

CSIR UGC NET EXAM (June 2013), Q.101

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Question

Let X_1, X_2, \dots be independent random variables each following exponential distribution with mean 1. Then which of the following statements are correct?

- ① $P(X_n > \log n \text{ for infinitely many } n \geq 1) = 1$
- ② $P(X_n > 2 \text{ for infinitely many } n \geq 1) = 1$
- ③ $P(X_n > \frac{1}{2} \text{ for infinitely many } n \geq 1) = 0$
- ④ $P(X_n > \log n, X_{n+1} > \log(n+1) \text{ for infinitely many } n \geq 1) = 0$

Prerequisite

Borel-Cantelli Lemma

Let E_1, E_2, \dots be a sequence of events in some probability space. The Borel–Cantelli lemma states that, if the sum of the probabilities of the events E_n is finite

$$\sum_{n=1}^{\infty} \Pr(E_n) < \infty \quad (0.1)$$

then the probability that infinitely many of them occur is 0

$$\Pr\left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} E_i\right) = 0 \quad (0.2)$$

Prerequisite

Second Borel-Cantelli Lemma

If the events E_n are independent and the sum of the probabilities of the E_n diverges to infinity, then the probability that infinitely many of them occur is 1. If for independent events E_1, E_2, \dots

$$\sum_{n=1}^{\infty} \Pr(E_n) = \infty \quad (0.3)$$

Then

$$\Pr\left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} E_i\right) = 1 \quad (0.4)$$

Solution

Formulating useful expression

PDF of X_i is

$$f_{X_i}(x) = \begin{cases} \lambda_i e^{-\lambda_i x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Mean of X_i is expressed as

$$\begin{aligned} E[X_i] &= \int_{-\infty}^{\infty} x f_{X_i}(x) dx = \int_{-\infty}^0 0 dx + \int_0^{\infty} x \lambda_i e^{-\lambda_i x} \\ &= \frac{1}{\lambda_i} \end{aligned} \tag{0.1}$$

From (0.1) and $E[X_i] = 1$, we have $\lambda_i = 1 \forall i \geq 1$

Solution

Formulating useful expression

Now, PDF of $X_i \forall i \geq 1$ is

$$f_{X_i}(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Now, for some constant $c \geq 0$

$$\begin{aligned} \Pr(X_n > c) &= \int_c^{\infty} f_{X_n}(x) dx \\ &= \int_c^{\infty} e^{-x} dx \\ &= e^{-c} \end{aligned} \tag{0.2}$$

Option 1

Option 1: $P(X_n > \log n \text{ for infinitely many } n \geq 1) = 1$

We can say the events $X_n > \log n$ are independent $\forall n \geq 1$ as X_n are independent random variable.

From (0.2)

$$\begin{aligned}\sum_{n=1}^{\infty} \Pr(X_n > \log n) &= \sum_{n=1}^{\infty} e^{-\log n} \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \\ &= \infty \text{ (Cauchy's Criterion)}\end{aligned}$$

Option 1

Option 1: $P(X_n > \log n \text{ for infinitely many } n \geq 1) = 1$

Now, from second Borel-Cantelli lemma

$$\begin{aligned} & \Pr(X_n > \log n \text{ for infinitely many } n \geq 1) \\ &= \Pr\left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} (X_n > \log n)\right) \\ &= 1 \end{aligned}$$

\therefore Option 1 is correct.

Option 2

Option 2: $P(X_n > 2 \text{ for infinitely many } n \geq 1) = 1$

We can say the events $X_n > 2$ are independent $\forall n \geq 1$ as X_n are independent random variable.

From (0.2)

$$\begin{aligned} \sum_{n=1}^{\infty} \Pr(X_n > 2) &= \sum_{n=1}^{\infty} e^{-2} \\ &= \infty \end{aligned}$$

Option 2

Option 2: $P(X_n > 2 \text{ for infinitely many } n \geq 1) = 1$

Now, from second Borel-Cantelli lemma

$$\begin{aligned} & \Pr(X_n > 2 \text{ for infinitely many } n \geq 1) \\ &= \Pr\left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} (X_i > 2)\right) \\ &= 1 \end{aligned}$$

\therefore Option 2 is correct.

Option 3

Option 3: $(X_n > \frac{1}{2} \text{ for infinitely many } n \geq 1) = 0$

We can say the events $X_n > \frac{1}{2}$ are independent $\forall n \geq 1$ as X_n are independent random variable.

From (0.2)

$$\sum_{n=1}^{\infty} \Pr\left(X_n > \frac{1}{2}\right) = \sum_{n=1}^{\infty} e^{-\frac{1}{2}} = \infty$$

Option 3

Option 3: $(X_n > \frac{1}{2} \text{ for infinitely many } n \geq 1) = 0$

Now, from second Borel-Cantelli lemma

$$\begin{aligned} & \Pr\left(X_n > \frac{1}{2} \text{ for infinitely many } n \geq 1\right) \\ &= \Pr\left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} \left(X_n > \frac{1}{2}\right)\right) \\ &= 1 \end{aligned}$$

\therefore Option 3 is incorrect.

Option 4

Opt 4: $P(X_n > \log n, X_{n+1} > \log(n+1) \text{ for infinitely many } n \geq 1) = 0$

We can say the events $X_n > \log n$ are independent $\forall n \geq 1$ as X_n are independent random variable.

Let the event $X_n > \log n, X_{n+1} > \log(n+1)$ be represented by E_n ,

$$\begin{aligned}\sum_{n=1}^{\infty} \Pr(E_n) &= \sum_{n=1}^{\infty} \Pr(X_n > \log n) \Pr(X_{n+1} > \log(n+1)) \\ &= \sum_{n=1}^{\infty} e^{-\log n} e^{-\log(n+1)} \text{ (from (0.2))} \\ &= \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} \\ &= 1\end{aligned}\tag{4.1}$$

Option 4

Opt 4: $P(X_n > \log n, X_{n+1} > \log(n+1) \text{ for infinitely many } n \geq 1) = 0$

Now, from Borel-Cantelli lemma

$$\begin{aligned} & \Pr(E_n \text{ for infinitely many } n \geq 1) \\ &= \Pr\left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} (X_n > \log n, X_{n+1} > \log(n+1))\right) \\ &= 0 \end{aligned}$$

\therefore Option 4 is correct.

Answer

Correct Answer

Hence, we conclude that the correct option are

Options 1, 2 and 4