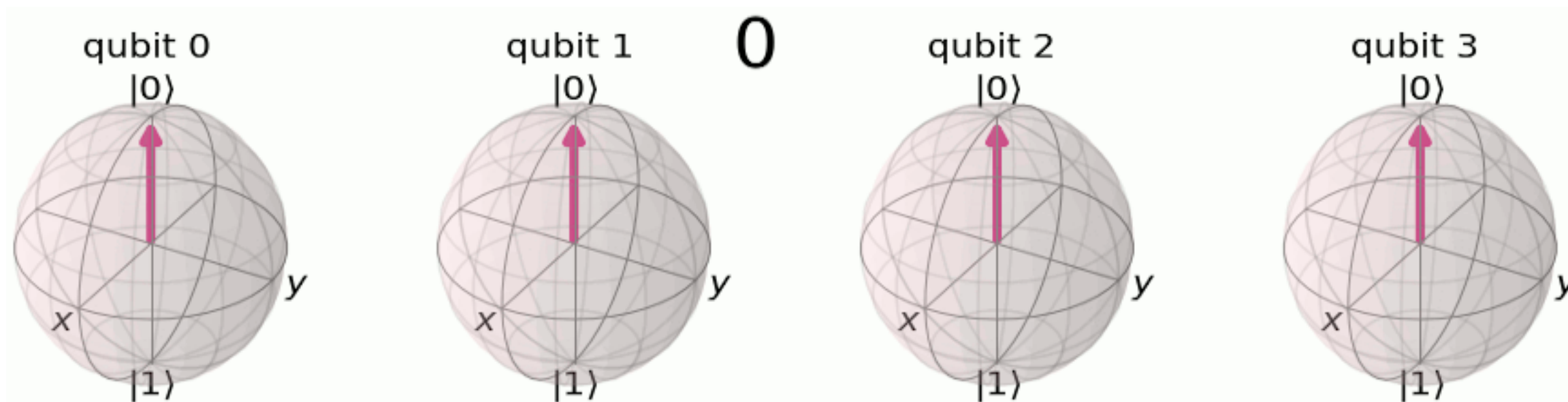
$$CROT_k = \begin{bmatrix} I & 0 \\ 0 & UROT_k \end{bmatrix}, \text{ where } UROT_k = \begin{bmatrix} 1 & 0 \\ 0 & \exp\left(\frac{2\pi i}{2^k}\right) \end{bmatrix}$$

$$CROT_k |0x_j\rangle = |0x_j\rangle \text{ and } CROT_k |1x_j\rangle = \exp\left(\frac{2\pi i}{2^k}x_j\right) |1x_j\rangle$$

The quantum circuit

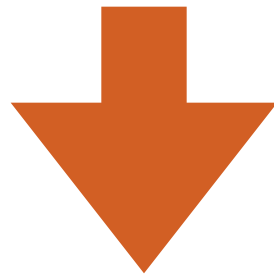


Counting in different basis



Counting in quantum basis

$$x = 2^{n-1}x_1 + 2^{n-2}x_2 + \dots + 2^1x_{n-1} + 2^0x_n$$



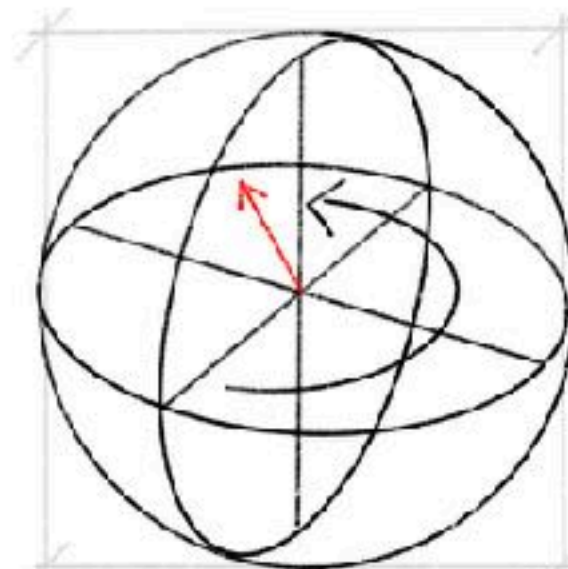
$$\frac{1}{\sqrt{2}} \left[|0\rangle + \exp\left(\frac{2\pi i}{2^n}x\right) |1\rangle \right] \otimes |x_2x_3\dots x_n\rangle$$

