

# AI1103 Assignment-7

SRIVATSAN T - CS20BTECH11062

Download all python codes from

<https://github.com/CS20BTECH11062/AI1103/tree/main/Assignment-7/codes>

and latex-tikz codes from

<https://github.com/CS20BTECH11062/AI1103/tree/main/Assignment-7/Assignment-7.tex>

QUESTION (CSIR UGC NET JUNE 2013 Q.59)

Let  $U_1, U_2, \dots, U_n$  be independent and identically distributed random variables each having a uniform distribution on  $(0,1)$ . Then,

$$\lim_{n \rightarrow +\infty} \Pr \left( U_1 + U_2, \dots, U_n \leq \frac{3}{4}n \right)$$

- 1) does not exist
- 2) exists and equals 0
- 3) exists and equals 1
- 4) exists and equals  $\frac{3}{4}$

SOLUTION

We use Weak law for large numbers to solve this problem. Let the collection of identically distributed random variables  $U_1, U_2, \dots, U_n$  have a finite mean  $\mu$  and finite variance  $\sigma^2$ .

$$\mu = E[U_i] \text{ for } i \in (1, 2, 3, \dots, n) \quad (0.0.1)$$

Since the distribution is uniform on  $(0,1)$ ,  $\mu = 0.5$ . Let  $M_n$  be the sample mean

$$M_n = \frac{U_1 + U_2 + U_3 + \dots + U_n}{n} \quad (0.0.2)$$

Expected value of  $M_n$  (using (0.0.2) and (0.0.1)) is

$$E[M_n] = \frac{E[U_1 + U_2 + U_3 + \dots + U_n]}{E[n]} \quad (0.0.3)$$

$$= \frac{E[U_1] + E[U_2] + \dots + E[U_n]}{n} \quad (0.0.4)$$

$$= \frac{n \times \mu}{n} \quad (0.0.5)$$

$$= \mu \quad (0.0.6)$$

Variance of M

$$Var(M_n) = \frac{Var(U_1 + U_2 + U_3 + \dots + U_n)}{n^2} \quad (0.0.7)$$

$$= \frac{Var(U_1) + Var(U_2) + \dots + Var(U_n)}{n^2} \quad (0.0.8)$$

$$= \frac{n \times \sigma^2}{n^2} \quad (0.0.9)$$

$$= \frac{\sigma^2}{n} \quad (0.0.10)$$

From Chebyshev inequality, for any  $\epsilon > 0$

$$\Pr(|M_n - \mu| \geq \epsilon) \leq \frac{Var(M_n)}{\epsilon^2} \quad (0.0.11)$$

From (0.0.1) and (0.0.10)

$$\Pr\left(\left|\frac{U_1 + U_2 + \dots + U_n}{n} - \mu\right| \geq \epsilon\right) \leq \frac{\sigma^2}{n \times \epsilon^2}$$

$$\lim_{n \rightarrow \infty} \Pr\left(\left|\frac{U_1 + U_2 + \dots + U_n}{n} - \mu\right| \geq \epsilon\right) \leq \lim_{n \rightarrow \infty} \frac{\sigma^2}{n \times \epsilon^2} \leq 0 \text{ for fixed } \epsilon > 0 \quad (0.0.12)$$

But since Probabilities are always non-negative,

$$\lim_{n \rightarrow \infty} \Pr\left(\left|\frac{U_1 + U_2 + \dots + U_n}{n} - \mu\right| \geq \epsilon\right) \rightarrow 0 \quad (0.0.13)$$

This is known as the weak law of large numbers

The inverse of (0.0.13) is also true

$$\lim_{n \rightarrow \infty} \Pr\left(\left|\frac{U_1 + U_2 + \dots + U_n}{n} - \mu\right| \leq \epsilon\right) \rightarrow 1 \quad (0.0.14)$$

$$\left|\frac{U_1 + U_2 + \dots + U_n}{n} - \mu\right| \leq \epsilon \text{ as } n \rightarrow \infty \quad (0.0.15)$$

From  $\epsilon$ ,  $n$  definition of limits, it is clear that

$$\frac{U_1 + U_2 + \dots + U_n}{n} \rightarrow \mu \quad (0.0.16)$$

$$U_1 + U_2 + \dots + U_n \rightarrow n \times \mu \text{ as } n \rightarrow \infty \quad (0.0.17)$$

Since  $\mu = \frac{1}{2}$ ,

$$\lim_{n \rightarrow +\infty} U_1 + U_2 \dots U_n = \frac{1}{2}n < \frac{3}{4}n \quad (0.0.18)$$

So

$$\lim_{n \rightarrow +\infty} \Pr\left(U_1 + U_2 \dots, U_n \leq \frac{3}{4}n\right) = 1 \quad (0.0.19)$$

Correct Option : C