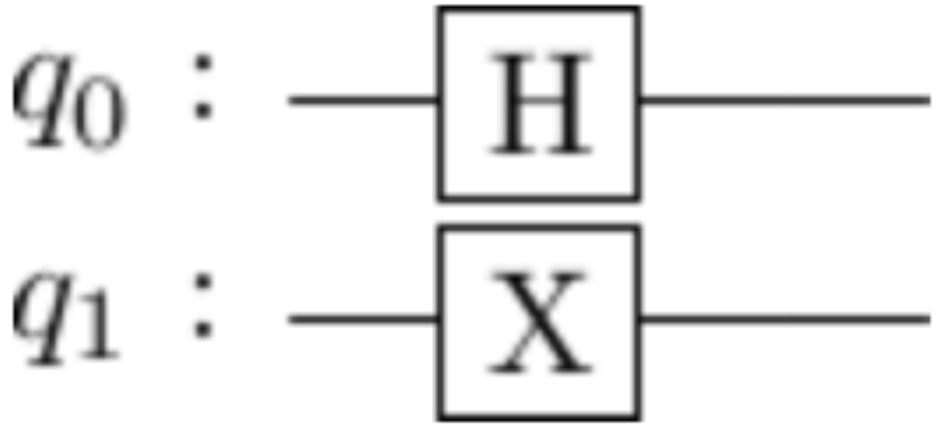
Quantum Algorithms

Hung-Wei Tseng

Recap: we can make the whole quantum circuit as a transition matrix

```
qc = QuantumCircuit(2)
qc.h(0)
qc.x(1)
qc.draw()
```



$$X|q_1\rangle \otimes H|q_0\rangle = (X \otimes H)|q_1 q_0\rangle$$

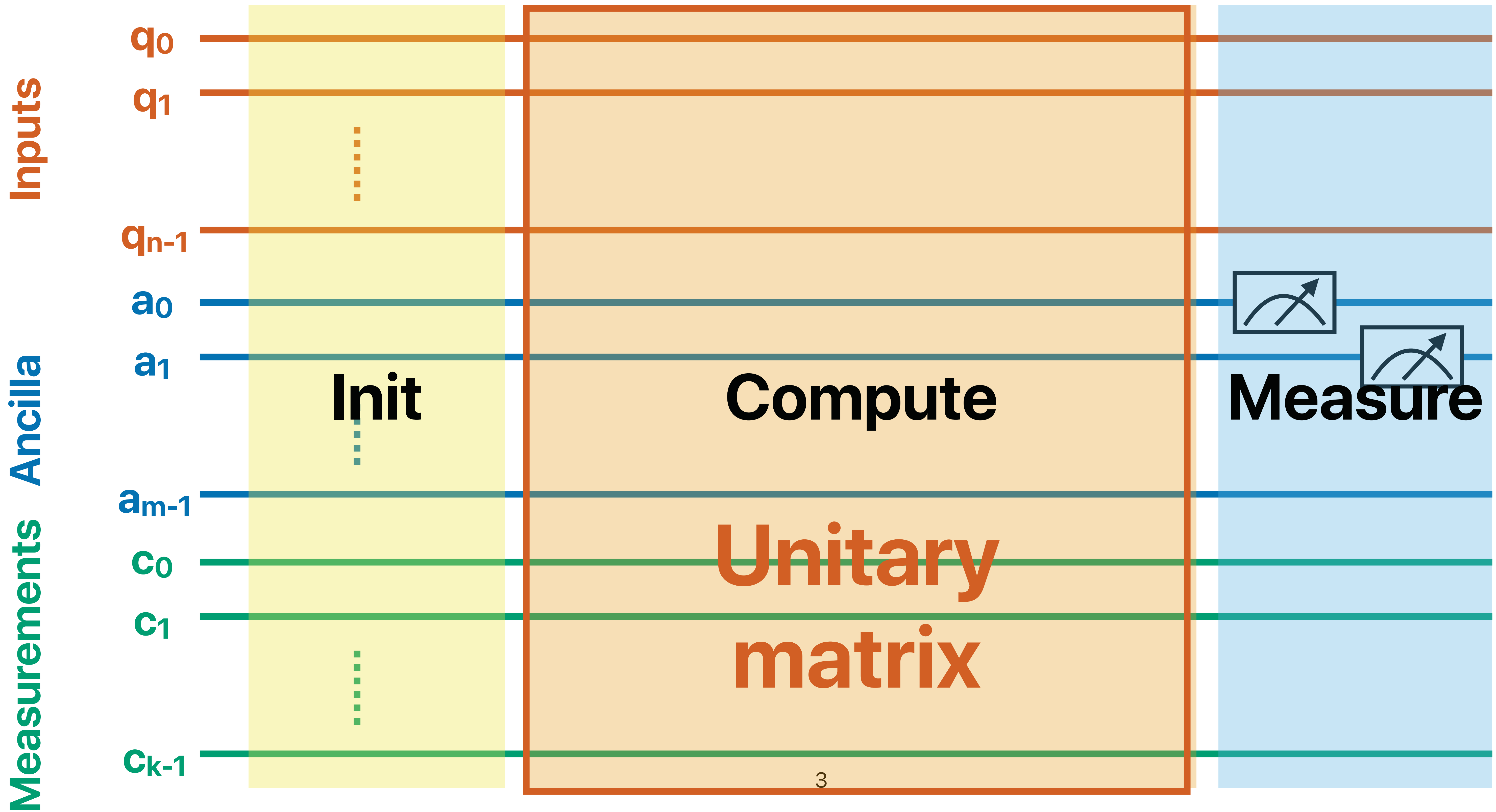
$$X \otimes H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \times \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & 1 \times \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ 1 \times \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & 0 \times \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{bmatrix}$$

