AI1103 - Assignment 4

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Latex codes :

https://github.com/CS20BTECH11004/AI1103/blob/main/Assignment%204/Assignment%204.tex

Question: CSIR UGC NET EXAM (June 2013), Q.101

Let X_1,X_2,... be independent random variables each following exponential distribution with mean 1. Then which of the following statements are correct?

1) P($X_n > logn$ for infinitely many $n \geq 1$) = 1
2) P($X_n > 2$ for infinitely many $n \geq 1$) = 1
3) P($X_n > \frac{1}{2}$ for infinitely many $n \geq 1$) = 0
4) P($X_n > logn, X_{n+1} > log(n+1)$ for infinitely many $n \geq 1$) = 0

Solution

PDF of X_i is

$$f_{X_i}(x) = \begin{cases} \lambda_i e^{-\lambda_i x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Mean of X_i is expressed as

$$\begin{aligned} E[X_i] &= \int_{-\infty}^{\infty} x f_{X_i}(x) dx \\ &= \int_{-\infty}^{0} 0 dx + \int_{0}^{\infty} x \lambda_i e^{-\lambda_i x} \\ &= \frac{1}{\lambda_i} \end{aligned} \tag{0.0.1}$$

From (0.0.1)and $E[X_i] = 1$, we have $\lambda_i = 1 \forall i \geq 1$

Now,for some constant $c \geq 0$

$$\begin{aligned} \Pr(X_n > c) &= \int_{c}^{\infty} f_{X_n}(x) dx \\ &= \int_{c}^{\infty} e^{-x} dx \\ &= e^{-c} \end{aligned} \tag{0.0.2}$$

Option 1: We can say the events $X_n > logn$ are independent $\forall n \geq 1$ as X_n are independent random variable.

From (0.0.2)

$$\begin{aligned} \sum_{n=1}^{\infty} \Pr(X_n > logn) &= \sum_{n=1}^{\infty} e^{-logn} \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \\ &= \infty \text{ (Cauchy's Criterion)} \end{aligned}$$

Now,from second Borel-Cantelli lemma we can say that

$$\begin{aligned} &\Pr(X_n > logn \text{ for infinitely many } n \geq 1) \\ &= \Pr(\lim_{n\to\infty}\sup X_n > logn) \\ &= 1 \end{aligned}$$

$\therefore$ Option 1 is correct.

Option 2: We can say the events $X_n > 2$ are independent $\forall n \geq 1$ as X_n are independent random variable.

From (0.0.2)

$$\begin{aligned} \sum_{n=1}^{\infty} \Pr(X_n > 2) &= \sum_{n=1}^{\infty} e^{-2} \\ &= \infty \end{aligned}$$

Now,from second Borel-Cantelli lemma we can say that

$$\begin{aligned} &\Pr(X_n > 2 \text{ for infinitely many } n \geq 1) \\ &= \Pr(\lim_{n\to\infty}\sup X_n > 2) \\ &= 1 \end{aligned}$$

$\therefore$ Option 2 is correct.

Option 3: We can say the events $X_n > \frac{1}{2}$ are independent $\forall n \geq 1$ as X_n are independent random variable.

From (0.0.2)

$$\sum_{n=1}^{\infty} \Pr\left(X_n > \frac{1}{2}\right) = \sum_{n=1}^{\infty} e^{-\frac{1}{2}}$$
$$= \infty$$

Now,from second Borel-Cantelli lemma we can say that

$$\Pr\left(X_n > \frac{1}{2} \text{ for infinitely many } n \geq 1\right)$$
$$= \Pr\left(\lim_{n\to\infty}\sup X_n > \frac{1}{2}\right)$$
$$= 1$$

$\therefore$ Option 3 is incorrect.

Option 4: We can say the events $X_n > logn$ are independent $\forall n \geq 1$ as X_n are independent random variable.
Let the event $X_n > logn, X_{n+1} > log(n+1)$ be represented by E_n'
From (0.0.2)

$$\sum_{n=1}^{\infty} \Pr(E_n)$$
$$= \sum_{n=1}^{\infty} \Pr(X_n > logn)\Pr(X_{n+1} > log(n+1))$$
$$= \sum_{n=1}^{\infty} e^{-logn} e^{-log(n+1)}$$
$$= \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$
$$= \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$
$$= 1 \tag{0.0.3}$$

Now,from Borel-Cantelli lemma we can say that

$$\Pr(E_n \text{ for infinitely many } n \geq 1)$$
$$= \Pr(\lim_{n\to\infty}\sup X_n > logn, X_{n+1} > log(n+1))$$
$$= 0$$

$\therefore$ Option 4 is correct.

$\boxed{\textbf{Solution: } \text{Options 1, 2, 4}}$