AI1103 - Assignment 5

Aman Panwar - CS20BTECH11004

Latex codes :

https://github.com/CS20BTECH11004/AI1103/blob/main/Assignment%205/Assignment%205.tex

QUESTION: UGC/MATH (mathA_June 2017), Q.118

Suppose the random variable X has the following probability density funtion

$$f(x) = \begin{cases} \alpha\,(x-\mu)^{\alpha-1}\,e^{-(x-\mu)^{\alpha}}; & x > \mu \\ 0 & x \le \mu \end{cases}$$

where $\alpha > 0, -\infty < \mu < \infty$. Which of the following are correct? The hazard function of X is

1) an increasing function for all $\alpha > 0$
2) a decreasing function for all $\alpha > 0$
3) an increasing function for some $\alpha > 0$
4) a decreasing function for some $\alpha > 0$

SOLUTION

For the random variable X, the CDF is

$$F(x) = \int_0^x f(y)dy \tag{0.0.1}$$

$$= \int_0^{\mu} 0dy + \int_{\mu}^{x} \alpha\,(y-\mu)^{\alpha-1}\,e^{-(y-\mu)^{\alpha}} \tag{0.0.2}$$

$$= 0 - e^{-(y-\mu)^{\alpha}}\Big|_{\mu}^{x} \tag{0.0.3}$$

$$= 1 - e^{-(x-\mu)^{\alpha}} \tag{0.0.4}$$

For X, the hazard function $H(y)$ is defined as

$$H(y) = \frac{f(y)}{1-F(y)}$$

$$\implies H(y) = \begin{cases} \frac{\alpha(y-\mu)^{\alpha-1}e^{-(y-\mu)^{\alpha}}}{1-\left(1-e^{-(y-\mu)^{\alpha}}\right)}; & y > \mu \\ 0 & y \le \mu \end{cases}$$

$$= \begin{cases} \alpha\,(y-\mu)^{\alpha-1}; & y > \mu \\ 0 & y \le \mu \end{cases}$$

Differentiating $H(y)$ w.r.t. y

$$H'(y) = \begin{cases} \alpha\,(\alpha-1)\,(y-\mu)^{\alpha-2}; & y > \mu \\ 0 & y \le \mu \end{cases}$$

When $y \le \mu$ then $H'(y)$ is non-negative. When $y > \mu$ then $(y-\mu)^{\alpha-2}$ is positive. This implies that the sign for $F'(y)$ for $y > \mu$ is decided by the sign of $\alpha\,(\alpha-1)$.

$$\alpha\,(1-\alpha) < 0 \implies 0 < \alpha < 1 \tag{0.0.5}$$

So, $H'(y) < 0$ for $y > \mu$ and $\alpha \in (0,1)$ and $H'(y) \ge 0$ for all other cases.
$\therefore$ For $\alpha > 0$, the Hazard function of X can be both increasing and decreasing.

Solution: Options 3, 4

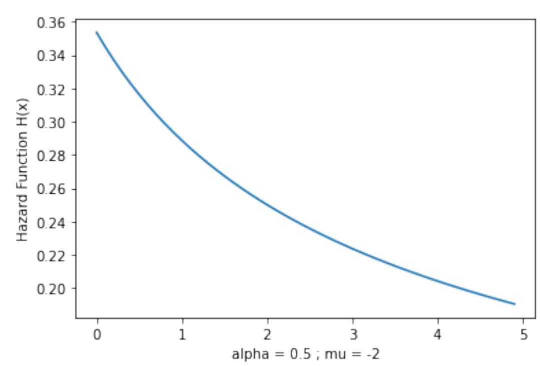

Fig. 4: Decreasing Hazard Function

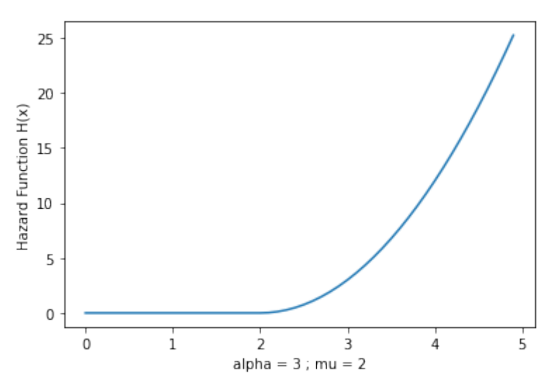

Fig. 4: Increasing Hazard Function