AI1103 Assignment-7

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Download all python codes from

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https://github.com/CS20BTECH11062/AI1103/tree/
    main/Assignment-7/codes
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and latex-tikz codes from

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https://github.com/CS20BTECH11062/AI1103/tree/
    main/Assignment-7/Assignment-7.tex
```

QUESTION (CSIR UGC NET June 2013 Q.59)

Let $U_1, U_2 \dots, U_n$ be independent and identically distributed random variables each having a uniform distribution on (0,1). Then,

$$\lim_{n\to+\infty} \Pr\left(U_1 + U_2 \dots, U_n \leq \frac{3}{4}n\right)$$

1) does not exist
2) exists and equals 0
3) exists and equals 1
4) exists and equals $\frac{3}{4}$

SOLUTION

We use Weak law for large numbers to solve this problem. Let the collection of identically distributed random variables $U_1, U_2 \dots, U_n$ have a finite mean μ and finite variance σ^2.

$$\mu = E(U_i) \quad \text{for i} \in (1,2,3\dots,\text{n}) \tag{0.0.1}$$

Since the distribution is uniform on (0,1), $\mu = 0.5$. Let M_n be the sample mean

$$M_n = \frac{U_1 + U_2 + U_3 \cdots + U_n}{n} \tag{0.0.2}$$

Expected value of M_n (using (0.0.2) and (0.0.1))is

$$E(M_n) = \frac{E(U_1 + U_2 + U_3 + \cdots + U_n)}{E(n)} \tag{0.0.3}$$

$$= \frac{E(U_1) + E(U_2) + \cdots + E(U_n)}{n} \tag{0.0.4}$$

$$= \frac{n \times \mu}{n} \tag{0.0.5}$$

$$= \mu \tag{0.0.6}$$

Variance of M

$$Var(M_n) = \frac{Var(U_1 + U_2 + U_3 \cdots + U_n)}{n^2} \tag{0.0.7}$$

$$= \frac{Var(U_1) + Var(U_2) \cdots + Var(U_n)}{n^2} \tag{0.0.8}$$

$$= \frac{n \times \sigma^2}{n^2} \tag{0.0.9}$$

$$= \frac{\sigma^2}{n} \tag{0.0.10}$$

From Chebyshev inequality, for any $\epsilon > 0$

$$\Pr(|M_n - \mu| \geq \epsilon) \leq \frac{Var(M_n)}{\epsilon^2} \tag{0.0.11}$$

From (0.0.1) and (0.0.10)

$$\Pr\left(\left|\frac{U_1 + U_2 \cdots + U_n}{n} - \mu\right| \geq \epsilon\right) \leq \frac{\sigma^2}{n \times \epsilon^2}$$

$$\begin{aligned} &\lim_{n\to\infty} \Pr\left(\left|\frac{U_1 + U_2 \cdots + U_n}{n} - \mu\right| \geq \epsilon\right) \\ &\leq \lim_{n\to\infty} \frac{\sigma^2}{n \times \epsilon^2} \leq 0 \;\; \text{for fixed } \epsilon > 0 \end{aligned} \tag{0.0.12}$$

But since Probabilities are always non-negative,

$$\lim_{n\to\infty} \Pr\left(\left|\frac{U_1 + U_2 \cdots + U_n}{n} - \mu\right| \geq \epsilon\right) \to 0 \tag{0.0.13}$$

This is known as the weak law of large numbers
The inverse of (0.0.13) is also true

$$\lim_{n\to\infty} \Pr\left(\left|\frac{U_1 + U_2 \cdots + U_n}{n} - \mu\right| \leq \epsilon\right) \to 1 \tag{0.0.14}$$

$$\left|\frac{U_1 + U_2 \cdots + U_n}{n} - \mu\right| \leq \epsilon \;\; \text{as n} \to \infty \tag{0.0.15}$$

From ϵ, n definition of limits, it is clear that

$$\frac{U_1 + U_2 \cdots + U_n}{n} \to \mu \tag{0.0.16}$$

$$U_1 + U_2 \dots U_n \to n \times \mu \;\; \text{as n} \to \infty \tag{0.0.17}$$

Since $\mu = \frac{1}{2}$,

$$\lim_{n\to\infty} U_1 + U_2 \ldots U_n = \frac{1}{2}n < \frac{3}{4}n \qquad (0.0.18)$$

So

$$\lim_{n\to+\infty} \Pr\left(U_1 + U_2 \ldots, U_n \leq \frac{3}{4}n\right) = 1 \qquad (0.0.19)$$

Correct Option : C