Quantum Phase Estimation

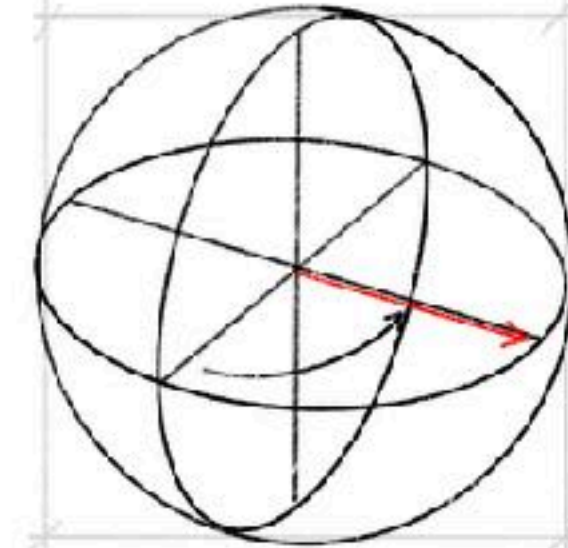
QPE

- Given a unitary operator U , the algorithm estimates θ in $U|\psi\rangle = e^{2\pi i\theta}|\psi\rangle$.
 - $|\psi\rangle$ is an eigenvector and $e^{2\pi i\theta}$ is the corresponding eigenvalue. Since U is unitary, all of its eigenvalues have a norm of 1.

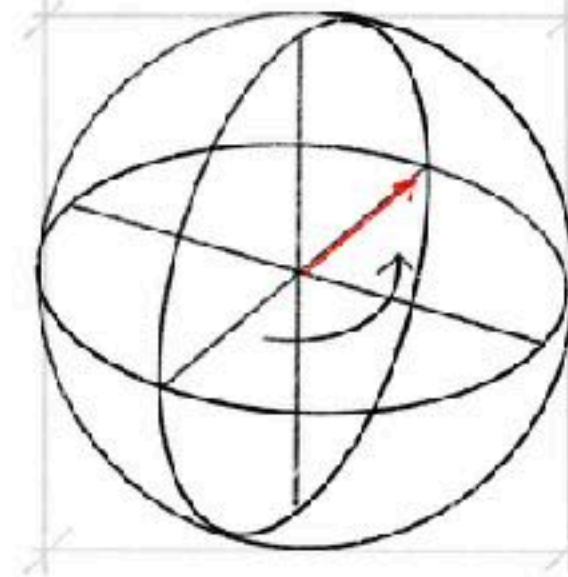
5 in the
fourier basis
(on 3 qubits)



