CHAPTER 1

Haar's Simple Wavelets

1.0 INTRODUCTION

This chapter explains the nature of the simplest wavelets and an algorithm to compute a fast wavelet transform. Such wavelets have been called "Haar's wavelets" since Haar's publication in 1910 (reference [19] in the bibliography). To analyze and synthesize a signal—which can be any array of data—in terms of simple wavelets, this chapter employs shifts and dilations of mathematical functions, but does *not* involve either calculus or linear algebra.

The first step in applying wavelets to any signal or physical phenomenon consists in representing the signal under consideration by a mathematical function, as in Figure 1.1(a). The usefulness of mathematical functions lies in their efficiency and versatility in representing various types of signals or phenomena. For instance, the horizontal axis in Figure 1.1(a)–(c) may correspond to time ($r = t$), while the vertical axis may correspond to the intensity of a signal ($s = f(r)$), for example, a sound; the values $s = f(r) = f(t)$ measure the sound at each time t at a fixed location. Alternatively, the horizontal axis may correspond to a spatial dimension ($r = x$), and then the values $s = f(r) = f(x)$ measure the intensity of the sound at each location x at a common time. Similarly, the same function f may represent the intensity of light along a cross section of an image.

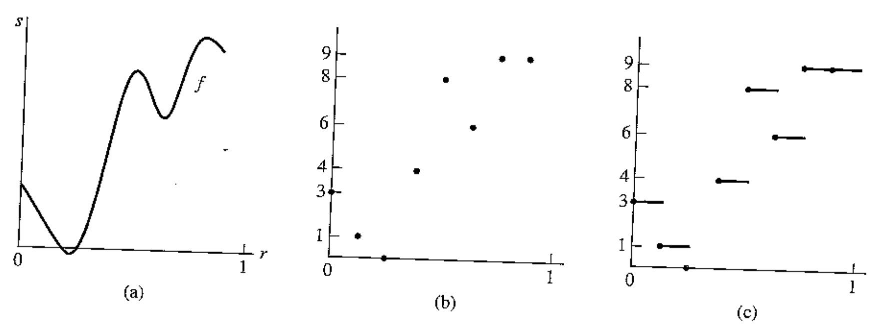


Figure 1.1 (a) Signal. (b) Sample. (c) Approximation.

In any event, because the same type of mathematical function f can represent many types of signals or phenomena, the same type of analysis or synthesis of f, in terms of wavelets or otherwise, will apply to all the signals or phenomena represented by f.

1.1 SIMPLE APPROXIMATION

Because practical measurements of real phenomena require time and resources, they provide not all values but only a finite sequence of values, called a **sample,** of the function representing the phenomenon under consideration, as in Figure 1.1(b). Therefore, the first step in the analysis of a signal with wavelets consists in approximating its function by means of the sample alone. One of the simplest methods of approximation uses a horizontal stair step extended through each sample point, as in Figure 1.1(c). The resulting steps form a new function, denoted here by $\tilde{f}$ and called a **simple function** or **step function,** which approximates the sampled function f. Although approximations more accurate than simple steps exist, they demand more sophisticated mathematics, so this chapter restricts itself to simple steps. A precise notation will prove useful to indicate the location of such steps. (The following notation is consistent with Y. Meyer's books on wavelets [31, p. 94].)

Definition 1.1 For all numbers u and w, the notation $[u, w[$ represents the **interval** of all numbers from u *included* to w *excluded*:

$$[u, w[\; = \{r : u \leq r < w\}. \qquad \square$$

(The symbol $\square$ marks the end of a definition or other formal unit.)

The analysis of the approximating function $\tilde{f}$ in terms of wavelets requires a precise labeling of each step, by means of shifts and dilations of the basic **unit step** function, denoted by $\varphi_{[0,1[}$ and exhibited in Figure 1.2(a). The unit step function $\varphi_{[0,1[}$ has the values (with the symbol := defining the left-hand side in terms of the right-hand side)

$$\varphi_{[0,1[}(r) := \begin{cases} 1 & \text{if } 0 \leq r < 1, \\ 0 & \text{otherwise.} \end{cases}$$

For a step at the same unit height 1 but with a narrower width w, Figure 1.2(b) shows the step function $\varphi_{[0,w[}$, defined by

$$\varphi_{[0,w[}(r) := \begin{cases} 1 & \text{if } 0 \leq r < w, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, for a step at the same unit height 1, but starting at a different location $r = u$ instead of 0, Figure 1.2(c) shows the step function $\varphi_{[u,w[}$, defined by

$$\varphi_{[u,w[}(r) := \begin{cases} 1 & \text{if } u \leq r < w, \\ 0 & \text{otherwise.} \end{cases}$$

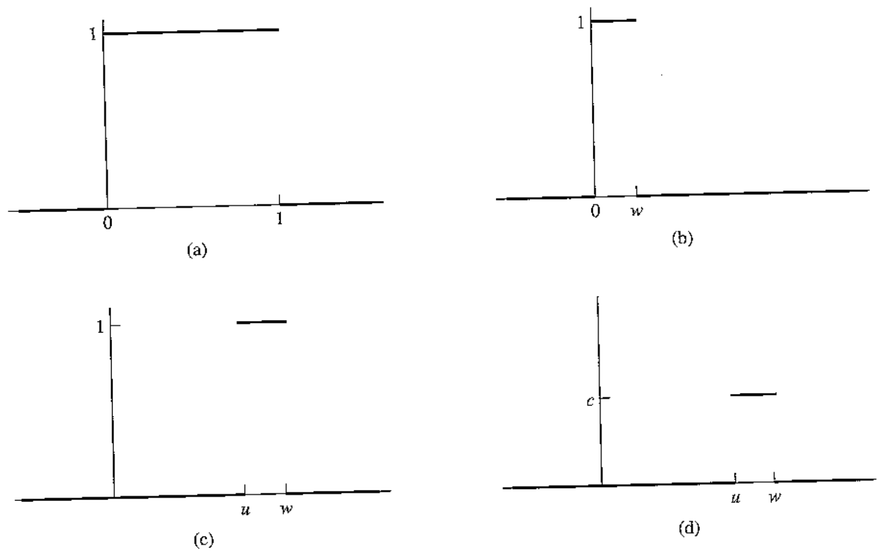


Figure 1.2 (a) $\varphi_{[0,1[}$; (b) $\varphi_{[0,w[}$; (c) $\varphi_{[u,w[}$; (d) $c \cdot \varphi_{[u,w[}$.

Finally, to construct a step function at a different height c, starting at the location u and ending at w, Figure 1.2(d) shows $c \cdot \varphi_{[u,w[}$, a scalar multiple by c of the function $\varphi_{[u,w[}$, so that

$$c \cdot \varphi_{[u,w[}(r) = \begin{cases} c & \text{if } u \leq r < w, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, if a sample point (r_j, s_j) includes a value $s_j = f(r_j)$ at height s_j and at abscissa (time or location) r_j, then that sample point corresponds to the step function

$$s_j \cdot \varphi_{[r_j, r_{j+1}[},$$

which approximates f at height s_j on the interval $[r_j, r_{j+1}[$ from r_j (included) to r_{j+1} (not included). Adding all the step functions corresponding to all the points in the sample yields a formula approximating the simple step function shown in Figure 1.1(c):

$$\begin{aligned} \tilde{f} &= s_0 \cdot \varphi_{[r_0,r_1[} + s_1 \cdot \varphi_{[r_1,r_2[} + \cdots + s_{n-1} \cdot \varphi_{[r_{n-1},r_n[} \\ &= \sum_{j=0}^{n-1} s_j \cdot \varphi_{[r_j,r_{j+1}[}. \end{aligned}$$

(The notation $\sum_{j=0}^{n-1} s_j \cdot \varphi_{[r_j,r_{j+1}[}$ represents the sum of all the terms $s_j \cdot \varphi_{[r_j,r_{j+1}[}$ from $s_0 \cdot \varphi_{[r_0,r_1[}$ through $s_{n-1} \cdot \varphi_{[r_{n-1},r_n[}$.)

To facilitate comparisons between different signals, and to allow for the use of common algorithms, simple wavelets pertain to the interval where $0 \leq r < 1$, so that one unit corresponds to the entire length of the signal. Thus, $r = \frac{1}{2}$ denotes the middle of the signal, and $r = \frac{7}{8}$ denotes the location at the seventh eighth of the signal.

Example 1.2 Table 1.1 lists two sample points, $(r_0, s_0) = (0, 9)$ and $(r_1, s_1) = (\frac{1}{2}, 1)$, from an otherwise unknown function. The sample in Table 1.1 corresponds to the *approximating* simple step function $\tilde{f}$, displayed in Figure 1.3(a) and specified by the formula

Table 1.1

j	0	1
r_j	0	$\frac{1}{2}$
s_j	**9**	**1**

$$\tilde{f} = \sum_{j=0}^{1} s_j \cdot \varphi_{[r_j, r_{j+1}[} = \mathbf{9} \cdot \varphi_{[0,\frac{1}{2}[} + \mathbf{1} \cdot \varphi_{[\frac{1}{2},1[}.$$

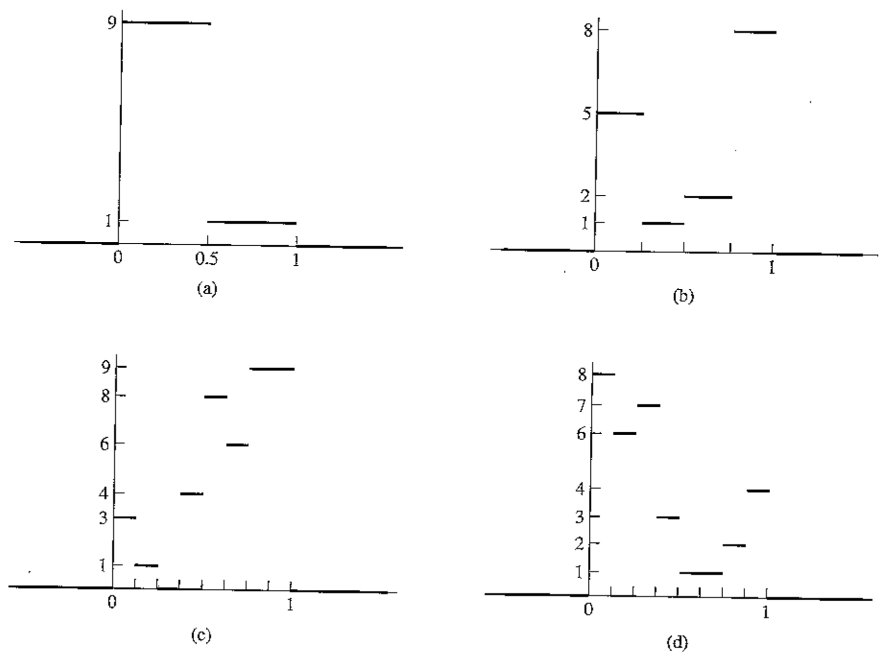


Figure 1.3 Examples of simple step functions.