$$= \boxed{\frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}} = \begin{bmatrix} 0 & H \\ H & 0 \end{bmatrix}$$

Unitary matrix

Recap: Quantum circuits



Recap: a state vector can be composed by two qubits

$$|ba\rangle = \begin{bmatrix} b_0 a_0 \\ b_0 a_1 \\ b_1 a_0 \\ b_1 a_1 \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$b_0 a_0 = \frac{1}{2}$$

$$b_0 a_1 = -\frac{1}{2}$$

$$b_1 a_0 = -\frac{1}{2}$$

$$b_1 a_1 = \frac{1}{2}$$

$$\frac{a_0}{a_1} = -1$$

$$|a_0|^2 + |a_1|^2 = 1$$

$$\frac{b_0}{b_1} = -1$$

$$|b_0|^2 + |b_1|^2 = 1$$

$$a_0 = \frac{1}{\sqrt{2}}$$

$$a_1 = \frac{-1}{\sqrt{2}}$$

$$b_0 = \frac{-1}{\sqrt{2}}$$

$$b_1 = \frac{1}{\sqrt{2}}$$

Recap: not all state vectors can be composed by two qubits

$$|ba\rangle = \begin{bmatrix} b_0a_0 \\ b_0a_1 \\ b_1a_0 \\ b_1a_1 \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$b_0a_0 = \frac{1}{\sqrt{2}}$$

$$b_0a_1 = 0$$

$$b_1a_0 = 0$$

$$b_1a_1 = \frac{1}{\sqrt{2}}$$

$$\frac{a_0}{a_1} = \infty$$

$$\frac{b_0}{b_1} = NaN$$

This is not possible to be a product of two states!

Recap: entanglement

- The state is not a product of qubits
- Measuring one will tell us the state of the other and collapse its superposition — because the state cannot stand by itself
- Two or more quantum systems (or quantum particles) have a non-classical correlation, or shared quantum state, even if they are separated by a large distance.

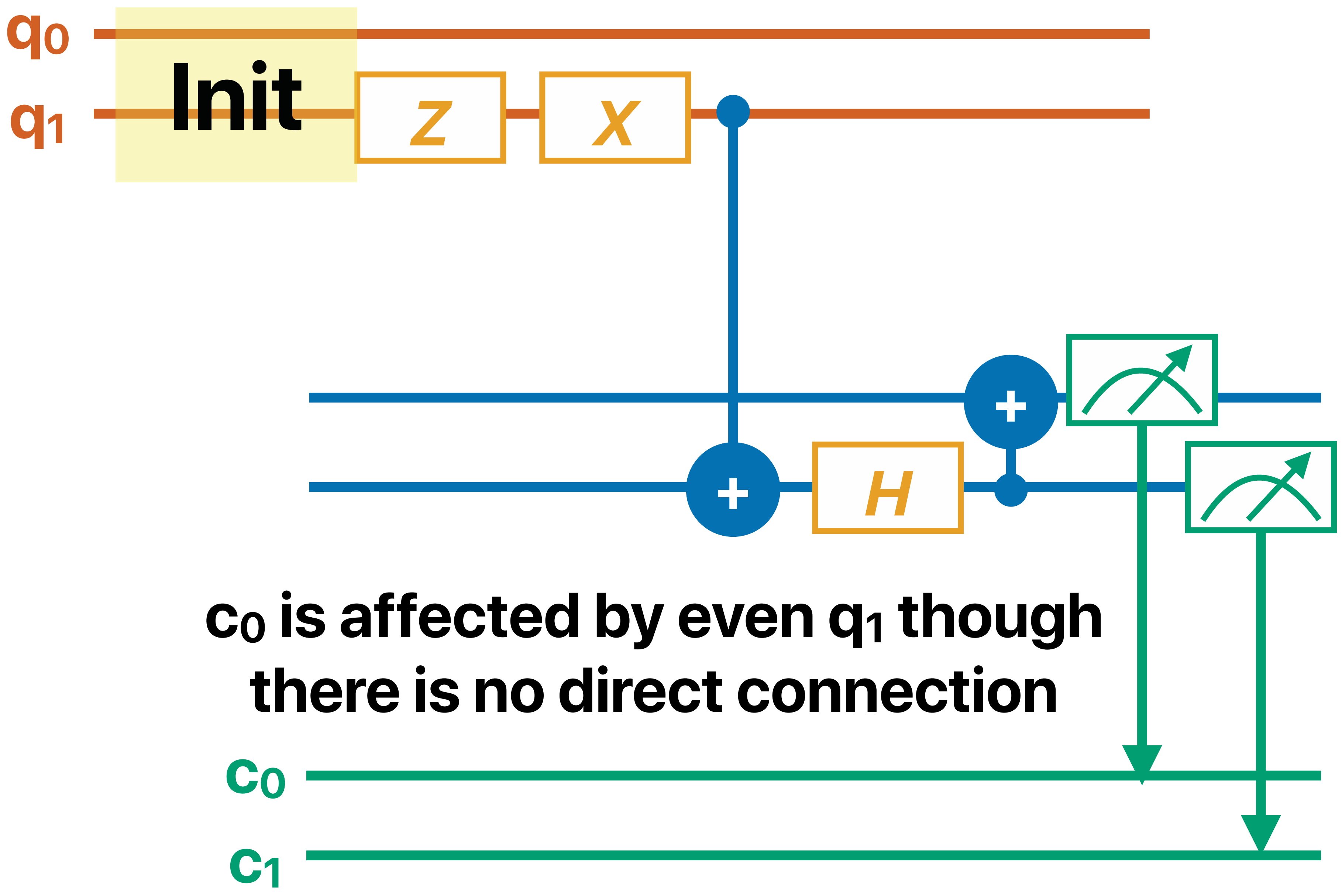
If we initialize the state as $|\Phi^+\rangle$

