Basic Elements of Quantum Computing (2)

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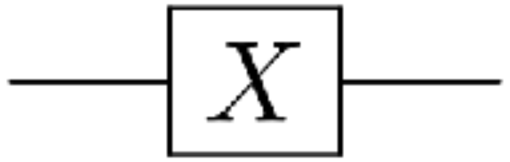
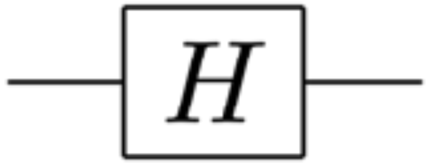
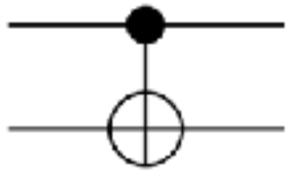
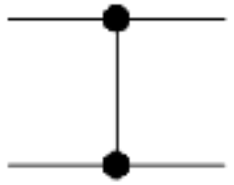
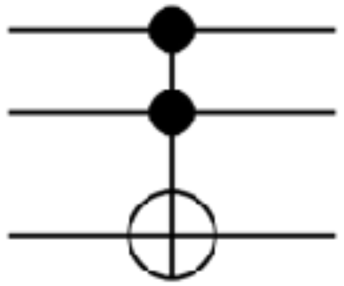
Recap: basis

- Another popular basis
  - $|+\rangle = \sqrt{\frac{1}{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
  - $|-\rangle = \sqrt{\frac{1}{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Recap: important quantum logical gates

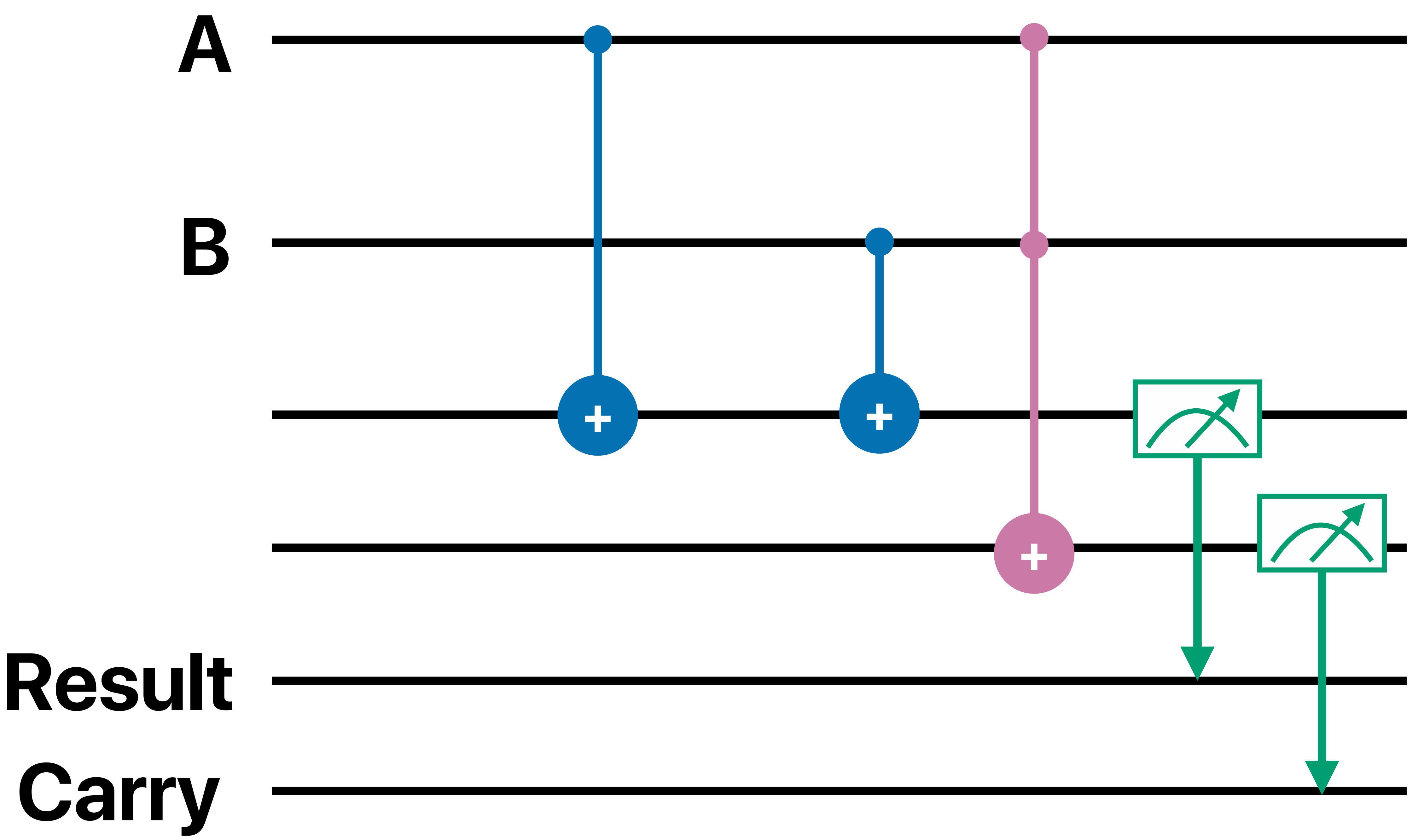
Pauli-X / NOT / Bit-flip	1-bit		$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Pauli Y	1-bit		$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
Pauli Z / Phase flip	1-bit		$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Hadamard	1-bit		$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
Controlled NOT	2-bit		$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
Controlled Z	2-bit		$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$
Toffoli	3-bit		$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