The first step, $\mathbf{9} \cdot \varphi_{[0,\frac{1}{2}[}$, has height **9** over the interval $[0, \frac{1}{2}[$ starting at 0 (included) and ending at $\frac{1}{2}$ (not included). The second step, $\mathbf{1} \cdot \varphi_{[\frac{1}{2},1[}$, has height **1** over the interval $[\frac{1}{2}, 1[$ starting at $\frac{1}{2}$ (included) and ending at 1 (not included). The notation $[0, \frac{1}{2}[$ and $[\frac{1}{2}, 1[$ shows that the value of $\tilde{f}$ at $\frac{1}{2}$ arises from $\mathbf{1} \cdot \varphi_{[\frac{1}{2},1[}$, which *includes* $\frac{1}{2}$, but not from $\mathbf{9} \cdot \varphi_{[0,\frac{1}{2}[}$, which *excludes* $\frac{1}{2}$. □

Example 1.3 The sample in Table 1.2 corresponds to the approximating simple step function $\tilde{g}$, displayed in Figure 1.3(b) and specified by the formula

$$\mathbf{5}\cdot\varphi_{[0,\frac{1}{4}[}+\mathbf{1}\cdot\varphi_{[\frac{1}{4},\frac{1}{2}[}+\mathbf{2}\cdot\varphi_{[\frac{1}{2},\frac{3}{4}[}+\mathbf{8}\cdot\varphi_{[\frac{3}{4},1[}.$$

Table 1.2

j	0	1	2	3
r_j	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
s_j	**5**	**1**	**2**	**8**

□

Example 1.4 The sample in Table 1.3 corresponds to the approximating simple step function $\tilde{h}$, displayed in Figure 1.3(c) and specified by the formula

Table 1.3

j	0	1	2	3	4	5	6	7
r_j	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$
s_j	**3**	**1**	**0**	**4**	**8**	**6**	**9**	**9**

$$\tilde{h}=\mathbf{3}\cdot\varphi_{[0,\frac{1}{8}[}+\mathbf{1}\cdot\varphi_{[\frac{1}{8},\frac{1}{4}[}+\mathbf{0}\cdot\varphi_{[\frac{1}{4},\frac{3}{8}[}+\mathbf{4}\cdot\varphi_{[\frac{3}{8},\frac{1}{2}[}$$
$$+\mathbf{8}\cdot\varphi_{[\frac{1}{2},\frac{5}{8}[}+\mathbf{6}\cdot\varphi_{[\frac{5}{8},\frac{3}{4}[}+\mathbf{9}\cdot\varphi_{[\frac{3}{4},\frac{7}{8}[}+\mathbf{9}\cdot\varphi_{[\frac{7}{8},1[}.$$

□

Slight variations exist for approximations by step functions. For instance, instead of steps extending on only one side of each sample point, other methods may use steps centered at the sample points, extending equally far on both sides of each sample point.

EXERCISES

Exercise 1.1. Write a formula for the step function $\tilde{p}$ plotted in Figure 1.3(d) and corresponding to the sample in Table 1.4.

Table 1.4

j	0	1	2	3	4	5	6	7
r_j	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$
s_j	**8**	**6**	**7**	**3**	**1**	**1**	**2**	**4**

Exercise 1.2. Plot and write a formula for the step function $\tilde{q}$ corresponding to the sample in Table 1.5.

Table 1.5

j	0	1	2	3	4	5	6	7
r_j	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$
s_j	**3**	**1**	**9**	**7**	**7**	**9**	**5**	**7**

Exercise 1.3. Verify, through algebra, logic, or cases, that for every number r,

$$\varphi_{[0,w[}(r)=\varphi_{[0,1[}(r/w),$$
$$\varphi_{[u,w[}(r)=\varphi_{[0,1[}\left(\frac{r-u}{w-u}\right).$$

Exercise 1.4. Define Haar's "wavelet" function $\psi_{[0,1[}$ by

$$\psi_{[0,1[} := \varphi_{[0,\frac{1}{2}[} - \varphi_{[\frac{1}{2},1[}.$$

Verify, through algebra, logic, or cases, that for every number r,

$$\psi_{[0,1[}(r) = \begin{cases} 1 & \text{if } 0 \leq r < \frac{1}{2}, \\ -1 & \text{if } \frac{1}{2} \leq r < 1, \\ 0 & \text{otherwise.} \end{cases}$$

1.2 APPROXIMATION WITH SIMPLE WAVELETS

1.2.1 The Basic Haar Wavelet Transform

Haar's basic transformation expresses the approximating function $\tilde{f}$ with wavelets by replacing an adjacent pair of steps by one wider step and one wavelet. The wider step measures the average of the initial pair of steps, while the wavelet, formed by two alternating steps, measures the difference of the initial pair of steps.

For instance, the *sum* of two adjacent steps with width 1/2 produces the basic unit step function $\varphi_{[0,1[}$, as in Figure 1.4: Indeed,

$$\varphi_{[0,1[} = \varphi_{[0,\frac{1}{2}[} + \varphi_{[\frac{1}{2},1[}.$$

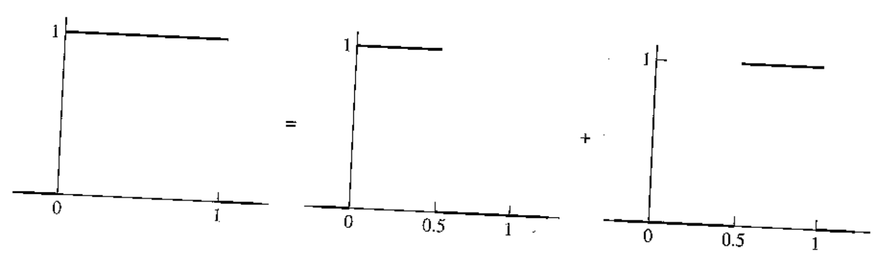


Figure 1.4 $\varphi_{[0,1[} = \varphi_{[0,\frac{1}{2}[} + \varphi_{[\frac{1}{2},1[}$.

Similarly, the *difference* of two such narrower steps gives the corresponding **basic wavelet,** denoted by $\psi_{[0,1[}$ and defined by

$$\psi_{[0,1[} = \varphi_{[0,\frac{1}{2}[} - \varphi_{[\frac{1}{2},1[}.$$

The wavelet $\psi_{[0,1[}$ so defined is a simple step function, with a first step at height 1 followed by a second step at height -1. Thus, from its first step to its second step, the values of the wavelet $\psi_{[0,1[}$ undergo a jump of size -2, as in Figure 1.5.

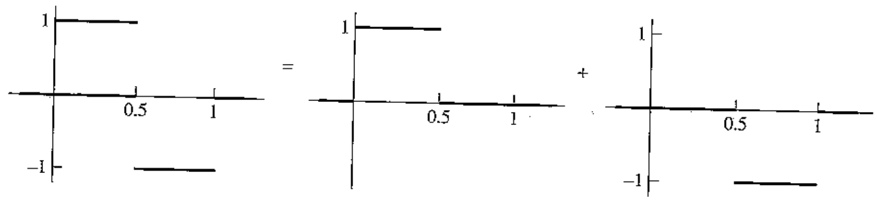


Figure 1.5 $\psi_{[0,1[} = \varphi_{[0,\frac{1}{2}[} - \varphi_{[\frac{1}{2},1[}$.

Adding and subtracting the two equations just obtained,

$$\begin{cases} \varphi_{[0,1[} = \varphi_{[0,\frac{1}{2}[} + \varphi_{[\frac{1}{2},1[}, \\ \psi_{[0,1[} = \varphi_{[0,\frac{1}{2}[} - \varphi_{[\frac{1}{2},1[}, \end{cases}$$

produces the inverse relation, which expresses the narrower steps $\varphi_{[0,\frac{1}{2}[}$ and $\varphi_{[\frac{1}{2},1[}$ in terms of the basic unit step $\varphi_{[0,1[}$ and wavelet $\psi_{[0,1[}$, as shown in Figure 1.6:

$$\begin{cases} \frac{1}{2}\left(\varphi_{[0,1[} + \psi_{[0,1[}\right) = \varphi_{[0,\frac{1}{2}[}, \\ \frac{1}{2}\left(\varphi_{[0,1[} - \psi_{[0,1[}\right) = \varphi_{[\frac{1}{2},1[}. \end{cases}$$

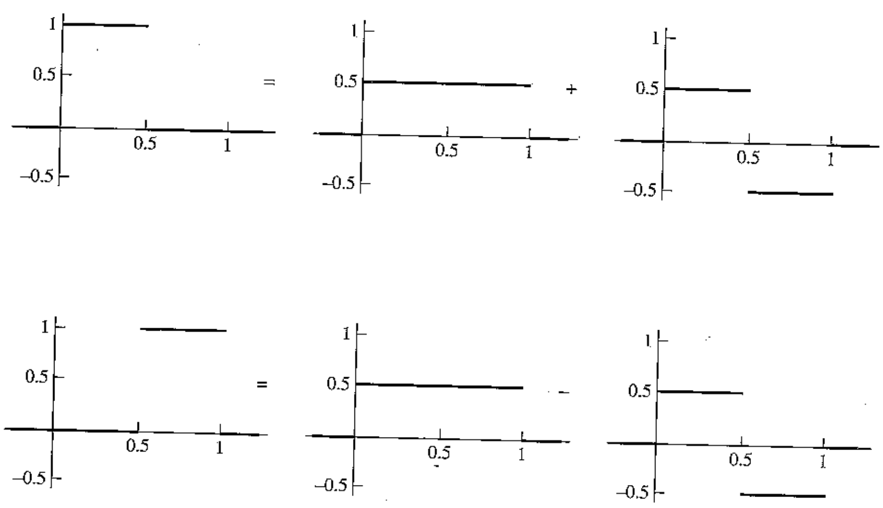


Figure 1.6 *Top:* $\varphi_{[0,\frac{1}{2}[} = \frac{1}{2}\left(\varphi_{[0,1[} + \psi_{[0,1[}\right)$. *Bottom:* $\varphi_{[\frac{1}{2},1[} = \frac{1}{2}\left(\varphi_{[0,1[} - \psi_{[0,1[}\right)$.

For two adjacent steps at heights s_0 and s_1, the equations just derived yield the following representation with one wider step and one wavelet:

$$\begin{aligned}\tilde{f} &:= s_0 \cdot \varphi_{[0,\frac{1}{2}[} + s_1 \cdot \varphi_{[\frac{1}{2},1[}\\ &= s_0 \cdot \frac{1}{2}\left(\varphi_{[0,1[} + \psi_{[0,1[}\right) + s_1 \cdot \frac{1}{2}\left(\varphi_{[0,1[} - \psi_{[0,1[}\right)\\ &= \frac{s_0+s_1}{2} \cdot \varphi_{[0,1[} + \frac{s_0-s_1}{2} \cdot \psi_{[0,1[}.\end{aligned}$$

1.2.2 Significance of the Basic Haar Wavelet Transform

Two sample values s_0 and s_1 measure the value (amplitude, height) of the function $\tilde{f}$ at r_0 and at r_1. In contrast, the results from the basic transform have the following significance.

- The number $(s_0 + s_1)/2$ measures the *average* of the function $\tilde{f}$.
- The number $(s_0 - s_1)/2$ measures the *change* in the function $\tilde{f}$.

The basic transform preserves all the information in the sample, since, while the transform describes the sample differently from the sample values, it also reproduces the sample exactly:

$$s_0 \cdot \varphi_{[0,\frac{1}{2}[} + s_1 \cdot \varphi_{[\frac{1}{2},1[} = \tilde{f} = \frac{s_0+s_1}{2} \cdot \varphi_{[0,1[} + \frac{s_0-s_1}{2} \cdot \psi_{[0,1[}.$$

Example 1.5 Table 1.6 lists two sample points, $(r_0, s_0) = (0, 9)$ and $(r_1, s_1) = (\frac{1}{2}, 1)$, from Example 1.3. For two such adjacent steps at heights $s_0 = 9$ and $s_1 = 1$, as displayed in Figure 1.7,

Table 1.6

j	0	1
r_j	0	$\frac{1}{2}$
s_j	**9**	**1**

$$\mathbf{9}\varphi_{[0,\frac{1}{2}[} + \mathbf{1}\varphi_{[\frac{1}{2},1[} = \frac{9+1}{2}\varphi_{[0,1[} + \frac{9-1}{2}\psi_{[0,1[} = \mathbf{5}\varphi_{[0,1[} + \mathbf{4}\psi_{[0,1[}.$$

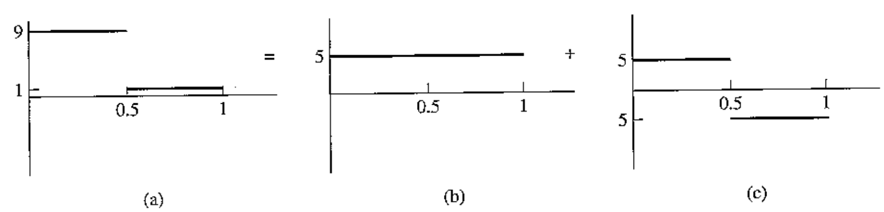


Figure 1.7 Example of a basic wavelet transform.