$$|\Phi^+\rangle = \sqrt{\frac{1}{2}}(|0\overset{X}{\boxed{0}}\rangle + |1\overset{X}{\boxed{1}}\rangle) \qquad 01 \rightarrow |\Psi^+\rangle = \sqrt{\frac{1}{2}}(|01\rangle + |10\rangle)$$

$$|\Phi^+\rangle = \sqrt{\frac{1}{2}}(|0\overset{Z}{\boxed{0}}\rangle + |1\overset{Z}{\boxed{1}}\rangle) \qquad 10 \rightarrow |\Phi^-\rangle = \sqrt{\frac{1}{2}}(|00\rangle - |11\rangle)$$

$$|\Phi^+\rangle = \sqrt{\frac{1}{2}}(|0\overset{ZX}{\boxed{0}}\rangle + |1\overset{ZX}{\boxed{1}}\rangle) \qquad 11 \rightarrow |\Psi^-\rangle = \sqrt{\frac{1}{2}}(|10\rangle - |01\rangle)$$

The new implementation



$$00 \rightarrow |\Phi^+\rangle = \sqrt{\frac{1}{2}}(|00\rangle + |11\rangle)$$

$$01 \rightarrow |\Psi^+\rangle = \sqrt{\frac{1}{2}}(|01\rangle + |10\rangle)$$

$$10 \rightarrow |\Phi^-\rangle = \sqrt{\frac{1}{2}}(|00\rangle - |11\rangle)$$

$$11 \rightarrow |\Psi^-\rangle = \sqrt{\frac{1}{2}}(|10\rangle - |01\rangle)$$

The Deutsch-Jozsa Algorithm

The Deutsch-Jozsa Problem

- Given a hidden Boolean function f, which takes as input a string of bits, and returns either $\mathbf{0}$ or $\mathbf{1}$, that is:

$f(\{x_0, x_1, x_2, \ldots\}) \rightarrow 0$ or 1 , where x_n is 0 or 1

- The given Boolean function is that it is guaranteed to either be balanced or constant
  - A constant function returns all 0s or all 1s for any input
  - A balanced function returns 0s for exactly half of all inputs and 1s for the other half
- Our task is to determine whether the given function is balanced or constant

David Deutsch and Richard Jozsa (1992). "Rapid solutions of problems by quantum computation". Proceedings of the Royal Society of London A. 439: 553–558. doi:10.1098/rspa.1992.0167

The classical solution

- Let's start by choosing two numbers and test their outputs
  - if $f(0,0,0,\ldots) \rightarrow 0$ and $f(1,0,0,\ldots) \rightarrow 1$, then we know the given one is a balanced one!
  - What if $f(0,0,0,\ldots) \rightarrow 0$ and $f(1,0,0,\ldots) \rightarrow 0$? We have to try one more run…
- The worst case will need to go through exactly half of the input space + 1, that is $2^{n-1} + 1$, numbers
- The classical solution is therefore