Recap: Multi-qubit system

- If we have an n-qubit system, we can have 2^n potential outcomes —
  $|\psi\rangle = \sum_{i=0}^{2^{n-1}} c_i |x_i\rangle$
  $$\sum_{i=0}^{2^{n-1}} |c_i|^2 = 1$$
- We can also express the states with a vector with 2^n elements
- For a 2-qubit system —
  $|\psi\rangle = c_0|00\rangle + c_1|01\rangle + c_2|10\rangle + c_3|11\rangle$
  $$= c_0 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

computational basis

Recap: 1-bit half adder



Representing separate qubits as a multi-qubit system

Kronecker product to express separate qubits together

$$|a\rangle = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}, \quad |b\rangle = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

$$|ba\rangle = |b\rangle \otimes |a\rangle = \begin{bmatrix} b_0 \times \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \\ \\ b_1 \times \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} b_0 a_0 \\ b_0 a_1 \\ b_1 a_0 \\ b_1 a_1 \end{bmatrix}$$

$$|cba\rangle = \begin{bmatrix} c_0 b_0 a_0 \\ c_0 b_0 a_1 \\ c_0 b_1 a_0 \\ c_0 b_1 a_1 \\ c_1 b_0 a_0 \\ c_1 b_0 a_1 \\ c_1 b_1 a_0 \\ c_1 b_1 a_1 \end{bmatrix}$$

Practice

- $|+\rangle = \sqrt{\frac{1}{2}}\begin{bmatrix}1\\1\end{bmatrix}$
- $|-\rangle = \sqrt{\frac{1}{2}}\begin{bmatrix}1\\-1\end{bmatrix}$

$$|0\rangle|1\rangle$$

$$\begin{bmatrix} 1\times\begin{bmatrix}0\\1\end{bmatrix} \\ 0\times\begin{bmatrix}0\\1\end{bmatrix} \end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

$$|+\rangle|1\rangle$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}}\times\begin{bmatrix}0\\1\end{bmatrix} \\ \frac{1}{\sqrt{2}}\times\begin{bmatrix}0\\1\end{bmatrix} \end{bmatrix} = \begin{bmatrix}0\\\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix}$$

$$|0\rangle|+\rangle$$

$$\begin{bmatrix} 1\times\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \\ 0\times\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\\0\end{bmatrix}$$

$$|-\rangle|+\rangle$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{2}}\times\begin{bmatrix}1\\1\end{bmatrix} \\ \frac{-1}{\sqrt{2}}\times\frac{1}{\sqrt{2}}\times\begin{bmatrix}1\\1\end{bmatrix} \end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\end{bmatrix}$$

What are the possible b and a?