$\frac{5}{8}$ Full turn



$\frac{10}{8}$ Full turn

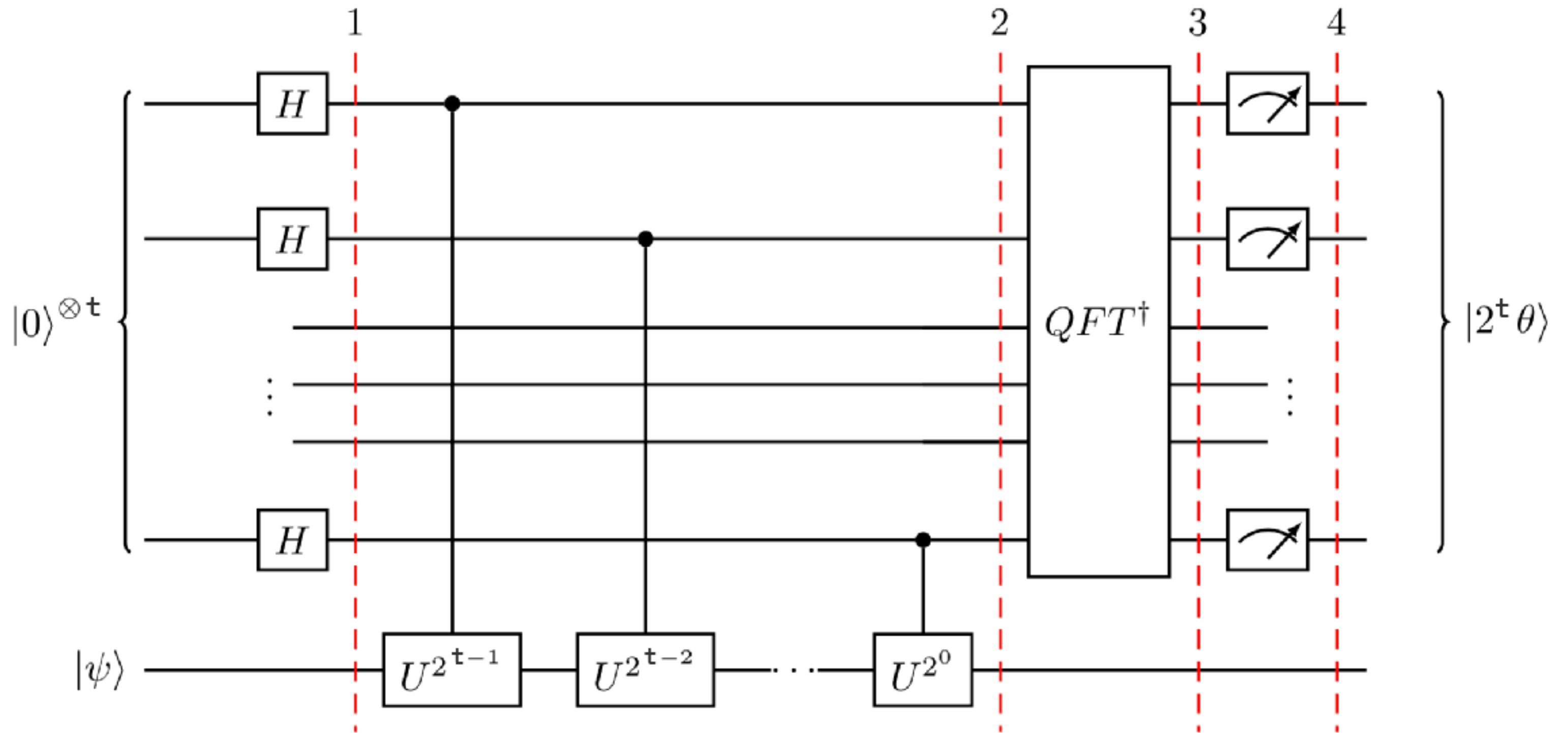


$\frac{20}{8}$ Full turn

The idea

- Uses phase kickback to write the phase of U (in the Fourier basis) to the t qubits in the counting register.
 - In the Fourier basis the topmost qubit completes one full rotation when counting between 0 and 2^t .
 - To count to a number, x between 0 and 2^t , we rotate this qubit by x around the z-axis.
 - For the next qubit we rotate by $\frac{2x}{2^t}$, then $\frac{4x}{2^t}$ for the third qubit.
 - When we use a qubit to control the U -gate, the qubit will turn (due to kickback) proportionally to the phase $e^{2i\pi\theta}$.
 - Use successive CU -gates to repeat this rotation an appropriate number of times until we have encoded the phase theta as a number between 0 and 2^t in the Fourier basis.
 - Use QFT to convert this into the computational basis

