AI1103 Assignment-8

SRIVATSAN T - CS20BTECH11062

Download all python codes from

```
https://github.com/CS20BTECH11062/AI1103/tree/main/Assignment−8/codes
```

and latex-tikz codes from

```
https://github.com/CS20BTECH11062/AI1103/tree/main/Assignment−8/Assignment−8.tex
```

QUESTION
(UGC MATH (MATHA) JUNE 2017 Q.52)

X and Y are independent random variables each having the density

$$f(t) = \frac{1}{\pi}\frac{1}{1+t^2} - \infty < t < +\infty \tag{0.0.1}$$

Then the density function of $\frac{X+Y}{3}$ for $-\infty <$ t $< +\infty$ is

1) $\frac{6}{\pi}\frac{1}{4+9t^2}$
2) $\frac{6}{\pi}\frac{1}{9+4t^2}$
3) $\frac{3}{\pi}\frac{1}{1+9t^2}$
4) $\frac{3}{\pi}\frac{1}{9+t^2}$

SOLUTION

Let us consider the random variables X and Y. The Characteristic function of the probability density $f(t)$ is

$$g(w) = \int_{-\infty}^{\infty} f(t)e^{iwt}dt \tag{0.0.2}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\pi}\frac{1}{1+t^2}e^{iwt}dt \tag{0.0.3}$$

$$= e^{-|w|} \ , -\infty < w < \infty \tag{0.0.4}$$

The product of the Characteristic function of probability density of X and Y is

$$h(w) = g_1(w) \times g_2(w) = e^{-2|w|} \tag{0.0.5}$$

To get the probability density of X+Y, we find the inverse characteristic function of h(w). But since there is a one to one correspondence between a function and its fourier transform, the inverse characteristic function is

$$F_{X+Y}(t) = \ f(\frac{t}{2}) \tag{0.0.6}$$

$$= \ \frac{1}{\pi}\frac{4}{4+t^2} \ , -\infty < t < \infty \tag{0.0.7}$$

But since F_{X+Y} is a probability distribution function, $\int_{-\infty}^{\infty} F_{X+Y}(t)dt = 1$.

$$\text{But } \int_{-\infty}^{\infty} F_{X+Y}(t)dt = \int_{-\infty}^{\infty} \frac{1}{\pi}\frac{4}{4+t^2}dt = 2 \tag{0.0.8}$$

So we plug in the normalisation factor $\frac{1}{2}$ and the new F_{X+Y} becomes

$$F_{X+Y}(t) = \frac{2}{\pi}\frac{1}{4+t^2} \ , -\infty < t < \infty \tag{0.0.9}$$

We know that if a random variable M has a probability density $f_M(x)$, then the probability density of random variable kM is

$$f_{kM}(x) = \frac{1}{|k|} f_M\left(\frac{x}{|k|}\right) \tag{0.0.10}$$

Probability density of $Z = \frac{X+Y}{3}$ given $F_{X+Y}(t)$ is

$$F_Z(t) = 3 \times f_{X+Y}(3t) \tag{0.0.11}$$

$$= \frac{6}{\pi}\frac{1}{4+9t^2} \tag{0.0.12}$$

Correct Option : 1

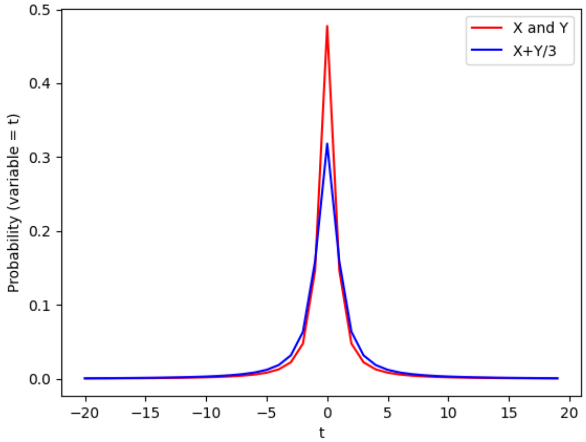

Fig. 4: Graph showing X,Y and $\frac{X+Y}{3}$ probability densities.Area under both the curves = 1