$$|ba\rangle = \begin{bmatrix} b_0a_0 \\ b_0a_1 \\ b_1a_0 \\ b_1a_1 \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$b_0a_0 = \frac{1}{2}$$

$$b_0a_1 = -\frac{1}{2}$$

$$b_1a_0 = -\frac{1}{2}$$

$$b_1a_1 = \frac{1}{2}$$

$$\frac{a_0}{a_1} = -1$$

$$|a_0|^2 + |a_1|^2 = 1$$

$$\frac{b_0}{b_1} = -1$$

$$|b_0|^2 + |b_1|^2 = 1$$

$$a_0 = \frac{1}{\sqrt{2}}$$

$$a_1 = \frac{-1}{\sqrt{2}}$$

$$b_0 = \frac{-1}{\sqrt{2}}$$

$$b_1 = \frac{1}{\sqrt{2}}$$

What are the possible b and a?

$$|ba\rangle = \begin{bmatrix} b_0a_0 \\ b_0a_1 \\ b_1a_0 \\ b_1a_1 \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$b_0a_0 = \frac{1}{\sqrt{2}}$$

$$b_0a_1 = 0$$

$$b_1a_0 = 0$$

$$b_1a_1 = \frac{1}{\sqrt{2}}$$

$$\frac{a_0}{a_1} = \infty$$

$$\frac{b_0}{b_1} = NaN$$

This is not possible to be a product of two states!

Which of the following can be a product of two qubits

- Which of the followings can be a product of two qubit states?

A. $\sqrt{\frac{1}{2}}(|01\rangle + |10\rangle)$

B. $\sqrt{\frac{1}{2}}(|00\rangle - |11\rangle)$

C. $\sqrt{\frac{1}{2}}(|10\rangle - |01\rangle)$

D. $\boxed{\sqrt{\frac{1}{2}}(|00\rangle + |01\rangle)}$

What are the possible b and a?

$$|ba\rangle = \begin{bmatrix} b_0a_0 \\ b_0a_1 \\ b_1a_0 \\ b_1a_1 \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$b_0a_0 = \frac{1}{\sqrt{2}}$$

$$b_0a_1 = \frac{1}{\sqrt{2}}$$

$$b_1a_0 = 0$$

$$b_1a_1 = 0$$

$$\frac{a_0}{a_1} = 1$$

$$|a_0|^2 + |a_1|^2 = 1$$

$$\frac{b_0}{b_1} = \infty$$

$$|b_0|^2 + |b_1|^2 = 1$$

$$a_0 = \frac{1}{\sqrt{2}}$$

$$a_1 = \frac{1}{\sqrt{2}}$$

$$b_0 = 1$$

$$b_1 = 0$$

Entanglement

- The two-qubit state is not a product of any two qubits
- Measuring one will tell us the state of the other and collapse its superposition — because the state cannot stand by itself

Four frequently used entangled states

- $00 \rightarrow |\Phi^+\rangle = \sqrt{\frac{1}{2}}(|00\rangle + |11\rangle)$
- $01 \rightarrow |\Psi^+\rangle = \sqrt{\frac{1}{2}}(|01\rangle + |10\rangle)$
- $10 \rightarrow |\Phi^-\rangle = \sqrt{\frac{1}{2}}(|00\rangle - |11\rangle)$
- $11 \rightarrow |\Psi^-\rangle = \sqrt{\frac{1}{2}}(|10\rangle - |01\rangle)$

Reversibility of gates

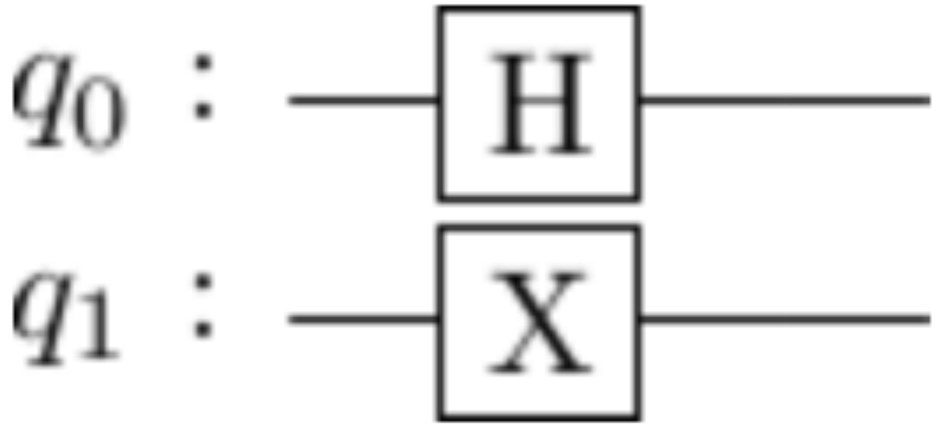
Unitary operations in quantum computing

- All introduced quantum gates are unitary operators
- Their transition matrices are all unitary matrix
- $UU^* = U^*U = I$ — All U has an inverse matrix
- You can always reverse the input from the output if appropriate gates are used