- The term $\mathbf{5}\varphi_{[0,1[}$ means that that the whole sample has an average value (average height) equal to **5**.
- The term $\mathbf{4}\psi_{[0,1[}$ means that from its first value to its second value, the sample changes as do **4** basic wavelets: It undergoes a jump of size $4 \cdot (-2) = -8$, effectively from 9 to 1. □

1.2.3 Shifts and Dilations of the Basic Haar Transform

To apply the basic transform starting at a different location u instead of 0, and over an interval extending to w instead of 1, define the shifted and dilated wavelet $\psi_{[u,w[}$ by the midpoint $v := (u+w)/2$:

$$\psi_{[u,w[}(r) := \begin{cases} 1 & \text{if } u \leq r < v, \\ -1 & \text{if } v \leq r < w. \end{cases}$$

Again, the sum and the difference of two narrower steps give a wider step and a wavelet:

$$\begin{cases} \varphi_{[u,w[} = \varphi_{[u,v[} + \varphi_{[v,w[}, \\ \psi_{[u,w[} = \varphi_{[u,v[} - \varphi_{[v,w[}. \end{cases}$$

Also, adding and subtracting the two equations just obtained yields the inverse relation, expressing the two narrower steps in terms of the wider step and the wavelet:

$$\begin{cases} \frac{1}{2}\left(\varphi_{[u,w[} + \psi_{[u,w[}\right) = \varphi_{[u,v[}, \\ \frac{1}{2}\left(\varphi_{[u,w[} - \psi_{[u,w[}\right) = \varphi_{[v,w[}. \end{cases}$$

The shifted and dilated basic transform just described applies to all the consecutive pairs of values, separated here by semicolons for convenience, in a sample with $2n$ values:

$$s_0, s_1; s_2, s_3; \ldots; s_{2k}, s_{2k+1}; \ldots; s_{2(n-1)}, s_{2n-1}.$$

Example 1.6 Table 1.7 lists four sample points corresponding to the approximating step function from Example 1.3,

$$\tilde{f} = \mathbf{5} \cdot \varphi_{[0,\frac{1}{4}[} + \mathbf{1} \cdot \varphi_{[\frac{1}{4},\frac{1}{2}[} + \mathbf{2} \cdot \varphi_{[\frac{1}{2},\frac{3}{4}[} + \mathbf{8} \cdot \varphi_{[\frac{3}{4},1[}.$$

Table 1.7

j	0	1	2	3
r_j	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
s_j	**5**	**1**	**2**	**8**

The basic transform applied to the first pair of steps gives

$$\mathbf{5} \cdot \varphi_{[0,\frac{1}{4}[} + \mathbf{1} \cdot \varphi_{[\frac{1}{4},\frac{1}{2}[} = \frac{5+1}{2}\varphi_{[0,\frac{1}{2}[} + \frac{5-1}{2}\psi_{[0,\frac{1}{2}[}.$$

Similarly, after a shift by two sample points to the right, the basic transform applied to the second pair gives

$$\mathbf{2} \cdot \varphi_{[\frac{1}{2},\frac{3}{4}[} + \mathbf{8} \cdot \varphi_{[\frac{3}{4},1[} = \frac{2+8}{2}\varphi_{[\frac{1}{2},1[} + \frac{2-8}{2}\psi_{[\frac{1}{2},1[}.$$

Thus,

$$\begin{aligned}\tilde{f} &= \mathbf{5}\cdot\varphi_{[0,\frac{1}{4}[} + \mathbf{1}\cdot\varphi_{[\frac{1}{4},\frac{1}{2}[} + \mathbf{2}\cdot\varphi_{[\frac{1}{2},\frac{3}{4}[} + \mathbf{8}\cdot\varphi_{[\frac{3}{4},1[}\\ &= \mathbf{3}\varphi_{[0,\frac{1}{2}[} + 2\psi_{[0,\frac{1}{2}[} + \mathbf{5}\varphi_{[\frac{1}{2},1[} + (-3)\psi_{[\frac{1}{2},1[}.\end{aligned}$$

The coefficients **3**, **5**, 2, and −3, have the following significance:

- $\mathbf{3}\,\varphi_{[0,\frac{1}{2}[}$ indicates that $\tilde{f}$ has an average value 3 over the first half of the interval, from 0 to $\frac{1}{2}$.
- $\mathbf{5}\,\varphi_{[\frac{1}{2},1[}$ indicates that $\tilde{f}$ has an average value 5 over the second half of the interval, from $\frac{1}{2}$ to 1.
- $2\psi_{[0,\frac{1}{2}[}$ indicates that $\tilde{f}$ undergoes a jump of size 2 times that of $\psi_{[0,\frac{1}{2}[}$, which jumps down from 1 to −1, for a total of $2\cdot(-2) = -4$ over the first half of the interval, indeed from 5 to 1.
- $(-3)\psi_{[\frac{1}{2},1[}$ indicates that $\tilde{f}$ undergoes a jump of size −3 times that of $\psi_{[\frac{1}{2},1[}$, which jumps down from 1 to −1, for a total of $(-3)\cdot(-2) = 6$ over the second half of the interval, from 2 to 8. □

Example 1.7 Table 1.8 reproduces the eight sample points of the function h from Example 1.4.

Table 1.8

j	0	1	2	3	4	5	6	7
r_j	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$
s_j	**3**	**1**	**0**	**4**	**8**	**6**	**9**	**9**

Applied to consecutive pairs of sample values $(s_0, s_1), (s_2, s_3), \dots, (s_{2k}, s_{2k+1}), \dots, (s_6, s_7)$, the basic simple-wavelet transform gives

$$\begin{aligned}\tilde{h} = {}& \frac{3+1}{2}\varphi_{[0,\frac{1}{4}[} + \frac{3-1}{2}\psi_{[0,\frac{1}{4}[} + \frac{0+4}{2}\varphi_{[\frac{1}{4},\frac{1}{2}[} + \frac{0-4}{2}\psi_{[\frac{1}{4},\frac{1}{2}[}\\ &+ \frac{8+6}{2}\varphi_{[\frac{1}{2},\frac{3}{4}[} + \frac{8-6}{2}\psi_{[\frac{1}{2},\frac{3}{4}[} + \frac{9+9}{2}\varphi_{[\frac{3}{4},1[} + \frac{9-9}{2}\psi_{[\frac{3}{4},1[}.\end{aligned}$$ □

Remark 1.8 Uppercase letters beginning words in technical phrases will indicate a specific technical meaning. For example, the phrase "Haar Wavelet Transform" designates the specific transform with the specific wavelets described in this chapter. □

Remark 1.9 Because the Haar Wavelet Transform does not use the abscissa—the first coordinate r_j of the data point (r_j, s_j)—data for the Haar Wavelet Transform can list the ordinates (values) s_j without the abscissae. For example, the data $(s_0, s_1) = (\mathbf{2}, \mathbf{8})$ can replace the entire Table 1.9. □

Table 1.9

j	0	1
r_j	0	$\frac{1}{2}$
s_j	**2**	**8**

EXERCISES

Exercise 1.5. Calculate the Haar Wavelet Transform for the data $(s_0, s_1) = (\mathbf{2}, \mathbf{8})$.

Exercise 1.6. Calculate the Haar Wavelet Transform for the data $(s_0, s_1) = (\mathbf{7}, \mathbf{3})$.

Exercise 1.7. Calculate the basic Haar Wavelet Transform for the first pair and for the last pair in the data $(s_0, s_1, s_2, s_3) = (\mathbf{2}, \mathbf{4}, \mathbf{8}, \mathbf{6})$.

Exercise 1.8. Calculate the basic Haar Wavelet Transform for the first pair and for the last pair in the data $(s_0, s_1, s_2, s_3) = (\mathbf{5}, \mathbf{7}, \mathbf{3}, \mathbf{1})$.

Exercise 1.9. Calculate the basic Haar Wavelet Transform for each pair (s_{2k}, s_{2k+1}) in the array $\vec{\mathbf{s}} = (8, 6, 7, 3, 1, 1, 2, 4)$.

Exercise 1.10. Calculate the basic Haar Wavelet Transform for each pair (s_{2k}, s_{2k+1}) in the array $\vec{\mathbf{s}} = (3, 1, 9, 7, 7, 9, 5, 7)$.

Exercise 1.11. For each array with four entries $\vec{\mathbf{s}} = (s_0, s_1, s_2, s_3)$, consider the averages of the first pair and last pair of entries,

$$(s_0 + s_1)/2,$$
$$(s_2 + s_3)/2.$$

Verify algebraically that the average of both averages produces the average of all the entries in $\vec{\mathbf{s}}$:

$$\frac{[(s_0 + s_1)/2] + [(s_2 + s_3)/2]}{2} = \frac{s_0 + s_1 + s_2 + s_3}{4}.$$

Exercise 1.12. For arrays with eight entries $\vec{\mathbf{s}} = (s_0, s_1, \ldots, s_6, s_7)$, consider the averages

$$(s_0 + s_1)/2,$$
$$(s_2 + s_3 + s_4 + s_5 + s_6 + s_7)/6.$$

Investigate whether

$$\frac{[(s_0 + s_1)/2] + [(s_2 + s_3 + s_4 + s_5 + s_6 + s_7)/6]}{2} \stackrel{?}{=} \frac{s_0 + s_1 + s_2 + s_3 + s_4 + s_5 + s_6 + s_7}{8}.$$

Either verify such an identity algebraically, or produce an example with specific numbers for which it fails.

Exercise 1.13. For each array $\vec{\mathbf{s}} = (s_0, s_1, \ldots, s_6, s_7)$, verify that

$$\frac{\frac{[(s_0+s_1)/2]+[(s_2+s_3)/2]}{2} + \frac{[(s_4+s_5)/2]+[(s_6+s_7)/2]}{2}}{2} = \frac{s_0 + s_1 + s_2 + s_3 + s_4 + s_5 + s_6 + s_7}{8}.$$

Thus, the average equals the averages of the averages of the averages.

Exercise 1.14. Generalize and verify, for instance by mathematical induction, a result analogous to that of Exercise 1.13, for arrays with an integral power of two number of entries.

1.3 THE ORDERED FAST HAAR WAVELET TRANSFORM

To analyze a signal or function in terms of wavelets, the Fast Haar Wavelet Transform begins with the initialization of an array with 2^n entries, and then proceeds with n iterations of the basic transform explained in the preceding section.

For each index $\ell \in \{1, \ldots, n\}$, *before* iteration number ℓ, the array will consist of $2^{n-(\ell-1)}$ coefficients of $2^{n-(\ell-1)}$ step functions $\varphi_k^{(n-[\ell-1])}$, defined below. *After* iteration number ℓ, the array will consist of half as many, $2^{n-\ell}$, coefficients of $2^{n-\ell}$ step functions $\varphi_k^{(n-\ell)}$, and $2^{n-\ell}$ coefficients of wavelets $\psi_k^{(n-\ell)}$.

Definition 1.10 For each positive integer n and each index $\ell \in \{0, \ldots, n\}$, define the step functions $\varphi_k^{(n-\ell)}$ and wavelets $\psi_k^{(n-\ell)}$ by

$$\begin{aligned}
\varphi_k^{(n-\ell)}(r) &:= \varphi_{[0,1[}\left(2^{n-\ell}\left[r - k2^{\ell-n}\right]\right) \\
&= \begin{cases} 1 & \text{if } k2^{\ell-n} \leq r < (k+1)2^{\ell-n}, \\ 0 & \text{otherwise,} \end{cases} \\
\psi_k^{(n-\ell)}(r) &:= \psi_{[0,1[}\left(2^{n-\ell}\left[r - k2^{\ell-n}\right]\right) \\
&= \begin{cases} 1 & \text{if } k2^{\ell-n} \leq r < \left(k + \left[\frac{1}{2}\right]\right)2^{\ell-n}, \\ -1 & \text{if } \left(k + \left[\frac{1}{2}\right]\right)2^{\ell-n} \leq r < (k+1)2^{\ell-n}, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

□

In the foregoing definition, the frequency increases with the index n, as in references [20] and [49]. By contrast, in such references as [7] and [31], the frequency *decreases* as the index *increases.*

1.3.1 Initialization

For Haar's wavelets, the initialization consists only in establishing a one-dimensional *array* $\vec{\mathbf{a}}^{(n)}$, also called a *vector* or a *finite sequence,* of sample values, of the form

$$\begin{aligned}
\vec{\mathbf{a}}^{(n)} &= \left(a_0^{(n)}, a_1^{(n)}, \ldots, a_j^{(n)}, \ldots, a_{2^n-2}^{(n)}, a_{2^n-1}^{(n)}\right) \\
&:= \vec{\mathbf{s}} = \left(s_0, s_1, \ldots, s_j, \ldots, s_{2^n-2}, s_{2^n-1}\right),
\end{aligned}$$

with a total number of sample values equal to an integral power of two, 2^n, as indicated by the superscript $^{(n)}$. Though indices ranging from 1 through 2^n would also serve the same purpose, indices ranging from 0 through $2^n - 1$ will accommodate a binary encoding with only n binary digits, and will also offer notational simplifications in the exposition. The array corresponds to the sampled step function

$$\tilde{f}^{(n)} = \sum_{j=0}^{2^n-1} a_j^{(n)} \varphi_j^{(n)}.$$

1.3.2 The Ordered Fast Haar Wavelet Transform

The preceding section has demonstrated how a first sweep of the basic transform applies to all the consecutive pairs (s_{2k}, s_{2k+1}) of the initial array of sample values $\vec{\mathbf{a}}^{(n)} = \vec{\mathbf{s}}$.

