

AI1103 - Assignment 5

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Latex codes :

<https://github.com/CS20BTECH11004/AI1103/blob/main/Assignment%205/Assignment%205.tex>

QUESTION: UGC/MATH (MATHA_JUNE 2017), Q.118
Suppose the random variable X has the following probability density function

$$f(x) = \begin{cases} \alpha (x - \mu)^{\alpha-1} e^{-(x-\mu)^\alpha}; & x > \mu \\ 0 & x \leq \mu \end{cases}$$

where $\alpha > 0, -\infty < \mu < \infty$. Which of the following are correct? The hazard function of X is

- 1) an increasing function for all $\alpha > 0$
- 2) a decreasing function for all $\alpha > 0$
- 3) an increasing function for some $\alpha > 0$
- 4) a decreasing function for some $\alpha > 0$

SOLUTION

For the random variable X, the CDF is

$$F(x) = \int_0^x f(y) dy \quad (0.0.1)$$

$$= \int_0^\mu 0 dy + \int_\mu^x \alpha (y - \mu)^{\alpha-1} e^{-(y-\mu)^\alpha} \quad (0.0.2)$$

$$= 0 - e^{-(y-\mu)^\alpha} \Big|_\mu^x \quad (0.0.3)$$

$$= 1 - e^{-(x-\mu)^\alpha} \quad (0.0.4)$$

For X, the hazard function $H(y)$ is defined as

$$\begin{aligned} H(y) &= \frac{f(y)}{1 - F(y)} \\ \Rightarrow H(y) &= \begin{cases} \frac{\alpha(y-\mu)^{\alpha-1} e^{-(y-\mu)^\alpha}}{1 - (1 - e^{-(y-\mu)^\alpha})}; & y > \mu \\ 0 & y \leq \mu \end{cases} \\ &= \begin{cases} \alpha (y - \mu)^{\alpha-1}; & y > \mu \\ 0 & y \leq \mu \end{cases} \end{aligned}$$

Differentiating $H(y)$ w.r.t. y

$$H'(y) = \begin{cases} \alpha(\alpha - 1)(y - \mu)^{\alpha-2}; & y > \mu \\ 0 & y \leq \mu \end{cases}$$

When $y \leq \mu$ then $H'(y)$ is non-negative. When $y > \mu$ then $(y - \mu)^{\alpha-2}$ is positive. This implies that the sign for $F'(y)$ for $y > \mu$ is decided by the sign of $\alpha(\alpha - 1)$.

$$\alpha(1 - \alpha) < 0 \implies 0 < \alpha < 1 \quad (0.0.5)$$

So, $H'(y) < 0$ for $y > \mu$ and $\alpha \in (0, 1)$ and $H'(y) \geq 0$ for all other cases.

\therefore For $\alpha > 0$, the Hazard function of X can be both increasing and decreasing.

Solution: Options 3, 4

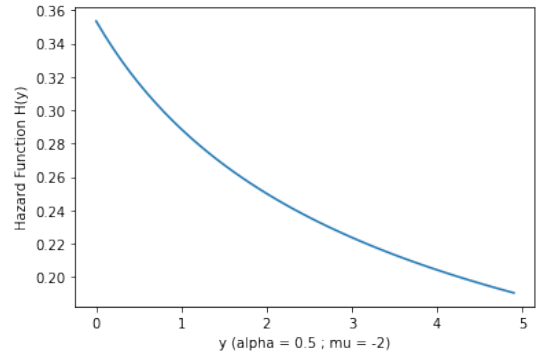


Fig. 4: Decreasing Hazard Function

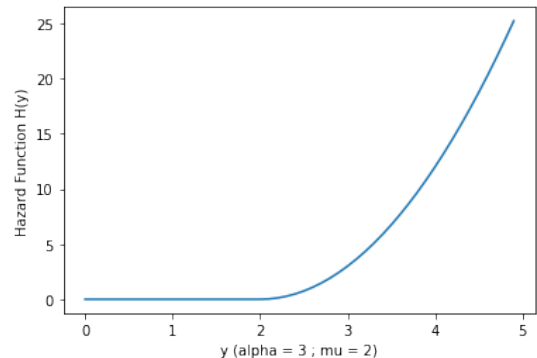


Fig. 4: Increasing Hazard Function