Expressing circuits as state vectors

For the following circuit

```
qc = QuantumCircuit(2)
qc.h(0)
qc.x(1)
qc.draw()
```



$$X|q_1\rangle \otimes H|q_0\rangle = (X \otimes H)|q_1 q_0\rangle$$

$$X \otimes H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \times \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & 1 \times \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ 1 \times \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & 0 \times \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & H \\ H & 0 \end{bmatrix}$$

For the following circuit

```
qc = QuantumCircuit(2)
qc.h(1)
```

$$H|q_1\rangle \otimes I|q_0\rangle = (H \otimes I)|q_1 q_0\rangle$$

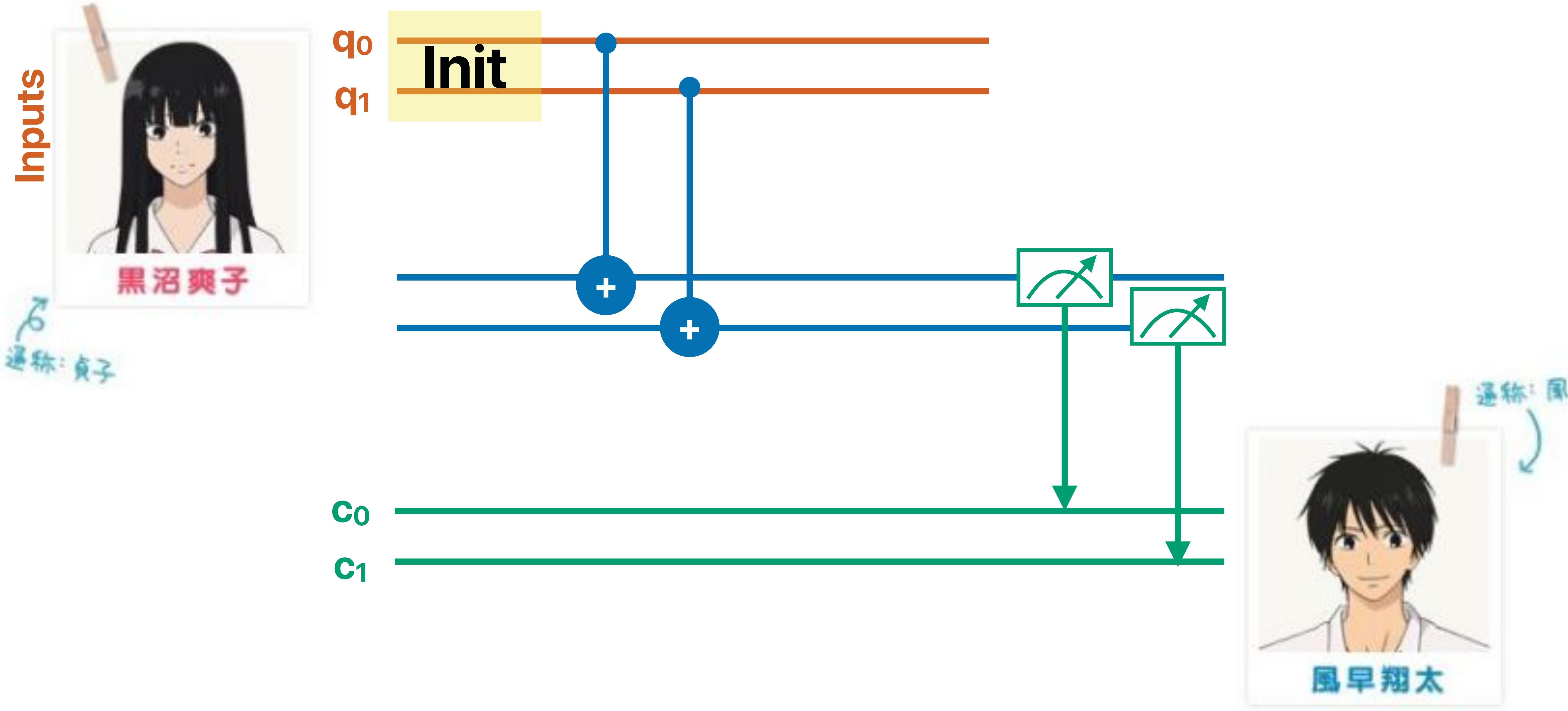
$$H \otimes I = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 1 \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ 1 \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & -1 \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

Quantum teleportation problem

Alice wants to tell Bob ...