In general, the ℓth sweep of the basic transform begins with an array of $2^{n-(\ell-1)}$ values

$$\vec{\mathbf{a}}^{(n-[\ell-1])} = \left(a_0^{(n-[\ell-1])}, \dots, a_{2^{n-(\ell-1)}-1}^{(n-[\ell-1])}\right),$$

and applies the basic transform to each pair $(a_{2k}^{(n-[\ell-1])}, a_{2k+1}^{(n-[\ell-1])})$, which gives two new wavelet coefficients

$$a_k^{(n-\ell)} := \frac{a_{2k}^{(n-[\ell-1])} + a_{2k+1}^{(n-[\ell-1])}}{2},$$
$$c_k^{(n-\ell)} := \frac{a_{2k}^{(n-[\ell-1])} - a_{2k+1}^{(n-[\ell-1])}}{2}.$$

These $2^{(n-\ell)}$ pairs of new coefficients represent the *result* of the ℓth sweep, a result that can also be reassembled into two arrays:

$$\vec{\mathbf{a}}^{(n-\ell)} := \left(a_0^{(n-\ell)}, a_1^{(n-\ell)}, \dots, a_k^{(n-\ell)}, \dots, a_{2^{n-\ell}-1}^{(n-\ell)}\right),$$
$$\vec{\mathbf{c}}^{(n-\ell)} := \left(c_0^{(n-\ell)}, c_1^{(n-\ell)}, \dots, c_k^{(n-\ell)}, \dots, c_{2^{n-\ell}-1}^{(n-\ell)}\right).$$

The arrays related to the ℓth sweep have the following significance.

$\vec{\mathbf{a}}^{(n-[\ell-1])}$: The beginning array,

$$\vec{\mathbf{a}}^{(n-[\ell-1])} = \left(a_0^{(n-[\ell-1])}, \dots, a_{2^{n-(\ell-1)}-1}^{(n-[\ell-1])}\right),$$

lists the values $a_k^{(n-[\ell-1])}$ of a simple step function $\tilde{f}^{(n-[\ell-1])}$ that approximates the initial function f with $2^{n-(\ell-1)}$ steps of narrower width $2^{(\ell-1)-n}$:

$$\tilde{f}^{(n-[\ell-1])} = \sum_{j=0}^{2^{n-(\ell-1)}-1} a_j^{(n-[\ell-1])} \varphi_j^{(n-[\ell-1])}.$$

$\vec{\mathbf{a}}^{(n-\ell)}$: The first array produced by the ℓth sweep,

$$\vec{\mathbf{a}}^{(n-\ell)} = \left(a_0^{(n-\ell)}, \ldots, a_{2^{n-\ell}-1}^{(n-\ell)}\right),$$

lists the values $a_k^{(n-\ell)}$ of a simple step function $\tilde{f}^{(n-\ell)}$ that approximates the initial function f with $2^{n-\ell}$ steps of wider width $2^{\ell-n}$,

$$\tilde{f}^{(n-\ell)} = \sum_{j=0}^{2^{n-\ell}-1} a_j^{(n-\ell)} \varphi_j^{(n-\ell)}.$$

$\vec{\mathbf{c}}^{(n-\ell)}$: The second array produced by the ℓth sweep,

$$\vec{\mathbf{c}}^{(n-\ell)} = \left(c_0^{(n-\ell)}, \ldots, c_{2^{n-\ell}-1}^{(n-\ell)}\right),$$

lists the coefficients $c_k^{(n-\ell)}$ of simple wavelets $\psi_j^{(n-\ell)}$ also of wider width $2^{\ell-n}$,

$$\dot{f}^{(n-\ell)} = \sum_{j=0}^{2^{n-\ell}-1} c_j^{(n-\ell)} \psi_j^{(n-\ell)}.$$

The wavelets given by the second new array, $\vec{\mathbf{c}}^{(n-\ell)}$, represent the difference between the finer steps of the initial approximation $\tilde{f}^{(n-[\ell-1])}$ and the coarser steps of $\tilde{f}^{(n-\ell)}$. Thus, each sweep of basic transforms expresses the previous finer approximation as the sum of a new, coarser approximation and a new, lower-frequency, set of wavelets. Nevertheless, because the basic step of Haar's transform does not alter the sampled function but merely expresses it with different wavelets, it follows that the initial approximation $\tilde{f}^{(n-[\ell-1])}$ still equals the sum of the two new approximations, $\tilde{f}^{(n-\ell)}$ and $\dot{f}^{(n-\ell)}$:

$$\tilde{f}^{(n-[\ell-1])} = \tilde{f}^{(n-\ell)} + \dot{f}^{(n-\ell)}.$$

Example 1.11 The array $(s_0, s_1, s_2, s_3) = (\mathbf{5}, \mathbf{1}, \mathbf{2}, \mathbf{8})$ reproduces data from Examples 1.3 and 1.6. To illustrate a common usage [7], the present example will store the final result—the Haar Wavelet Transform of the data—ordered by increasing frequencies: from the lowest frequencies produced last but stored in the first (left) part of the array, through to highest frequencies produced first but stored in the last (right) part of the array.

1.3.2.1 Initialization. The initial array $\vec{\mathbf{a}}^{(2)} = \vec{\mathbf{s}} = (5, 1, 2, 8)$ contains the $2^2 = 4$ values of the sample, as in Figure 1.8(a).

1.3.2.2 First Sweep. Begin with $\vec{\mathbf{a}}^{(2)} = (5, 1, 2, 8)$.

$$\vec{\mathbf{a}}^{(2-1)} = \left(\frac{5+1}{2}, \frac{2+8}{2}\right) = (3, 5),$$

$$\vec{\mathbf{c}}^{(2-1)} = \left(\frac{5-1}{2}, \frac{2-8}{2}\right) = (2, -3),$$

which can be stored in the form

$$\vec{\mathbf{s}}^{(2-1)} = \left(\vec{\mathbf{a}}^{(2-1)}; \vec{\mathbf{c}}^{(2-1)}\right) = (\mathbf{3}, \mathbf{5}; 2, -3).$$

The first array, $\vec{\mathbf{a}}^{(2-1)} = (\mathbf{3}, \mathbf{5})$, represents a coarse approximation of the initial sample $\vec{\mathbf{a}}^{(2)}$, and means that the first half of the sample, $(5, 1)$, has an average value of 3, and that the second half of the sample, $(2, 8)$, has an average value of 5. The second array, $\vec{\mathbf{c}}^{(2-1)} = (2, -3)$, means that on the first half of the sample, the values jump downward by 2 times the jump of a wavelet, hence by a total jump of $2 * (-2) = -4$, whereas on the second half of the sample, the values jump by -3 times the jump of a wavelet, hence by a total jump of $(-3) * (-2) = 6$.

1.3.2.3 Second Sweep. Keep $\vec{\mathbf{c}}^{(2-1)}$ and continue with $\vec{\mathbf{a}}^{(2-1)} = (3, 5)$.

$$\vec{\mathbf{a}}^{(2-2)} = \left(\frac{3+5}{2}\right) = (\mathbf{4}),$$

$$\vec{\mathbf{c}}^{(2-2)} = \left(\frac{3-5}{2}\right) = (-1),$$

which can be stored in the form

$$\vec{\mathbf{s}}^{(2-2)} = \left(\vec{\mathbf{a}}^{(2-2)}; \vec{\mathbf{c}}^{(2-2)}; \vec{\mathbf{c}}^{(2-1)}\right) = (\mathbf{4}; -1;\ 2, -3).$$

The first array, $\vec{\mathbf{a}}^{(2-2)} = (4)$, means that the whole sample, $(5, 1, 2, 8)$, has an average value of 4. The second array, $\vec{\mathbf{c}}^{(2-2)} = (-1)$, means that at the middle of the sample, the average of the first half, 3, jumps up toward the average of the second half, 5, as does -1 basic wavelet $\psi_{[0,1[}$, in effect a jump of size $(-1) \cdot (-2) = 2$.

1.3.2.4 Results. The final result from two consecutive sweeps takes the following form:

$$\begin{aligned}\tilde{f} &= 4 \cdot \varphi_{[0,1[} + (-1) \cdot \psi_{[0,1[} + 2 \cdot \psi_{[0,\frac{1}{2}[} + (-3) \cdot \psi_{[\frac{1}{2},1[} \\ &= 4 \cdot \varphi_0^{(0)} + (-1) \cdot \psi_0^{(0)} + 2 \cdot \psi_0^{(1)} + (-3) \cdot \psi_1^{(1)}.\end{aligned}$$

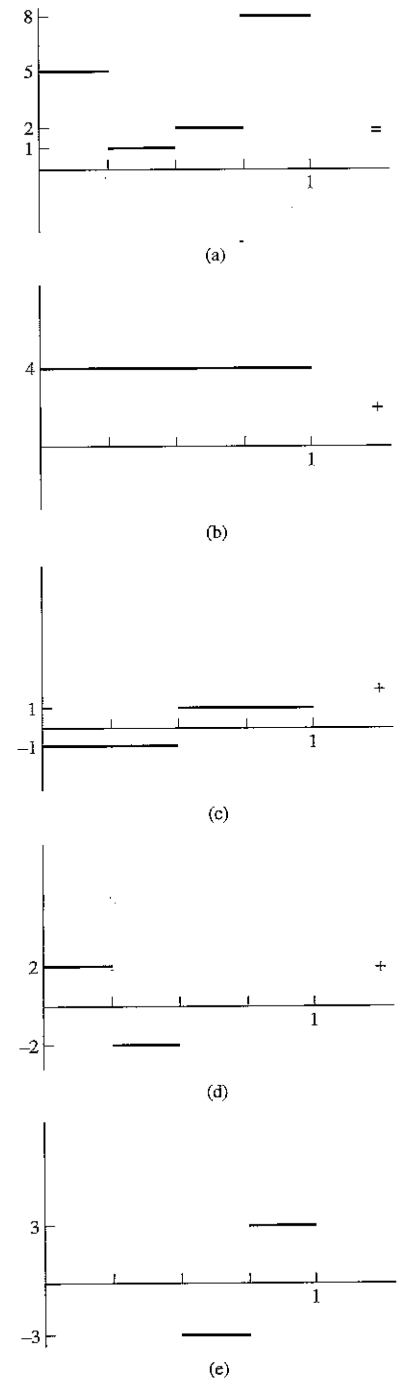
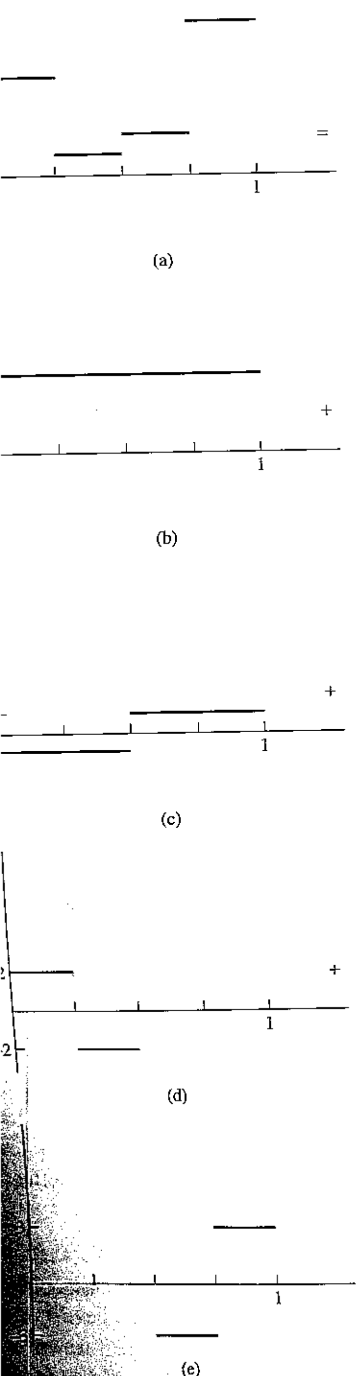


Figure 1.8 Example of a wavelet transform. (a) Data. (b)–(e) Haar Wavelet Transform.



velet transform. (a) Data. (b)–(e) Haar Wavelet Transform.