Recap: we can make the whole quantum circuit as a transition matrix

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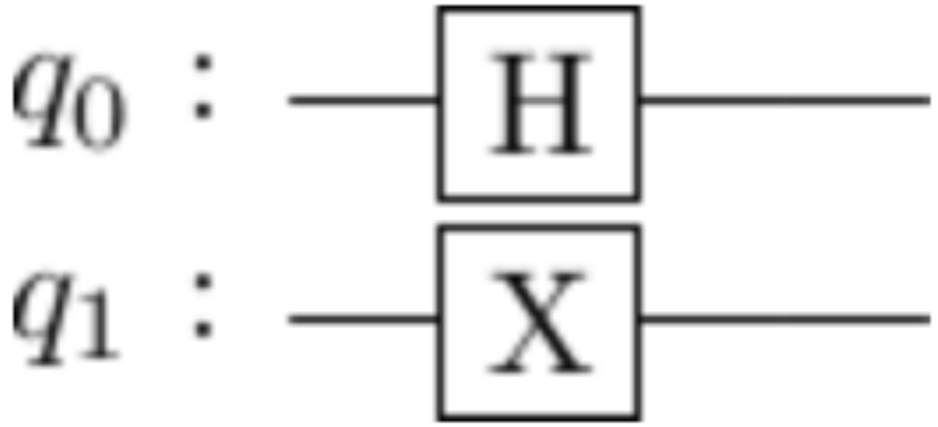

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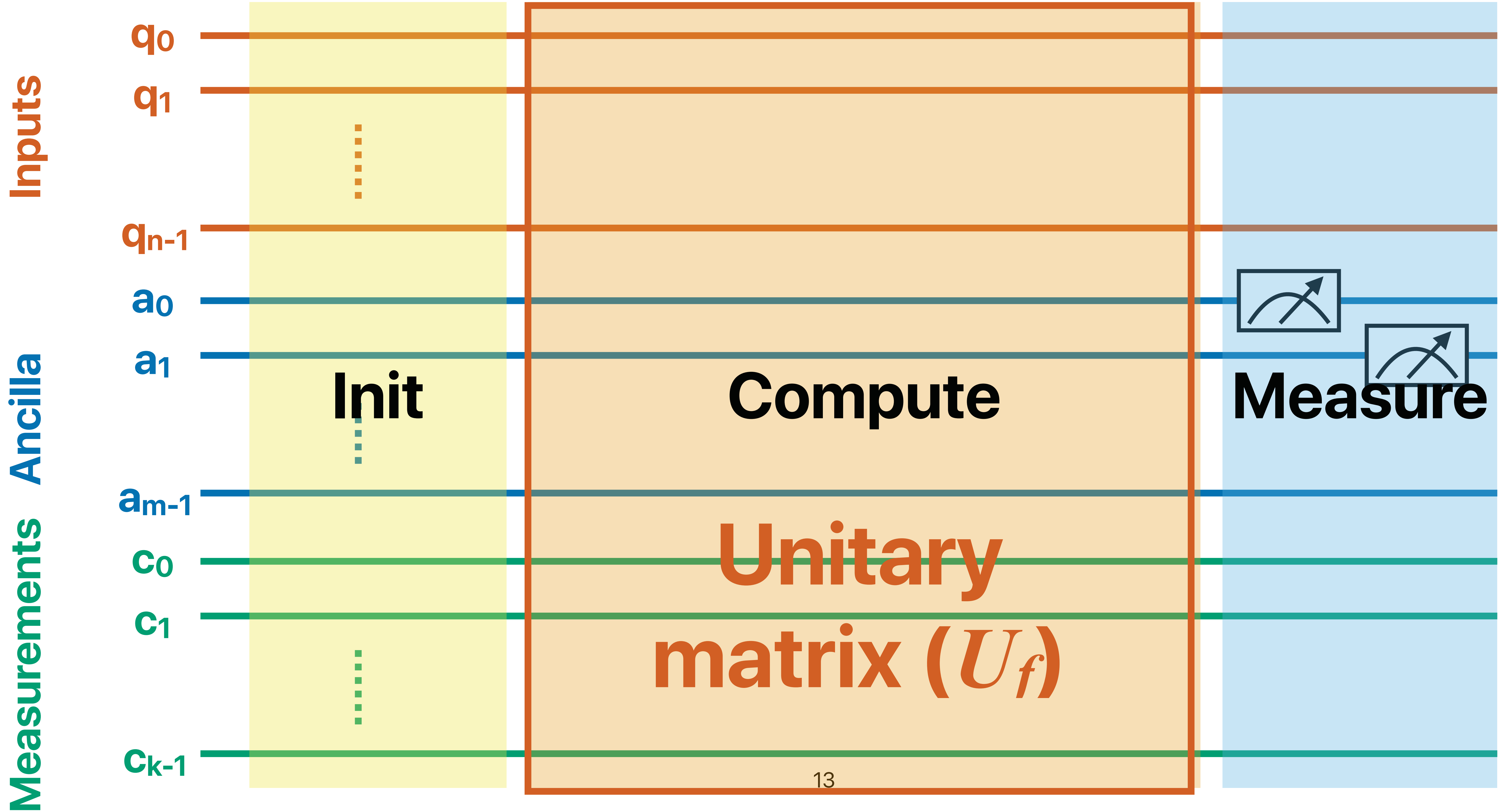
$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \times \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & 1 \times \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ 1 \times \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & 0 \times \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{bmatrix}$$

$$= \boxed{\frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}} = \begin{bmatrix} 0 & H \\ H & 0 \end{bmatrix}$$

