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AI1103 - Assignment 4

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Latex codes:

https://github.com/CS20BTECH11004/AI1103/blob/main/Assignment%204/Assignment%204.tex

QUESTION: CSIR UGC NET EXAM (JUNE 2013), Q.101

Let $X_1, X_2,...$ be independent random variables each following exponential distribution with mean 1. Then which of the following statements are correct?

- 1) $P(X_n > \log n \text{ for infinitely many } n \ge 1) = 1$
- 2) $P(X_n > 2 \text{ for infinitely many } n \ge 1) = 1$
- 3) $P(X_n > \frac{1}{2} \text{ for infinitely many } n \ge 1) = 0$
- 4) $P(X_n > \log n, X_{n+1} > \log(n+1)$ for infinitely many $n \ge 1$ = 0

SOLUTION

PDF of X_i is

$$f_{X_i}(x) = \begin{cases} \lambda_i e^{-\lambda_i x}, & x \ge 0\\ 0, & x < 0 \end{cases}$$

Mean of X_i is expressed as

$$E[X_i] = \int_{-\infty}^{\infty} x f_{X_i}(x) dx$$

$$= \int_{-\infty}^{0} 0 dx + \int_{0}^{\infty} x \lambda_i e^{-\lambda_i x}$$

$$= \frac{1}{\lambda_i}$$
(0.0.1)

From (0.0.1)and $E[X_i] = 1$, we have $\lambda_i = 1 \forall i \ge 1$ Now,for some constant $c \ge 0$

$$\Pr(X_n > c) = \int_{c}^{\infty} f_{X_n}(x) dx$$
$$= \int_{c}^{\infty} e^{-x} dx$$
$$= e^{-c} \qquad (0.0.2)$$

1) **Option 1:** We can say the events $X_n > \log n$ are independent $\forall n \geq 1$ as X_n are independent random variable.

From (0.0.2)

$$\sum_{n=1}^{\infty} \Pr(X_n > \log n) = \sum_{n=1}^{\infty} e^{-\log n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}$$

$$= \infty \text{ (Cauchy's Criterion)}$$

Now,from second Borel-Cantelli lemma we can say that

$$\Pr(X_n > \log n \text{ for infinitely many } n \ge 1)$$

= $\Pr(\lim_{n\to\infty} \sup X_n > \log n)$
= 1

.. Option 1 is correct.

2) **Option 2:** We can say the events $X_n > 2$ are independent $\forall n \geq 1$ as X_n are independent random variable.

From (0.0.2)

$$\sum_{n=1}^{\infty} \Pr(X_n > 2) = \sum_{n=1}^{\infty} e^{-2}$$
$$= \infty$$

Now,from second Borel-Cantelli lemma we can say that

$$Pr(X_n > 2 \text{ for infinitely many } n \ge 1)$$

= $Pr(\lim_{n\to\infty} \sup X_n > 2)$
= 1

.. Option 2 is correct.

3) **Option 3:** We can say the events $X_n > \frac{1}{2}$ are independent $\forall n \geq 1$ as X_n are independent random variable.

From (0.0.2)

$$\sum_{n=1}^{\infty} \Pr\left(X_n > \frac{1}{2}\right) = \sum_{n=1}^{\infty} e^{-\frac{1}{2}}$$

$$= \infty$$

Now,from second Borel-Cantelli lemma we can say that

$$\Pr\left(X_n > \frac{1}{2} \text{ for infinitely many } n \ge 1\right)$$

= $\Pr\left(\lim_{n \to \infty} \sup X_n > \frac{1}{2}\right)$
= 1

- :. Option 3 is incorrect.
- 4) **Option 4:** We can say the events $X_n > \log n$ are independent $\forall n \geq 1$ as X_n are independent random variable.

Let the event $X_n > \log n, X_{n+1} > \log(n+1)$ be represented by E_n '

From (0.0.2)

$$\sum_{n=1}^{\infty} \Pr(E_n)$$

$$= \sum_{n=1}^{\infty} \Pr(X_n > \log n) \Pr(X_{n+1} > \log(n+1))$$

$$= \sum_{n=1}^{\infty} e^{-\log n} e^{-\log(n+1)}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$

$$= 1 \qquad (0.0.3)$$

Now,from Borel-Cantelli lemma we can say that

Pr
$$(E_n \text{ for infinitely many } n \ge 1)$$

= Pr $(\lim_{n\to\infty} \sup X_n > \log n, X_{n+1} > \log(n+1))$
= 0

:. Option 4 is correct.

Solution: Options 1, 2, 4