Implementation of the idea



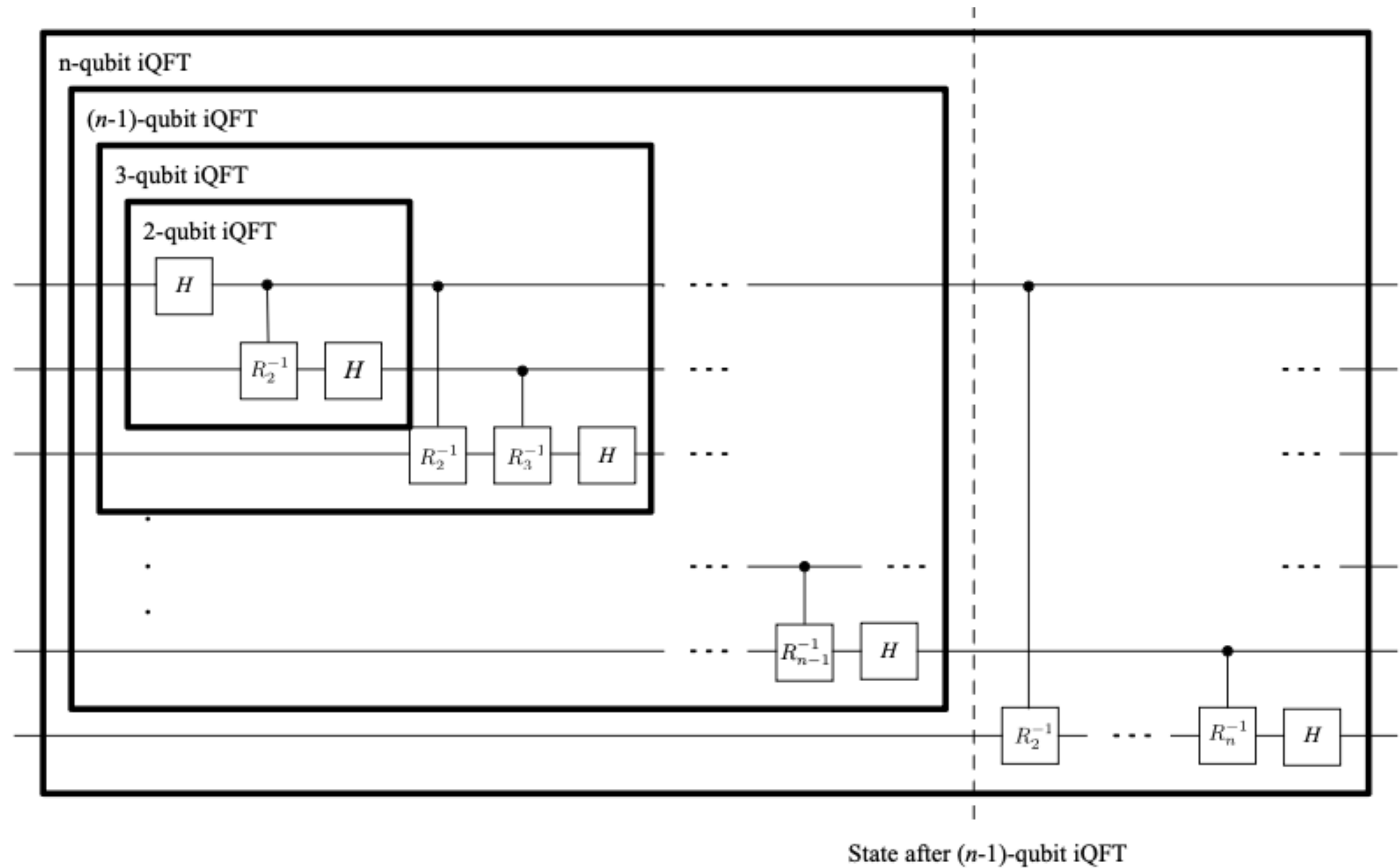


Figure 3.2: Inverse Quantum Fourier Transform (iQFT) circuit.