Let's try something different

What if we encode data using entangled states?

Binary	Qubits state
00	$\|\Phi^+\rangle = \sqrt{\frac{1}{2}}(\|00\rangle + \|11\rangle)$
01	$\|\Psi^+\rangle = \sqrt{\frac{1}{2}}(\|01\rangle + \|10\rangle)$
10	$\|\Phi^-\rangle = \sqrt{\frac{1}{2}}(\|00\rangle - \|11\rangle)$
11	$\|\Psi^-\rangle = \sqrt{\frac{1}{2}}(\|10\rangle - \|01\rangle)$

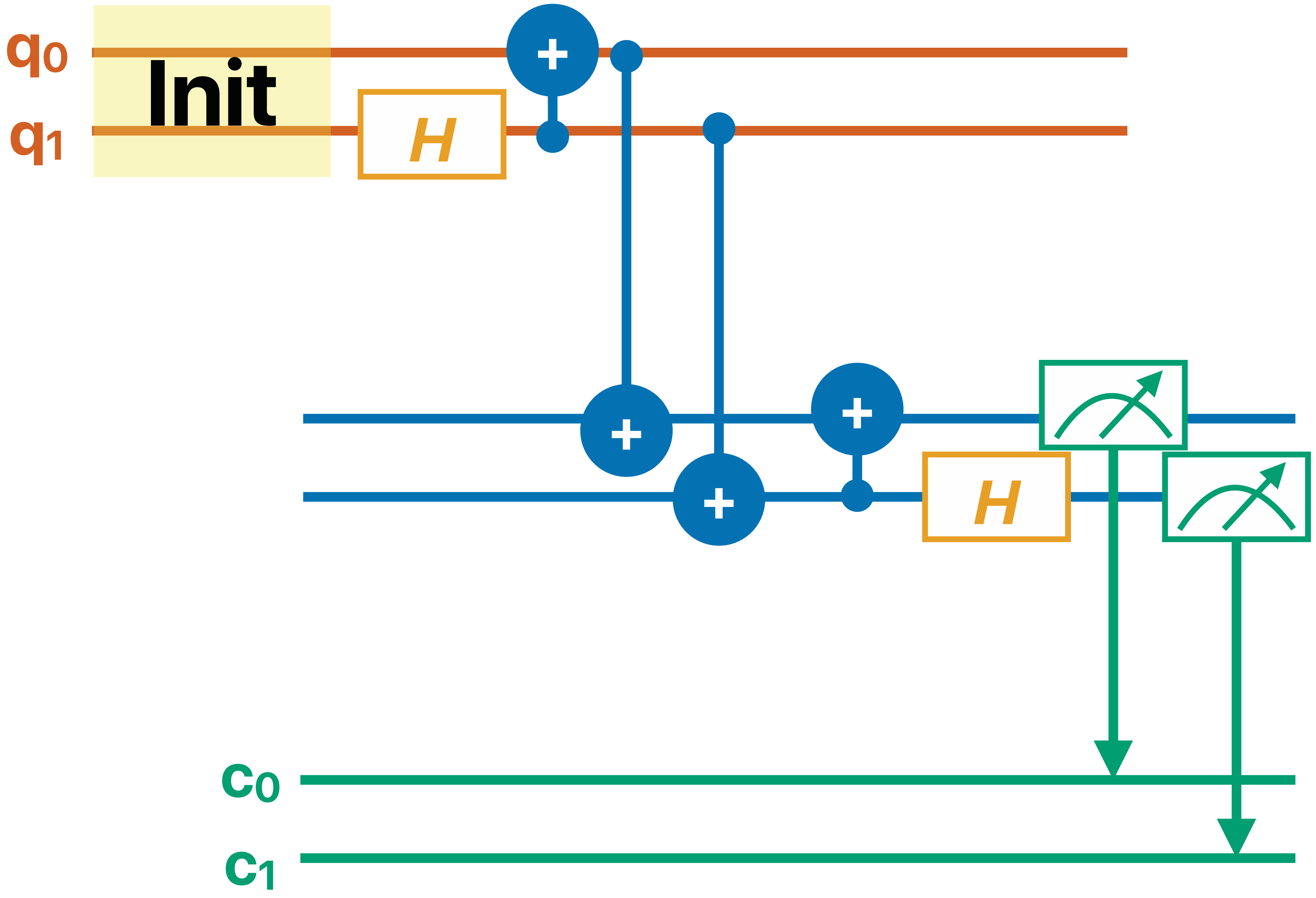
How to encode data in these states?

