3.2 - The Growth of Functions

One important note: The author considers $\log x$ to be $\log_2 x$

Def: We say $f(x)$ is $O(g(x))$ if there exists C, k constants such that $|f(x)| \leq C|g(x)|$ where $x > k$

As x grows without bound, $f(x)$ grows slower than some multiple of $g(x)$

Note that C and k aren't unique

Ex: $f(x) = 7x^3$

$$f(x) = O(x^3)$$

It is known that for $n > m$, as $x \to \infty$, x^n grows faster than x^m

So $f(x) = 3x^2 = O(x^3)$

(but $f(x) = x^3 \neq O(x^2)$)

Ex: Show $f(x) = 5x^3 + 4x^2 + 2x - 3$ is $O(x^3)$

$$\leq 5x^3 + 4x^3 + 2x^3 \quad \text{(for } x \text{ suitably large)}$$
$$= 11x^3$$

So $f(x) \leq 11x^3$, so $f(x) = O(x^3)$ some constant times

(All the components are bounded above by x^3, so the whole thing is bounded above by some constant times x^3)

In general, if $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, then $f(x) = O(x^n)$

Ex: Is $f(x) = x^3$ $O(x^2)$?

If so, then $x^3 \le Cx^2$ for some C and for $x >$ some k)
Then (since $x \ne 0$) $x \le C$, which is not true for all $x > k$, no matter what k is,
So $f(x)$ is not $O(x^2)$

Other known bounds

1) For $f(x) = n!$, $f(x)$ is $O(n^n)$
2) For $f(x) = \log(n!)$, $f(x)$ is $O(n \log n)$
3) For $f(x) = \log n$, $f(x)$ is $O(n)$

Pf: 1) If $f(x) = n!$, $f(x) = 1 \cdot 2 \cdot \dots \cdot n \le n \cdot n \cdot \dots \cdot n = n^n$

So $n! \le n^n$, so $f(x) = O(n^n)$

2) From 1)

$$n! \le n^n$$
$$\log n! \le \log n^n$$
$$\log n! \le n \log n$$

3) First, we need to prove (by induction) that $n<2^n$

Prove $n<2^n$ for n a positive integer

Basis: $n=1$

$1<2$

Assume true for some $k\geq 1$

$2^{k+1}=2^k\cdot 2>k\cdot 2=2k$

Since $k\geq 1$, $k+k\geq k+1$

So $2^{k+1}>k+1$ $\boxtimes$

Now, to prove 3)

$n<2^n$

$\log n<\log 2^n=n\log 2\leq Cn$

Growth of Combinations

Th: If $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$, then

1) $(f_1+f_2)(x)$ is $O(\max(|g_1(x)|,|g_2(x)|))$

2) $(f_1f_2)(x)=O(g_1(x)g_2(x))$

Ex: Show $f(x) = x^2 \log x$ is $O(x^3)$

$$x^2 \leq C_1 x^2$$

$$\log x \leq C_2 x$$

So $f(x) \leq C x^3$

Ex: Find the smallest n so that $f(x) = x^2 + 2x^2 \log x + 5x$ is $O(x^n)$

$$x^2 \leq C_1 x^2 \leq C_1 x^3$$

$$2x^2 \log x \leq C_2 x^3$$

$$5x \leq C_3 x \leq C_3 x^3$$

So $f(x) \leq C x^3$

So $n = 3$

Def: We say $f(x)$ is $\Omega(g(x))$ (big-Omega of $g(x)$) is $|f(x)| \geq C|g(x)|$ for some C and for all $x >$ some k

Def: We say $f(x)$ is $\Theta(g(x))$ (big-Theta) if
$f(x)$ is $O(g(x))$ and $\Omega(g(x))$

$$C_1|g(x)| \le |f(x)| \le C_2|g(x)|$$

In this case, we say $f(x)$ is of <u>order</u> $g(x)$

Note: If $f(x)$ is $\Theta(g(x))$, then $g(x)$ is $\Theta(f(x))$

Ex: Show $f(x) = 3x^2 + 8x\log x$ is $\Theta(x^2)$

$$3x^2 \le 3x^2 + 8x\log x$$

$$3x^2 + 8x\log x \le 3x^2 + 8x \cdot C_1 x = 3x^2 + C_2 x^2 \le Cx^2$$

So $f(x)$ is $\Omega(x^2)$ and $O(x^2)$, so it's $\Theta(x^2)$

Ex: Show $\frac{x^3+2x}{2x+1}$ is $O(x^2)$

$$\frac{x^3+2x}{2x+1} < \frac{x^3+2x}{2x} = \frac{x^3}{2x} + \frac{2x}{2x} = \frac{x^2}{2} + 1$$

$$\le \frac{x^2}{2} + x^2$$

$$= \frac{3}{2}x^2$$

So $\frac{x^3+2x}{2x+1}$ is $O(x^2)$

Ex: Show $(\log x)^2$ is $O(x^2)$

$$(\log x)^2 = (\log x)(\log x) \leq Cx \cdot x = Cx^2$$

Ex: Find big-O for $(n^3 + n^2 \log n)(\log n + 1) + (12 \log n + 19)(n^3 + 2)$

$$f(x) = n^3 \log n + n^2 (\log n)^2 + n^3 + n^2 \log n + 12 n^3 \log n + 19 n^3$$
$$+ 24 \log n + 38$$

$$= 13 n^3 \log n + n^2 (\log n)^2 + 20 n^3 + n^2 \log n + 24 \log n + 38$$
$$\leq 13 n^4 + n^4 + 20 n^3 + n^3 + 24 n + 38$$
$$\leq 13 n^4 + n^4 + 20 n^4 + n^3 + 24 n^4 + 38 n^4$$
$$= O(n^4)$$

Ex: Find the smallest nonnegative integer n so that

$$f(x) = \frac{x^3 + 5 \log x}{x^4 + 1} \text{ is } O(x^n)$$

$$\frac{x^3 + 5 \log x}{x^4 + 1} < \frac{x^3 + 5 \log x}{x^4} = \frac{x^3}{x^4} + \frac{5 \log x}{x^4}$$
$$= \frac{1}{x} + \frac{5 \log x}{x^4}$$
$$< \frac{1}{x} + \frac{5x}{x^4}$$

$$= \frac{1}{x} + \frac{5}{x^3}$$

$$\le C_1 + C_2$$

$$= C$$

So $n = 0$

So $f(x) = O(1)$