Implementation

- Setup: $|\psi\rangle$ is in one set of qubit registers. An additional set of n qubits form the counting register on which we will store the value $2^n\theta$: $|\psi_0\rangle = |0\rangle^{\otimes n}|\psi\rangle$
- Superposition: Apply a n -bit Hadamard gate operation $H^{\otimes n}$ on the counting register:

$$|\psi_1\rangle = \frac{1}{2^{\frac{n}{2}}} (|0\rangle + |1\rangle)^{\otimes n} |\psi\rangle$$

- Controlled Unitary Operations — CU that applies the unitary operator U on the target register only if its corresponding control bit is $|1\rangle$. Since U is a unitary operator with eigenvector $|\psi\rangle$ such that $U|\psi\rangle = e^{2\pi i\theta}|\psi\rangle$, this means $U^{2^j}|\psi\rangle = U^{2^j-1}U|\psi\rangle = U^{2^j-1}e^{2\pi i\theta}|\psi\rangle = \dots = e^{2\pi i2^j\theta}|\psi\rangle$
- Applying all the n controlled operations CU^{2^j} with $0 \leq j \leq n-1$, and using the relation

$$|0\rangle \otimes |\psi\rangle + |1\rangle \otimes e^{2\pi i\theta}|\psi\rangle = (|0\rangle + e^{2\pi i\theta}|1\rangle) \otimes |\psi\rangle$$

$$\begin{aligned} |\psi_2\rangle &= \frac{1}{2^{\frac{n}{2}}} \left(|0\rangle + e^{2\pi i\theta 2^{n-1}} |1\rangle \right) \otimes \dots \otimes \left(|0\rangle + e^{2\pi i\theta 2^1} |1\rangle \right) \otimes \left(|0\rangle + e^{2\pi i\theta 2^0} |1\rangle \right) \otimes |\psi\rangle \\ &= \frac{1}{2^{\frac{n}{2}}} \sum_{k=0}^{2^n-1} e^{2\pi i\theta k} |k\rangle \otimes |\psi\rangle \end{aligned}$$

where k denotes the integer representation of n -bit binary numbers.

Implementation (2)

- Inverse Fourier Transform

Recall that QFT maps an n-qubit input state $|x\rangle$ into an output as

$$QFT|x\rangle = \frac{1}{2^{\frac{n}{2}}} \left(|0\rangle + e^{\frac{2\pi i}{2}x} |1\rangle \right) \otimes \left(|0\rangle + e^{\frac{2\pi i}{2^2}x} |1\rangle \right) \otimes \dots \otimes \left(|0\rangle + e^{\frac{2\pi i}{2^{n-1}}x} |1\rangle \right) \otimes \left(|0\rangle + e^{\frac{2\pi i}{2^n}x} |1\rangle \right)$$

- Replacing x by $2^n\theta$ in the above expression gives exactly the expression of $|\psi_2\rangle$. Therefore, to recover the state $|2^n\theta\rangle$, apply an inverse Fourier transform on the auxiliary register. Doing so, we find

$$|\psi_3\rangle = \frac{1}{2^{\frac{n}{2}}} \sum_{k=0}^{2^n-1} e^{2\pi i\theta k} |k\rangle \otimes |\psi\rangle \xrightarrow{QFT_n^{-1}} \frac{1}{2^n} \sum_{x=0}^{2^n-1} \sum_{k=0}^{2^n-1} e^{-\frac{2\pi i k}{2^n}(x-2^n\theta)} |x\rangle \otimes |\psi\rangle$$

- Measurement: The above expression peaks near $x = 2^n\theta$. For the case when $2^n\theta$ is an integer, measuring in the computational basis gives the phase in the auxiliary register with high probability:

$$|\psi_4\rangle = |2^n\theta\rangle \otimes |\psi\rangle$$

For the case when $2^n\theta$ is not an integer, it can be shown that the above expression still peaks near $x = 2^n\theta$ with probability better than $4/\pi^2 \approx 40\%$.

Example: T-Gate

- T -gate adds a phase of $e^{\frac{i\pi}{4}}$ to the state $|1\rangle$

$$T|1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e^{\frac{i\pi}{4}} |1\rangle$$

- Since QPE will give us θ where

$$T|1\rangle = e^{2i\pi\theta} |1\rangle$$

we expect to find

$$\theta = \frac{1}{8}$$