Unitary matrix

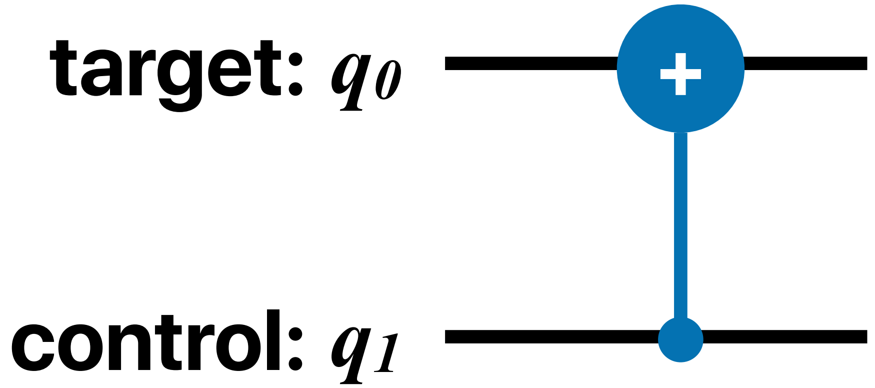
Recap: Quantum circuits



Let’s revisit CNOT gate, again

What's CNOT

- Flip the state of q_0 if q_1 is set



$$q_1 q_0\rangle$$

$$|00\rangle \rightarrow |00\rangle$$

$$|01\rangle \rightarrow |01\rangle$$

$$|10\rangle \rightarrow |11\rangle$$

$$|11\rangle \rightarrow |10\rangle$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- Or can we say it's swapping the amplitudes of $|10\rangle$ and $|11\rangle$

What's CNOT

- Flip the state of q_1 if q_0 is set

$$q_1 q_0\rangle$$

$$|00\rangle \rightarrow |00\rangle$$
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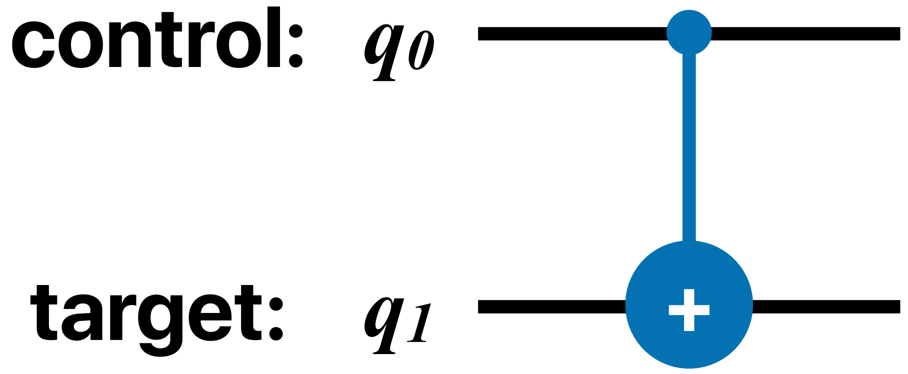
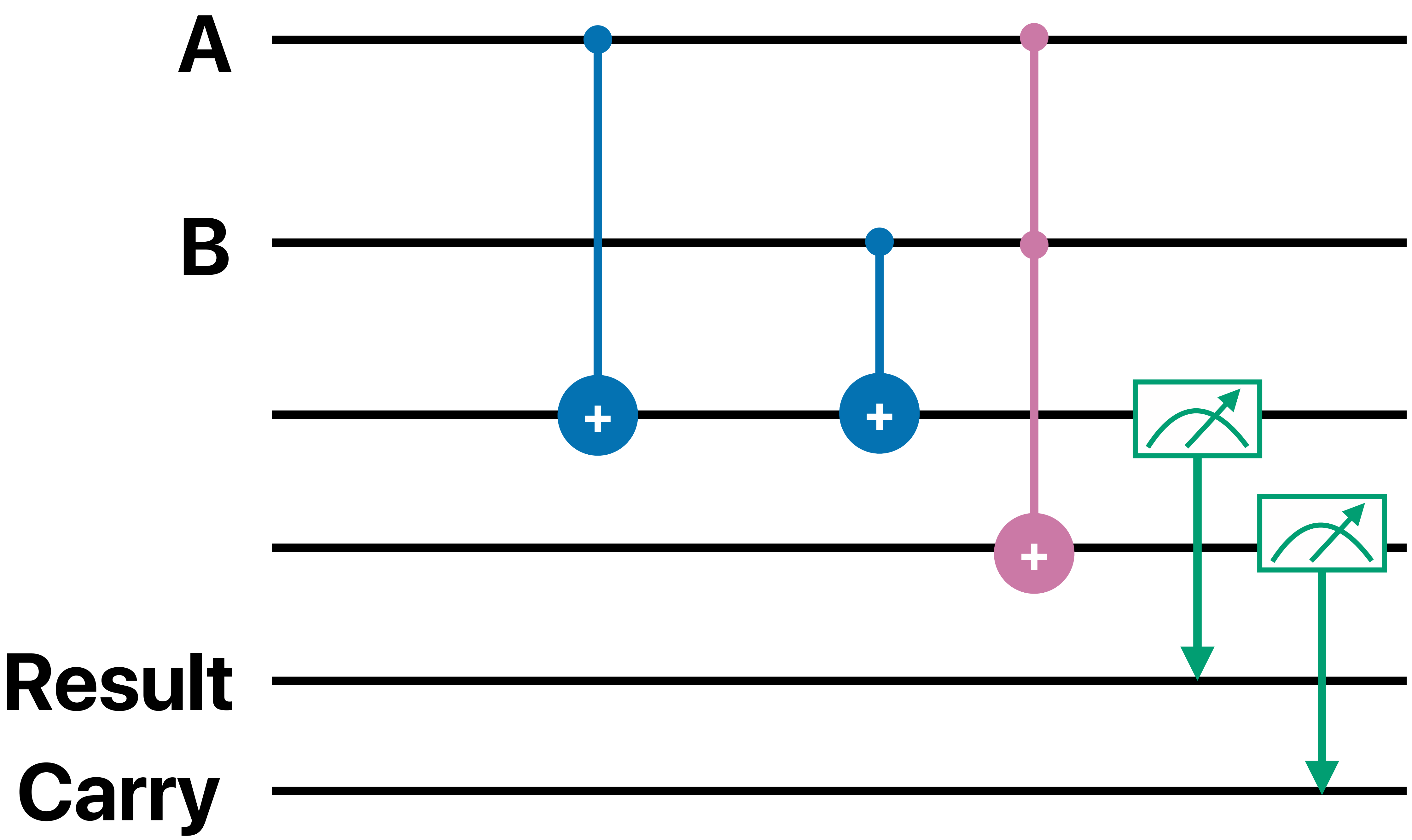
$$|10\rangle \rightarrow |10\rangle$$
$$|11\rangle \rightarrow |01\rangle$$