Binary	Qubits state
00	$\lvert\Phi^+\rangle = \sqrt{\frac{1}{2}}(\lvert 00\rangle + \lvert 11\rangle)$
01	$\lvert\Psi^+\rangle = \sqrt{\frac{1}{2}}(\lvert 01\rangle + \lvert 10\rangle)$
10	$\lvert\Phi^-\rangle = \sqrt{\frac{1}{2}}(\lvert 00\rangle - \lvert 11\rangle)$
11	$\lvert\Psi^-\rangle = \sqrt{\frac{1}{2}}(\lvert 10\rangle - \lvert 01\rangle)$

$$\begin{bmatrix} a_{00} \to b_{00} & a_{01} \to b_{00} & a_{10} \to b_{00} & a_{11} \to b_{00} \\ a_{00} \to b_{01} & a_{01} \to b_{01} & a_{10} \to b_{01} & a_{11} \to b_{01} \\ a_{00} \to b_{10} & a_{01} \to b_{10} & a_{10} \to b_{10} & a_{11} \to b_{10} \\ a_{00} \to b_{11} & a_{01} \to b_{11} & a_{10} \to b_{11} & a_{11} \to b_{11} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix}$$

How to encode data to this new encoding?



$$00 \rightarrow |\Phi^+\rangle = \sqrt{\frac{1}{2}}(|00\rangle + |11\rangle)$$

$$01 \rightarrow |\Psi^+\rangle = \sqrt{\frac{1}{2}}(|01\rangle + |10\rangle)$$

$$10 \rightarrow |\Phi^-\rangle = \sqrt{\frac{1}{2}}(|00\rangle - |11\rangle)$$

$$11 \rightarrow |\Psi^-\rangle = \sqrt{\frac{1}{2}}(|10\rangle - |01\rangle)$$

Four states

- $00 \rightarrow |\Phi^{+}\rangle = \sqrt{\frac{1}{2}}(|00\rangle \boxed{+} |11\rangle)$
- $01 \rightarrow |\Psi^{+}\rangle = \sqrt{\frac{1}{2}}(|01\rangle + |10\rangle)$
- $10 \rightarrow |\Phi^{-}\rangle = \sqrt{\frac{1}{2}}(|00\rangle \boxed{-} |11\rangle)$
- $11 \rightarrow |\Psi^{-}\rangle = \sqrt{\frac{1}{2}}(|10\rangle \boxed{-} |01\rangle)$