The initial array $\mathbf{a}^{(2)} = (5, 1, 2, 8)$ contained the values of the sample and of the approximating function $\tilde{f}$. In contrast, the formula just obtained expresses the same $\tilde{f}$ as a sum of higher-frequency wavelets from the first sweep, followed by a lower-frequency wavelet and a global average from the second sweep, as in Figure 1.8(b)–(e). □

Example 1.12 The array $\vec{\mathbf{s}} = (\mathbf{3}, \mathbf{1}, \mathbf{0}, \mathbf{4}, \mathbf{8}, \mathbf{6}, \mathbf{9}, \mathbf{9})$ reproduces the sample from Example 1.4.

1.3.2.5 Initialization. $\vec{\mathbf{a}}^{(3)} := \vec{\mathbf{s}} = (3, 1, 0, 4, 8, 6, 9, 9)$.

1.3.2.6 First Sweep.

$$\vec{\mathbf{a}}^{(3-1)} = \left(\frac{3+1}{2}, \frac{0+4}{2}, \frac{8+6}{2}, \frac{9+9}{2}\right) = (2, 2, 7, 9),$$

$$\vec{\mathbf{c}}^{(3-1)} = \left(\frac{3-1}{2}, \frac{0-4}{2}, \frac{8-6}{2}, \frac{9-9}{2}\right) = (1, -2, 1, 0),$$

which can be stored in the form

$$\vec{\mathbf{s}}^{(3-1)} = \left(\vec{\mathbf{a}}^{(3-1)}; \vec{\mathbf{c}}^{(3-1)}\right) = (\mathbf{2}, \mathbf{2}, \mathbf{7}, \mathbf{9};\ 1, -2, 1, 0)\,.$$

1.3.2.7 Second Sweep.

$$\vec{\mathbf{a}}^{(3-1)} = (\mathbf{2}, \mathbf{2}, \mathbf{7}, \mathbf{9}),$$

$$\vec{\mathbf{a}}^{(3-2)} = \left(\frac{2+2}{2}, \frac{7+9}{2}\right) = (\mathbf{2}, \mathbf{8}),$$

$$\vec{\mathbf{c}}^{(3-2)} = \left(\frac{2-2}{2}, \frac{7-9}{2}\right) = (0, -1),$$

which can be stored in the form

$$\vec{\mathbf{s}}^{(3-2)} = \left(\vec{\mathbf{a}}^{(3-2)}; \vec{\mathbf{c}}^{(3-2)}; \vec{\mathbf{c}}^{(3-1)}\right) = (\mathbf{2}, \mathbf{8};\ 0, -1;\ 1, -2, 1, 0)\,.$$

1.3.2.8 Third Sweep.

$$\vec{\mathbf{a}}^{(3-2)} = (\mathbf{2}, \mathbf{8}),$$

$$\vec{\mathbf{a}}^{(3-3)} = \left(\frac{2+8}{2}\right) = (\mathbf{5}),$$

$$\vec{\mathbf{c}}^{(3-3)} = \left(\frac{2-8}{2}\right) = (-3),$$

which can be stored in the form

$$\vec{\mathbf{s}}^{(3-3)} = \left(\vec{\mathbf{a}}^{(3-3)}; \vec{\mathbf{c}}^{(3-3)}; \vec{\mathbf{c}}^{(3-2)}; \vec{\mathbf{c}}^{(3-1)}\right) = (\mathbf{5}; -3; 0, -1; 1, -2, 1, 0)\,.$$

1.3.2.9 Results. The initial array $\vec{\mathbf{a}}^{(3)} = \vec{\mathbf{s}}$ represents the approximating function $\tilde{f}$ by its sample values,

$$\begin{aligned}\tilde{f} = 3 \cdot \varphi_{[0,\frac{1}{8}[} + 1 \cdot \varphi_{[\frac{1}{8},\frac{1}{4}[} + 0 \cdot \varphi_{[\frac{1}{4},\frac{3}{8}[} + 4 \cdot \varphi_{[\frac{3}{8},\frac{1}{2}[}\\ + 8 \cdot \varphi_{[\frac{1}{2},\frac{5}{8}[} + 6 \cdot \varphi_{[\frac{5}{8},\frac{3}{4}[} + 9 \cdot \varphi_{[\frac{3}{4},\frac{7}{8}[} + 9 \cdot \varphi_{[\frac{7}{8},1[}.\end{aligned}$$

In contrast, the wavelet coefficients $\vec{\mathbf{c}}^{(3-\ell)}$ produced by the consecutive sweeps of basic transforms express the same approximating function $\tilde{f}$ in terms of consecutively lower frequencies, ending with a constant step across the entire interval,

$$\begin{aligned}\tilde{f} = 1 \cdot \psi_{[0,\frac{1}{4}[} + (-2) \cdot \psi_{[\frac{1}{4},\frac{1}{2}[} + 1 \cdot \psi_{[\frac{1}{2},\frac{3}{4}[} + 0 \cdot \psi_{[\frac{3}{4},1[} &\quad \textit{1st Sweep}\\ + 0 \cdot \psi_{[0,\frac{1}{2}[} + (-1) \cdot \psi_{[\frac{1}{2},1[} &\quad \textit{2nd Sweep}\\ + (-3) \cdot \psi_{[0,1[} + \mathbf{5} \cdot \varphi_{[0,1[} &\quad \textit{3rd Sweep}\end{aligned}$$

stored in the form $\vec{\mathbf{s}}^{(3-3)} = (\mathbf{5}; -3; 0, -1; 1, -2, 1, 0)$.

1.3.2.10 Significance. The term produced last, $\mathbf{5} \cdot \varphi_{[0,1[}$, means that the sample has an average value equal to $\mathbf{5}$.

The penultimate term, $-3 \cdot \psi_{[0,1[}$, indicates that the sample undergoes a jump 3 times the size of, and in the opposite direction from, the wavelet $\psi_{[0,1[}$ (which jumps downward by 2 at the middle of the interval). Indeed, the sample jumps upward by 6 on average at the middle of the interval: The array $\vec{\mathbf{a}}^{(3-2)} = (2, 8)$ shows that the average jumps from 2 on the left-hand half of the interval to 8 on the right-hand half of the interval.

The two terms $0 \cdot \psi_{[0,\frac{1}{2}[} + (-1) \cdot \psi_{[\frac{1}{2},1[}$ mean that the sample does not exhibit any average jump at the first quarter of the interval, and exhibits an average jump of $(-1) \cdot (-1) = 1$ at the third quarter.

The four terms $1 \cdot \psi_{[0,\frac{1}{4}[} + (-2) \cdot \psi_{[\frac{1}{4},\frac{1}{2}[} + 1 \cdot \psi_{[\frac{1}{2},\frac{3}{4}[} + 0 \cdot \psi_{[\frac{3}{4},1[}$ reveal that the sample oscillates, as do the fastest wavelets, with jumps of sizes -2, 4, -2, and 0. □

EXERCISES

Exercise 1.15. Calculate the Haar Wavelet Transform for the data $(s_0, s_1) = (\mathbf{3}, \mathbf{9})$.

Exercise 1.16. Calculate the Haar Wavelet Transform for the data $(s_0, s_1) = (\mathbf{1}, \mathbf{7})$.

Exercise 1.17. Calculate the Haar Wavelet Transform for the data $\vec{\mathbf{s}} = (\mathbf{2}, \mathbf{4}, \mathbf{8}, \mathbf{6})$.

Exercise 1.18. Calculate the Haar Wavelet Transform for the data $\vec{\mathbf{s}} = (\mathbf{5}, \mathbf{7}, \mathbf{3}, \mathbf{1})$.

Exercise 1.19. Calculate the Haar Wavelet Transform for the data $\vec{\mathbf{s}} = (8, 6, 7, 3, 1, 1, 2, 4)$.

Exercise 1.20. Calculate the Haar Wavelet Transform for the data $\vec{\mathbf{s}} = (3, 1, 9, 7, 7, 9, 5, 7)$.

Exercise 1.21. Assume that the Haar Wavelet Transform of a sample $\vec{\mathbf{s}} = (s_0, s_1)$ produces the results $a_0^{(1-1)} := 7$ and $c_0^{(1-1)} := 2$.

(a) Explain how $a_0^{(1-1)} = 7$ relates to the sample (s_0, s_1).

(b) Explain how $c_0^{(1-1)} = 2$ relates to the sample (s_0, s_1).

Exercise 1.22. Assume that the Haar Wavelet Transform of a sample $\vec{\mathbf{s}} = (s_0, s_1)$ produces the results $a_0^{(1-1)} := 6$ and $c_0^{(1-1)} := -3$.

(a) Explain how $a_0^{(1-1)} = 6$ relates to the sample (s_0, s_1).

(b) Explain how $c_0^{(1-1)} = -3$ relates to the sample (s_0, s_1).

Exercise 1.23. Assume that the Haar Wavelet Transform of a sample $\vec{\mathbf{s}} = (s_0, s_1, s_2, s_3)$ produces the results $\vec{\mathbf{c}}^{(2-1)} = (2, 2)$, $\vec{\mathbf{c}}^{(2-2)} = (1)$, and $\vec{\mathbf{a}}^{(2-2)} = (6)$.

(a) Explain how $a_0^{(2-2)} = 6$ relates to the sample.

(b) Explain how $c_0^{(2-2)} = 1$ relates to the sample.

(c) Explain how $c_0^{(2-1)} = 2$ relates to the sample.

(d) Explain how $c_1^{(2-1)} = 2$ relates to the sample.

Exercise 1.24. Assume that the Haar Wavelet Transform of a sample $\vec{\mathbf{s}} = (s_0, s_1, s_2, s_3)$ produces the results $\vec{\mathbf{c}}^{(2-1)} = (2, 0)$, $\vec{\mathbf{c}}^{(2-2)} = (2)$, and $\vec{\mathbf{a}}^{(2-2)} = (4)$.

(a) Explain how $a_0^{(2-2)} = 4$ relates to the sample.

(b) Explain how $c_0^{(2-2)} = 2$ relates to the sample.

(c) Explain how $c_0^{(2-1)} = 2$ relates to the sample.

(d) Explain how $c_1^{(2-1)} = 0$ relates to the sample.

1.4 THE IN-PLACE FAST HAAR WAVELET TRANSFORM

Whereas the presentation in the preceding section conveniently lays out all the steps of the Fast Haar Wavelet Transform, it requires additional arrays at each sweep, and it assumes that the whole sample is known at the start of the algorithm. In contrast, some applications require real-time processing as the signal proceeds, which precludes any knowledge of the whole sample, and some applications involve arrays so large that they do not allow sufficient space for additional arrays at each sweep. The two problems just described, lack of time or space, have a common solution in the In-Place Fast Haar Wavelet Transform presented here, which differs from the preceding algorithm only in its indexing scheme.

1.4.1 In-Place Basic Sweep

For each pair $(a_{2k}^{(n-[\ell-1])}, a_{2k+1}^{(n-[\ell-1])})$, instead of placing its results in two additional arrays, the ℓth sweep of the in-place transform merely *replaces* the pair $(a_{2k}^{(n-[\ell-1])}, a_{2k+1}^{(n-[\ell-1])})$ by the new entries $(a_k^{(n-\ell)}, c_k^{(n-\ell)})$:

1.4.1.1 Initialization. Consider the pair $(a_{2k}^{(n-[\ell-1])}, a_{2k+1}^{(n-[\ell-1])})$.

1.4.1.2 Calculation. Perform the basic transform

$$a_k^{(n-\ell)} := \frac{a_{2k}^{(n-[\ell-1])} + a_{2k+1}^{(n-[\ell-1])}}{2},$$

$$c_k^{(n-\ell)} := \frac{a_{2k}^{(n-[\ell-1])} - a_{2k+1}^{(n-[\ell-1])}}{2}.$$

1.4.1.3 Replacement. Replace the initial pair $(a_{2k}^{(n-[\ell-1])}, a_{2k+1}^{(n-[\ell-1])})$ by the transformed pair $(a_k^{(n-\ell)}, c_k^{(n-\ell)})$.