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

- Or can we say it's swapping the amplitudes of $|01\rangle$ and $|11\rangle$

1-bit half adder



A	B	carry	result
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

Let's look into this circuit

- $|+\rangle = \sqrt{\frac{1}{2}}\begin{bmatrix}1\\1\end{bmatrix}$
- $|-\rangle = \sqrt{\frac{1}{2}}\begin{bmatrix}1\\-1\end{bmatrix}$

control: q_0

target: q_1

CNOT $|-1\rangle = -|-1\rangle$

$$= \begin{bmatrix}0\\1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\0\\1\end{bmatrix}$$

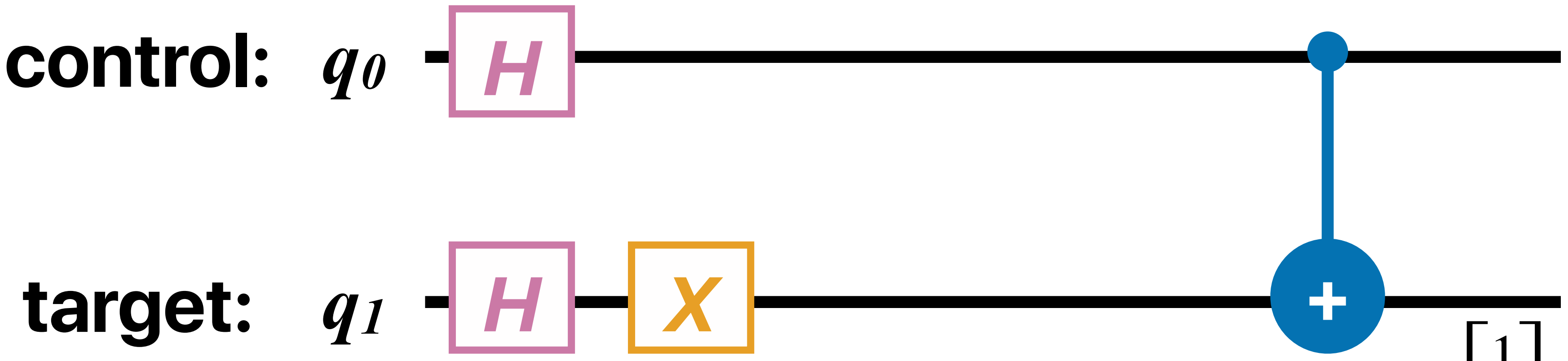
q_0 is changed! *-1* goes to q_0!

q_1 is not changed!

