CSIR UGC NET EXAM (June 2013), Q.101

Aman Panwar - CS20BTECH11004

June 24, 2021

CSIR UGC NET EXAM (June 2013), Q.101

Question

Let X_1,X_2,... be independent random variables each following exponential distribution with mean 1. Then which of the following statements are correct?

1. $P(X_n > \log n$ for infinitely many $n \geq 1) = 1$
2. $P(X_n > 2$ for infinitely many $n \geq 1) = 1$
3. $P(X_n > \frac{1}{2}$ for infinitely many $n \geq 1) = 0$
4. $P(X_n > \log n, X_{n+1} > \log(n+1)$ for infinitely many $n \geq 1) = 0$

Prerequisite

Borel-Cantelli Lemma

Let E_1,E_2,... be a sequence of events in some probability space. The Borel–Cantelli lemma states that, if the sum of the probabilities of the events E_n is finite

$$\sum_{n=1}^{\infty} \Pr\left(E_n\right) < \infty \tag{0.1}$$

then the probability that infinitely many of them occur is 0

$$\Pr\left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} E_i\right) = 0 \tag{0.2}$$

Prerequisite

Second Borel-Cantelli Lemma

If the events E_n are independent and the sum of the probabilities of the E_n diverges to infinity, then the probability that infinitely many of them occur is 1. If for independent events $E_1, E_2, ...$

$$\sum_{n=1}^{\infty} \Pr(E_n) = \infty \tag{0.3}$$

Then

$$\Pr\left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} E_i\right) = 1 \tag{0.4}$$

Solution

Formulating useful expression

PDF of X_i is

$$f_{X_i}(x) = \begin{cases} \lambda_i e^{-\lambda_i x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Mean of X_i is expressed as

$$\begin{aligned} E[X_i] &= \int_{-\infty}^{\infty} x f_{X_i}(x) dx = \int_{-\infty}^{0} 0 dx + \int_{0}^{\infty} x \lambda_i e^{-\lambda_i x} \\ &= \frac{1}{\lambda_i} \end{aligned} \tag{0.1}$$

From (0.1)and $E[X_i] = 1$, we have $\lambda_i = 1 \forall i \geq 1$

Solution

Formulating useful expression

PDF of X_i is

$$f_{X_i}(x) = \begin{cases} \lambda_i e^{-\lambda_i x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Now, for some constant $c \geq 0$

$$\begin{aligned} \Pr\left(X_n > c\right) &= \int_c^\infty f_{X_n}(x)dx \\ &= \int_c^\infty e^{-x}dx \\ &= e^{-c} \end{aligned} \tag{0.2}$$

Option 1

Option 1: P(X_n > $\log n$ for infinitely many $n \geq 1$) = 1

We can say the events $X_n > \log n$ are independent $\forall n \geq 1$ as X_n are independent random variable.
From (0.2)

$$\sum_{n=1}^{\infty} \Pr\left(X_n > \log n\right) = \sum_{n=1}^{\infty} e^{-\log n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}$$

$$= \infty \text{ (Cauchy's Criterion)}$$

Option 1

Option 1: P($X_n > \log n$ for infinitely many $n \geq 1$) = 1

Now, from second Borel-Cantelli lemma

$$\begin{aligned} &\Pr\left(X_n > \log n \text{ for infinitely many } n \geq 1\right) \\ &= \Pr\left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} (X_n > \log n)\right) \\ &= 1 \end{aligned}$$

$\therefore$ **Option 1 is correct.**

Option 2

Option 2: P($X_n > 2$ for infinitely many $n \geq 1$) = 1

We can say the events $X_n > 2$ are independent $\forall n \geq 1$ as X_n are independent random variable.
From (0.2)

$$\sum_{n=1}^{\infty} \Pr\left(X_n > 2\right) = \sum_{n=1}^{\infty} e^{-2}$$
$$= \infty$$

Option 2

Option 2: P($X_n > 2$ for infinitely many $n \geq 1$) = 1

Now, from second Borel-Cantelli lemma

$$\begin{aligned}
&\Pr\left(X_n > 2 \text{ for infinitely many } n \geq 1\right) \\
&= \Pr\left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} (X_n > 2)\right) \\
&= 1
\end{aligned}$$

$\therefore$ **Option 2 is correct.**

Option 3

Option 3: ($X_n > \frac{1}{2}$ for infinitely many $n \geq 1$) = 0

We can say the events $X_n > \frac{1}{2}$ are independent $\forall n \geq 1$ as X_n are independent random variable.
From (0.2)

$$\sum_{n=1}^{\infty} \Pr\left(X_n > \frac{1}{2}\right) = \sum_{n=1}^{\infty} e^{-\frac{1}{2}}$$
$$= \infty$$

Option 3

Option 3: ($X_n > \frac{1}{2}$ for infinitely many $n \geq 1$) = 0

Now, from second Borel-Cantelli lemma

$$\begin{aligned}
&\Pr\left(X_n > \frac{1}{2} \text{ for infinitely many } n \geq 1\right) \\
&= \Pr\left(\bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty}\left(X_n > \frac{1}{2}\right)\right) \\
&= 1
\end{aligned}$$

$\therefore$ **Option 3 is incorrect.**

Option 4

Opt 4: P($X_n > \log n, X_{n+1} > \log(n+1)$ for infinitely many $n \geq 1$) = 0

We can say the events $X_n > \log n$ are independent $\forall n \geq 1$ as X_n are independent random variable.

Let the event $X_n > \log n, X_{n+1} > \log(n+1)$ be represented by E_n'

$$\begin{aligned}
\sum_{n=1}^{\infty} \Pr\left(E_n\right) &= \sum_{n=1}^{\infty} \Pr\left(X_n > \log n\right) \Pr\left(X_{n+1} > \log(n+1)\right) \\
&= \sum_{n=1}^{\infty} e^{-\log n} e^{-\log(n+1)} \text{(from (0.2))} \\
&= \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} \\
&= 1
\end{aligned} \tag{4.1}$$

Option 4

Opt 4: $P(X_n > \log n, X_{n+1} > \log(n+1)$ for infinitely many $n \geq 1) = 0$

Now, from Borel-Cantelli lemma

$$\begin{aligned}
&\Pr\left(E_n \text{ for infinitely many } n \geq 1\right) \\
&= \Pr\left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} (X_n > \log n, X_{n+1} > \log(n+1))\right) \\
&= 0
\end{aligned}$$

$\therefore$ **Option 4 is correct.**

Answer

Correct Answer

Hence, we conclude that the correct option are

Options 1, 2 and 4