Example 1.13 For the initial array $\vec{\mathbf{s}}^{(1)} := \vec{\mathbf{s}} := (9, 1)$, the In-Place Haar Wavelet Transform gives

$$\vec{\mathbf{s}}^{(1-1)} = \left(\frac{9+1}{2}, \frac{9-1}{2}\right) = (\mathbf{5}, 4).$$ □

Example 1.14 For the initial array $\vec{\mathbf{s}}^{(2)} := \vec{\mathbf{s}} := (5, 1, 2, 8)$, the first In-Place basic sweep gives

$$\vec{\mathbf{s}}^{(2-1)} = \left(\frac{5+1}{2}, \frac{5-1}{2}, \frac{2+8}{2}, \frac{2-8}{2}\right) = (\mathbf{3}, 2, \mathbf{5}, -3).$$ □

Example 1.15 For the initial array $\vec{\mathbf{s}}^{(3)} = \vec{\mathbf{s}} = (3, 1, 0, 4, 8, 6, 9, 9)$, the first In-Place basic sweep yields

$$\vec{\mathbf{s}}^{(3-1)} = \left(\frac{3+1}{2}, \frac{3-1}{2}, \frac{0+4}{2}, \frac{0-4}{2}, \frac{8+6}{2}, \frac{8-6}{2}, \frac{9+9}{2}, \frac{9-9}{2}\right)$$
$$= (\mathbf{2}, 1, \mathbf{2}, -2, \mathbf{7}, 1, \mathbf{9}, 0).$$

For convenience, the entries in **boldface** show the starting array $\vec{\mathbf{a}}^{(3-1)}$ for the next sweep, described in the next subsection. □

1.4.2 The In-Place Fast Haar Wavelet Transform

The in-place basic sweep explained in the preceding subsection extends to a complete algorithm through mere record-keeping. The first few sweeps proceed as follows.

1.4.2.1 Initialization.

$$\vec{\mathbf{s}}^{(n)} := \vec{\mathbf{s}} = (s_0, s_1, s_2, s_3, \dots, s_{2k}, s_{2k+1}, \dots, s_{2^n-2}, s_{2^n-1}).$$

1.4.2.2 First Sweep.

$$\begin{aligned}\vec{\mathbf{s}}^{(n-1)} &= \Bigg(\frac{s_0+s_1}{2}, \frac{s_0-s_1}{2}, \frac{s_2+s_3}{2}, \frac{s_2-s_3}{2}, \dots, \frac{s_{2k}+s_{2k+1}}{2}, \frac{s_{2k}-s_{2k+1}}{2},\\ &\qquad \dots, \frac{s_{2^n-2}+s_{2^n-1}}{2}, \frac{s_{2^n-2}-s_{2^n-1}}{2}\Bigg)\\ &= \Big(\mathbf{a}_0^{(n-1)}, c_0^{(n-1)}, \mathbf{a}_1^{(n-1)}, c_1^{(n-1)}, \mathbf{a}_2^{(n-1)}, c_2^{(n-1)}, \mathbf{a}_3^{(n-1)}, c_3^{(n-1)},\\ &\qquad \dots, \mathbf{a}_k^{(n-1)}, c_k^{(n-1)}, \dots, \mathbf{a}_{2^{n-1}-1}^{(n-1)}, c_{2^{n-1}-1}^{(n-1)}\Big).\end{aligned}$$

1.4.2.3 Second Sweep. In the new array $\vec{\mathbf{s}}^{(n-1)}$, keep but skip over the wavelet coefficients $c_k^{(n-1)}$, and perform a basic sweep on the array $\vec{\mathbf{a}}^{(n-1)}$ at its new location, now occupying every other entry in $\vec{\mathbf{s}}^{(n-1)}$:

$$\begin{aligned}\vec{\mathbf{s}}^{(n-2)} &= \Bigg(\frac{\mathbf{a}_0^{(n-1)}+\mathbf{a}_1^{(n-1)}}{2}, c_0^{(n-1)}, \frac{\mathbf{a}_0^{(n-1)}-\mathbf{a}_1^{(n-1)}}{2}, c_1^{(n-1)},\\ &\qquad \dots, \frac{\mathbf{a}_2^{(n-1)}+\mathbf{a}_3^{(n-1)}}{2}, c_2^{(n-1)}, \frac{\mathbf{a}_2^{(n-1)}-\mathbf{a}_3^{(n-1)}}{2}, c_3^{(n-1)}, \dots\\ &\qquad \dots, \frac{\mathbf{a}_{2^{n-1}-2}^{(n-1)}+\mathbf{a}_{2^{n-1}-1}^{(n-1)}}{2}, c_{2^{n-1}-2}^{(n-1)}, \frac{\mathbf{a}_{2^{n-1}-2}^{(n-1)}-\mathbf{a}_{2^{n-1}-1}^{(n-1)}}{2}, c_{2^{n-1}-1}^{(n-1)}\Bigg)\\ &= \Big(\mathbf{a}_0^{(n-2)}, c_0^{(n-1)}, c_0^{(n-2)}, c_1^{(n-1)}, \mathbf{a}_1^{(n-2)}, c_2^{(n-1)}, c_1^{(n-2)}, c_3^{(n-1)},\\ &\qquad \mathbf{a}_2^{(n-2)}, c_4^{(n-1)}, c_2^{(n-2)}, c_5^{(n-1)}, \dots, c_{2^{n-2}-1}^{(n-2)}, c_{2^{n-1}-1}^{(n-1)}\Big).\end{aligned}$$

In general, the In-Place ℓth sweep begins with an array

$$\begin{aligned}\vec{\mathbf{s}}^{(n-[\ell-1])} &= \Big(\mathbf{a}_0^{(n-[\ell-1])}, c_0^{(n-1)}, c_0^{(n-2)}, c_1^{(n-1)},\\ &\qquad c_0^{(n-3)}, c_2^{(n-1)}, c_1^{(n-2)}, c_3^{(n-1)}, \dots, c_{2^{n-2}-1}^{(n-2)}, c_{2^{n-1}-1}^{(n-1)}\Big),\end{aligned}$$

which contains the array

$$\vec{\mathbf{a}}^{(n-[\ell-1])} = \left(\mathbf{a}_0^{(n-[\ell-1])}, \mathbf{a}_1^{(n-[\ell-1])}, \ldots, \mathbf{a}_{2^{n-(\ell-1)}-1}^{(n-[\ell-1])}\right)$$

at the locations $\mathbf{a}_k^{(n-[\ell-1])} = s_{2^{\ell-1}k}^{(n-[\ell-1])}$, in other words, at multiples of $2^{\ell-1}$ apart in $\vec{\mathbf{s}}^{(n-[\ell-1])}$, and which the ℓth sweep replaces by

$$\begin{aligned}
a_j^{(n-\ell)} &:= \frac{a_{2j}^{(n-[\ell-1])} + a_{2j+1}^{(n-[\ell-1])}}{2} = \frac{s_{2^{\ell-1}2j}^{(n-[\ell-1])} + s_{2^{\ell-1}(2j+1)}^{(n-[\ell-1])}}{2},\\
c_j^{(n-\ell)} &:= \frac{a_{2j}^{(n-[\ell-1])} - a_{2j+1}^{(n-[\ell-1])}}{2} = \frac{s_{2^{\ell-1}2j}^{(n-[\ell-1])} - s_{2^{\ell-1}(2j+1)}^{(n-[\ell-1])}}{2},\\
s_{2^{\ell-1}2j}^{(n-\ell)} &:= a_j^{(n-\ell)},\\
s_{2^{\ell-1}(2j+1)}^{(n-\ell)} &:= c_j^{(n-\ell)},
\end{aligned}$$

so that the new array $\vec{\mathbf{a}}^{(n-\ell)}$ occupies entries at multiples of 2^ℓ apart in $\vec{\mathbf{s}}^{(n-\ell)}$, because $a_j^{(n-\ell)} = s_{2^{\ell-1}2j}^{(n-\ell)} = s_{2^\ell j}^{(n-\ell)}$. Hence, the foregoing considerations lead to the following algorithm.

Algorithm 1.16 In-Place Fast Haar Wavelet Transform.

DATA:	
n	(nonnegative integer)
$\vec{\mathbf{s}}$	(array of 2^n numbers)
START.	
$I := 1$	(index increment)
$J := 2$	(increment between pairs)
$M := 2^n$	(number of sample values)
FOR $L := 1, \ldots, n$ DO	(loop of basic sweeps)
$M := M/2$	(halve M)
FOR $K := 0, \ldots, M-1$ DO	(loop of values)
$a_k^{(n-\ell)} := (s_{J\cdot K} + s_{J\cdot K+I})/2$	
$c_k^{(n-\ell)} := (s_{J\cdot K} - s_{J\cdot K+I})/2$	

```
        s_{J·K} := a_k^{(n-ℓ)}
        s_{J·K+I} := c_k^{(n-ℓ)}
      END                       (end of the loop of values)
      I := J                    (double I)
      J := 2 * J                (double J)
    END                         (end of basic sweeps)
    STOP.
    RESULT:
```

$$\vec{\mathbf{s}} = \left(a_0^{(n)}, c_0^{(n-1)}, c_0^{(n-2)}, c_1^{(n-1)}, \ldots\right),$$

$$c_j^{(n-\ell)} = s_{2^{\ell-1}+2^\ell j} = s_{2^{\ell-1}(2j+1)} \text{ for } j \in \{0, \ldots, 2^{n-\ell} - 1\}.$$ □

Example 1.17 For the initial array $\vec{\mathbf{s}}^{(2)} := \vec{\mathbf{s}} := (5, 1, 2, 8)$, the In-Place Fast Haar Wavelet Transform proceeds as follows.

1.4.2.4 Initialization. $\vec{\mathbf{s}}^{(2)} := \vec{\mathbf{s}} := (5, 1, 2, 8) = \left(\mathbf{a}_0^{(2)}, \mathbf{a}_1^{(2)}, \mathbf{a}_2^{(2)}, \mathbf{a}_3^{(2)}\right)$.

1.4.2.5 First Sweep. The first sweep operates on all the entries of $\vec{\mathbf{s}}$:

$$\begin{aligned}\vec{\mathbf{s}}^{(2-1)} &= \left(\mathbf{a}_0^{(2-1)}, c_0^{(2-1)}, \mathbf{a}_1^{(2-1)}, c_1^{(2-1)}\right)\\ &= \left(\frac{5+1}{2}, \frac{5-1}{2}, \frac{2+8}{2}, \frac{2-8}{2}\right) = (\mathbf{3}, 2, \mathbf{5}, -3).\end{aligned}$$

1.4.2.6 Second Sweep. The second sweep operates on the even-indexed entries:

$$\begin{aligned}\vec{\mathbf{s}}^{(2-1)} &= (\mathbf{3}, 2, \mathbf{5}, -3),\\ \vec{\mathbf{s}}^{(2-2)} &= \left(\mathbf{a}_0^{(2-2)}, c_0^{(2-1)}, c_0^{(2-2)}, c_1^{(2-1)}\right)\\ &= \left(\frac{3+5}{2}, 2, \frac{3-5}{2}, -3\right) = (\mathbf{4}, 2, -1, -3).\end{aligned}$$ □

Example 1.18 The array $\vec{\mathbf{s}} = (\mathbf{3}, \mathbf{1}, \mathbf{0}, \mathbf{4}, \mathbf{8}, \mathbf{6}, \mathbf{9}, \mathbf{9})$ reproduces the data from Example 1.4.