Phase kickback

Let's look into this circuit

- $|+\rangle = \sqrt{\frac{1}{2}}\begin{bmatrix}1\\1\end{bmatrix}$
- $|-\rangle = \sqrt{\frac{1}{2}}\begin{bmatrix}1\\-1\end{bmatrix}$



$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix}$$

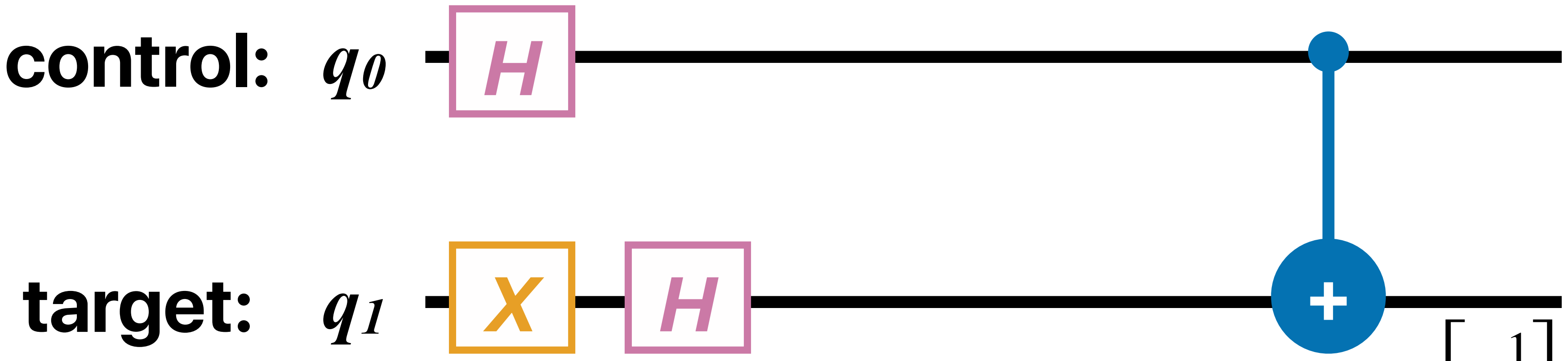
$$= \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

$$\text{CNOT}\,|++\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

$$= |++\rangle$$

Let's look into another circuit

- $|+\rangle = \sqrt{\frac{1}{2}}\begin{bmatrix}1\\1\end{bmatrix}$
- $|-\rangle = \sqrt{\frac{1}{2}}\begin{bmatrix}1\\-1\end{bmatrix}$



$$= \begin{bmatrix}0\\1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$$

$$\text{CNOT}\,|-+\rangle = \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle)$$

$$= |--\rangle \text{CNOT}\,|-+\rangle$$

$$= \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle) = |--\rangle$$

Phase kickback

- Kickback is where the eigenvalue added by a gate to a qubit is 'kicked back' into a different qubit via a controlled operation.
  - For example, we saw that performing an X-gate on a $|-\rangle$ qubit gives it the phase -1
  - When our control qubit is in either $|0\rangle$ or $|1\rangle$, this phase affects the whole state, however it is a global phase and has no observable effects

$$\text{CNOT}|-0\rangle = |-\rangle \otimes |0\rangle$$
$$= |-0\rangle$$
$$\text{CNOT}|-1\rangle = X|-\rangle \otimes |1\rangle$$
$$= -|-\rangle \otimes |1\rangle$$
$$= -|-1\rangle$$

  - This can then be written as the two separable qubit states:

$$\text{CNOT}|-+\rangle = |-\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$
$$= |--\rangle$$

An overview of Deutsch-Jozsa Algorithm

- Initialize n + 1 qubits
- Transform these qubits into Hadamard basis: making each qubit 50%-50% of being 0 or 1
- Encode the given function as an unitary matrix (i.e., oracle)
- Return the qubits for measurements
- Measure the qubits to obtain the solution

An overview of Deutsch-Jozsa Algorithm

- Initialize n + 1 qubits
- Transform these qubits into Hadamard basis: making each qubit 50%-50% of being 0 or 1

$$|\psi_1 = \sum_{x\in\{0,1\}^n} H|x\rangle = \sum_{x\in\{0,1\}^n} \frac{1}{\sqrt{2^{n+1}}}|x\rangle(|0\rangle - |1\rangle)$$

- Encode the given function as an unitary matrix (i.e., oracle)

$$U_f|\psi_1\rangle = |\psi_2 = \sum_{x\in\{0,1\}^n} \frac{1}{\sqrt{2^{n+1}}}(-1)^{f(x)}|x\rangle(|0\rangle - |1\rangle)$$

