

Characteristic Function

Srivatsan T

IITH

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Question

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X and Y are independent random variables each having the density

$$f(t) = \frac{1}{\pi} \frac{1}{1+t^2} \quad -\infty < t < +\infty$$

Then the density function of $\frac{X+Y}{3}$ for $-\infty < t < +\infty$ is

1 $\frac{6}{\pi} \frac{1}{4+9t^2}$

3 $\frac{3}{\pi} \frac{1}{1+9t^2}$

2 $\frac{6}{\pi} \frac{1}{9+4t^2}$

4 $\frac{3}{\pi} \frac{1}{9+t^2}$

Characteristic Function

- 1 If a random variable admits a probability density function, then the characteristic function is the Fourier transform of the probability density function.
- 2 It provides an alternate way to deal with probabilities of random variables other than PDF and CDF.
- 3 It has particularly simpler results in case of sum of independent random variables.

PDF of $X+Y$

PDF of sum of random variables X and Y given their individual PDFs can be calculated using

- 1 Convolution
- 2 Characteristic Function

PDF of $X+Y$ using characteristic function

Property

Characteristic function of sum of independent random variables is the product of characteristic function of those random variables.

And we obtain the PDF of $X+Y$ by calculating the inverse characteristic function of $X+Y$.

CF of X and Y

Given PDF of X and Y

$$f(t) = \frac{1}{\pi} \frac{1}{1+t^2}, \quad -\infty < t < +\infty \quad (1)$$

We then calculate the fourier transform of $f(t)$ to get the CF of X and Y.

CF of X and Y

$$g(w) = \int_{-\infty}^{\infty} f(t) e^{iwt} dt \quad (2)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{1}{1+t^2} e^{iwt} dt \quad (3)$$

$$= e^{-|w|}, \quad -\infty < w < \infty \quad (4)$$

CF-contd.

Let CF of $Z = X + Y$ be $h(w)$

$h(w)$

$$h(w) = \text{CF of } x \times \text{CF of } Y \quad (5)$$

$$= e^{-|w|} \times e^{-|w|} \quad (6)$$

$$= e^{-2|w|} \quad (7)$$

Inverse CF of Z

Now to find the PDF of Z, we calculate the inverse fourier transform of $h(w)$.

Finding $F_{X+Y}(t)$

$$F_{X+Y}(t) = \int_{-\infty}^{\infty} h(w)e^{-iwt}dw \quad (8)$$

$$= \int_{-\infty}^{\infty} e^{-iwt-2|w|}dw \quad (9)$$

$$= \frac{4}{4+t^2}, -\infty < t < \infty \quad (10)$$

Solution contd.

$$F_{X+Y}(t)$$

$$\text{But } \int_{-\infty}^{\infty} F_{X+Y}(t) dt = \int_{-\infty}^{\infty} \frac{4}{4+t^2} dt = 2\pi \quad (11)$$

So we plug in the normalisation factor $\frac{1}{2\pi}$ and the new F_{X+Y} becomes

$$F_{X+Y}(t) = \frac{2}{\pi} \frac{1}{4+t^2}, \quad -\infty < t < \infty \quad (12)$$

Fun Approach

Recap

$$g(w) = \int_{-\infty}^{\infty} f(t)e^{iwt} dt \quad (13)$$

$$= e^{-|w|}, -\infty < w < \infty \quad (14)$$

$$h(w) = e^{-2|w|}, -\infty < w < \infty \quad (15)$$

Notice that $g(2w) = h(w)$

Fourier transform obeys one to one correspondence

Solution contd.

Replacing w with $2w$ in (13)

$g(2w)$

$$g(2w) = \int_{-\infty}^{\infty} f(t) e^{i(2w)t} dt \quad (16)$$

$$= \int_{-\infty}^{\infty} f\left(\frac{t}{2}\right) e^{i2w \frac{t}{2}} d\left(\frac{t}{2}\right) \quad (17)$$

$$= \int_{-\infty}^{\infty} \frac{1}{2} f\left(\frac{t}{2}\right) e^{iwt} dt \quad (18)$$

Solution contd.

Since $g(2w) = h(w)$

$h(w)$

$$h(w) = \int_{-\infty}^{\infty} \frac{1}{2} f\left(\frac{t}{2}\right) e^{iwt} dt \quad (19)$$

Let $\frac{1}{2} f\left(\frac{t}{2}\right)$ be any function of t whose Characteristic function is $h(w)$.

Solution contd.

Recall from (1) $f(t) = \frac{1}{\pi} \frac{1}{1+t^2}$, $-\infty < t < +\infty$

$$F_Z(t) = \frac{1}{2} f\left(\frac{t}{2}\right) \quad (20)$$

$$= \frac{1}{2} \frac{1}{\pi} \frac{4}{4+t^2} \quad (21)$$

$$= \frac{1}{\pi} \frac{2}{4+t^2}, \quad -\infty < t < +\infty \quad (22)$$

$$\frac{Z}{3} = \frac{X+Y}{3}$$

We know that if a random variable M has a probability density $f_M(x)$, then the probability density of random variable kM is

$$f_{kM}(x) = \frac{1}{|k|} f_M\left(\frac{x}{|k|}\right) \quad (23)$$

Probability density of $\frac{Z}{3}$ given $F_Z(t)$ is

$$F_{\frac{Z}{3}}(t) = 3 \times f_{X+Y}(3t) \quad (24)$$

$$= \frac{6}{\pi} \frac{1}{4 + 9t^2}, \quad -\infty < t < +\infty \quad (25)$$

Thus option 1 is correct

Figures

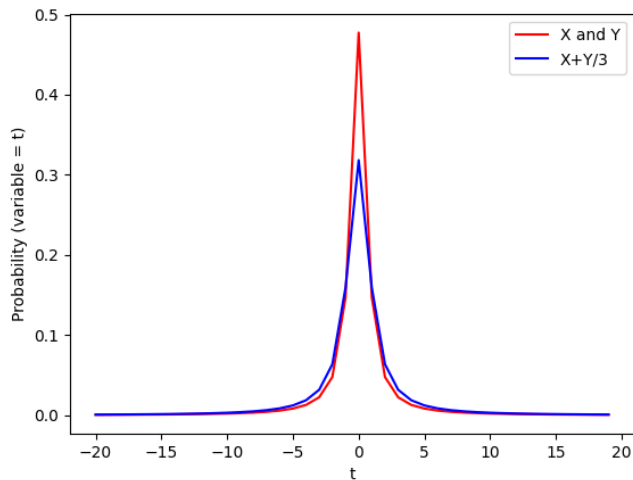


Figure: PDF of X, Y and $\frac{X + Y}{3}$