1.4.2.7 Initialization. $\vec{\mathbf{s}}^{(3)} = \vec{\mathbf{s}} = (3, 1, 0, 4, 8, 6, 9, 9)$.

1.4.2.8 In-Place Fast Haar Wavelet Transform.

$$\begin{aligned}\vec{\mathbf{s}}^{(3-1)} &= \left(\mathbf{a}_0^{(3-1)}, c_0^{(3-1)}, \mathbf{a}_1^{(3-1)}, c_1^{(3-1)}, \mathbf{a}_2^{(3-1)}, c_2^{(3-1)}, \mathbf{a}_3^{(3-1)}, c_3^{(3-1)}\right)\\ &= \left(\frac{3+1}{2}, \frac{3-1}{2}, \frac{0+4}{2}, \frac{0-4}{2}, \frac{8+6}{2}, \frac{8-6}{2}, \frac{9+9}{2}, \frac{9-9}{2}\right)\end{aligned}$$

$$
\begin{aligned}
&= (\mathbf{2}, 1, \mathbf{2}, -2, \mathbf{7}, 1, \mathbf{9}, 0),\\
\vec{\mathbf{s}}^{(3-2)} &= \left(\mathbf{a}_0^{(3-2)}, c_0^{(3-1)}, c_0^{(3-2)}, c_1^{(3-1)}, \mathbf{a}_1^{(3-2)}, c_2^{(3-1)}, c_1^{(3-2)}, c_3^{(3-1)}\right)\\
&= \left(\frac{2+2}{2}, 1, \frac{2-2}{2}, -2, \frac{7+9}{2}, 1, \frac{7-9}{2}, 0\right)\\
&= (\mathbf{2}, 1, 0, -2, \mathbf{8}, 1, -1, 0),\\
\vec{\mathbf{s}}^{(3-3)} &= \left(\mathbf{a}_0^{(3-3)}, c_0^{(3-1)}, c_0^{(3-2)}, c_1^{(3-1)}, c_0^{(3-3)}, c_2^{(3-1)}, c_1^{(3-2)}, c_3^{(3-1)}\right)\\
&= \left(\frac{2+8}{2}, 1, 0, -2, \frac{2-8}{2}, 1, -1, 0\right)\\
&= (\mathbf{5}, 1, 0, -2, -3, 1, -1, 0).
\end{aligned}
$$

□

EXERCISES

Exercise 1.25. Calculate the In-Place Fast Haar Wavelet Transform for the data $\vec{\mathbf{s}} = (\mathbf{2}, \mathbf{4}, \mathbf{8}, \mathbf{6})$.

Exercise 1.26. Calculate the In-Place Fast Haar Wavelet Transform for the data $\vec{\mathbf{s}} = (\mathbf{5}, \mathbf{7}, \mathbf{3}, \mathbf{1})$.

Exercise 1.27. Calculate the In-Place Fast Haar Wavelet Transform for the data $\vec{\mathbf{s}} = (8, 6, 7, 3, 1, 1, 2, 4)$.

Exercise 1.28. Calculate the In-Place Fast Haar Wavelet Transform for the data $\vec{\mathbf{s}} = (3, 1, 9, 7, 7, 9, 5, 7)$.

Exercise 1.29. Assume that for a sample $\vec{\mathbf{s}} = (s_0, s_1, s_2, s_3)$, the *In-Place* Fast Haar Wavelet Transform gives $\vec{\mathbf{s}}^{(2-2)} = (5, -1, 2, 0)$.

(a) In the result $\vec{\mathbf{s}}^{(2-2)} = (5, -1, 2, 0)$, identify the entry that measures the average of the whole sample.

(b) In the result $\vec{\mathbf{s}}^{(2-2)} = (5, -1, 2, 0)$, identify the entry that measures the change from the average over the first half of the sample to the average over the second half.

(c) In the result $\vec{\mathbf{s}}^{(2-2)} = (5, -1, 2, 0)$, identify the entry that measures the change from s_0 to s_1.

(d) In the result $\vec{\mathbf{s}}^{(2-2)} = (5, -1, 2, 0)$, identify the entry that measures the change from s_2 to s_3.

Exercise 1.30. Assume that for a sample $\vec{\mathbf{s}} = (s_0, s_1, s_2, s_3)$, the In-Place Fast Haar Wavelet Transform gives $\vec{\mathbf{s}}^{(2-2)} = (6, 1, -2, -1)$.

(a) In the result $\vec{\mathbf{s}}^{(2-2)} = (6, 1, -2, -1)$, identify the entry that measures the average of the whole sample.

(b) In the result $\vec{\mathbf{s}}^{(2-2)} = (6, 1, -2, -1)$, identify the entry that measures the change from the average over the first half of the sample to the average over the second half.

(c) In the result $\vec{\mathbf{s}}^{(2-2)} = (6, 1, -2, -1)$, identify the entry that measures the change from s_0 to s_1.

(d) In the result $\vec{\mathbf{s}}^{(2-2)} = (6, 1, -2, -1)$, identify the entry that measures the change from s_2 to s_3.

Exercise 1.31. For each sample with four entries $\vec{\mathbf{s}} = (s_0, s_1, s_2, s_3)$, express each entry of its In-Place Haar Wavelet Transform

$$\left(a_0^{(2-2)}, c_0^{(2-1)}, c_0^{(2-2)}, c_1^{(2-1)}\right)$$

with algebraic formulae in terms of the sample (s_0, s_1, s_2, s_3). For example, $a_0^{(2-2)} = (s_0 + s_1 + s_2 + s_3)/4$; derive similar formulae for the remaining three entries, $c_0^{(2-1)}, c_0^{(2-2)}, c_1^{(2-1)}$.

Exercise 1.32. For each sample with four entries $\vec{\mathbf{s}} = (s_0, s_1, s_2, s_3)$, assume that the In-Place Fast Haar Wavelet Transform produces

$$\left(a_0^{(2-2)}, c_0^{(2-1)}, c_0^{(2-2)}, c_1^{(2-1)}\right).$$

Derive an algebraic formula in terms of the result for the average of the first half of the sample, (s_0, s_1). In other words, explain how to compute the average of s_0 and s_1 in terms of

$$\left(a_0^{(2-2)}, c_0^{(2-1)}, c_0^{(2-2)}, c_1^{(2-1)}\right).$$

Exercise 1.33. Assume that for some sample with eight entries

$$\vec{\mathbf{s}} = (s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7),$$

the In-Place Fast Haar Wavelet Transform produces the final result $\vec{\mathbf{s}}^{(3-3)} := (4, -1, -1, 2, 0, 1, -2, -2)$.

(a) Determine the average of the whole sample $\vec{\mathbf{s}}$.

(b) In the array of results, identify the value of $c_1^{(3-2)}$.

(c) In the array of results, identify the indices k and ℓ such that the entry 0 represents $c_k^{(3-\ell)}$. In other words, determine k and ℓ such that $c_k^{(3-\ell)} = 0$.

(d) Determine the average of the second half of the sample, (s_4, s_5, s_6, s_7).

Exercise 1.34. Assume that for some sample with eight entries $\vec{\mathbf{s}} = (s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7)$, the In-Place Fast Haar Wavelet Transform produces the final result $\vec{\mathbf{s}}^{(3-3)} := (5, 1, 1, 0, -3, -1, 0, 1)$.

(a) Determine the average of the whole sample $\vec{\mathbf{s}}$.

(b) In the array of results, identify the value of $c_0^{(3-3)}$.

(c) In the array of results, identify the indices k and ℓ such that the entry -1 represents $c_k^{(3-\ell)}$. In other words, determine k and ℓ such that $c_k^{(3-\ell)} = -1$.

(d) Determine the average of the second half of the sample, (s_4, s_5, s_6, s_7).

Exercise 1.35. Write and test a computer program to compute the In-Place Fast Haar Wavelet Transform.

Exercise 1.36. Write and test a computer program to compute the Ordered Fast Haar Wavelet Transform.

1.5 THE IN-PLACE FAST INVERSE HAAR WAVELET TRANSFORM

As described in the preceding section, the Fast Haar Wavelet Transform neither alters nor diminishes the information contained in the initial array $\vec{\mathbf{s}} = (s_0, \ldots, s_{2^n-1})$, because each basic transform

$$\begin{cases} a_k^{(\ell)} = (\frac{1}{2})\left(a_{2k}^{(\ell-1)} + a_{2k+1}^{(\ell-1)}\right), \\ c_k^{(\ell)} = (\frac{1}{2})\left(a_{2k}^{(\ell-1)} - a_{2k+1}^{(\ell-1)}\right) \end{cases}$$

admits an inverse transform:

$$\begin{cases} a_{2k}^{(\ell-1)} = a_k^{(\ell)} + c_k^{(\ell)}, \\ a_{2k+1}^{(\ell-1)} = a_k^{(\ell)} - c_k^{(\ell)}. \end{cases}$$

Repeated applications of the basic inverse transform just given, beginning with the wavelet coefficients

$$\vec{\mathbf{s}}^{(0)} = \left(a_0^{(n)}, c_0^{(1)}, \ldots, c_{2^{(n-1)}-1}^{(1)}\right),$$

reconstruct the initial array $\vec{\mathbf{s}}^{(n)} = \vec{\mathbf{s}} = (s_0, \ldots, s_{2^n-1})$.

Algorithm 1.19 In-Place Inverse Haar Wavelet Transform.

DATA:	
n	(nonnegative integer)
$\vec{\mathbf{s}} = \vec{\mathbf{s}}^{(0)}$	(array of 2^n numbers)
START.	
$I := 2^{(n-1)}$	(low-pass index increment)
$J := 2 * I$	(pair index increment)
$M := 1$	(pairs of lowest frequency)
FOR $L := n, \ldots, 1$ DO	(loop of basic sweeps)
FOR $K := 0, \ldots, M-1$ DO	(loop of coefficients)

$a_{2k}^{(\ell-1)} := s_{J\cdot K} + s_{J\cdot K+I}$
$a_{2k+1}^{(\ell-1)} := s_{J\cdot K} - s_{J\cdot K+I}$
$s_{J\cdot K} := a_{2k}^{(\ell-1)}$
$s_{J\cdot K+I} := a_{2k+1}^{(\ell-1)}$
END (loop of coefficients)
$J := I$ (halve J)
$I := I/2$ (halve I)
$M := 2 * M$ (double M)
END (loop of basic sweeps)
STOP.
RESULT:
$\vec{\mathbf{s}} = \vec{\mathbf{s}}^{(n)} = (s_0, s_1, s_2, s_3, \ldots, s_{2^n-1})$. □

Example 1.20 For the array of coefficients $\vec{\mathbf{s}}^{(0)} = (5, 4)$, the Fast Inverse Haar Wavelet Transform gives

$$\begin{aligned}
\vec{\mathbf{s}} = \vec{\mathbf{s}}^{(0)} &= (5, 4),\\
I &:= 1,\\
J &:= 2,\\
K &:= 0,\\
a_{2\cdot 0}^{(1)} &= s_{2\cdot 0} + s_{2\cdot 0+1} = 5 + 4 = 9,\\
a_{2\cdot 0+1}^{(1)} &= s_{2\cdot 0} - s_{2\cdot 0+1} = 5 - 4 = 1,\\
\vec{\mathbf{s}} = \vec{\mathbf{s}}^{(1)} &= (a_0^{(1)}, a_1^{(1)}) = (9, 1),
\end{aligned}$$

which correctly reproduces the initial array $\vec{\mathbf{s}}^{(1)} = (9, 1)$. □

Example 1.21 For the wavelet coefficients

$$\vec{\mathbf{s}}^{(0)} = (4, 2, -1, -3),$$

the In-Place Fast Inverse Haar Wavelet Transform gives

$$\begin{aligned}
\vec{\mathbf{s}} = \vec{\mathbf{s}}^{(0)} &= (\mathbf{4}, 2, \mathbf{-1}, -3),\\
I &:= 2,\\
J &:= 4,\\
K &:= 0,\\
a_{2\cdot 0}^{(1)} &= s_{4\cdot 0}^{(0)} + s_{4\cdot 0+2}^{(0)} = 4 + (-1) = 3,\\
a_{2\cdot 0+1}^{(1)} &= s_{4\cdot 0}^{(0)} - s_{4\cdot 0+2}^{(0)} = 4 - (-1) = 5,
\end{aligned}$$

$$
\begin{aligned}
\vec{\mathbf{s}}^{(1)} &= (3, 2, 5, -3),\\
I &:= 1,\\
J &:= 2,\\
K &:= 0,\\
a^{(2)}_{2\cdot 0} &= s^{(1)}_{2\cdot 0} + s^{(1)}_{2\cdot 0+1} = 3 + 2 = 5,\\
a^{(2)}_{2\cdot 0+1} &= s^{(1)}_{2\cdot 0} - s^{(1)}_{2\cdot 0+1} = 3 - 2 = 1,\\
K &:= 1,\\
a^{(2)}_{2\cdot 1} &= s^{(1)}_{2\cdot 1} + s^{(1)}_{2\cdot 1+1} = 5 + (-3) = 2,\\
a^{(2)}_{2\cdot 1+1} &= s^{(1)}_{2\cdot 1} - s^{(1)}_{2\cdot 1+1} = 5 - (-3) = 8,\\
\vec{\mathbf{s}}^{(2)} &= (a^{(2)}_0, a^{(2)}_1, a^{(2)}_2, a^{(2)}_3) = (5, 1, 2, 8),
\end{aligned}
$$

which correctly reproduces the initial array $\vec{\mathbf{s}}^{(2)} = (5, 1, 2, 8)$. □

EXERCISES

Exercise 1.37. Assume that the In-Place Fast Haar Wavelet Transform of a sample $\vec{\mathbf{s}} = (s_0, s_1)$ produces the results $(7, 2)$. Apply the inverse transform to reconstruct the values of the sample, s_0 and s_1.