Flip one bit

Different phase

Flip one bit

Different phase

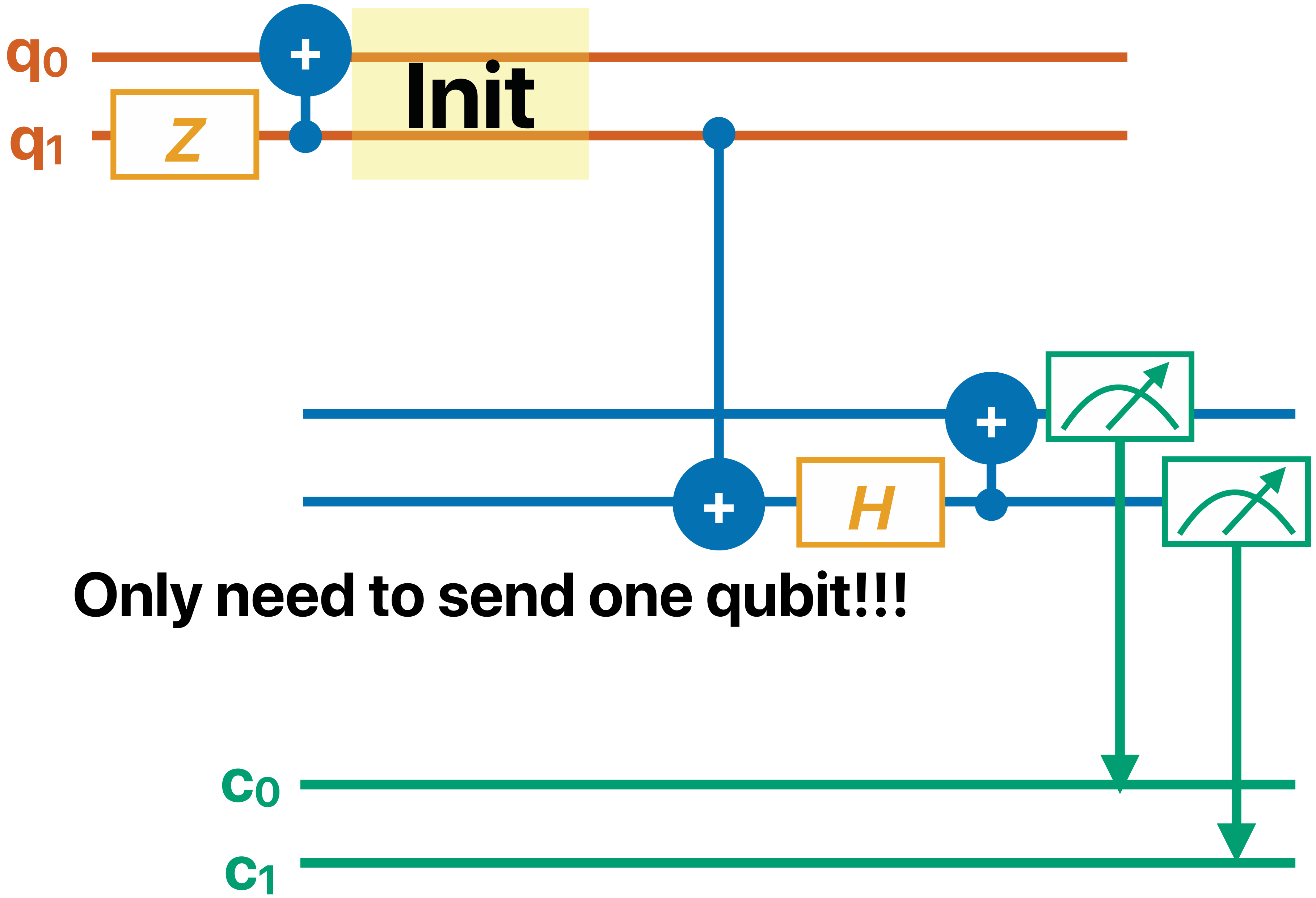
If we initialize the state as $|\Phi^+\rangle$

$$|\Phi^+\rangle = \sqrt{\frac{1}{2}}(|0\underset{X}{\boxed{0}}\rangle + |1\underset{X}{\boxed{1}}\rangle) \qquad 01 \rightarrow |\Psi^+\rangle = \sqrt{\frac{1}{2}}(|01\rangle + |10\rangle)$$

$$|\Phi^+\rangle = \sqrt{\frac{1}{2}}(|0\underset{Z}{\boxed{0}}\rangle + |1\underset{Z}{\boxed{1}}\rangle) \qquad 10 \rightarrow |\Phi^-\rangle = \sqrt{\frac{1}{2}}(|00\rangle - |11\rangle)$$

$$|\Phi^+\rangle = \sqrt{\frac{1}{2}}(|0\underset{ZX}{\boxed{0}}\rangle + |1\underset{ZX}{\boxed{1}}\rangle) \qquad 11 \rightarrow |\Psi^-\rangle = \sqrt{\frac{1}{2}}(|10\rangle - |01\rangle)$$

The new implementation



Only need to send one qubit!!!

