# AI1103 - Assignment 4

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Latex codes :

https://github.com/CS20BTECH11004/AI1103/blob
/main/Assignment%204/Assignment%204.tex

QUESTION: CSIR UGC NET EXAM (JUNE 2013),
Q.101

Let $X_1, X_2,...$ be independent random variables each following exponential distribution with mean 1. Then which of the following statements are correct?

1) $P(X_n > logn$ for infinitely many $n \geq 1) = 1$
2) $P(X_n > 2$ for infinitely many $n \geq 1) = 1$
3) $P(X_n > \frac{1}{2}$ for infinitely many $n \geq 1) = 0$
4) $P(X_n > logn, X_{n+1} > log(n + 1)$ for infinitely many $n \geq 1) = 0$

SOLUTION

PDF of $X_i$ is

$$f_{X_i}(x) = \begin{cases} \lambda_i e^{-\lambda_i x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Mean of $X_i$ is expressed as

$$E[X_i] = \int\limits_{-\infty}^{\infty} x f_{X_i}(x) dx$$

$$= \int\limits_{-\infty}^{0} 0 dx + \int\limits_{0}^{\infty} x\lambda_i e^{-\lambda_i x}$$

$$= \frac{1}{\lambda_i} \qquad (0.0.1)$$

From (0.0.1)and $E[X_i] = 1$, we have $\lambda_i = 1 \forall i \geq 1$
Now,for some constant $c \geq 0$

$$\Pr(X_n > c) = \int\limits_{c}^{\infty} f_{X_n}(x) dx$$

$$= \int\limits_{c}^{\infty} e^{-x} dx$$

$$= e^{-c} \qquad (0.0.2)$$

1) **Option 1:** We can say the events $X_n > logn$ are independent $\forall n \geq 1$ as $X_n$ are independent random variable.
From (0.0.2)

$$\sum_{n=1}^{\infty} \Pr(X_n > logn) = \sum_{n=1}^{\infty} e^{-logn}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}$$

$$= \infty \text{ (Cauchy's Criterion)}$$

Now,from second Borel-Cantelli lemma we can say that

$$\Pr(X_n > logn \text{ for infinitely many } n \geq 1)$$
$$= \Pr(\lim_{n\to\infty}\sup X_n > logn)$$
$$= 1$$

∴ Option 1 is correct.

2) **Option 2:** We can say the events $X_n > 2$ are independent $\forall n \geq 1$ as $X_n$ are independent random variable.
From (0.0.2)

$$\sum_{n=1}^{\infty} \Pr(X_n > 2) = \sum_{n=1}^{\infty} e^{-2}$$

$$= \infty$$

Now,from second Borel-Cantelli lemma we can say that

$$\Pr(X_n > 2 \text{ for infinitely many } n \geq 1)$$
$$= \Pr(\lim_{n\to\infty}\sup X_n > 2)$$
$$= 1$$

∴ Option 2 is correct.

3) **Option 3:** We can say the events $X_n > \frac{1}{2}$ are independent $\forall n \geq 1$ as $X_n$ are independent random variable.

From (0.0.2)

$$\sum_{n=1}^{\infty} \Pr\left(X_n > \frac{1}{2}\right) = \sum_{n=1}^{\infty} e^{-\frac{1}{2}}$$
$$= \infty$$

Now,from second Borel-Cantelli lemma we can say that

$$\Pr\left(X_n > \frac{1}{2} \text{ for infinitely many } n \geq 1\right)$$
$$= \Pr\left(\lim_{n \to \infty} \sup X_n > \frac{1}{2}\right)$$
$$= 1$$

∴ Option 3 is incorrect.

4) **Option 4:** We can say the events $X_n > logn$ are independent $\forall n \geq 1$ as $X_n$ are independent random variable.
Let the event $X_n > logn, X_{n+1} > log(n+1)$ be represented by $E_n$'
From (0.0.2)

$$\sum_{n=1}^{\infty} \Pr(E_n)$$
$$= \sum_{n=1}^{\infty} \Pr(X_n > logn) \Pr(X_{n+1} > log(n+1))$$
$$= \sum_{n=1}^{\infty} e^{-logn} e^{-log(n+1)}$$
$$= \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$
$$= \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$
$$= 1 \qquad\qquad (0.0.3)$$

Now,from Borel-Cantelli lemma we can say that

$$\Pr(E_n \text{ for infinitely many } n \geq 1)$$
$$= \Pr(\lim_{n \to \infty} \sup X_n > logn, X_{n+1} > log(n+1))$$
$$= 0$$

∴ Option 4 is correct.

**Solution:** Options 1, 2, 4