Exercise 1.38. Assume that the In-Place Fast Haar Wavelet Transform of a sample $\vec{\mathbf{s}} = (s_0, s_1)$ produces the results $(6, -3)$. Apply the inverse transform to reconstruct the values of the sample, s_0 and s_1.

Exercise 1.39. Assume that the In-Place Fast Haar Wavelet Transform of a sample $\vec{\mathbf{s}} = (s_0, s_1, s_2, s_3)$ produces the results $(6, 2, 1, 2)$. Apply the inverse transform to reconstruct the sample $\vec{\mathbf{s}}$.

Exercise 1.40. Assume that the In-Place Fast Haar Wavelet Transform of a sample $\vec{\mathbf{s}} = (s_0, s_1, s_2, s_3)$ produces the results $(4, 2, 2, 0)$. Apply the inverse transform to reconstruct the sample $\vec{\mathbf{s}}$.

Exercise 1.41. Assume that for some sample with eight entries $\vec{\mathbf{s}} = (s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7)$, the In-Place Fast Haar Wavelet Transform produces the result $(4, -1, -1, 2, 0, 1, -2, -2)$. Apply the inverse transform to reconstruct the sample $\vec{\mathbf{s}}$.

Exercise 1.42. Assume that for some sample with eight entries $\vec{\mathbf{s}} = (s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7)$, the In-Place Fast Haar Wavelet Transform produces the result $(5, 1, 1, 0, -3, -1, 0, 1)$. Apply the inverse transform to reconstruct the sample $\vec{\mathbf{s}}$.

Exercise 1.43. Assume that the In-Place Fast Haar Wavelet Transform stops at the end of the ℓth sweep. Explain how to reconstruct the initial sample from the result of the ℓth sweep.

Exercise 1.44. Assume that for some sample with eight entries $\vec{\mathbf{s}} = (s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7)$, the In-Place Fast Haar Wavelet Transform stops at the end of the second sweep and gives

$$\vec{\mathbf{s}}^{(3-2)} := (3, 1, -1, 1, 7, 1, 1, -1).$$

Reconstruct all the values of the sample.

Exercise 1.45. Write a computer program to compute the In-Place Fast Inverse Haar Wavelet Transform. Test the program by computing the In-Place Fast Haar Wavelet Transform and then the In-Place Fast Inverse Haar Wavelet Transform.

Exercise 1.46. Write a computer program to compute the Ordered Fast Inverse Haar Wavelet Transform. Test the program by computing the Ordered Fast Haar Wavelet Transform and then the Ordered Fast Inverse Haar Wavelet Transform.

1.6 EXAMPLES

This section provides a first demonstration of the practical significance of mathematical wavelets with real data. Any other finite sequence of numbers—including random numbers—might serve the same purpose, but the specific contexts demonstrated here may help in providing suggestions for further applications.

1.6.1 Creek Water Temperature Analysis

This example serves mainly to explain the practical significance of wavelet coefficients.

The following sixteen numbers—also plotted in Figure 1.9—represent semi-weekly measurements of temperature, in degrees Fahrenheit, for December 1992 and January 1993 at a fixed common location along Hangman Creek, during a study of riverbank erosion by Mr. Jim Fox, in Spokane, Washington.

$$\begin{array}{c} 32.0, 10.0, 20.0, 38.0, 37.0, 28.0, 38.0, 34.0, \\ 18.0, 24.0, 18.0, 9.0, 23.0, 24.0, 28.0, 34.0 \end{array}$$

The In-Place Fast Haar Wavelet Transform produces the result

$$\begin{array}{rrrr} 25.9375, & 11.0, & -4.0, & -9.0; \\ -4.625, & 4.5, & -1.75, & 2.0; \\ 3.6875, & -3.0, & 3.75, & 4.5, \\ -5.0, & -0.5, & -3.75, & -3.0. \end{array}$$

Equivalently, a rearrangement in increasing frequencies yields the Ordered Fast Haar Wavelet Transform:

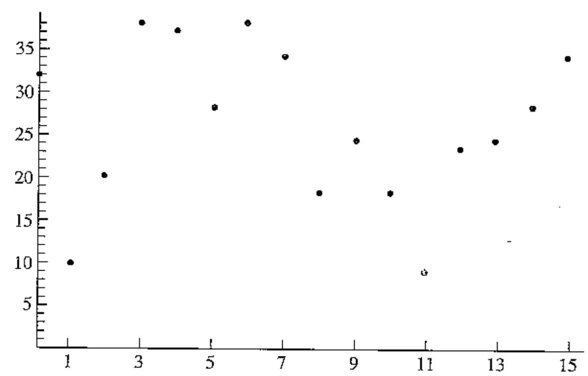


Figure 1.9 Temperature (°F) versus time (half weeks).

$$
\begin{array}{llllllll}
25.9375; \\
3.6875; \\
-4.625 & -1.75; \\
-4.0 & -4.625 & -1.75 & 3.6875; \\
11.0 & -9.0 & 4.5 & 2.0 & -3.0 & 4.5 & -0.5 & -3.0.
\end{array}
$$

The first coefficient, 25.9375, represents the average temperature for the whole two-month period.

The second coefficient, 3.6875, is the coefficient of the longest wavelet over the whole period, which means that the temperature changed by $3.6875 * (-2) = -7.375$, a *decrease* of 7.375°F, from December to January.

The next two coefficients, -4.625 and -1.75, represent similar changes of temperature over the first half (first two quarters) and over the second half (last two quarters) of the period. The coefficient -4.625 corresponds to a change of $-4.625 * (-2) = 9.25$, an *increase* of 9.25°F from the first two weeks to the last two weeks in December. The coefficient -1.75 corresponds to a change of $-1.75 * (-2) = 3.5$, an *increase* of 3.5°F from the first two weeks to the last two weeks in January.

Each of the next four coefficients, -4.0, -4.625, -1.75, and 3.6875, represents a change of temperature over two weeks. For instance, the coefficient -4 means that the temperature *increased* by $-4 * (-2) = 8$°F from the first week to the second week of December.

Finally, each of the last eight coefficients,

$$11.0 \quad -9.0 \quad 4.5 \quad 2.0 \quad -3.0 \quad 4.5 \quad -0.5 \quad -3.0$$

represents a change of temperature over one week. For instance, the coefficient 11 means that the temperature changed by $11 * (-2) = -22$°F during the first week of December; indeed, the data show a drop from 32°F down to 10°F.

As a verification, the In-Place Fast Inverse Haar Wavelet Transform reproduced the data exactly.

EXERCISES

Exercise 1.47. Analyze the following measurements of the ground frost depth, in centimeters, at Qualchan on Hangman Creek, for the same period (also by Mr. Jim Fox).

$$22.0,\ 27.0,\ 48.8,\ 47.5,\ 47.0,\ 48.5,\ 48.0,\ 47.0,$$
$$43.0,\ 41.0,\ 41.0,\ 38.0,\ 36.0,\ 47.1,\ 34.0,\ 32.0.$$

Exercise 1.48. Analyze the following measurements of the ground frost depth, in centimeters, at Kracher on Hangman Creek, for the same period (also by Mr. Jim Fox).

$$12.0,\ 16.0,\ 27.0,\ 32.8,\ 33.5,\ 33.5,\ 39.0,\ 39.0,$$
$$40.0,\ 41.3,\ 41.3,\ 42.0,\ 43.0,\ 45.0,\ 35.5,\ 49.0.$$

Exercise 1.49. Analyze the following measurements of river flow, in cubic feet per second, at the US Geological Survey Data Station 1242400 on Hangman Creek, for the same period.

$$10.0,\ 12.0,\ 12.0,\ 7.0,\ 8.0,\ 9.1,\ 8.2,\ 9.4,$$
$$16.0,\ 15.0,\ 13.0,\ 11.0,\ 6.4,\ 9.0,\ 19.,\ 118.0.$$

Exercise 1.50. Obtain data of any kind and analyze them with the Haar Wavelet Transform.

1.6.2 Financial Stock Index Event Detection

This example demonstrates the automated use of wavelet transforms—here the automated search for coefficients with large magnitudes—to detect events in large data sets.

The top panel in Figure 1.10 displays on the vertical axis the New York Stock Exchange (NYSE) Composite Index, and on the horizontal axis the date, from 2 January 1981 (business day 0) through 7 February 1988 (business day 2047).

The middle panel in Figure 1.10 shows the coefficients of the In-Place Haar Wavelet Transform. The first coefficient, 111.15, in position 0 (superimposed on the vertical axis), represents the average of the index for the entire period: the coefficient of the slowest function, $\varphi_{[0,1[}$, extending over the whole period. Because in this example the value 111.15 has an order of magnitude larger than the values of the other coefficients, the scale of the graph does not reveal the other coefficients as well as does the bottom panel.

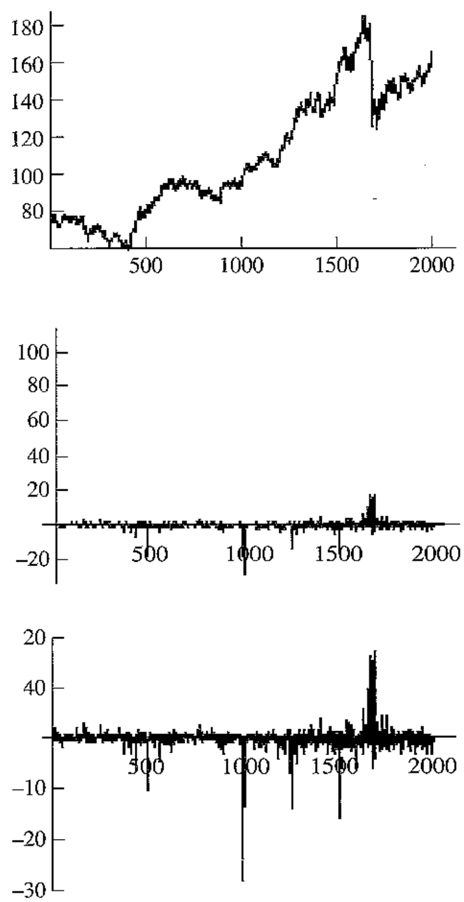


Figure 1.10 New York Stock Exchange Composite Index for 1981–1987. *Top.* Data: index (vertical axis) vs. day (horizontal axis). *Middle.* In-Place Haar Wavelet Transform, average first. *Bottom.* Same transform without average, for details.

The bottom panel in Figure 1.10 shows the coefficients of the In-Place Fast Haar Wavelet Transform in positions from 1 through 2047, both included, but not the average in position 0.

The coefficient with the largest magnitude, -28.96, in position 1024, at the middle of the array of coefficients, corresponds to the slowest wavelet, $\psi_{[0,1[}$, extending over the whole period. Thus, the value -28.96 reflects a *rise* by $(-28.96) * (-2) \approx 58$ points from the first half of the period to the second half of the period (with each half about 3.5-year long).

The next-largest magnitude, 15.255, in position 1717, reflects the *drop* by $15.255 * (-2) \approx 30.5$ points between business days 1716 and 1717: Friday 16 and Monday 19 October 1987.

EXERCISES

Identify the significance of the following wavelet coefficients.

Exercise 1.51. Identify the significance of -10.59 in position 513.

Exercise 1.52. Identify the significance of -14.41 in position 1281.

Exercise 1.53. Identify the significance of -16.26 in position 1537.

Exercise 1.54. Identify the significance of $+15.30$ in position 1716.