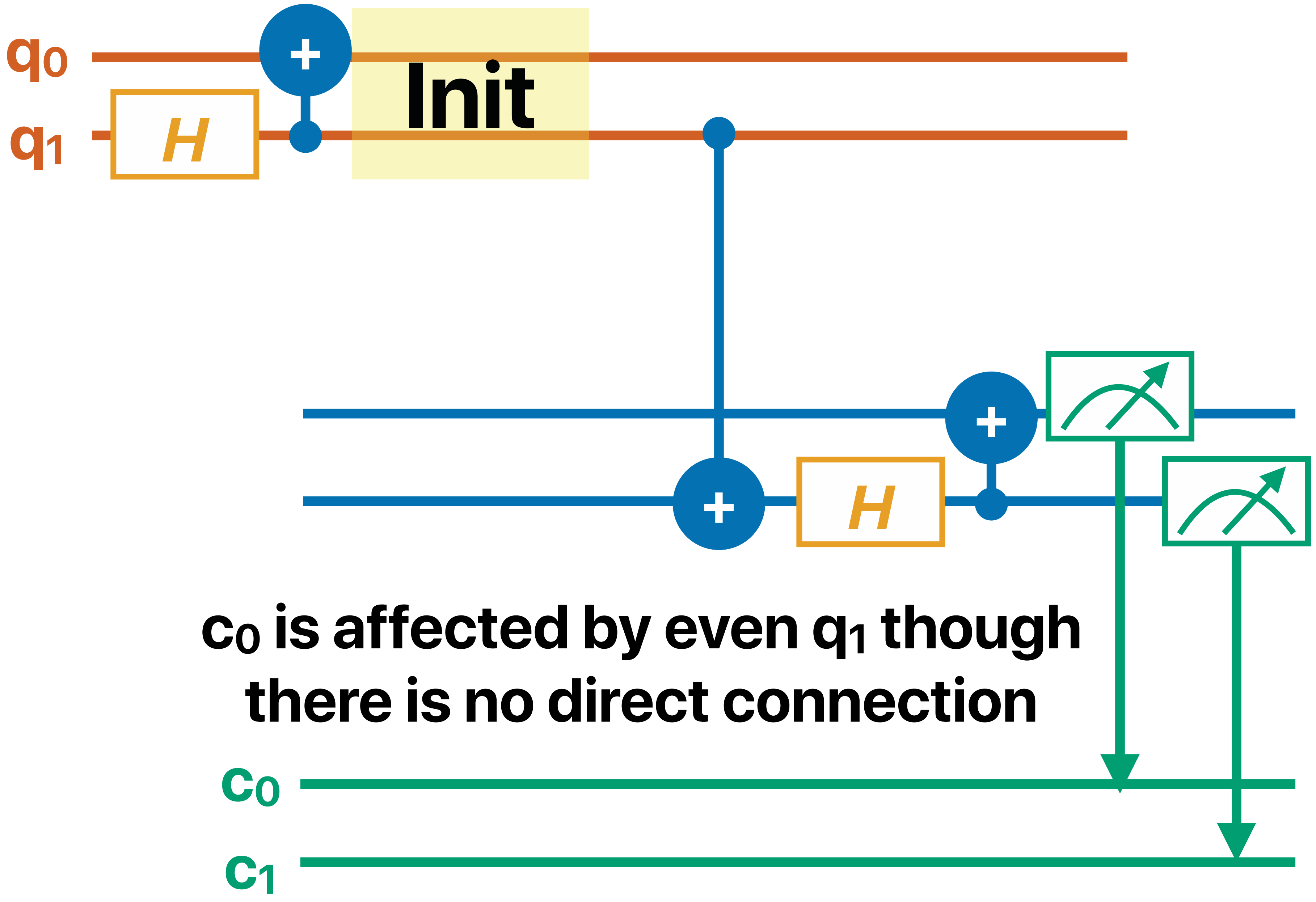
$$00 \rightarrow |\Phi^+\rangle = \sqrt{\frac{1}{2}}(|00\rangle + |11\rangle)$$

$$01 \rightarrow |\Psi^+\rangle = \sqrt{\frac{1}{2}}(|01\rangle + |10\rangle)$$

$$10 \rightarrow |\Phi^-\rangle = \sqrt{\frac{1}{2}}(|00\rangle - |11\rangle)$$

$$11 \rightarrow |\Psi^-\rangle = \sqrt{\frac{1}{2}}(|10\rangle - |01\rangle)$$

The new implementation



$$00 \rightarrow |\Phi^+\rangle = \sqrt{\frac{1}{2}}(|00\rangle + |11\rangle)$$

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$$10 \rightarrow |\Phi^-\rangle = \sqrt{\frac{1}{2}}(|00\rangle - |11\rangle)$$

$$11 \rightarrow |\Psi^-\rangle = \sqrt{\frac{1}{2}}(|10\rangle - |01\rangle)$$

Entanglement

Entanglement

- The two-qubit state is not a product of any two qubits
- Measuring one will tell us the state of the other and collapse its superposition — because the state cannot stand by itself
- Two or more quantum systems (or quantum particles) have a non-classical correlation, or shared quantum state, even if they are separated by a large distance.