- Return the qubits for measurements

- Measure the qubits to obtain the solution

$$|\psi_3\rangle = \frac{1}{2^n}\sum_{x=0}^{2^n-1}(-1)^{f(x)}\left[\sum_{y=0}^{2^n-1}(-1)^{x\cdot y}|y\rangle\right]$$

$$= \frac{1}{2^n}\sum_{y=0}^{2^n-1}\left[\sum_{x=0}^{2^n-1}(-1)^{f(x)}(-1)^{x\cdot y}\right]|y\rangle$$

Why this works

- Constant: When the oracle is constant, it has no effect (up to a global phase) on the input qubits, and the quantum states before and after querying the oracle are the same. Since the H-gate is its own inverse, in $|\psi_3$ we reverse $|\psi_1$ to obtain the initial quantum state of $|00...0\rangle$ in the first register.

$$H^{\otimes n}\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix}=\frac{1}{\sqrt{2^n}}\begin{bmatrix}1\\1\\1\\\vdots\\1\end{bmatrix} \quad \xrightarrow{\text{after } U_f} \quad H^{\otimes n}\frac{1}{\sqrt{2^n}}\begin{bmatrix}1\\1\\1\\\vdots\\1\end{bmatrix}=\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix}$$

- Balanced: After creating $|\psi_1$, our input register is an equal superposition of all the states in the computational basis. When the oracle is balanced, phase kickback adds a negative phase to exactly half these states

$$U_f\frac{1}{\sqrt{2^n}}\begin{bmatrix}1\\1\\1\\\vdots\\1\end{bmatrix}=\frac{1}{\sqrt{2^n}}\begin{bmatrix}-1\\1\\-1\\\vdots\\1\end{bmatrix}$$

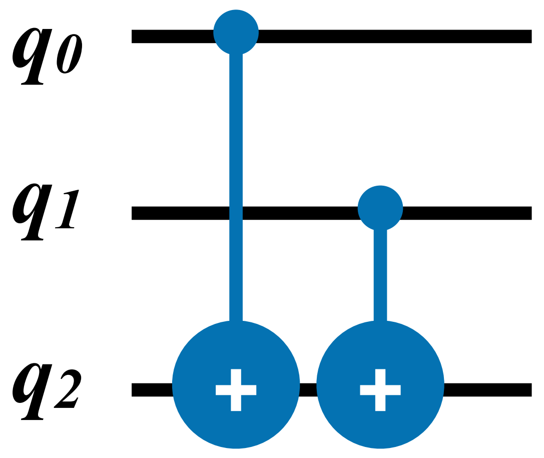
An example of the algorithm

$$f(0,0) = 0$$

$$f(0,1) = 1$$

$$f(1,0) = 1$$

$$f(1,1) = 0$$

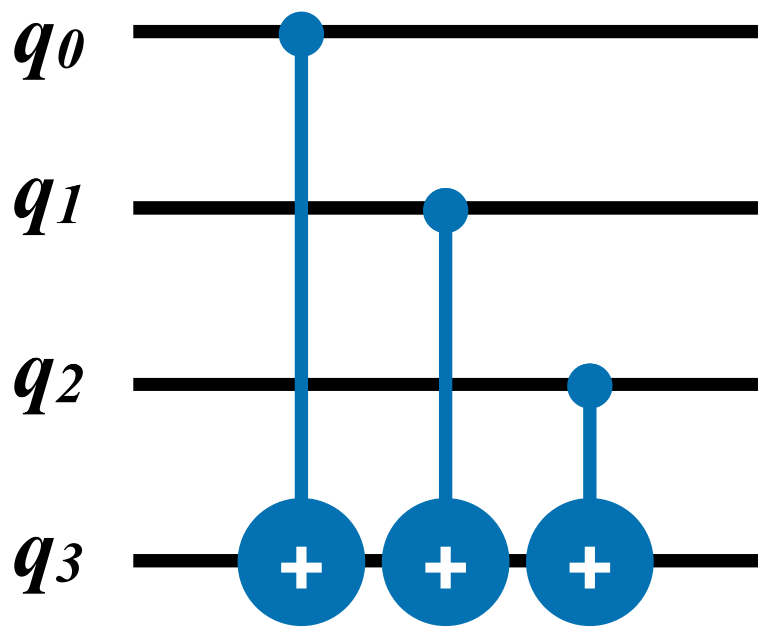


$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Create the oracle

Create the oracle for Deutsch-Jozsa algorithm

- Constant
  - if f(x) = 0, then apply the I gate to the qubit in register 2.
  - if f(x) = 1, then apply the X gate to the qubit in register 2.
- Balanced
  - Many solutions. One is performing a CNOT for each qubit in register 1, with the qubit in register 2 as the target.



Announcement

- Assignments will be reduced from 4 to 2
  - You may consider them as take-home midterm and final
  - Assignment 1 — release on 2/11, due 2/18
  - Assignment 2 — release on 3/4, due 3/11
- No lectures on 2/13, 3/4, and 3/6